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Abstract. The framework of stationarity testing is extended to allow a generic smooth trend function estimated nonparametrically. The asymptotic behavior of the pseudo-Lagrange Multiplier test is analyzed in this setting. The proposed implementation delivers a consistent test whose limiting null distribution is standard normal. Theoretical analyses are complemented with simulation studies and some empirical applications.

Keywords. Time series, stationarity testing, limiting distribution, nonparametric regression, nonparametric hypothesis testing.

JEL classifications: C14, C12, C22.

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1 Introduction and scope of the paper

Since the influential paper by Nelson and Plosser (1982), a considerable amount of research has focused on developing both unit-root and stationarity tests, capable of distinguishing between integrated and stationary-around-a-trend/level stochastic processes. The importance of this topic in economics stems from the fact that a number of statistical and practical policy implications are associated with this distinction, which becomes crucial in applied time series forecasting, where it is well known that difference stationary and trend stationary processes often imply very different forecasts (e.g., Diebold and Killian, 2000). Other application fields include analysis of economic/financial time series, health economics (studies on health expenditure and gross domestic product, e.g. Jewell *et al.*, 2003), hydrology (where testing for stationarity is an important topic, e.g., Wang, 2006, Gelder *et al.*, 2007), or climate change studies (air pollutant emissions —e.g. Dore and Johnston, 2000—, temperatures —e.g. Gay-Garcia *et al.*, 2009—). Furthermore, both unit root and stationarity tests are also regularly applied for model selection.

Most research has focused on unit root testing (a recent overview can be consulted in Harvey *et al.*, 2009). The pioneering contribution by Perron (1989) provided evidence that the standard Dickey-Fuller test can be quite misleading when structural breaks that affect the time series are ignored. Since this evidence, a fruitful research line has produced modified tests that incorporate dummy variables in order to cope with the potential presence of one or more breaks in the series (see Perron, 2006 for a review). Besides, a number of research efforts have focused on smooth transition models. This relates to the widespread recognition that changes which affect many series (e.g., the effects of economic policy, the broadcasting of a news, or the difussion of an epidemy) indeed do not happen instantaneously, but rather produce more or less rapid gradual variations in the series. So, several authors (e.g., Leybourne *et al.*, 1998; Kapetanios *et al.*, 2003; Enders and Granger, 1998; Caner and Hansen, 2001; Sollis, 2004) have proposed tests for the null hypothesis of unit root against the alternative of stationarity around a deterministic trend which experiences gradual adjustment between different regimes.

A complementary approach for distinguishing unit root from stationary series is stationarity testing. Derivation of the latter kind of test is important, as in many cases (e.g., cointegration analysis) it is more interesting to test the null of stationarity. Furthermore, as unit root tests are known to have low power under stationary but highly persistent processes, stationarity tests provide a useful means to confirm results from unit root tests. Stationarity tests have been developed for a number of trend specifications: these include linear models (e.g., Kwiatkowski *et al.*, 1992; Leybourne and McCabe, 1994), polynomials (e.g., Nabeya and Tanaka, 1988), linear trends with breaks (e.g., Lee and Strazicich, 2001; Busetti and Harvey, 2001; Kurozumi, 2002), logistic transition functions (Harvey and Mills, 2004), a generic –piecewise continuous– exogenous trend (Presno and Landajo, 2010) or a non-linear time series model including smooth trend components with unknown parameters (Landajo and Presno, 2010).

A potential drawback of mainstream unit root and stationarity tests stems from their lack of robustness to misspecification of the trend function. When the researcher chooses an incorrect specification for the deterministic trend function, the tests generally produce incorrect results. For instance, under misspecification of the trend component, standard stationarity tests typically diverge in probability as sample size goes to infinity, so a spurious unit root is detected with probability approaching one. This problem may have serious implications for empirical research, as it is widely recognized that trend specification is difficult in practice, and it is fairly optimistic to assume that the researcher always makes the correct guess.

This recognition has lead some authors to device *flexible* tests, that do not depend critically on trend specification. To our knowledge, the earliest proposal was due to Bierens (1997), who developed a unit root test that considers as alternative a random process that is stationary around a trend which belongs to a very general class of functions of time. Bierens's approach relies on approximating the trend function by Chebyshev polynomials, and the test becomes more flexible as more complex Chebyshev polynomials are used. The limiting null distribution of the test statistic, which depends on the complexity of the chosen approximant, is computed by Monte Carlo simulations. Other proposals are due to Enders and Lee (2004), who consider a Fourier approximation using a single frequency component, and to Rodrigues and Taylor (2009), who apply this strategy to generalise the local GLS de-trending unit root test.

In the field of stationarity testing, Becker *et al.* (2006) proposed two *flexible* tests. The first test relies on approximating the unknown trend function by a model including an intercept, a linear trend and two trigonometric components, namely a sine and cosine function, whose frequency is chosen to maximize goodness of fit. The limiting null distribution of the test, which is nonstandard, is derived in the same paper. The test performs quite well in simulations, providing robustness against breaks of unknown form and number, as the chosen specification is capable of mimicking a great many different trend forms. A potential limitation stems from the fact that most smooth functions have Fourier expansions with an infinite number of frequencies, so the approximation capabilities of the chosen specification are limited, at least in theory. The same paper also proposes the *cumulative frequency* test, which provides further flexibility as it relies on (finite) trigonometric polynomials. The null distribution of the test, which as in Bierens's approach, depends on the order of the trigonometric polynomial, is computed in the same paper by Monte Carlo simulations. The cumulative frequency test relies on a *fixed* (though possibly high order) trigonometric polynomial, that may be enough for many applications; however, as the test relies on a single fixed parametric structure, its representation capabilities are limited to functions with a finite number of nonnull terms in their Fourier expansion, so in this implementation the test is flexible, but not nonparametric.[2] This draw-

[2]A forceful limitation of the allowed model complexities may be helpful in order to enhance the power of the test in small samples, although it may also induce oversize in large samples, as the

back is well known in nonparametric statistics. A classical remedy is provided by the method of sieves (Grenander, 1981): by using an increasing sequence of parametric models, whose complexity grows with sample size at appropriately limited rates, the method delivers consistent nonparametric estimation and hypothesis testing in very general settings (e.g., Hong and White, 1995).

The sieve principle can also be exploited in stationarity testing. Briefly stated, a *flexible* parametric stationarity test may be rendered properly nonparametric by a suitable modification, namely, by nesting stationarity testing into an appropriately sieved structure. This is the main goal of the paper: a nonparametric stationarity test is proposed, and its asymptotics and empirical behavior is studied. The conclusions of the analysis are fairly general. At first glance, the outcome of the analysis is simple, as it closely resembles its parametric counterparts, including derivation of the limiting null distribution of the test and asymptotic power analyses. The differences with respect to conventional stationarity testing are of technical nature, as relevant statistical inferences are now carried out upon the basis of an increasing sequence of approximate models for the trend function, whose complexity is indexed by sample size, instead of relying on a single fixed structure, as was the case in standard stationarity tests. From the standpoint of empirical research, the proposed test appeals as its implementation is simple, and the test has a standard limiting null distribution, unlike most stationarity and unit root tests. As to the scope of applications, the nonparametric test is, in our view, most suitable for long series sampled at high frequency as available, e.g., in financial econometrics.

The proposed test relies on nonparametric least squares estimation of the trend component, which is carried out through trigonometric series regression. Our approach has the following features: (**a**) we focus on smooth trends which can be approximated with arbitrary accuracy, in mean-squared sense, by linear combinations

effects of specification bias asymptotically are nonvanishing. It is the researcher's choice which of these effects he/she wishes to control best in a specific application.

of the elements of a cosine basis (in principle, any squared integrable function on $[0,1]$ has this property). (**b**) The stochastic part of the null model is generated by a linear filter process (the performance of the test under several nonlinear time series models is also studied in simulations). (**c**) The behavior of the test when the long-run variance of the process is estimated upon the residuals of the nonparametric regression is studied. Finally, (**d**) the proposed test asymptotically has the correct size, and is consistent under unit root alternatives, with its limiting null distribution being standard normal, which enables a fairly simple implementation in practice.

As compared with previous stationarity tests, this proposal provides a number of interesting features: **(i)** the proposed test is fully nonparametric. It relies on a sieve mechanism which ensures both consistent estimation of the trend function and asymptotically correct behavior of the stationarity test. **(ii)** The limiting null distribution of the (suitably rescaled) test is standard, unlike those of most unit root and stationarity tests, whose distributions are nonstandard as well as model-dependent, in the sense that they range with each trend specification. The expedient of rescaling the test statistic avoids the burden of computing (usually, by Monte Carlo simulations) a different set of critical values for each choice of model complexity. (**iii**) The analytical results in this paper are valid for the case when the data are driven by linear processes, so our analyses are not limited to the i.i.d. context. **(iv)** The complicated issue of estimating the long-run variance of the process in the nonparametric environment is addressed analytically. A considerable amount of –both theoretical and empirical– research has been devoted to this topic both in the unit root and stationarity testing literature (see references in Sections 3 and 5 below). It seems natural to rely on these efforts as the starting point in the nonparametric context. Unfortunately, validity of the available results on long-run variance estimation does not follow automatically in the nonparametric context. This is a consequence of the fact that the nonparametric estimators for the trend function generally converge at slower rates than their parametric counterparts. This affects the properties of usual estimators for the long-run variance, which have stochastic orders that may differ from those appearing in the parametric unit root/stationarity testing literature. In practice, this means that some strategies which are valid in the parametric context may perform poorly in the nonparametric setting, or even induce inconsistency of the tests. A proposal (namely, a class of kernel estimators for the long-run variance of the process) is provided in the paper, and its theoretical behavior is analysed, including derivation of appropriate rates for bandwidth increase in the nonparametric setting. The estimator for the long-run variance of the process is computed upon the residuals of nonparametric regression. We provide further flexibility by the expedient of allowing possibly different model complexities for the numerator and denominator of the test statistic. To our knowledge, the possibility of this separate treatment of both components in the test statistic has not been exploited in the literature, although our simulations indicate that this (slightly unconventional) strategy may be useful in certain cases, where a more complex model may be advisable in the numerator of the statistic —e.g., this may induce undersmoothing, so reducing the bias of the nonparametric trend estimator, and allowing better control of the test's size— than in the denominator, where stronger complexity control may be useful to better estimate the long-run variance of the process. In practice it is difficult to provide "optimal" rules for such a general testing problem, as performance depends on features such as the oscillatory behavior of the trend function and the stochastic properties of the error processes involved. In Section 3 below some empirical rules of thumb are proposed which conform with the allowed theoretical rates, and appear to provide sensible performance in a number of time series processes.

The rest of the paper is structured as follows: in Section 2 the nonparametric test is introduced, and its limiting behavior is analysed. In Section 3 the finite sample properties of the test are investigated and, in Section 4, some empirical applications are presented. The paper closes with a summary of conclusions. All the mathematical derivations are collected in the Appendix.

2 A nonparametric stationarity test

The model

The following error-components model may be used as a general framework:

$$y_{t,T} = \mu_t + \theta^* (t/T) + \varepsilon_t,$$
$$\mu_t = \mu_{t-1} + u_t\,; \;\; t = 1, ..., T; \;\; T = 1, 2, ... \tag{1}$$

with $\theta^* : [0,1] \to \mathbb{R}$ being a smooth function (i.e., a trend). We consider approximants to θ^* of the form $\theta_m(u) = \sum_{j=0}^{m} \beta_{j,m}\varphi_j(u)$, with $\beta_m = (\beta_{0,m}, ..., \beta_{m,m})' \in \mathbb{R}^{m+1}$, $\varphi_0(u) \equiv 1, \varphi_j(u) = \sqrt{2}\cos(j\pi u),\ j \geq 1,\ u \in [0,1]$. (The basis $\{\varphi_j, j = 0, 1, ...\}$ is a complete and orthonormal in $L_2[0,1]$.) We let $m = m_T$ grow to infinity with sample size T at an appropriate rate and assume that θ^* is the limit of $\{\theta_{m_T}\}$ under the metric $d_T(\theta_{m_T}, \theta^*) \equiv \sqrt{T^{-1}\sum_{t=1}^{T}\left[\theta_{m_T}(t/T) - \theta^*(t/T)\right]^2}$. This holds for any function in $L_2[0,1]$, although further smoothness conditions on θ^* will be imposed below.
In addition, $\{\varepsilon_t\}$ and $\{u_t\}$ are independent zero-mean error processes with characteristics to be detailed below and respective (finite) variances $E\left(\varepsilon_t^2\right) = \sigma_\varepsilon^2 > 0$ and $E\left(u_t^2\right) = \sigma_u^2 \geq 0$; $\{\mu_t\}$ starts with μ_0, which is assumed to be zero.

Under the assumption that the parameters entering the model nonlinearly are known, Lagrange Multiplier (LM) stationarity testing relies on the following setting:

$$H_0 : q \equiv \frac{\sigma_u^2}{\sigma_\varepsilon^2} = 0, \qquad H_1 : q > 0 \tag{2}$$

In standard stationarity testing, a parametric model for the trend function, such as $\theta^*(u) = \beta_0 + \beta_1 u$, is specified in advance, and the LM statistic to test (2) has the well-known expression:

$$S_T = \widehat{\sigma}^{-2}T^{-2}\widehat{\boldsymbol{\varepsilon}}'\boldsymbol{A}_T\widehat{\boldsymbol{\varepsilon}} = \widehat{\sigma}^{-2}T^{-2}\sum_{t=1}^{T}E_t^2, \tag{3}$$

where $\widehat{\boldsymbol{\varepsilon}}$ is the $T \times 1$ vector of OLS residuals (we suppress double indexing for notational simplicity), $\boldsymbol{A_T} = [a_{jk}]$, with $a_{jk} = \min(j,k)$; $j,k = 1,\ldots,T$. Below we apply the standard decomposition $\boldsymbol{A}_T = \boldsymbol{C}_T\boldsymbol{C}_T'$, with $\boldsymbol{C}_T$ being a $T \times T$ lower triangular matrix of ones. $E_t = \sum_{i=1}^{t}\widehat{\varepsilon}_i$ denotes the forward partial sum of the residuals, and $\widehat{\sigma}^2$ is a suitable estimator for the long-run variance of $\{\varepsilon_t\}$, to be denoted as σ^2 and assumed non-null.

We will analyze the behavior of the above stationarity test when the trend function θ^* is estimated nonparametrically and the test is carried out upon the residuals of this regression. We consider the estimator $\widehat{\theta}_{m_T}(u) = \sum_{j=0}^{m_T}\widehat{\beta}_j\varphi_j(u)$, with $\widehat{\beta}_{m_T} = (\widehat{\beta}_0,...,\widehat{\beta}_{m_T})'$, which (given m_T) is computed by OLS regression, i.e., $\widehat{\beta}_{m_T} = (\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'\mathbf{y}$, with $\boldsymbol{\Phi} = [\varphi_{t,j}]$, $\varphi_{t,j} = \varphi_j\,(t/T)$, $t = 1,...,T; j = 0,...,m_T$; we may write $\boldsymbol{\Phi} = [\varphi_1,...,\varphi_m]$, with $\varphi_j = [\varphi_j\,(1/T)\,,...\varphi_j\,(T/T)]'$.

The pseudo-LM test statistic has the usual form:

$$\widehat{S}_T = \widehat{\sigma}^{-2}T^{-2}\boldsymbol{e}'\boldsymbol{A}_T\boldsymbol{e} \tag{4}$$

with $\boldsymbol{e} = (e_1,\ldots,e_T)'$ being the vector of OLS residuals from the above nonparametric regression.

We shall follow the usual conventions: the symbol " $\xrightarrow{L}$ " indicates convergence in distribution, " $\xrightarrow{p}$ " denotes convergence in probability, and symbols O_p and o_p are used with their usual probability order meanings, as $T \to \infty$ with respect to the probability measure P.

Asumptions

We consider model (1) under the following assumptions:

Assumption 1. (i) The underlying probability space $(\Omega, \mathbb{F}, P)$ is complete, and the unobservable error process $\{\varepsilon_t\}$ is generated as $\varepsilon_t = \sum_{j=0}^{\infty} \alpha_j v_{t-j}$, with $\sum_{j=1}^{\infty} |\alpha_j| < \infty$ and $\alpha \equiv \sum_{j=0}^{\infty} \alpha_j \neq 0$. (ii) The process $\{v_t \mid t = 1, 2, ...\}$ is independent identically distributed (i.i.d.), with $E(v_t) = 0$, $var\,(v_t) = \sigma_v^2 > 0$ and $E\,|v_t|^r < \infty$ for some $r > 2$. (iii) The process $\{u_t\}$ is independent of $\{v_t\}$, has $E(u_t) = 0$, $Var\,(u_t) = \sigma_u^2 \geq 0$, $E\,|u_t|^{2+\delta} < \infty$ for some $\delta > 0$, and $\sum_{t=1}^{T} u_t = O_p(T^{1/2})$.

Assumption 2. As $T \to \infty$, (i) $m_T^{3/2} T \; d_T\,(\theta_{m_T}, \theta^*) \to 0$, with (ii) $m_T \to \infty$ and $m_T^{9/2} T^{-1} \to 0$.

Assumption 3. $\widehat{\sigma}^2 \geq 0$ and, as $T \to \infty$, (i) $m_T^{1/2}\,(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{p} 0$ under H_0, and (ii) under H_1, $\widehat{\sigma}^2 = O_p\left(T^{\zeta}\right)$, $0 \leq \zeta < 2$. $\square$

Results

First we derive the limiting behavior of the (standardized) test when the long-run variance is known.

Proposition 1. Under Assumptions 1 to 2, let $Z_T = s_{m_T}^{-1}\,(\sigma^{-2} S_T - \mu_{m_T})$, where $S_T = T^{-2} \mathbf{e}' \mathbf{C}_T' \mathbf{C}_T \mathbf{e}$, $\mathbf{e} = \left(\mathbf{I}_T - \mathbf{\Phi}\,(\mathbf{\Phi}'\mathbf{\Phi})^{-1}\,\mathbf{\Phi}'\right) \mathbf{y}$, $\mathbf{\Phi} = [\varphi_{t,j}]$, $\varphi_{t,j} = \varphi_j\,(t/T)$, $t = 1, ..., T$, $j = 0, ..., m_T$, $\mu_{m_T} = \sum_{j=m_T+1}^{\infty} (j\pi)^{-2}$ and $s_{m_T}^2 = 2 \sum_{j=m_T+1}^{\infty} (j\pi)^{-4}$. Then as $T \to \infty$:

(a) under H_0, $Z_T \xrightarrow{L} N(0,1)$, and

(b) under H_1, $P\,(Z_T > \kappa_T) \to 1$ for any nonstochastic sequence $\{\kappa_T\}$ with $\kappa_T m_T^{-3/2} T^{-2} \to 0$. $\square$

An analogous result follows when σ^2 is estimated from data.

Proposition 2. Under Assumptions 1 to 3, let $\widehat{Z}_T = s_{m_T}^{-1}\left(\widehat{\sigma}^{-2}S_T - \mu_{m_T}\right)$. Then as $T \to \infty$:
(**a**) under H_0, $\widehat{Z}_T \xrightarrow{L} N(0,1)$, and
(**b**) under H_1, $P\left(\widehat{Z}_T > \kappa_T\right) \to 1$ if $\kappa_T m_T^{-3/2} T^{-(2-\zeta)} \to 0$. $\square$

Assumption 3 requires a suitable estimator for σ^2. We follow Pötscher and Prucha (1991), and results are stated for nonparametric estimators with kernel W belonging to the class $\mathbb{W}_\rho$, of functions $W : \mathbb{R} \to [-1,1]$ satisfying $W(0) = 1$, $W(-x) = W(x)$ for all x, $W(x) = 0$ for $|x| > 1$ and $\lim_{x\to 0} |W(x) - 1| / x^\rho < \infty$ for some $\rho > 0$.

The following result states that Proposition 2 holds if σ^2 is replaced by a nonparametric estimator with kernel W belonging to the class $\mathbb{W}_\rho$.

(The truncation estimator is embedded into the scheme of Proposition 2 by using $|\widehat{\sigma}^2|$ instead of $\widehat{\sigma}^2$.)

Proposition 3. Under Assumptions 1 to 3, let $\widehat{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}\widehat{\sigma}_i \geq 0$, with $\widehat{\sigma}_i = T^{-1}\sum_{t=1+|i|}^{T} e_t^{(d)} e_{t-|i|}^{(d)}$, $\mathbf{e}^{(d)} = \left(e_1^{(d)}, ..., e_T^{(d)}\right)' = \left(\mathbf{I}_T - \mathbf{\Phi}_d\left(\mathbf{\Phi}_d'\mathbf{\Phi}_d\right)^{-1}\mathbf{\Phi}_d'\right)\mathbf{y}$, $\mathbf{\Phi}_d = [\varphi_{t,j}]$, $\varphi_{t,j} = \varphi_j(t/T)$, $t = 1, ..., T$, $j = 0, ..., m_T^{(d)}$, and $w_{i,T} = W\left(i/\left(1+\ell_T\right)\right)$, with kernel $W \in \mathbb{W}_\rho$. If the following conditions hold: (*i*) either (*i.1*) $E|v_t|^4 < \infty$ or (*i.2*) $E|v_t|^r < \infty$, with $2 < r < 4$ and $\alpha_j = O\left(j^{-(1+q_v+\epsilon)}\right)$, where $\epsilon > 0$ and $-q_v \leq \min\{-2(r-1)/(r-2), -(\rho+1)\}$, (*ii*) $\left(m_T^{(d)}\right)^{9/2} T^{-1} \to 0$, (*iii*) $m_T \ell_T^2 m_T^{(d)} T^{-1} \to 0$, (*iv*) $m_T\ \ell_T^3 T^{-1} \to 0$, $m_T \ell_T^{-2\rho} \to 0$, (*v*) $m_T^d \to \infty$, $\ell_T \to \infty$, then as $T \to \infty$:
(**a**) under H_0, $m_T^{1/2}\left(\widehat{\sigma}^2 - \sigma^2\right) \xrightarrow{p} 0$, and
(**b**) under H_1, $\widehat{\sigma}^2 = O_p\left(\ell_T T\right)$ and $P\left(\widehat{Z}_T > \kappa_T\right) \to 1$ if $\kappa_T \ell_t m_T^{-3/2} T^{-1} \to 0$. $\square$

3 Monte Carlo study

In this section we first provide computer simulation results for the performance of the test under i.i.d. errors, and afterwards the research is extended to time series.

3.1 Simulations in i.i.d. environment

We analyse the following trend specifications:

(**A**) $\theta^*(u) \equiv 0$.

(**B**) $\theta^*(u) \equiv 1 + 2u + 3u^2$.

(**C**) $\theta^*(u) \equiv 1 + 2u + 3\left[1 + \exp\{-50(u - 0.3)\}\right]^{-1} - 4\left[1 + \exp\{-40(u - 0.6)\}\right]^{-1}$.

(D) $\theta^*(u) \equiv 1 + 2u + 2\left[1 + \exp\{-\gamma(u - 0.3)\}\right]^{-1}$, with $\gamma = 20, 50, 100$.

(E) $\theta^*(u) \equiv 1 + 2u + 2\left[1 + \exp\{-\gamma(u - 0.3)(u - 0.6)\}\right]^{-1}$, for $\gamma = 20, 50, 100$.

(F) $\theta^*(u) \equiv 1 + 2u + 2\left[1 - \exp\{-\gamma(u - 0.3)^2\}\right]$, where $\gamma = 20, 50, 100$.

(G) $\theta^*(u) \equiv 1 + 2u + 2(u) \times \mathbf{1}(u > 0.3)$.

(H) $\theta^*(u) \equiv 1 + 2u - 1(u) \times \mathbf{1}(u > 0.3)$.

(I) $\theta^*(u) \equiv 1 + 2u - 3(u - 0.3) \times \mathbf{1}(u > 0.3) + 4(u - 0.6) \times \mathbf{1}(u > 0.6) - 5(u - 0.8) \times \mathbf{1}(u > 0.8)$.

with $u \in [0, 1]$ and $\mathbf{1}(\bullet)$ denoting the indicator function.

Specification (A) corresponds to a level plus noise model under the null hypothesis, and a random walk with noise model under the alternative, while (B) allows us to analyze an explosive deterministic trend. Specification (C) represents an artificial neural network trend, or equivalently, a linear trend affected by two smooth transitions of large magnitude which are modelled by logistic sigmoids.

Sigmoid functions are very flexible and allow the analysis of series with gradual changes, which are very common in fields such as economics and climatology. So, several classes of sigmoid curves are considered in specifications (D)-(F). In particular, we used a transition function of the form $\phi_k(u) = \left[1 + \exp\left\{-\gamma \prod_{i=1}^{k}(u - \lambda_i)\right\}\right]^{-1}, \gamma > 0, k = 1, 2$, where γ controls the speed of change, and the λ_is determine the location of the transition. When $k = 1$ this curve allows us to characterize asymmetric behavior of two regimes, with the transition between them being more rapid as γ increases. For $k = 2$ the behavior of the function is roughly similar both for large and small values of u, and different around the midpoint $(\lambda_1 + \lambda_2)/2$, displaying a symmetric change about it. We also considered the transition function $[1 - \exp\{-\gamma(u - \lambda)^2\}], \gamma > 0$, which approximates $\phi_k(u)$ when $k = 2$ and $\lambda_1 = \lambda_2$. All these structures have been widely used in smooth transition regression models, especially in economics, where they have been applied to research the validity of the purchasing power parity hypothesis, and to model issues such as nonlinear behavior of inflation, money demand or asymmetric behavior of macroeconomic variables (such as industrial production and unemployment rates). Smooth transition models have also been applied to analyse the usefulness of the interest rate spread in predicting output growth (for a review of smooth transition models and their application in economics see Teräsvirta, 2005).

Finally, specifications (G)-(I) correspond to trends with breaks.

Figure 1 below displays all the above trend specifications.

—INSERT FIGURE 1 ABOUT HERE—

In simulations we considered sample sizes $T = 100, 300, 500, 1000, 1500, 2000$, and signal-to-noise ratio values $q = 0, 0.01, 0.1$. In the i.i.d. analysis, simulations were based on 5,000 replications, with the processes u_t and ε_t being $N(0, 1)$. We applied the deterministic rule $m_T = m_T^d = \left[5T^{1/5}\right]$. This rate of increase is somewhat slower than others which appear in related (mainly cross-sectional) nonparametric regression contexts, although the combined requirements of Assumptions 2 and 3 advised us against being too liberal in this respect. The variance of the process was estimated by using the "unbiased" estimator $\widehat{\sigma}^2 = \left(T - m_T^d - 1\right)^{-1} e^{(d)\prime}e^{(d)}$, as our simulations indicated that, in small samples, this estimator outperformed the (asymptotically equivalent) plug-in estimator $\widehat{\sigma}^2 = T^{-1}e^{(d)\prime}e^{(d)}$ when included in the nonparametric stationarity test. (It can be readily checked that the asymptotic results in Proposition

3 remains valid if these "unbiased" estimators for the autocovariances are used; we omit derivations for brevity.)

Table 1 displays the rejection rates at 5% significance level, with the critical value provided by the $N(0,1)$ limiting distribution.

—INSERT TABLE 1 ABOUT HERE—

Results indicate that size is close to the nominal significance level (excepting case D with $\gamma = 100$, where a slight oversize is observed) and power figures are very similar for all the trend specifications considered. So, the test seems to perform suitably under a wide spectrum of smooth trend specifications, as well as some structural break models, thus being free from the overrejection problems caused by misspecification of the trend function.

Next we compared the performance of the nonparametric test with that of the two flexible stationarity tests (hereafter BEL1 and BEL2) proposed by Becker *et al.* (2006). As commented above, BEL1 approximates the unknown trend function (the paper focuses mainly on smooth breaks of unknown form and number) by a single selected frequency component from its classical Fourier expansion. The recommendation is to select the frequency (with a maximum of 5) which minimizes the sum of squared residuals. The BEL2 (or cumulative frequency) test also relies on classical Fourier series: the trend function is estimated by least squares regression on a basis of sines and cosines, and in order to avoid power loss, Becker *et al.* (2006) recommend that at most the first two frequencies be included. We checked the performance of the BEL tests —including a linear trend component, τ_τ test— for trend specifications (A)-(I) and sample sizes: $T = 100, 500, 1000, 2000$. Results are reported in Table 2 below.

—INSERT TABLE 2 ABOUT HERE—

The BEL2 test displays better control of the test size than BEL1, but less power. These results agree with Becker *et al.* (2006) conclusions. However, for the trend specifications considered in this paper, even the BEL2 strategy suffers size distortions in some of the studied cases. The magnitude of oversize is reasonable in small sample sizes, but becomes large as sample size increases (indeed, size approaches 1). On the other hand, the BEL tests shows more power than the nonparametric test for $T \leq 1000$, but for larger sample sizes the power of both tests becomes very similar. This could be expected, as the nonparametric test includes more frequencies terms in order to more tightly control test size.

3.2 Simulation analysis in time series

Then we investigated the finite sample properties of the test under more general error processes. In particular, ε_t was generated according to the following DGPs:

1. AR(1) model: $\varepsilon_t = \rho\varepsilon_{t-1} + \upsilon_t$,with $\rho = 0.5, 0.2, 0, -0.2$.

2. MA(1) model: $\varepsilon_t = \upsilon_t + \rho\upsilon_{t-1}$, for $\rho = 0.5, 0.2, -0.2$.

   We also investigated four nonlinear processes analyzed by Hong and Lee (2003) and Escanciano (2006), namely:

3. AR(1) model with heteroskedasticity (ARHET): $\varepsilon_t = \rho_1\varepsilon_{t-1} + h_t\upsilon_t; h_t^2 = 0.1 + 0.1\varepsilon_{t-1}^2 + \rho_2\varepsilon_{t-2}^2$.

4. AR(1) model plus a bilinear term (AR-BIL): $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-1}\upsilon_t + \upsilon_t$.

5. Bilinear model (BIL): $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2}\upsilon_{t-1} + \upsilon_t$.

6. Nonlinear moving average model (NLMA): $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\upsilon_{t-1}\upsilon_{t-2} + \upsilon_t$.

For these last models we considered $\rho_1 = 0.5$ and $\rho_2 = 0.1$. Again, the basis processes $\{u_t\}$ and $\{v_t\}$ were i.i.d. $N(0, 1)$.

For time series only (A)-(C) trend specifications were analysed and 2,000 replications were carried out. Also, the slightly more conservative rule $m_T = \left[4T^{1/5}\right]$,

$m_T^{(d)} = \left[0.85 \times 4T^{1/5}\right]$ was applied. The choice $m_T^d \leq m_T$ was not mandatory in light of the theoretical results above, although in our simulations it enabled easier control of size.

In this case it is necessary to treat autocorrelation. A number of papers have proposed several methods for long-run variance estimation, and analysed the finite sample behavior of the stationarity test under these proposals (e.g., Kwitkowski *et al.*, 1992; Kurozumi, 2002; Hobijn *et al.*, 2004; Sul *et al.*, 2005). In our simulations these methods did not provide satisfactory results. This is not surprising considering Propositions 1 to 3 above, as the probability orders of most magnitudes differ somewhat from their analogues in parametric stationarity testing, this mainly being a consequence of the slower convergence rates of the nonparametric estimators for the trend function, as compared with their parametric counterparts. This directly affects the probability orders which are relevant in long-run variance estimation, which implies that standard corrections for autocorrelation may be generally invalid in the nonparametric setting. Theoretical analyses are required to establish the validity of (and the required modifications in) these standard approaches in nonparametric stationarity testing.

The poor performance of mainstream autocorrelation treatments in our nonparametric setting led us to devise a new strategy. We combined (in a somewhat *ad hoc* fashion) the technical apparatus of Proposition 3 above with methods adapted from previous approaches. Our proposal is oriented to ensure a suitable performance of the nonparametric test in most common applications, although it should only be seen as a reasonable starting point, and further refinements (both in asymptotic theory and empirical research) should be pursued in future research. As estimator for σ^2 we used the following truncation estimator (i.e., a kernel estimator with rectangular kernel): $\widehat{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} \left(T - |i| - m_T^d - 1\right)^{-1} \sum_{t=1+|i|}^{T} e_t^{(d)} e_{t-|i|}^{(d)}$. In order to select the bandwidth parameter ℓ_T, a data-driven rule was applied. We considered values in the interval $\ell_T^{(-)} \leq \ell_T \leq \ell_T^{(+)}$, with $\ell_T^{(-)}$ and $\ell_T^{(+)}$ being deterministic limits fixed in advance. By imposing $\ell_T^{(-)} \to \infty$ and (e.g.) $\ell_T^{(+)} = [cT^{1/5}]$ $(c > 0)$, the rate $\ell_T = O_p\left(T^{1/5}\right)$ is achieved and consistency of the test is ensured in many common applications. The following scheme, adequate for AR(1) or MA(1) error processes, is then applied to obtain ℓ_T (as Kurozumi's (2002) rule, the scheme uses a tuning parameter, k; in our simulations we set $k = 0.5$; higher values are recommended in case of stronger autocorrelation):

1. Set $\ell_T^{(-)}$ and $\ell_T^{(+)}$. (Here, $\ell_T^{(-)} = 0$, $\ell_T^{(+)} = \left[2T^{1/5}k\right]$ were fixed.). Set K_T, the maximum lag order permitted in AR fitting (here, $K_T = \left[2T^{1/5}k\right]$ was used).

2. Fit AR(p) models to the residual vector $\mathbf{e}^{(d)}$, with $p = 0, ..., K_T$, and select the order p^* that minimizes Schwarz's information criterion (SIC). Go to step 3.

3.a. If $p^* = 0$, set $\ell_T = \ell_T^{(-)}$.

3.b. If $p^* = 1$, set $\ell_T = \min\left([20\,|b_T|\,k]\,, \ell_T^{(+)}\right)$, with b_T being the regression coefficient of the fitted AR(1) model.

3.c. If $p^* > 1$, compute sample autocorrelations (r_i) of $\mathbf{e}^{(d)}$, with $i = 1, ..., \ell_T^{(+)}$, and select i^* such that $|r_{i^*}| = \max_{i=1,...,\ell_T^{(+)}}\{r_{i^*}\}$. Set $\ell_T = \min\left(\max(i^*, p^*), \ell_T^{(+)}\right)$.

The above procedure performed well under a wide range of circumstances (several specifications of the trend, stochastic characteristics of the process, autocorrelation levels). Further refinements would be available in specific cases where more detailed knowledge of the nature of the error process is available *a priori*. Results are reported in Tables 3 and 4 below.

—INSERT TABLES 3 AND 4 ABOUT HERE—

As compared with the i.i.d. case, some impairment of the finite sample perfomance of the test is observed, with slight distortions in size and loss of power. As expected, the behavior of the test improves as sample size increases (it starts to be reasonable for series with $T \geq 500$). Regarding the error processes considered, we observed that the test works better in series with nonnegative autocorrelations.

4 Empirical applications

To illustrate the application of the test, we investigated three economic time series: the daily series of the Japanese yen/US dollar exchange rate, the FTSE Eurotop 100 index, and the labor force participation rate. For all of them large samples are avalaible, which is desirable in order to attain good properties in terms of power, as seen in the previous Section. In addition, some of these series are known to display nonlinear features which distort the behavior of standard unit root/stationarity tests. These series have been analysed in previous works, which allows us to make comparisons.

The daily Japanese yen/US dollar exchange rate series

The data range from July 7, 2002 to July 7, 2007 (1,827 observations) and were analyzed, together with other exchange rate series, by Brooks (2008) in the context of vector autoregressive estimation. These financial series —or their first differences— appear to exhibit nonlinear patterns (e.g., Mills and Markellos, 2008) of the kind analyzed in the above Section.

First we outline the details of stationarity testing for this exchange rate series. The deterministic rule $m_T = \left[4T^{1/5}\right]$ was applied, according to results from Section 3. As sample size is $T = 1,827$, this gave $m_T = 17$. Figure 2 below displays the series and its fitted trend, under this model complexity. The numerator of the test statistic is straightforwardly computed upon the residuals of the OLS regression $y_t = \sum_{j=0}^{17} \widehat{\beta}_j \sqrt{2} \cos(j\pi t/T) + e_t$. This gives $S_T = 0.377$ for the raw "KPSS" statistic.

—INSERT FIGURE 2 ABOUT HERE—

For estimation of σ^2 we used the rule outlined in previous Section. In order to have better control over the size of the test, the residuals used to compute $\widehat{\sigma}^2$ were obtained from a slightly simpler cosine regression with $m_T^{(d)} = \left[0.85 \times 4T^{1/5}\right]$, and

the (rectangular-kernel) estimator was applied, i.e.,
$\widehat{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} \left(T - |i| - m_T^d - 1\right)^{-1} \sum_{t=1+|i|}^{T} e_t^{(d)} e_{t-|i|}^{(d)}$, with $\ell_T = 3$ obtained by applying the rule outlined in Section 3 above. This gives $\widehat{\sigma}^2 = 17.9819$.

In order to facilitate the application of the test, Table 5 below displays the rescaling factors μ_{m_T} and s_{m_T} for a representative range of values of m_T. In our case, as $m_T = 17$, the table gives $\mu_{m_T} = 0.00579$ and $s_{m_T} = 0.00113$. Therefore, the value of the rescaled statistic is $\widehat{Z}_T = s_{m_T}^{-1} (\widehat{\sigma}^{-2} S_T - \mu_{m_T}) = 13.446$ which, by checking the $N(0, 1)$ distribution, indicates that the null of stationarity around a deterministic trend is rejected at the critical level $p = 0.000$.

—INSERT TABLE 5 ABOUT HERE—

In a second stage the analysis was extended to the first difference of the series, for which the fitted trend was computed again under $m_T = 17$. In this case $\widehat{Z}_T = 0.122$ (critical level $p = 0.451$), with $\ell_T = 0$ selected by the data-driven device. So, the null of stationarity cannot be rejected.

In empirical applications it is strongly advisable to carry out a sensitivity analysis in order to assess the robustness of the test's results under moderate variations of m_T and ℓ_T. Table 6 displays this analysis for the daily Japanese yen/US dollar exchange rate series, which indicates that conclusions remain unaffected.

—INSERT TABLE 6 ABOUT HERE—

These results, together with the output from mainstream unit root tests, coincide to suggest that the daily Japanese yen/US dollar exchange rate series has a single unit root. This conclusion is also in accordance with predictions from financial theory.

The daily series of the FTSE Eurotop 100 index

Then we analysed the close prices of the daily series of the FTSE Eurotop 100 index spanning the period from 17 December 2002 to 30 October 2009 (1,739 observations) (avalaible at http://www.fin-rus.com/analysis/export_eng_/default.asp). This index represents the performance of the 100 most highly capitalised blue chip companies on european stock markets.

Figure 3 plots the logs of the series and the fitted trend, which was computed for $m_T = 17$. For estimation of the long-run variance the data driven device selected $\ell_T = 5$, and the observed value of the test statistic was $\widehat{Z}_T = 5.449$ (critical level $p = 0.000$). On the contrary, the extension of the analysis to the first difference of the series (i.e., the return series) indicated that the null of stationarity around a deterministic trend cannot be rejected for returns: the value of the test statistic is $\widehat{Z}_T = -0.257$ —critical level $p = 0.601$—, with $\ell_T = 5$ selected by above procedure to estimate σ^2. The robustness of these conclusions under moderate changes of m_T and ℓ_T can be checked in Table 6 above.

—INSERT FIGURE 3 ABOUT HERE—

These results are in accordance with both financial theories and a large amount of empirical research, all of them indicating that logarithms of asset prices contain a unit root, while asset return series do not. More precisely, it is commonly argued that return series are long memory processes.

The labor force participation rate

Finally, we completed our illustration of the application of the test with a study on the monthly data of labor force participation rate (LFPR) in the United States between January 1948 and August 2007 (716 observations) (the data come from the Federal Reserve Bank of Saint-Louis database, avalaible at http://research.stlouisfed.org/fred2, and have been analysed by Gustavsson and Österholm, 2010). The LFPR is measured each month by the Bureau of Labor Statitistics as the fraction of the civilian, non-institutional population 16 years or older who are either working or actively seeking work. It provides a useful complement to other indicators, such as employment and the unemployment rate, in assessing labor market conditions. Recently, Gustavsson and Österholm (2006, 2010) and Madsen *et al.* (2008) examined the time-series properties of the LFPR, as they are more revealing about potential hysteresis in unemployment than the unemployment rate. In addition, the presence of a unit root in the LFPR would have implications for the degree of uncertainty about future pension and social security payments.

The most noticeable feature of the LFPR is its increase over the post-World War II period (mostly between the early 1960s and 2000) due to women entering the labor force, the increase in the racial and ethnic diversity of the U.S. populations and to the aging of the baby boomers (DiCecio *et al.*, 2008). However, in recent years the LFPR has suffered a modest drop, which has sparked some debate, as some economists argue that this fall reflects a change in the trend whereas others view it as a cyclical deviation of the trend. Also, some studies (e.g. Madsen *et al.*, 2008) emphasize the non-linear behavior of the LFPR, which exhibits an asymmetric response as it responds differently when employment prosprects weaken than when they improve .

In Figure 4 below the plot of the series and the fitted trend (computed for $m_T = 14$) suggest both the presence of multiple changes and non-linear behavior in the series. The observed value of the test statistic is $\widehat{Z}_T = 0.406$ (critical level $p = 0.342$; the long run variance was estimated using $\ell_T = 6$). The sensitivity analysis in Table 6 above would confirm non-rejection of the hypothesis of stationarity around a non-linear trend, with this conclusion remaining unaffected for several m_T values, though the conclusions are somewhat sensitive to small ℓ_T values.

This conclusion differs from Gustavsson and Österholm (2006, 2010), who applied univariate and panel unit root tests to the aggregate and disaggregated (by combinations of gender, race and age) LFPR in the U.S, and concluded that the series is non-stationary; however, the non-inclusion of structural breaks in their study may bias the results of the unit root and stationarity tests towards non-stationarity. Madsen *et al.* (2008) extend the analysis applying the Caner and Hansen (2001) unit root test in the presence of a non-linear threshold and a LM unit root test with one and two structural breaks (Lee and Strazicich, 2003), and obtain results that are more in accordance with ours.

—INSERT FIGURE 4 ABOUT HERE—

5 Concluding remarks and further research

We have proposed a nonparametric stationarity test which allows stationarity testing to be carried out without relying on *a priori* specification of the trend component, that tends to be problematic in practice. The test is consistent under unit root alternatives and its limiting null distribution is standard normal. Simulation analyses indicate that the test performs suitably in a wide range of circumstances (trend shapes, stochastic dependence structures), providing a safeguard against misspecification of the trend function, particularly in large series, as those typically available in finance.

The issue of long-run variance estimation has also been addressed. The theoretical results allow nonparametric stationarity testing under nonparametric (kernel) estimation of the long-run variance, with a deterministic rule for bandwidth selection. In practice, data-driven bandwidth selection often tends to outperform deterministic rules in estimating long-run variances. A data-driven procedure to treat autocorrelation —with deterministic brackets that ensure the appropriate stochastic order for the estimator— has been outlined. Simulations indicate that this procedure performs suitably in common applications of the nonparametric stationarity test.

The above results suggest a number of interesting research avenues. First, more extensive analyses on autocorrelation treatment in nonparametric stationarity testing is clearly indicated. The asymptotic validity of a number of procedures proposed in the literature, including the most recent ones (e.g., Kiefer and Vogelsang, 2005; Phillips, 2005; Sun e*t al.,* 2008; Hashimzade and Vogelsang, 2008; Amsler e*t al.,* 2009), still has to be established in the nonparametric case. Simulation studies are required in order to assess the empirical performance of the various methods of treating for autocorrelation in this new setting.

Finally, the proposed nonparametric approach relies on trigonometric series estimation of the trend function. This allowed a relatively simple mathematical analysis. Computer simulations suggest that analogue results may be obtained for other classes of series estimators, particularly for algebraic polynomials. A confirmation of this conjecture would be desirable (results in a classical paper by MacNeill, 1978, are crucial for this extension), although the technical burden tends to increase dramatically when the cosine basis is replaced by other classes of polynomials.

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Appendix. Mathematical Proofs

Notational issues and previous remarks

1. In the proof of limiting normality the following kernel is used: for any (positive integer) m, let

$$K_m(u,v) = \min(u,v) - uv - \sum_{j=1}^{m} 2(j\pi)^{-2} \sin(j\pi u)\sin(j\pi v);\ u,v \in [0,1]$$

This kernel has the (uniformly convergent) Mercer expansion

$$K_m(u,v) = \sum_{j=m+1}^{\infty} 2(j\pi)^{-2} \sin(j\pi u)\sin(j\pi v),$$

with (reciprocal) eigenvalues $\eta_j = (m+j)^{-2}\pi^{-2}$, $j = 1, 2, ...,$ $\mu_m = \int_0^1 K_m(u,u)du = \sum_{j=m+1}^{\infty}(j\pi)^{-2}$ and $s_m^2 = 2\int_0^1\int_0^1 K_m^2(u,v)\,du\,dv = 2\sum_{j=m+1}^{\infty}(j\pi)^{-4}$ (for further details, see Tanaka, 1996, Chapter 5, page 153). It is readily checked that $s_m^2 \geq 2\pi^{-4}\int_{m+1}^{\infty} x^{-4}dx = 2/3\pi^{-4}(m+1)^{-3}$, so $s_{m_T}^{-1} \leq \sqrt{3/2}\pi^2 (m+1)^{3/2}$.

2. Let $r_{m_T}(u) = \theta^*(u) - \theta_{m_T}(u)$, $u \in [0,1]$. Given T, the OLS residuals have the decomposition $e_t = \varepsilon_t + r_{m_T}(t/T) + \theta_{m_T}(t/T) - \widehat{\theta}_{m_T}(t/T)$. In matrix form, $\mathbf{e} = (e_1, ..., e_T)' = \mathbf{\Pi}_{m_T}\varepsilon + \mathbf{\Pi}_{m_T}\mathbf{r}_{m_T} + \mathbf{\Pi}_{m_T}\mu$, with $\mathbf{\Pi}_{m_T} = (\mathbf{I}_T - \mathbf{\Phi}(\mathbf{\Phi}'\mathbf{\Phi})^{-1}\mathbf{\Phi}')$, $\mathbf{\Phi} = [\varphi_{t,j}]$, $t = 1, ..., T$, $j = 0, ..., m_T$; $\varepsilon = (\varepsilon_1, ..., \varepsilon_T)'$, $\mu = (\mu_1, ..., \mu_T)'$ and $\mathbf{r}_{m_T} = (r_{m_T}(1/T), ..., r_{m_T}(T/T))'$.

3. In this Appendix the symbols for order ($O(\cdot)$) and probability order ($O_p(\cdot)$) are intended uniformly in m. The following previous lemmas are required.

Lemma A.1. Let $\widetilde{\mathbf{B}}_m = \left[\widetilde{b}^{(m)}_{s,t}\right] = T^{-1}\mathbf{C}_T\mathbf{\Pi}_m\mathbf{C}'_T$, with m fixed. Let $\widetilde{S}_T = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\widetilde{b}^{(m)}_{s,t}u_s u_t$ and $S_T = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}K_m(s/T, t/T)\,u_s u_t$, where $\mathbf{u} = (u_1, ..., u_T)'$ is a sequence of i.i.d. random variables with $E(u_i) = 0$ and $var(u_i) = \sigma_u^2 < \infty$. Then, for some $c < \infty$ not depending on m or T, as $T \to \infty$:
(**a**) $\sup_{s,t=1,...,T}\left|\widetilde{b}_{s,t} - K_m(s/T, t/T)\right| \leq cm^3T^{-1}$, (**b**) $E\left|\widetilde{S}_T - S_T\right| \leq cm^3T^{-1}$. □

Proof. As to part (**a**), (if $T \geq m+1$) we have $\widetilde{b}^{(m)}_{s,t} = \min(s/T, t/T) - \widetilde{\mathbf{g}}'_s\widetilde{\mathbf{H}}^{-1}_{m+1}\widetilde{\mathbf{g}}_t$, with $\widetilde{\mathbf{H}}_{m+1} = \left[\widetilde{h}_{j,k}\right] = T^{-1}\mathbf{\Phi}'\mathbf{\Phi}$, $j, k = 0, ..., m$, and $\widetilde{\mathbf{g}}_t = [\widetilde{g}_{o,t}, ..., \widetilde{g}_{m,t}]'$, with $\widetilde{g}_{j,t} = T^{-1}\sum_{i=1}^{t}\varphi_j(i/T)$. We also have $K_m(s/T, t/T) = \min(s/T, t/T) - \mathbf{g}_s{}'\mathbf{H}^{-1}_{m+1}\mathbf{g}_t$, where $\mathbf{H}_{m+1} = [h_{j,k}]$, with $h_{j,k} = \int_0^1\varphi_j(u)\varphi_k(u)du$, $j, k = 0, ..., m$, and $\mathbf{g}_s = [g_{o,t}, ..., g_{m,t}]'$, with $g_{j,t} = \int_0^{t/T}\varphi_j(u)du$. Orthonomality of the basis ensures $\mathbf{H}_{m+1} = \mathbf{I}_{m+1}$, so $K_m(s/T, t/T) = \min(s/T, t/T) - \mathbf{g}'_s\mathbf{g}_t$ and $\widetilde{b}^{(m)}_{s,t} - K_m(s/T, t/T) = \mathbf{g}'_s\mathbf{g}_t - \widetilde{\mathbf{g}}'_s\widetilde{\mathbf{H}}^{-1}_{m+1}\widetilde{\mathbf{g}}_t = A_1 + A_2$, with $A_1 = \mathbf{g}'_s\mathbf{g}_t - \widetilde{\mathbf{g}}'_s\widetilde{\mathbf{g}}_t$ and $A_2 = \widetilde{\mathbf{g}}'_s\widetilde{\mathbf{g}}_t - \widetilde{\mathbf{g}}'_s\widetilde{\mathbf{H}}^{-1}_{m+1}\widetilde{\mathbf{g}}_t$. Discretization arguments and standard inequalities for eigenvalues ensure $|A_1| \leq c(1+m)^2T^{-1}$ and $A_2 = O(m^3T^{-1})$. Part (**b**) directly follows from part (**a**) and Lemma 3 in Nabeya and Tanaka (1988). ■

Lemma A.2. Let $S_{1T} = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}K_{m_T}(s/T, t/T)\,v_s v_t$, where the components of $\mathbf{v} = (v_1, ..., v_T)'$ are i.i.d. random variables with $E(v_i) = 0, var(v_i) = \sigma_v^2 > 0$ and $E|v_i|^{2+\delta} < \infty, \delta > 0$. Let $Z_{1T} = s^{-1}_{m_T}(\sigma_v^{-2}S_{1T} - \mu_{m_T})$, with $\mu_{m_T} = \int_0^1 K_{m_T}(u, u)\,du = \sum_{j=m_T+1}^{\infty}(j\pi)^{-2}$, $s^2_{m_T} = 2\int_0^1\int_0^1 K^2_{m_T}(u, v)\,du\,dv = 2\sum_{j=m_T+1}^{\infty}(j\pi)^{-4}$. If $m_T \to \infty$ and $m_T^3T^{-1} \to 0$, then $Z_{1T} \xrightarrow{L} N(0, 1)$ as $T \to \infty$. □

Proof. Without loss of generality we assume $\sigma_v^2 = 1$. It suffices to check that a central limit theorem for quadratic forms (with nonvanishing diagonal) in i.i.d. random variables holds. We apply Theorem 2.1.(iii) in Bhansali *et al.* (2007). We have $S_{1T} = \mathbf{u}'\mathbf{D}\mathbf{u} = \sum_{s=1}^{T}\sum_{t=1}^{T}d_{s,t}v_s v_t$, with $\mathbf{D} = [d_{s,t}]$ and $d_{s,t} = T^{-1}K_{m_T}(s/T, t/T)$. Let $\|\mathbf{D}\|_{2,T} = \sqrt{\sum_{s=1}^{T}\sum_{t=1}^{T}d^2_{s,t}}$ and $\|\mathbf{D}\|_{sp,T} = \widetilde{\eta}_1$, with $\widetilde{\eta}_1$ being the highest eigenvalue of $\mathbf{D}$. We start by deriving limiting normality for $\widetilde{Z}_{1T} = \left(\sqrt{2}\,\|\mathbf{D}\|_{2,T}\right)^{-1}(S_{1T} - E(S_{1T}))$. This follows under the conditions (1) $\|\mathbf{D}\|_{sp,T}/\|\mathbf{D}\|_{2,T} \to 0$ as $T \to \infty$, and (2) $\sum_{t=1}^{T}d^2_{t,t} = o(\|\mathbf{D}\|^2_{2,T})$. These requirements can be readily checked upon the basis of the (Euclidean, spectral) norms of kernel $K_{m_T}(\cdot,\cdot)$, namely,

$$\|K_{m_T}\|_2 = \sqrt{\int_0^1\int_0^1 K_{m_T}^2(u,v)\,du\,dv} = \sqrt{\sum_{j=m_T+1}^{\infty}(j\pi)^{-4}}$$

—this implies $\|K_{m_T}\|_2^{-1} = O\left(m_T^{3/2}\right)$—, and $\|K_{m_T}\|_{sp} = \eta_1 = (m_T+1)^{-2}\pi^{-2} = O\left(m_T^{-2}\right)$. Discretization arguments and Aronszajn Theorem give $\|\mathbf{D}\|_{sp,T} = \widetilde{\eta}_1 \leq \eta_1 + |\widetilde{\eta}_1 - \eta_1| = O(m_T^{-2})$ and $\|\mathbf{D}\|_{2,T} = O\left(m_T^{-3/2}\right)$. Hence, we obtain $\|\mathbf{D}\|_{sp,T} / \|\mathbf{D}\|_{2,T} = O\left(m_T^{-1/2}\right)$. It is directly checked as $\sum_{t=1}^T d_{t,t}^2 = O(m_T^{-2}T^{-1})$ and $\|\mathbf{D}\|_{2,T}^2 = O\left(m_T^{-3}\right)$. So, $\widetilde{Z}_{1T} \xrightarrow{L} N(0,1)$, and as $Z_{1T} = \widetilde{Z}_{1T} + O_p\left(m_T^{5/2}T^{-1}\right)$ the conclusion follows. ■

Lemma A.3. Let $S_{1T} = T^{-1}\sum_{s=1}^T\sum_{t=1}^T K_{m_T}(s/T,t/T)\,\varepsilon_s\varepsilon_t$, with $\varepsilon_t = \sum_{i=0}^\infty \alpha_i v_{t-i}$, under Assumption 1 and $\sigma^2 = \alpha^2\sigma_v^2$. Let $Z_T = s_{m_T}^{-1}(\sigma^{-2}S_{1T} - \mu_{m_T})$. If $m_T \to \infty$ and $m_T^3 T^{-1} \to 0$ then $Z_T \xrightarrow{L} N(0,1)$ as $T \to \infty$. □

Proof. Without loss of generality we assume $\sigma_v^2 = 1$, so $\sigma^2 = \sum_{i=0}^\infty\sum_{k=0}^\infty \alpha_i\alpha_k > 0$, and proceed as in Tanaka (1990, Theorem 1, Appendix). First, we have

$$Z_T = s_{m_T}^{-1}\sigma^{-2}\left(T^{-1}\sum_{s=1}^T\sum_{t=1}^T K_{m_T}(s/T,t/T)\,\varepsilon_s\varepsilon_t - \mu_{m_T}\sigma^2\right) = \sigma^{-2}\sum_{i=0}^\infty\sum_{k=0}^\infty \alpha_i\alpha_k w_{T,i,k}$$

with $w_{T,i,k} = s_{m_T}^{-1}\left[T^{-1}\sum_{s=1}^T\sum_{t=1}^T K_{m_T}(s/T,t/T)\,v_{s-i}v_{t-k} - \mu_{m_T}\right]$. The following decomposition is applicable: $Z_T = Z_{T,M} + V_{T,M}$, where $Z_{T,M} = \sigma^{-2}\sum_{i=0}^M\sum_{k=0}^M \alpha_i\alpha_k w_{T,i,k}$ and $V_{T,M}$ is the remainder term. It is readily checked that $E|V_{T,M}| \leq c_M$ for all T large, with $c_M \equiv 3\sigma^{-2}(1+\delta)\sum_{i=0}^\infty|\alpha_i|\cdot\sum_{k=M+1}^\infty|\alpha_i| \to 0$ as $M \to \infty$.

As to $Z_{T,M}$, first fix M and let $T \to \infty$. We have $Z_{T,M} = \widetilde{Z}_{T,M} + R_{T,M}$, with $\widetilde{Z}_{T,M} = w_{T,0,0}\left(\sigma^{-2}\sum_{i=0}^M\sum_{k=0}^M \alpha_i\alpha_k\right)$ and $R_{T,M} = \sigma^{-2}\sum_{i=0}^M\sum_{k=0}^M \alpha_i\alpha_k\,(w_{T,i,k} - w_{T,0,0})$. As $w_{T,0,0} \xrightarrow{L} N(0,1)$ by Lemma A.2, we have (for any fixed M, as $T \to \infty$) that $\widetilde{Z}_{T,M} \xrightarrow{L} \widetilde{Z}_M$, which is Gaussian with mean zero and variance $\left(\sigma^{-2}\sum_{i=0}^M\sum_{k=0}^M \alpha_i\alpha_k\right)^2$.

As to the remainder $R_{T,M}$, for fixed M as $T \to \infty$, it holds $E|w_{T,i,k} - w_{T,0,0}| \leq \acute{c}M^2 m_T^{5/2}T^{-1}$, with $\acute{c}$ depending neither on $0 \leq i,k \leq M$, nor on M or T. Hence,

$\limsup_{T\to\infty} E\left|w_{T,i,k} - w_{T,0,0}\right| = 0$ and the same applies, for any fixed M, to $R_{T,M}$. Hence, we have, for any M, $\widetilde{Z}_{T,M} \xrightarrow{L} \widetilde{Z}_M$ as $T \to \infty$. It also holds $\widetilde{Z}_M \xrightarrow{L} N(0,1)$ as $M \to \infty$, by the continuous mapping Theorem. Since $Z_T = Z_{T,M} + V_{T,M} = \widetilde{Z}_{T,M} + R_{T,M} + V_{T,M}$ and $E\left|Z_T - \widetilde{Z}_{T,M}\right| \leq E\left|R_{T,M}\right| + E\left|V_{T,M}\right|$, Theorem 4.2 in Billingsley (1968) and Tchebyshev's inequality give $Z_T \xrightarrow{L} N(0,1)$. ■

Proof of Proposition 1. As to part (**a**), since $\mathbf{e} = \mathbf{\Pi}_{m_T}\varepsilon + \mathbf{\Pi}_{m_T}\mathbf{r}$, the decomposition $S_T = S_{1T}+A_1+A_2$, with $S_{1T} = T^{-2}\varepsilon'\mathbf{\Pi}'_{m_T}\mathbf{C}'_T\mathbf{C}_T\mathbf{\Pi}_{m_T}\varepsilon$, $A_1 = T^{-2}\mathbf{r}'\mathbf{\Pi}'_{m_T}\mathbf{C}'_T\mathbf{C}_T\mathbf{\Pi}_{m_T}\varepsilon +$ $T^{-2}\varepsilon'\mathbf{\Pi}'_{m_T}\mathbf{C}'_T\mathbf{C}_T\mathbf{\Pi}_{m_T}\mathbf{r}$, and $A_2 = T^{-2}\mathbf{r}'\mathbf{\Pi}'_{m_T}\mathbf{C}'_T\mathbf{C}_T\mathbf{\Pi}_{m_T}\mathbf{r}$, is directly obtained. Hence, $Z_T = \widetilde{Z}_{1T} + s_{m_T}^{-1}\sigma^{-2}A_1 + s_{m_T}^{-1}\sigma^{-2}A_2$, with $\widetilde{Z}_{1T} = s_{m_T}^{-1}\left(\sigma^{-2}S_{1T} - \mu_{m_T}\right)$.

Limiting normality of $\widetilde{Z}_{1T}$ is readily checked. It suffices to derive limiting normality for $\widetilde{Z}_{2T} = s_{m_T}^{-1}\left(\sigma^{-2}\widetilde{S}_{2T} - \mu_{m_T}\right)$, with $\widetilde{S}_{2T} = T^{-1}\varepsilon'\mathbf{B}_T\varepsilon$ and $\mathbf{B}_T = [b_{s,t}] = T^{-1}\mathbf{C}_T\mathbf{\Pi}_{m_T}\mathbf{C}'_T$. First we approximate $\widetilde{S}_{2T}$ by $S_{2T} = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T} K_{m_T}(s/T, t/T)\,\varepsilon_s\varepsilon_t$. Let $R_T = \widetilde{S}_{2T} - S_{2T} = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\left[b_{s,t} - K_{m_T}(s/T,t/T)\right]\varepsilon_s\varepsilon_t$ and $\delta_T \equiv$ $\sup_{s,t=1,\ldots,T}\left|\widetilde{b}_{s,t} - K_m(s/T,t/T)\right|$. By Lemma A.1, $\delta_T = O(m_T^3 T^{-1})$. A probability inequality in Tanaka (1990, Appendix, Theorem 1) ensures, for any $x > 0$, that $P\left(|R_T| > x\right) \leq (c/x)\delta_T\left(\sum_{i=0}^{\infty}|\alpha_i|\right)^2$ for some constant $c > 0$ not depending on T. Hence, $R_T = O_p(\delta_T) = O_p(m_T^3 T^{-1})$. Therefore, $\widetilde{Z}_{2T} = Z_{2T} + s_{m_T}^{-1}\sigma^{-2}R_T$, with $Z_{2T} = s_{m_T}^{-1}(\sigma^{-2}S_{2T} - \mu_{m_T})$. As $s_{m_T}^{-1} = O\left(m_T^{3/2}\right)$, we obtain $s_{m_T}^{-1}\sigma^{-2}R_T = O_p\left(m_T^{9/2}T^{-1}\right)$, which is asymptotically negligible as $m_T^{9/2}T^{-1} \to 0$. Lemma A.3 gives $Z_{2T} \xrightarrow{L} N(0,1)$. Therefore, $\widetilde{Z}_{2T} \xrightarrow{L} N(0,1)$, which amounts to $\widetilde{Z}_{1T} \xrightarrow{L} N(0,1)$.

It is readily checked that the bias terms have lower probability orders than $\widetilde{Z}_T$. In particular, the basic projection inequality of least squares regression ensures $s_{m_T}^{-1}\sigma^{-2}A_1$ $= O_p\left(Tm_T^{3/2}d_T(\theta_{m_T},\theta^*)\right)$ and $s_{m_T}^{-1}\sigma^{-2}A_2 = O\left(Tm_T^{3/2}d_T^2(\theta_{m_T},\theta^*)\right)$. Both quantities are asymptotically negligible under H_0 given Assumption 2. As to (**b**), under H_1 we have $\mathbf{e} = \mathbf{\Pi}_{m_T}\varepsilon + \mathbf{\Pi}_{m_T}\mathbf{r} + \mathbf{\Pi}_{m_T}\mu$, that combined with standard inequalities gives $S_T = O_p(T^2)$. Hence, $Z_T = s_{m_T}^{-1}\sigma^{-2}S_T - s_{m_T}^{-1}\mu_{m_T} = O_p\left(m_T^{3/2}T^2\right) - O_p\left(m_T^{1/2}\right) = O_p\left(m_T^{3/2}T^2\right)$, so $P\left(Z_T > \kappa_T\right) \to 1$ if $\kappa_T = o\left(m_T^{3/2}T^2\right)$. ■

Proof of Proposition 2. As to (**a**), we have $\widetilde{Z}_T - Z_T = s_{m_T}^{-1}\left(\sigma^{-2}S_T - \mu_{m_T}\right)\widehat{\sigma}^{-2}\left(\sigma^2 - \widehat{\sigma}^2\right) +$

$\widehat{\sigma}^{-2}\left(\sigma^2-\widehat{\sigma}^2\right)s_{m_T}^{-1}\mu_{m_T}$, and as $s_{m_T}^{-1}\left(\sigma^{-2}S_T-\mu_{m_T}\right) = O_p(1)$ by Proposition 1 and $\widehat{\sigma}^2-\sigma^2 = o_p(1)$, it holds $\widetilde{Z}_T-Z_T = o_p(1)$ as Assumption 3 imposed $m_T^{1/2}\left(\widehat{\sigma}^2-\sigma^2\right) \xrightarrow{p} 0$ under H_0. As to (**b**), as $S_T = O_p\left(T^2\right)$ we have $\widetilde{Z}_T = s_{m_T}^{-1}\widehat{\sigma}^{-2}S_T - s_{m_T}^{-1}\mu_{m_T} = \widehat{\sigma}^{-2}O_p\left(m_T^{3/2}T^2\right) - O\left(m_T^{1/2}\right) = O_p\left(m_T^{3/2}T^{2-\zeta}\right)$ and the conclusion follows. ■

Lemma A.4. Let $\{\varphi_j, j=0,1,...\}$ be an orthonormal set in $L_2[0,1]$ and let $\widetilde{\mathbf{H}}_{m_T+1}^{-1} = (T^{-1}\mathbf{\Phi}'\mathbf{\Phi})^{-1}$, with $\mathbf{\Phi} = [\varphi_{t,j}]$, $\varphi_{t,j} = \varphi_j(t/T)$, $t=1,...,T$; $j=0,...,m_T$. Let $V_T = \mathbf{v}'\mathbf{\Phi}(\mathbf{\Phi}'\mathbf{\Phi})^{-1}\mathbf{\Phi}'\mathbf{v}$, with $\mathbf{v} = (v_1,...,v_T)'$, being a finite sample of the i.i.d. process in Assumption 1. Under the conditions: (i) $\sup_{j\geq 0}\|\varphi_j\|_\infty \leq \Delta < \infty$, and each φ_j satisfies the Lipschitz condition $|\varphi_j(u)-\varphi_j(u')| \leq cj\,|u-u'|$; $u,u' \in [0,1]$, with $c<\infty$ not depending on j, (ii) for any fixed m, $\left\|\widetilde{\mathbf{H}}_{m+1}-\mathbf{I}_{m+1}\right\|_{2,m+1} \leq cm^2T^{-1}$ as $T\to\infty$, with c not depending on m or T, and (iii) $m = m_T \to \infty$ and $m_T^{9/2}T^{-1} \to 0$, as $T\to\infty$. Then $Q_T = (2(1+m_T))^{-1/2}\left(\sigma_v^{-2}V_T-(1+m_T)\right) \xrightarrow{L} N(0,1)$ as $T\to\infty$. □

Proof. This case is analogous to Hong and White (1995, Appendix, Theorem A.1). Without loss of generality we assume $\sigma_v^2 = 1$. Let $V_T = T^{-1}\mathbf{v}'\mathbf{B}_T\mathbf{v}$, with $\mathbf{B}_T = [b_{s,t}] = \mathbf{\Phi}(T^{-1}\mathbf{\Phi}'\mathbf{\Phi})^{-1}\mathbf{\Phi}' = \mathbf{\Phi}\widetilde{\mathbf{H}}_{m_T+1}^{-1}\mathbf{\Phi}'$. For any T, we shall use the kernel $K'_{m_T}(u,v) = \sum_{j=0}^{m_T}\varphi_j(u)\varphi_j(v)$; $u,v\in[0,1]$, which is degenerate, with reciprocal eigenvalues $\eta'_1 = ... = \eta'_{m_T+1} = 1$, which implies $\left\|K'_{m_T}\right\|_{sp} = 1$ and $\left\|K'_{m_T}\right\|_2 = m_T+1$. The analysis proceeds as in Lemma A.1. First, we approximate V_T by $\widetilde{V}_T = T^{-1}\sum_{s=1}^T\sum_{t=1}^T K'_{m_T}(s/T,t/T)v_sv_t$ and obtain, by Lemma 3 in Nabeya and Tanaka (1988), $V_T = \widetilde{V}_T + O_p\left(m_T^5T^{-1}\right)$. The rest of the proof is analogous to that of Lemma A.2. ■

Lemma A.5. Under the assumptions of Lemma A.4, let $V_T = \varepsilon'\mathbf{\Phi}(\mathbf{\Phi}'\mathbf{\Phi})^{-1}\mathbf{\Phi}'\varepsilon$, with $\varepsilon = (\varepsilon_1,...,\varepsilon_T)'$ being a finite sample from the linear filter process in Assumption 1. Then $Q_T = (2(1+m_T))^{-1/2}\left(\sigma^{-2}V_T-(1+m_T)\right) \xrightarrow{L} N(0,1)$ as $T\to\infty$. □

Proof. It is analogous to the proof of Lemmas A.2 and A.4. Without loss of generality we assume $\sigma_v^2 = 1$, so $\sigma^2 = \sum_{i=0}^\infty\sum_{k=0}^\infty\alpha_i\alpha_k > 0$. Let $V_T = T^{-1}\varepsilon'\mathbf{B}_T\varepsilon$, with $\mathbf{B}_T = [b_{s,t}] = \mathbf{\Phi}(T^{-1}\mathbf{\Phi}'\mathbf{\Phi})^{-1}\mathbf{\Phi}' = \mathbf{\Phi}\widetilde{\mathbf{H}}_{m_T+1}^{-1}\mathbf{\Phi}'$. First, we approximate V_T by $\widetilde{V}_T = T^{-1}\sum_{s=1}^T\sum_{t=1}^T K'_{m_T}(s/T,t/T)\varepsilon_s\varepsilon_t$. Lemma A.4 and the same probability inequality used in Lemma A.2 (see Tanaka, 1990, Theorem 1, Appendix) give $V_T - \widetilde{V}_T = O_p\left(m_T^5 T^{-1}\right)$. The rest of the proof is analogous to that of Lemma A.3. ■

Proof of Proposition 3. We have the decomposition $\mathbf{e}^{(d)} = \widetilde{\varepsilon}^{(d)} + \widetilde{\mathbf{r}}^{(d)} + \widetilde{\mu}^{(d)}$, with $\widetilde{\varepsilon}^{(d)} = \left(\widetilde{\varepsilon}_t^{(d)}\right) = \mathbf{\Pi}_{m_T^{(d)}}\varepsilon$, $\widetilde{\mathbf{r}}^{(d)} = \widetilde{\mathbf{r}}_{m_T^{(d)}} = \left(\widetilde{r}_t^{(d)}\right) = \mathbf{\Pi}_{m_T^{(d)}}\mathbf{r}_{m_T^{(d)}}$, $\mathbf{r}_{m_T^{(d)}} = \left(r_{m_T^{(d)}}(1/T), ..., r_{m_T^{(d)}}(T/T)\right)'$, and $\widetilde{\mu}^{(d)} = \left(\widetilde{\mu}_t^{(d)}\right) = \mathbf{\Pi}_{m_T^{(d)}}\mu$. Hence, $\widehat{\sigma}_i = T^{-1}\sum_{t=1+|i|}^{T} e_t^{(d)} e_{t-|i|}^{(d)} = \sum_{k=1}^{6} A_{k,i}$, with $A_{1,i} = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\varepsilon}_t^{(d)}\widetilde{\varepsilon}_{t-|i|}^{(d)}$, $A_{2,i} = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{r}_t^{(d)}\widetilde{r}_{t-|i|}^{(d)}$,
$A_{3,i} = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\varepsilon}_t^{(d)}\widetilde{r}_{t-|i|}^{(d)} + T^{-1}\sum_{t=1+|i|}^{T}\widetilde{r}_t^{(d)}\widetilde{\varepsilon}_{t-|i|}^{(d)}$, $A_4 = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\mu}_t^{(d)}\widetilde{\mu}_{t-|i|}^{(d)}$, $A_5 = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\varepsilon}_t^{(d)}\widetilde{\mu}_{t-|i|}^{(d)} + T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\mu}_t^{(d)}\widetilde{\varepsilon}_{t-|i|}^{(d)}$, and $A_6 = T^{-1}\sum_{t=1+|i|}^{T}\widetilde{r}_t^{(d)}\widetilde{\mu}_{t-|i|}^{(d)} + T^{-1}\sum_{t=1+|i|}^{T}\widetilde{\mu}_t^{(d)}\widetilde{r}_{t-|i|}^{(d)}$. Cauchy-Schwarz inequality and the projection inequality of least squares regression give: $|A_{2,i}| \leq d_T^2\left(\theta_{m_T^{(d)}}, \theta^*\right)$, $A_{3,i} = O_p\left(d_T\left(\theta_{m_T^{(d)}}, \theta^*\right)\right)$, $|A_{4,i}| \leq T^{-1}\sum_{t=1}^{T}\mu_t^2 = O_p(T)$ and $A_{5,i} = O_p\left(d_T\left(\theta_{m_T^{(d)}}, \theta^*\right)T^{1/2}\right)$. We may write $\widehat{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}\widehat{\sigma}_i = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}A_{1,i} + R_T$, with
$R_T = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}\sum_{k=2}^{5} A_{k,i}$. As $|w_{i,T}| \leq 1$, we obtain:

(1) under H_0: $|R_T| \leq (2\ell_T + 1)\left\{d_T^2\left(\theta_{m_T^{(d)}}, \theta^*\right) + 2\sqrt{T^{-1}\sum_{t=1}^{T}\varepsilon_t^2}\; d_T\left(\theta_{m_T^{(d)}}, \theta^*\right)\right\} = O_p\left(\ell_T d_T\left(\theta_{m_T}, \theta^*\right)\right)$,

(2) under H_1: $|R_T| \leq \sum_{i=-\ell_T}^{\ell_T}|w_{i,T}|\sum_{k=2}^{5}|A_{k,i}| = O_p\left(\ell_T T^{-1}\sum_{t=1}^{T}\mu_t^2\right) = O_p\left(\ell_T T\right)$.

Now we analyze $\sum_{i=-\ell_T}^{\ell_T} w_{i,T}A_{1,i}$. As $A_{1,i} = T^{-1}\sum_{t=1+|j|}^{T}\widetilde{\varepsilon}_t^{(d)}\widetilde{\varepsilon}_{t-|j|}^{(d)}$ and $\widetilde{\varepsilon}_t^{(d)} = \varepsilon_t^{(d)} - h_t$, with $\mathbf{h} = (h_1, ..., h_T)' = \mathbf{\Phi}_d(\mathbf{\Phi}_d'\mathbf{\Phi}_d)^{-1}\mathbf{\Phi}_d'\varepsilon$, we obtain the decomposition $\sum_{i=-\ell_T}^{\ell_T} w_{i,T}A_{1,i} = \widetilde{\sigma}^2 + \sum_{i=-\ell_T}^{\ell_T} w_{i,T}\left(B_{1,i} + B_{2,i}\right)$ with $\widetilde{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}T^{-1}\sum_{t=1+|i|}^{T}\varepsilon_t\varepsilon_{t-|i|}$, $B_{1,i} = -T^{-1}\sum_{t=1+|i|}^{T}\varepsilon_t h_{t-|i|} - T^{-1}\sum_{t=1+|i|}^{T}h_t\varepsilon_{t-|i|}$, and $B_{2,i} = T^{-1}\sum_{t=1+|i|}^{T}h_t h_{t-|i|}$.
It is readily obtained $\sum_{i=-\ell_T}^{\ell_T} w_{i,T}\left(B_{1,i} + B_{2,i}\right) = O_p\left(\ell_T\sqrt{T^{-1}\sum_{t=1}^{T}h_t^2}\right)$. As $T^{-1}\sum_{t=1}^{T}h_t^2 = T^{-1}\varepsilon'\mathbf{\Phi}_d(\mathbf{\Phi}_d'\mathbf{\Phi}_d)^{-1}\mathbf{\Phi}_d'\varepsilon = T^{-1}V_T$, with V_T as in Lemma A.5 above, we have $V_T = \sigma^2\left(Q_T\sqrt{2(1 + m_T^{(d)})} + (1 + m_T^{(d)})\right) = O_p(m_T^{(d)})$, and Lemma A.5 gives $T^{-1}\sum_{t=1}^{T}h_t^2 = T^{-1}V_T = O_p\left(m_T^{(d)}T^{-1}\right)$.

Then we apply the decomposition $\widehat{\sigma}^2 = (\widehat{\sigma}^2 - \widetilde{\sigma}^2) + (\widetilde{\sigma}^2 - \overline{\sigma}^2) + (\overline{\sigma}^2 - \sigma^2) + \sigma^2$, with $\overline{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}T^{-1}\sum_{t=1+|i|}^{T}E\left(\varepsilon_t\varepsilon_{t-|i|}\right)$. As to the first difference, the above results give, under H_0, $\widehat{\sigma}^2 - \widetilde{\sigma}^2 = \sum_{i=-\ell_T}^{\ell_T} w_{i,T}\left(B_{1,i} + B_{2,i}\right) + R_T = O_p\left(\ell_T\left(m_T^{(d)}\right)^{1/2}T^{-1/2}\right)$

$+\, O_p\left(\ell_T d_T\left(\theta_{m_T^{(d)}}, \theta^*\right)\right) = O_p\left(\ell_T\left(m_T^{(d)}\right)^{1/2} T^{-1/2}\right)$ because of Assumption 2 and $\ell_T = o(T)$.

Corollary 6.3 in Pötscher and Prucha (1991) gives $\widehat{\sigma}^2 - \overline{\sigma}^2 = O_p\left(\ell_T^{3/2} T^{-1/2}\right)$ and $\overline{\sigma}^2 - \sigma^2 = O_p\left(\ell_T^{-\rho}\right)$. Hence, under H_0, $\widehat{\sigma}^2 - \sigma^2 = O_p\left(\ell_T\left(m_T^{(d)}\right)^{1/2} T^{-1/2}\right) + O_p\left(\ell_T^{3/2} T^{-1/2}\right) + O_p\left(\ell_T^{-\rho}\right)$. Under H_0 and the assumptions of this proposition all these terms vanish in probability, even when multiplied by $m_T^{1/2}$. These arguments are valid under assumption (*i.2*), but remain true under (*i.1*), i.e., when $E\left|\varepsilon\right|^4$ and $\sum_{j=0}^{\infty} |\alpha_j| < \infty$, as a consequence of Corollary 8.3.1 in Anderson (1971) and Corollary 6.3 in Pötscher and Prucha (1991).

Finally, under H_1 the dominant term in $\widehat{\sigma}^2$ is R_T, so $\widehat{\sigma}^2 = O_p\left(\ell_T T\right)$, as well as non-negative by construction, and the rate of divergence is derived as in Proposition 2. ■

Tables

Table 1. Size and power of the nonparametric stationarity test. i.i.d. $N(0,1)$ errors; 5% significance.

	Trend A						Trend B						Trend C					
$q\backslash T$	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000
0	0.036	0.045	0.053	0.057	0.057	0.058	0.042	0.046	0.056	0.06	0.056	0.055	0.041	0.068	0.066	0.069	0.063	0.063
0.01	0.036	0.139	0.343	0.882	0.992	1	0.047	0.145	0.341	0.876	0.993	1	0.047	0.159	0.360	0.886	0.993	1
0.1	0.133	0.874	0.997	1	1	1	0.116	0.871	0.997	1	1	1	0.105	0.872	0.996	1	1	1
	Trend D ($\gamma = 20$)						Trend D ($\gamma = 50$)						Trend D ($\gamma = 100$)					
0	0.042	0.050	0.058	0.062	0.049	0.051	0.032	0.054	0.058	0.045	0.058	0.061	0.049	0.075	0.096	0.107	0.132	0.101
0.01	0.060	0.140	0.311	0.878	0.991	1	0.040	0.142	0.331	0.858	0.992	1	0.073	0.168	0.396	0.874	0.995	1
0.1	0.142	0.838	0.991	1	1	1	0.089	0.831	0.994	1	1	1	0.115	0.841	0.995	1	1	1
	Trend E ($\gamma = 20$)						Trend E ($\gamma = 50$)						Trend E ($\gamma = 100$)					
0	0.049	0.050	0.057	0.050	0.057	0.045	0.033	0.049	0.057	0.059	0.054	0.051	0.034	0.054	0.053	0.059	0.058	0.059
0.01	0.049	0.126	0.321	0.869	0.994	1	0.041	0.131	0.326	0.891	0.993	1	0.030	0.139	0.329	0.878	0.992	0.998
0.1	0.137	0.836	0.992	1	1	1	0.134	0.832	0.991	1	1	1	0.075	0.835	0.994	1	1	1
	Trend F ($\gamma = 20$)						Trend F ($\gamma = 50$)						Trend F ($\gamma = 100$)					
0	0.044	0.050	0.059	0.054	0.057	0.059	0.041	0.046	0.056	0.054	0.049	0.053	0.029	0.049	0.055	0.054	0.057	0.048
0.01	0.037	0.136	0.336	0.867	0.992	1	0.034	0.135	0.323	0.882	0.993	1	0.048	0.130	0.341	0.899	0.992	1
0.1	0.109	0.831	0.992	1	1	1	0.105	0.821	0.991	1	1	1	0.083	0.841	0.992	1	1	1
	Trend G						Trend H						Trend I					
0	0.024	0.057	0.068	0.090	0.105	0.096	0.053	0.052	0.059	0.060	0.054	0.068	0.027	0.049	0.063	0.058	0.060	0.054
0.01	0.038	0.150	0.357	0.871	0.994	1	0.030	0.140	0.324	0.871	0.993	1	0.050	0.143	0.331	0.878	0.993	1
0.1	0.091	0.834	0.994	1	1	1	0.112	0.843	0.990	1	1	1	0.135	0.822	0.993	1	1	1

Table 2. Size and power of the Becker *et al.* (2006) test. i.i.d. $N(0, 1)$ errors; 5% significance.

	BEL1 test												BEL2 test											
	Trend A				Trend B				Trend C				Trend A				Trend B				Trend C			
$q\backslash T$	100	500	1000	2000	100	500	1000	2000	100	500	1000	2000	100	500	1000	2000	100	500	1000	2000	100	500	1000	2000
0	0.011	0.008	0.007	0.010	0.074	0.196	0.353	0.639	0.999	1	1	1	0.050	0.050	0.060	0.052	0.053	0.078	0.106	0.164	0.257	0.976	1	1
0.01	0.065	0.889	0.995	1	0.123	0.917	0.996	1	0.999	1	1	1	0.098	0.882	0.999	1	0.090	0.896	0.999	1	0.315	0.985	0.999	1
0.1	0.554	0.999	1	1	0.584	1	1	1	0.959	1	1	1	0.522	1	1	1	0.512	1	1	1	0.666	1	1	1
	Trend D ($\gamma = 20$)				Trend D ($\gamma = 50$)				Trend D ($\gamma = 100$)				Trend D ($\gamma = 20$)				Trend D ($\gamma = 50$)				Trend D ($\gamma = 100$)			
0	0.162	0.634	0.928	0.998	0.426	0.996	1	1	0.501	0.999	1	1	0.060	0.117	0.180	1	0.180	0.687	0.954	1	0.228	0.892	0.998	1
0.01	0.236	0.940	0.995	1	0.469	0.955	1	1	0.527	0.978	1	1	0.112	0.901	1	1	0.202	0.946	1	1	0.294	0.968	1	1
0.1	0.605	0.999	1	1	0.672	0.999	1	1	0.699	0.999	1	1	0.521	1	1	1	0.592	1	1	1	0.620	1	1	1
	Trend E ($\gamma = 20$)				Trend E ($\gamma = 50$)				Trend E ($\gamma = 100$)				Trend E ($\gamma = 20$)				Trend E ($\gamma = 50$)				Trend E ($\gamma = 100$)			
0	0.055	0.049	0.059	0.059	0.259	0.915	0.997	1	0.729	1	1	1	0.053	0.063	0.044	0.082	0.055	0.076	0.118	0.223	0.077	0.174	0.408	0.761
0.01	0.143	0.931	0.994	1	0.362	0.956	0.999	1	0.731	0.975	1	1	0.093	0.884	0.999	1	0.095	0.892	0.998	1	0.112	0.918	1	1
0.1	0.599	1	1	1	0.684	0.999	1	1	0.749	0.999	1	1	0.511	1	1	1	0.517	1	1	1	0.535	1	1	1
	Trend F ($\gamma = 20$)				Trend F ($\gamma = 50$)				Trend F ($\gamma = 100$)				Trend F ($\gamma = 20$)				Trend F ($\gamma = 50$)				Trend F ($\gamma = 100$)			
0	0.239	0.919	1	1	0.596	1	1	1	0.523	1	1	1	0.048	0.058	0.062	0.069	0.151	0.600	0.906	0.996	0.429	0.994	1	1
0.01	0.334	0.950	0.999	1	0.600	0.973	0.997	1	0.571	0.979	0.999	1	0.103	0.890	0.999	1	0.210	0.925	0.999	1	0.493	0.978	1	1
0.1	0.648	0.999	1	1	0.708	0.999	1	1	0.735	1	1	1	0.519	1	1	1	0.566	1	1	1	0.699	1	1	1
	Trend G				Trend H				Trend I				Trend G				Trend H				Trend I			
0	0.018	0.118	0.323	0.754	0.010	0.023	0.050	0.117	0.015	0.056	0.071	0.069	0.067	0.151	0.296	0.576	0.066	0.062	0.087	0.164	0.040	0.085	0.129	0.207
0.01	0.104	0.916	0.998	1	0.071	0.894	0.996	1	0.092	0.897	0.998	1	0.112	0.909	0.999	1	0.100	0.888	0.999	1	0.104	0.899	0.999	1
0.1	0.559	1	1	1	0.559	1	1	1	0.593	1	1	1	0.525	1	1	1	0.519	1	1	1	0.524	1	1	1

Note: BEL1 denotes the test where the frequency k is estimated by minimizing the SSR; k^{max}=5.BEL2 denotes the test where the first two cumulative frequencies are included.

Table 3. Size and power of the nonparametric stationarity test. Time Series; 5% significance.

			Trend A						Trend B						Trend C					
	ρ	$q\backslash T$	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000
AR(1)	0.5	0	0.256	0.056	0.06	0.079	0.076	0.104	0.213	0.059	0.063	0.074	0.090	0.075	0.185	0.047	0.055	0.088	0.084	0.081
		0.01	0.231	0.072	0.09	0.262	0.487	0.735	0.216	0.069	0.09	0.256	0.486	0.726	0.223	0.057	0.102	0.259	0.487	0.729
		0.1	0.186	0.101	0.247	0.692	0.943	0.999	0.196	0.105	0.256	0.693	0.935	0.996	0.223	0.100	0.260	0.709	0.949	0.996
	0.2	0	0.223	0.156	0.105	0.084	0.092	0.097	0.230	0.175	0.090	0.088	0.098	0.108	0.208	0.155	0.112	0.135	0.123	0.103
		0.01	0.236	0.154	0.244	0.705	0.914	0.983	0.238	0.166	0.258	0.717	0.908	0.985	0.184	0.137	0.249	0.738	0.905	0.983
		0.1	0.202	0.184	0.544	0.961	0.980	0.997	0.214	0.190	0.527	0.971	0.979	0.999	0.143	0.160	0.505	0.973	0.971	0.994
	0	0	0.151	0.075	0.063	0.058	0.061	0.071	0.157	0.068	0.067	0.059	0.063	0.068	0.138	0.087	0.105	0.147	0.094	0.069
		0.01	0.135	0.200	0.485	0.927	0.960	0.984	0.145	0.211	0.495	0.930	0.962	0.993	0.139	0.228	0.493	0.928	0.952	0.979
		0.1	0.185	0.471	0.700	0.974	0.980	0.994	0.213	0.433	0.713	0.975	0.977	0.997	0.188	0.444	0.683	0.968	0.975	0.998
	-0.2	0	0.586	0.285	0.165	0.114	0.117	0.101	0.620	0.292	0.187	0.129	0.135	0.120	0.324	0.234	0.204	0.270	0.185	0.119
		0.01	0.577	0.364	0.607	0.929	0.958	0.987	0.617	0.386	0.620	0.940	0.952	0.986	0.302	0.327	0.595	0.936	0.953	0.987
		0.1	0.241	0.436	0.680	0.978	0.972	0.998	0.258	0.449	0.686	0.970	0.977	0.998	0.188	0.420	0.676	0.978	0.976	0.996
MA(1)	0.5	0	0.211	0.053	0.058	0.055	0.069	0.092	0.240	0.047	0.05	0.058	0.076	0.078	0.180	0.051	0.054	0.094	0.092	0.078
		0.01	0.228	0.074	0.144	0.503	0.788	0.932	0.221	0.078	0.157	0.513	0.783	0.942	0.175	0.074	0.159	0.507	0.781	0.944
		0.1	0.134	0.189	0.493	0.959	0.980	0.997	0.124	0.176	0.507	0.949	0.985	0.998	0.105	0.157	0.496	0.955	0.987	0.996
	0.2	0	0.200	0.126	0.072	0.068	0.058	0.060	0.224	0.122	0.083	0.069	0.06	0.082	0.186	0.133	0.091	0.118	0.078	0.076
		0.01	0.218	0.147	0.250	0.736	0.921	0.985	0.184	0.142	0.242	0.742	0.933	0.982	0.190	0.136	0.258	0.735	0.926	0.985
		0.1	0.201	0.206	0.518	0.97	0.973	0.999	0.244	0.187	0.495	0.959	0.979	0.994	0.153	0.185	0.494	0.964	0.972	0.997
	-0.2	0	0.631	0.343	0.218	0.120	0.107	0.104	0.607	0.350	0.229	0.123	0.135	0.104	0.342	0.242	0.191	0.293	0.179	0.110
		0.01	0.618	0.402	0.632	0.955	0.954	0.991	0.659	0.390	0.629	0.959	0.960	0.993	0.313	0.349	0.592	0.950	0.955	0.99
		0.1	0.270	0.493	0.699	0.978	0.985	0.997	0.238	0.458	0.682	0.977	0.975	0.998	0.170	0.428	0.687	0.979	0.977	0.997

Table 4. Size and power of the nonparametric stationarity test. Time Series; 5% significance level.

			Trend A						Trend B						Trend C					
	ρ_1	$q\backslash T$	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000	100	300	500	1000	1500	2000
ARHET	0.5	0	0.185	0.056	0.067	0.075	0.080	0.098	0.198	0.075	0.080	0.093	0.102	0.109	0.083	0.03	0.046	0.241	0.140	0.067
		0.01	0.196	0.190	0.490	0.837	0.946	0.993	0.201	0.204	0.544	0.890	0.951	0.991	0.039	0.160	0.519	0.892	0.959	0.991
		0.1	0.043	0.289	0.569	0.908	0.990	0.999	0.036	0.310	0.654	0.945	0.988	0.999	0.027	0.282	0.637	0.935	0.983	0.998
ARBIL	0.5	0	0.209	0.065	0.06	0.069	0.080	0.082	0.215	0.070	0.057	0.069	0.089	0.070	0.197	0.055	0.065	0.097	0.077	0.084
		0.01	0.226	0.075	0.092	0.269	0.463	0.718	0.253	0.074	0.078	0.256	0.483	0.719	0.192	0.066	0.084	0.275	0.457	0.718
		0.1	0.214	0.095	0.253	0.669	0.939	0.990	0.215	0.118	0.262	0.681	0.946	0.992	0.178	0.093	0.267	0.703	0.918	0.994
BIL	0.5	0	0.214	0.067	0.057	0.071	0.080	0.09	0.213	0.058	0.056	0.079	0.081	0.088	0.198	0.054	0.055	0.099	0.088	0.085
		0.01	0.206	0.065	0.121	0.371	0.550	0.737	0.207	0.059	0.117	0.340	0.535	0.739	0.192	0.052	0.115	0.358	0.559	0.732
		0.1	0.186	0.249	0.627	0.916	0.969	0.996	0.191	0.249	0.603	0.906	0.973	0.994	0.153	0.244	0.612	0.922	0.969	0.998
NLMA	0.5	0	0.213	0.057	0.051	0.067	0.06	0.084	0.211	0.053	0.054	0.071	0.082	0.081	0.185	0.047	0.054	0.091	0.085	0.087
		0.01	0.210	0.077	0.113	0.356	0.569	0.747	0.211	0.067	0.112	0.346	0.597	0.769	0.159	0.065	0.108	0.384	0.588	0.783
		0.1	0.215	0.239	0.618	0.913	0.973	0.997	0.177	0.247	0.618	0.931	0.979	0.994	0.153	0.234	0.626	0.913	0.977	0.996

Table 5. Rescaling factors for the nonparametric stationarity test.

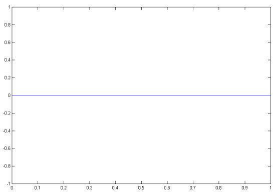
m_T	1	2	3	4	5	6	7	8	9	10
μ_{m_T}	0.06535	0.04002	0.02876	0.02242	0.01837	0.01556	0.01349	0.01191	0.01066	0.00964
s_{m_T}	0.04111	0.02017	0.01239	0.00856	0.00636	0.00496	0.00401	0.00333	0.00282	0.00243
m_T	11	12	13	14	15	16	17	18	19	20
μ_{m_T}	0.00881	0.00810	0.00750	0.00698	0.00653	0.00614	0.00579	0.00548	0.00519	0.00494
s_{m_T}	0.00212	0.00187	0.00167	0.00150	0.00135	0.00123	0.00113	0.00104	0.00096	0.00089
m_T	21	22	23	24	25	26	27	28	29	30
μ_{m_T}	0.00471	0.00450	0.00431	0.00413	0.00397	0.00382	0.00368	0.00355	0.00343	0.00332
s_{m_T}	0.00083	0.00077	0.00073	0.00068	0.00064	0.00061	0.00057	0.00054	0.00052	0.00049
m_T	31	32	33	34	35	36	37	38	39	40
μ_{m_T}	0.00322	0.00312	0.00302	0.00294	0.00285	0.00278	0.00270	0.00263	0.00256	0.00250
s_{m_T}	0.00047	0.00045	0.00043	0.00041	0.00039	0.00038	0.00036	0.00035	0.00033	0.00032

Table 6. Sensitivity analysis. Values of the test statistic under moderate variations of m_T and ℓ_T.

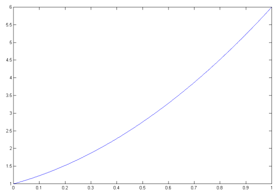
Japanese yen/US dollar exchange rate						Japanese yen/US dollar exchange rate (differences)					
	$\ell_T = 1$	$\ell_T = 2$	$\ell_T = 3$	$\ell_T = 4$	$\ell_T = 5$		–	–	$\ell_T = 0$	$\ell_T = 1$	$\ell_T = 2$
$m_T = 16$	32.840	18.290	12.077	8.652	6.482	$m_T = 16$	–	–	0.125	-0.0004	-0.355
$m_T = 17$	36.056	20.212	**13.446**	9.716	7.353	$m_T = 17$	–	–	**0.122**	-0.007	-0.372
$m_T = 18$	38.279	21.524	14.369	10.425	7.926	$m_T = 18$	–	–	0.433	0.293	-0.103
FTSE Eurotop 100 index (logs)						FTSE Eurotop 100 (returns)					
	$\ell_T = 3$	$\ell_T = 4$	$\ell_T = 5$	$\ell_T = 6$	$\ell_T = 7$		$\ell_T = 3$	$\ell_T = 4$	$\ell_T = 5$	$\ell_T = 6$	$\ell_T = 7$
$m_T = 16$	12.934	9.408	7.199	5.672	4.558	$m_T = 16$	-0.350	-1.440	-0.377	-0.185	0.943
$m_T = 17$	10.430	7.368	**5.449**	4.123	3.155	$m_T = 17$	-0.228	-1.382	**-0.257**	-0.054	1.140
$m_T = 18$	11.534	8.226	6.154	4.721	3.676	$m_T = 18$	0.030	-1.218	-0.001	0.219	1.512
Labor force participation rate											
	$\ell_T = 4$	$\ell_T = 5$	$\ell_T = 6$	$\ell_T = 7$	$\ell_T = 8$	–	–	–	–	–	–
$m_T = 13$	1.364	0.630	0.101	-0.311	-0.590	–	–	–	–	–	–
$m_T = 14$	1.797	0.989	**0.406**	-0.048	-0.355	–	–	–	–	–	–
$m_T = 15$	2.304	1.412	0.769	0.269	-0.070	–	–	–	–	–	–

Note: In bold type the value of the statistic for m_T and ℓ_T values obtained according to the data-driven rules. Critical values of the N(0,1) distribution: 1.282,1.645 and 2.326, at the 10%, 5% and 1% levels respectively.

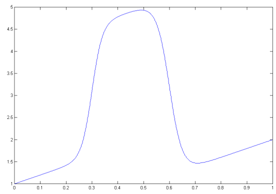
Figure 1. Trend specifications



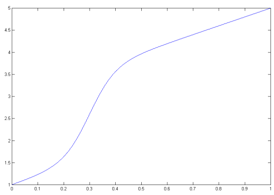
Panel 1. Trend A



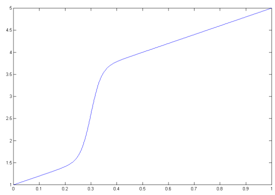
Panel 2. Trend B



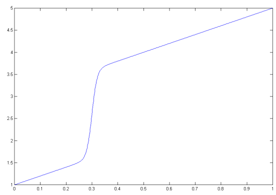
Panel 3. Trend C



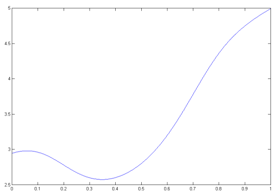
Panel 4. Trend D ($\gamma = 20$)



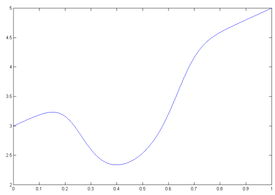
Panel 5. Trend D ($\gamma = 50$)



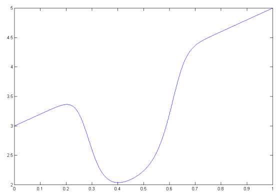
Panel 6. Trend D ($\gamma = 100$)



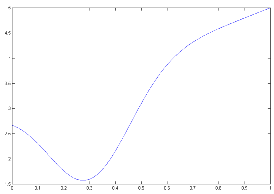
Panel 7. Trend E ($\gamma = 20$)



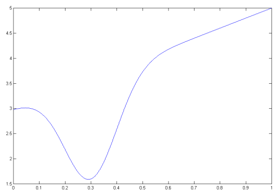
Panel 8. Trend E ($\gamma = 50$)



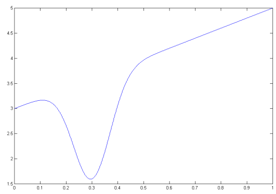
Panel 9. Trend E ($\gamma = 100$)



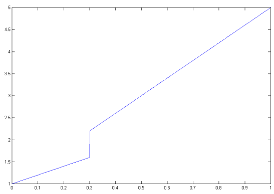
Panel 10. Trend F ($\gamma = 20$)



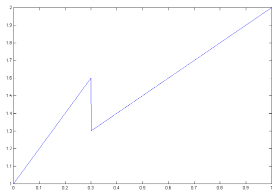
Panel 11. Trend F ($\gamma = 50$)



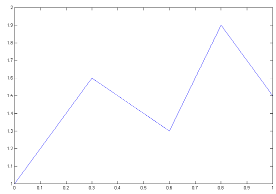
Panel 12. Trend F ($\gamma = 100$)



Panel 13. Trend G



Panel 14. Trend H



Panel 15. Trend I

Figure 2. Daily Japanese yen/US dollar exchange rate series (broken line) vs. nonparametric fitted trend (continuous line).

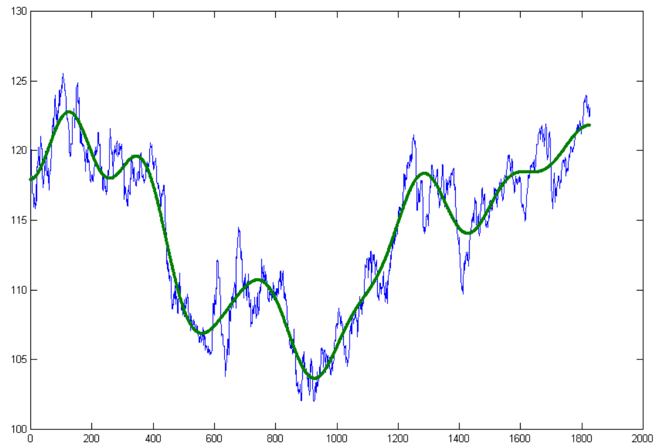


Figure 3. Daily FTSE Eurotop 100 index series, in logs (broken line) vs. nonparametric fitted trend (continuous line).

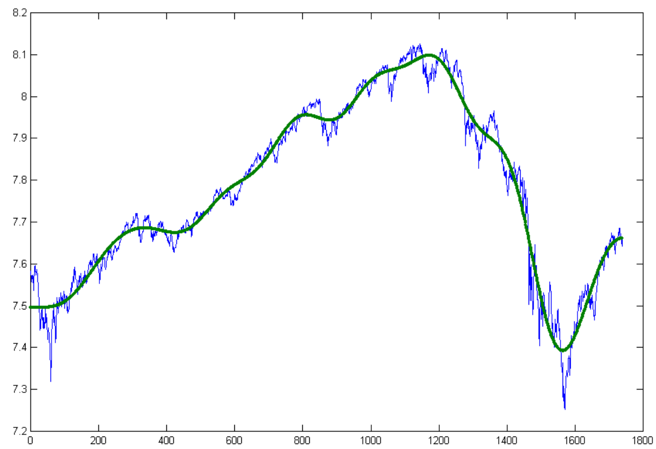


Figure 4. Monthly labor force participation rate series, (broken line) vs. nonparametric fitted trend (continuous line).