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Minimizing Conditional Value-at-Risk under Constraint on Expected Value

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Abstract

Conditional Value-at-Risk (CVaR) measures the expected loss amount beyond VaR. It has vast advantage over VaR because of its property of coherence. This paper gives an analytical solution in a complete market setting to the risk reward problem faced by a portfolio manager whose portfolio needs to be continuously rebalanced to minimize risk taken (measured by CVaR) while meeting the reward goal (measured by expected return). The optimal portfolio is identified whenever it exists, and the associated minimal risk is calculated. An example in the Black-Scholes framework is cited where dynamic hedging strategy is calculated and the efficient frontier is plotted.

Keywords: Conditional Value-at-Risk, Portfolio optimization, Risk minimization, Neyman-Pearson problem

JEL Classification: G11, G32, C61

Mathematics Subject Classification (2010): 91G10, 91B30, 90C46

1 Introduction

The portfolio selection problem studied by Markowitz [21] is formulated as an optimization problem with the objective of maximizing expected return, subject to the constraint of variance being bounded above. More recently, Bielecki et al. [6] solve the reverse problem in a dynamic setting with the objective of minimizing variance, subject to the constraint of expected return being bounded below. In both cases, the measure of risk of the portfolio is chosen as variance. However, it has been long noted that this dispersion measure is

*The findings and conclusions expressed are solely those of the author and do not represent views of the Federal Reserve Bank of New York, or the staff of the Federal Reserve System.

only valid as a risk measure when the loss distribution is symmetric, which is certainly not true for a typical loan portfolio where the distribution is left skewed. Much research has been done in developing risk measures that focus on extreme events in the tail. Value-at-Risk (VaR), a tail risk measure that answers the question of monetary loss over a specified time period with a certain probability, was introduce by JP Morgan [22] in its Riskmetrics system in 1994. It is regarded as the industrial standard in risk exposure measurement, and forms the basis for risk capital calculation. Research carried out by Campbell et al. [7], Consigli [9], Goldfarb and Iyengar [17], Gaivoronski and Pflug [15], Benati and Rizzi [5] centers around VaR, such as its measurement and risk reward optimization.

Although Value-at-Risk is the most dominant risk measure used in practice, it fails one of the four general properties, proposed by Artzner et al. [3] and [4], that a coherent risk measure should possess, namely the property of subadditivity. This property encourages diversification, and the lack of which "can destabilize an economy and induce crashes when they would not otherwise occur", see Danielsson et al. [11]. Conditional Value-at-Risk (CVaR), defined incrementally based on VaR, satisfies all the properties for coherence. For references on CVaR, see Acerbi and Tasche [1], and Rockafellar and Uryasev [23] and [24]. Its structural validity as a coherent risk measure, and its intuitive definition as the expected loss amount beyond VaR, attract great interest in both research community and industry. When historical simulation method is used for risk calculation, CVaR estimate is typically stabler than VaR estimate since the latter is a quantile number in the tail, which is highly sensitive to the updating of data set in the look-back period. Moreover, the convexity of CVaR observed by Rockafellar and Uryasev [23] provides great advantage over VaR in solving portfolio optimization problems.

Besides the choice of CVaR as the risk measure in this paper, the reward measure is chosen to be the expected return, as opposed to the expected utility on return. Gandy [16] and Zheng [30] focus on the problem of utility maximization with the constraint of CVaR being bounded from above. The dynamic solutions they derived are based on the assumption of a strictly concave utility function which excludes the expected return as a special case.

Along the line of mean-CVaR optimization, Rochafellar and Uryasev [23] and [24] propose a convex characterization of CVaR, which calls for the easily implemented tool of linear programming, and has been widely used as a simulation-based CVaR minimization technique in a static setting. Acerbi and Simonetti [2] extend this approach to a general spectral measure. However, no analytical solution is given, thus the technique cannot be adapted to a dynamic setting where the portfolio needs to be continuously rebalanced. Attempts have been made to cope with the dynamic case. Ruszczynsk and Shapiro [26] revise CVaR into a

dynamic risk measure, called the "conditional risk mapping for CVaR". Their paper leverages Rochafellar and Uryasev's static result for CVaR optimization at each time step, and rolls it backwards in time to achieve a dynamic version. In this paper, we extend Rochafellar and Uryasev's work, similar to the extension by Bielecki et al. [6] to the Markowitz [21] result in the mean-variance case, by measuring risk and reward at a fixed time horizon, while allowing dynamic portfolio management throughout the time period to achieve the mean-CVaR objective.

It has been observed in Kondor et al. [18] and Cherny [8], that the optimal portfolio normally does not exist for the mean-CVaR optimization in a static setting if the portfolio value is unbounded. When the solutions do exist in some limited cases, they take the form of a binary option. We confirm this result in a dynamic setting with a simple criterion in Theorem 3.17, where the portfolio value is allowed to be unbounded from above but restricted to be bounded from below since this is crucial in excluding arbitrage opportunities for continuous-time investment models. In the case the portfolio value is bounded both from below and from above, Schied [27], Sekine [28], and Li and Xu [20] find the optimal solution to be binary for CVaR minimization without the return constraint. We call this binary solution where the optimal portfolio either takes the value of the lower bound or a higher level 'Two-Line Configuration' in this paper. The key to finding the solution is the observation that the core part of CVaR minimization can be transformed into Shortfall risk minimization using the representation (CVaR is the Fenchel-Legendre dual of the Expected Shortfall) given by Rockafellar and Uryasev [23]. Föllmer and Leukert [13] characterize the solution to the latter problem in general semimartingale models to be binary ('Two-Line Configuration') where they have demonstrated its close relationship to the Neymann-Pearson lemma. The main contribution of our paper is that, when adding the return constraint to the CVaR minimization objective, we prove the optimal solution to be a 'Three-Line Configuration' in Theorem 3.15. This can be viewed in part as a generalization of the binary solution for Neymann-Pearson lemma with an additional constraint on expectation. The new solution can take not only the upper or the lower bound, but also a level in between.

This paper is organized as follows: Section 2 formulates the dynamic portfolio selection problem, gives a conceptual outline of the 'Three-Line Configuration' solution and shows an application of the Black-Scholes model; Section 3 details the analytic solution in general for both the case where portfolio value is bounded from above and the case where it is unbounded from above in a complete market setting; Section 4 lists possible future work. The Appendix records all the proofs.

2 Dynamic Portfolio Selection in the Mean-CVaR Plane

2.1 Conceptual Outline

Suppose the interest rate is a constant r and the risky asset S_t is a d-dimensional real-valued locally bounded semimartingale process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, P)$ that satisfies the usual conditions where $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. The value of a self-financing portfolio X_t which invests ξ_t shares in the risky asset evolves according to the dynamics

$$dX_t = \xi_t dS_t + r(X_t - \xi_t S_t)dt, \quad X_0 = x_0.$$

Here $\xi_t dS_t$ and $\xi_t S_t$ are interpreted as inner products if the risky asset is multidimensional $d > 1$ and as products if $d = 1$. In the above equation, we assume ξ_t is a d-dimensional predictable process such that the stochastic integral with respect to S_t is well-defined and we are looking for a strategy $(\xi_t)_{0\leq t\leq T}$ to minimize the conditional Value-at-Risk at confidence level $0 < \lambda < 1$ of the final portfolio value: $\inf_{\xi_t} CVaR_\lambda(X_T)$, while requiring the expected value to remain above a constant z: $E[X_T] \geq z$. In addition, we require uniform bounds on the value of the portfolio over time: $x_d \leq X_t \leq x_u \, a.s.$, $\forall t \in [0, T]$, where the constants satisfy $-\infty < x_d < x_0 < x_u \leq \infty$. Therefore, our **Main Problem** is

$$\begin{aligned} (1) \qquad & \inf_{\xi_t} CVaR_\lambda(X_T) \\ \text{subject to} \quad & E[X_T] \geq z, \\ & x_d \leq X_t \leq x_u \ a.s., \quad \forall t \in [0, T]. \end{aligned}$$

Note that the no-bankruptcy condition can be imposed by setting the lower bound $x_d = 0$, and the portfolio value can be unbounded from above by taking the upper bound x_u as infinity.

Assumption 2.1 *Assume there is No Free Lunch with Vanishing Risk (as defined in Delbaen and Schachermayer [10]) and the market is complete with a unique equivalent local martingale measure $\tilde{P}$ such that the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ has a continuous distribution.*

Under the above assumption any $\mathcal{F}$-measurable random variable can be replicated by a dynamic portfolio. Thus the dynamic optimization problem (1) can be reduced to a static problem

$$\text{(2)} \qquad \inf_{X\in\mathcal{F}} CVaR_\lambda(X)$$
$$\text{subject to} \quad E[X] \geq z, \quad \tilde{E}[X] = x_r, \quad x_d \leq X \leq x_u \ a.s.$$

Here the expectation E is taken under the physical probability measure P, and the expectation $\tilde{E}$ is taken under the risk neutral probability measure $\tilde{P}$, while $x_r = x_0 e^{rT}$ is assumed to satisfy $-\infty < x_d < x_0 \leq x_r < x_u \leq \infty$ in relation to the lower bound x_d, the upper bound x_u and the initial capital x_0.

Although in this paper we focus on the complete market solution, to solve the problem in an incomplete market setting, the exact hedging argument via Martingale Representation Theorem that translates the dynamic problem (1) into the static problem (2) has to be replaced by a super-hedging argument via Optional Decomposition developed by Kramkov [19], and Föllmer and Kabanov [12]. The detail is similar to the process carried out for Shortfall risk minimization in Föllmer and Leukert [13], and for convex risk minimization in Rudloff [25]. The second part of the assumption, namely the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ having a continuous distribution, is imposed for the simplification it brings to the presentation in the main theorems, instead of technical impossibility, for its lengthy discussion bring diminishing marginal new insight to our focus on an analytic solution for the main problem (2).

Using the equivalence between Conditional Value-at-Risk and the Fenchel-Legendre dual of the Expected Shortfall derived in Rockafellar and Uryasev [23],

$$\text{(3)} \qquad CVaR_\lambda(X) = \frac{1}{\lambda} \inf_{x\in\mathbb{R}} \left(E[(x-X)^+] - \lambda x \right), \quad \forall \lambda \in (0,1),$$

the static optimization problem (2) can be reduced to a two-step static optimization we name as

Two-Constraint Problem:

Step 1: Minimization of Expected Shortfall

$$\text{(4)} \qquad v(x) = \inf_{X\in\mathcal{F}} E[(x-X)^+]$$
$$\text{subject to} \quad E[X] \geq z, \quad (\textit{return constraint})$$
$$\tilde{E}[X] = x_r, \quad (\textit{capital constraint})$$
$$x_d \leq X \leq x_u \ a.s.$$

Step 2: Minimization of Conditional Value-at-Risk

$$\inf_{X\in\mathcal{F}} CVaR_\lambda(X) = \frac{1}{\lambda}\inf_{x\in\mathbb{R}}\left(v(x)-\lambda x\right). \tag{5}$$

Without the return constraint on the expectation $E[X] \geq z$, we name the problem as

One-Constraint Problem:

Step 1: Minimization of Expected Shortfall

$$\begin{aligned} v(x) &= \inf_{X\in\mathcal{F}} E[(x-X)^+] \\ \text{subject to} \quad & \tilde{E}[X] = x_r, \quad (\textit{capital constraint}) \\ & x_d \leq X \leq x_u \ a.s. \end{aligned} \tag{6}$$

Step 2: Minimization of Conditional Value-at-Risk

$$\inf_{X\in\mathcal{F}} CVaR_\lambda(X) = \frac{1}{\lambda}\inf_{x\in\mathbb{R}}\left(v(x)-\lambda x\right). \tag{7}$$

The solution to the *Step 1* of *One-Constraint Problem* (6) is given in Föllmer and Leukert [13] with a constant translation on the lower bound x_d; the solution to *Step 2* of *One-Constraint Problem* (7), and thus to the main problem in (1) and (2) without the return constraint is given in Schied [27], Sekine [28], and Li and Xu [20]. Schied [27] derives the solution to a more general law invariant risk measure which includes CVaR as a special case. Li and Xu [20] derive the solution to CVaR minimization without the assumption on the probability space being atomless and allowing the portfolio value to be unbounded from above.

In the rest of this subsection, we give a conceptual comparison between the solution to the *One-Constraint Problem* and the solution to the *Two-Constraint Problem*. To this end, we start with the specification of the solution to the *One-Constraint Problem*. A constant translation of the result from Föllmer and Leukert [13] yields the optimal solution to **Step 1** of the **One-Constraint Problem** under Assumption 2.1,

$$X(x) = x_d\mathbb{I}_A + x\mathbb{I}_{A^c}, \quad \text{for } x_d < x < x_u, \tag{8}$$

where we define the set $A = \left\{\omega\in\Omega : \frac{d\tilde{P}}{dP}(\omega) > a\right\}$ and $\mathbb{I}_{\cdot}(\omega)$ to be the indicator function. The optimality of this binary solution for X can be proved in various ways, but it is clearly a result of the Neyman-Pearson

Lemma once the connection between the problem of Minimization of Expected Shortfall and that of hypothesis testing between P and $\tilde{P}$ is established, see Föllmer and Leukert [13]. To view it as a solution from convex duality approach, see Theorem 1.19 in Xu [29]. A simplified version to the proof of Proposition 3.14 in this paper gives a direct method using Lagrange multiplier for convex optimization as a third approach.

Note that in (8), 'a' is computed from the budget constraint $\tilde{E}[X] = x_r$ for every fixed level 'x'. To proceed to **Step 2**, Li and Xu [20] vary the value of 'x' and look for the best x^* and its associated optimal a^*. Under some technical conditions, the solution to **Step 2** of the **One-Constraint Problem** is shown by Theorem 2.10 and Remark 2.11 in Li and Xu [20] to be

$$X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{A^{*c}}, \quad \textbf{(Two-Line Configuration)} \tag{9}$$

$$CVaR_\lambda(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)\left(P(A^*) - \lambda \tilde{P}(A^*)\right), \tag{10}$$

where (a^*, x^*) is the solution to the *capital constraint* ($\tilde{E}[X(x)] = x_r$) in **Step 1** and the *first order Euler condition* ($v'(x) = 0$) in **Step 2**:

$$x_d \tilde{P}(A) + x \tilde{P}(A^c) = x_r, \tag{11}$$

$$P(A) + \frac{\tilde{P}(A^c)}{a} - \lambda = 0. \tag{12}$$

We restate the full details of these results in general for convenience in Theorem 3.11 and Theorem 3.16 in Section 3. One interesting observation is that the optimal portfolio exists regardless whether the upper bound on the portfolio is finite $x_u < \infty$ or otherwise $x_u = \infty$, while we will see shortly that this is no longer true for the *Two-Constraint Problem*. More general solution when the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ is not restricted to have a continuous distribution is presented with computational examples in Li and Xu [20]. Another interesting aspect of the solution is that if the manager invests only in the riskless asset, then the portfolio value is constant $X = x_r$ and the risk is $CVaR = -x_r$. The possibility of investment in the risky asset is confirmed to decrease the risk as shown in (10).

The *Two-Line Configuration* in (9) as a final solution to the *One-Constraint Problem* is inherited from the structure of the solution to the Expected Shortfall Minimization in (8), thus possessing a direct link to the solution to the Neyman-Pearson Lemma. We show in this paper, particularly in Proposition 3.14 and Theorem 3.15, that when the upper bound is finite $x_u < \infty$, under some technical conditions, the solution to both **Step 1** and **Step 2** of the **Two-Constraint Problem**, and thus the **Main Problem** (1) and (2),

turns out to be a *Three-Line Configurations* of the form

$$X^{**} = x_d \mathbb{I}_{A^{**}} + x^{**} \mathbb{I}_{B^{**}} + x_u \mathbb{I}_{D^{**}}, \quad \textbf{(Three-Line Configuration)} \tag{13}$$

$$CVaR_\lambda(X_T^{**}) = \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right) \tag{14}$$

and (a^{**}, b^{**}, x^{**}) is the solution to the same two conditions as in the *Two-Line Configuration* case, namely the *capital constraint* and the *first order Euler condition*, plus the additional *return constraint* ($E[X(x)] = z$ where $X(x) = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$):

$$x_d P(A) + x P(B) + x_u P(D) = z, \tag{15}$$

$$x_d \tilde{P}(A) + x \tilde{P}(B) + x_u \tilde{P}(D) = x_r, \tag{16}$$

$$P(A) + \frac{\tilde{P}(B) - bP(B)}{a - b} - \lambda = 0, \tag{17}$$

where we define sets

$$A = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a\right\}, \quad B = \left\{\omega \in \Omega \,:\, b \le \tfrac{d\tilde{P}}{dP}(\omega) \le a\right\}, \quad D = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) < b\right\}.$$

This solution can be viewed as an extension of the binary solution for the Neyman-Pearson Lemma to a *Three-Line Configuration* where an extra constraint on the expected value is introduced.

We will see in Theorem 3.17 that the solution for the optimal portfolio most likely will not exist when the portfolio value is unbounded from above $x_u = \infty$, but the infimum of the CVaR can still be computed, and we can find a sequence of portfolios with *Three-Line Configuration* whose CVaR converge to the infimum.

Since we provide in this paper an analytical solution to the static CVaR minimization problem with the *Three-Line Configuration*, it is straight-forward to find the dynamic solution to the Main Problem (1) under the complete market assumption, even in multidimensional case for the risky asset. This result does not require the modification of CVaR measure, thus it is different from the solution given by Ruszczyński and Shapiro in [26].

2.2 Example: Mean-CVaR Portfolio Selection Problem with Bankruptcy Prohibition under the Black-Scholes Model

We illustrate the calculation of the optimal portfolio as a *Three-Line Configuration* (13) in the Black-Scholes Model. Suppose an agent is trading between a money market account with interest rate $r = 5\%$ and one stock that follows geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$ with parameter values $\mu = 0.2$, $\sigma = 0.1$ and $S_0 = 10$. The endowment starts at $x_0 = 10$ and bankruptcy is not allowed at any time, thus $x_d = 0$. The expected terminal value $E[X_T]$ at time horizon $T = 2$ is required to be above a fixed level 'z'. We first define two thresholds for the expected return: $z^* = E[X^*]$ where X^* comes from (9), is the expected return of the optimal portfolio for the *One Constraint Problem* (7); $\bar{z}$ is the highest expected value achievable by any self-financing portfolio starting with capital x_0 (see Definition 3.2 and Lemma 3.3). When 'z' is a low number, namely $z \leq z^*$, the additional *return requirement* is already satisfied by the *Two-Line Configuration* (9) and the optimal solutions for both the *One Constraint Problem* and the *Two Constraint Problem* coincide which we call the *'Star-System'* (9) and its calculation in the Black-Scholes Model is provided by Li and Xu [20]. When the *return requirement* becomes meaningful, i.e., $z \in (z^*, \bar{z}]$, we calculate the optimal *Three-Line Configuration* (13) which we call the *'Double-Star-System'*.

Since the stock price has log-normal distribution and the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ is a scaled power function of the final stock price, $P(A)$, $P(B)$, $P(D)$ and $\tilde{P}(A)$, $\tilde{P}(B)$, $\tilde{P}(D)$ can be computed as

$$P(A) = N(-\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln a}{\theta\sqrt{T}}), \quad P(D) = 1 - N(-\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln b}{\theta\sqrt{T}}), \quad P(B) = 1 - P(A) - P(D),$$
$$\tilde{P}(A) = N(\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln a}{\theta\sqrt{T}}), \quad \tilde{P}(D) = 1 - N(\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln b}{\theta\sqrt{T}}), \quad \tilde{P}(B) = 1 - \tilde{P}(A) - \tilde{P}(D),$$

where $N(\cdot)$ is the cumulative distribution function of a standard normal random variable and $\theta = \frac{\mu - r}{\sigma}$. Thus the solution (a^{**}, b^{**}, x^{**}) to equations (15)-(17) can be found numerically, and the final value of the optimal portfolio, the corresponding dynamic hedging strategy and the associated minimal CVaR are:

$$\begin{aligned}
X_t^{**} &= e^{-r(T-t)}[x^{**}N(d_+(a^{**}, S_t, t)) + x_d N(d_-(a^{**}, S_t, t))] \\
&\quad + e^{-r(T-t)}[x^{**}N(d_-(b^{**}, S_t, t)) + x_u N(d_+(b^{**}, S_t, t))] - e^{r(T-t)}x^{**}, \\
\xi_t^{**} &= \frac{x^{**} - x_d}{\sigma S_t\sqrt{2\pi(T-t)}} e^{-r(T-t) - \frac{d_-^2(a^{**}, S_t, t)}{2}} + \frac{x^{**} - x_u}{\sigma S_t\sqrt{2\pi(T-t)}} e^{-r(T-t) - \frac{d_+^2(b^{**}, S_t, t)}{2}}, \\
CVaR_\lambda(X_T^{**}) &= \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right),
\end{aligned}$$

where $d_-(a, s, t) = \frac{1}{\theta\sqrt{T-t}}[-\ln a + \frac{\theta}{\sigma}(\frac{\mu + r - \sigma^2}{2}t - \ln\frac{s}{S_0}) + \frac{\theta^2}{2}(T-t)]$, $\quad d_+(a, s, t) = -d_-(a, s, t)$.

One-Constraint Problem			Two-Constraint Problem			
x_u	30	50	x_u	30	30	50
			z	20	25	25
			$\bar{z}$	28.8866	28.8866	45.5955
			z^*	18.8742	18.8742	18.8742
x^*	19.0670	19.0670	x^{**}	19.1258	19.5734	19.1434
a^*	14.5304	14.5304	a^{**}	14.3765	12.5785	14.1677
			b^{**}	0.0068	0.1326	0.0172
$CVaR_{5\%}(X_T^*)$	-15.2118	-15.2118	$CVaR_{5\%}(X_T^{**})$	-15.2067	-14.8405	-15.1483

Table 1: Black-Scholes' Example for One-Constraint and Two-Constraint Problems

The numerical results are summarized in Table 1. As expected we observe that the upper bound on the portfolio value x_u has no impact on the 'Star-System', as (x^*, a^*) and $CVaR_\lambda(X_T^*)$ are optimal whenever $x_u \geq x^*$, including the case when $x_u = \infty$. However, the 'Double-Star-System' and the minimal $CVaR_\lambda(X_T^{**})$ are sensitive to x_u. The stricter the return requirement z and the higher the upper bound x_u, the more the *Three-Line Configuration* X^{**} deviates from the *Two-Line Configuration* X^*. The stricter return requirement (higher z) implies higher minimal $CVaR_\lambda(X_T^{**})$ compared to $CVaR_\lambda(X_T^*)$; the less strict upper bound (higher x_u) translates to decreased minimal $CVaR_\lambda(X_T^{**})$. As $x_u \to \infty$, $CVaR_\lambda(X_T^{**})$ approaches $CVaR_\lambda(X_T^*)$, in which case the 'Double-Star-System' is an approximation for the minimal $CVaR(X_T^*)$ achieved by the 'Star-System' while meeting the additional return requirement.

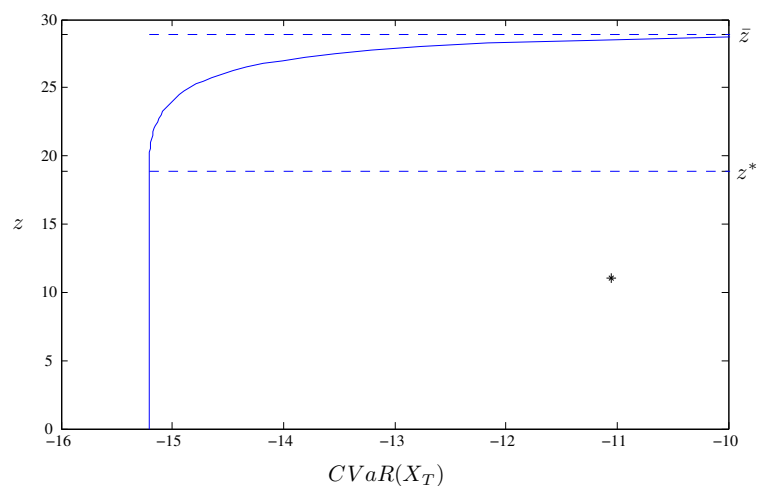

Figure 1: Efficient Frontier for Mean-CVaR Portfolio Selection

Figure 1 shows the efficient frontier of our mean-CVaR portfolio selection problem with bankruptcy prohibition $x_d = 0$ and upper bound $x_u = 30$. All the portfolios on the curve are efficient in the sense that the lowest risk as measured by CVaR is attained at each level of required expected return z; or conversely,

for every fixed level of risk, the portfolio achieves the highest expected return. When $z \leq z^*$, the straight line indicates that the optimal portfolio comes from the 'Star-System'. When $z \in (z^*, z_u]$, the 'Double-Star-System' forms the optimal portfolio and the minimal CVaR increases as return requirement z increases.

The star positioned at $(-x_r, x_r) = (-11.0517, 11.0517)$, where $x_r = x_0 e^{rT}$, corresponds to the portfolio that invests purely in the money market account. As a contrast to its position on the traditional Capital Market Line (the efficient frontier for a mean-variance portfolio selection problem), the pure money market account portfolio is no longer efficient in the mean-CVaR portfolio selection problem.

3 Analytical Solution to the Main Problem

Suppose Assumption 2.1 holds. We prove the solution to the mean-CVaR problem (2) in general, i.e., the *Two-Constraint Problem* (4) and (5), in this section in two seperate cases: when there is finite upper bound and when there is no upper bound on the portfolio value.

3.1 Case $x_u < \infty$: Finite Upper Bound

We first define the general Three-Line Configuration and some particular Two-Line Configurations as its degenerate forms. When the portfolio value is bounded from above, the constants satisfy $-\infty < x_d < x_r = x_0 e^{rT} < x_u < \infty$. Recall the definitions of the sets A, B, D are

$$A = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a\right\}, \quad B = \left\{\omega \in \Omega \,:\, b \leq \tfrac{d\tilde{P}}{dP}(\omega) \leq a\right\}, \quad D = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) < b\right\}. \tag{18}$$

Definition 3.1 *Suppose $x \in [x_d, x_u]$.*

1. *Any **Three-Line Configuration** has the structure $X = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$.*

2. *The **Two-Line Configuration** $X = x \mathbb{I}_B + x_u \mathbb{I}_D$ is associated to the above definition in the case $a = \infty$, $B = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) \geq b\right\}$ and $D = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) < b\right\}$.*
   *The **Two-Line Configuration** $X = x_d \mathbb{I}_A + x \mathbb{I}_B$ is associated to the above definition in the case $b = 0$, $A = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) > a\right\}$, and $B = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) \leq a\right\}$.*
   *The **Two-Line Configuration** $X = x_d \mathbb{I}_A + x_u \mathbb{I}_D$ is associated to the above definition in the case $a = b$, $A = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) > a\right\}$, and $D = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) < a\right\}$.*

Moreover,

1. ***General Constraints*** *are the* capital constraint *and the equality part of the* expected return constraint *for a* ***Three-Line Configuration*** $X = x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D$*:*

$$E[X] = x_d P(A) + xP(B) + x_u P(D) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x\tilde{P}(B) + x_u \tilde{P}(D) = x_r.$$

2. ***Degenerated Constraints 1*** *are the* capital constraint *and the equality part of the* expected return constraint *for a* ***Two-Line Configuration*** $X = x\mathbb{I}_B + x_u\mathbb{I}_D$*:*

$$E[X] = xP(B) + x_u P(D) = z,$$
$$\tilde{E}[X] = x\tilde{P}(B) + x_u \tilde{P}(D) = x_r.$$

Degenerated Constraints 2 *are the* capital constraint *and the equality part of the* expected return constraint *for a* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x\mathbb{I}_B$*:*

$$E[X] = x_d P(A) + xP(B) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x\tilde{P}(B) = x_r.$$

Degenerated Constraints 3 *are the* capital constraint *and the equality part of the* expected return constraint *for a* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x_u\mathbb{I}_D$*:*

$$E[X] = x_d P(A) + x_u P(D) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x_u \tilde{P}(D) = x_r.$$

Note that ***Degenerated Constraints 1*** *correspond to the* ***General Constraints*** *when* $a = \infty$*;* ***Degenerated Constraints 2*** *correspond to the* ***General Constraints*** *when* $b = 0$*; and* ***Degenerated Constraints 3*** *correspond to the* ***General Constraints*** *when* $a = b$*.*

We use the *Two-Line Configuration* $X = x_d\mathbb{I}_A + x_u\mathbb{I}_D$, where the value of the random variable X takes either the upper or the lower bound, as well as its capital constraint to define the 'Bar-System' from which we calculate the highest achievable return.

Definition 3.2 (The 'Bar-System') *For fixed* $-\infty < x_d < x_r < x_u < \infty$*, let* $\bar{a}$ *be a solution to the*

capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x_u\tilde{P}(D) = x_r$ *in* ***Degenerated Constraints 3*** *for the* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x_u\mathbb{I}_D$. *Consequently, the 'Bar-System'* $\bar{A}$, $\bar{D}$ *and* $\bar{X}$ *are associated to the constant* $\bar{a}$ *in the sense* $\bar{X} = x_d\mathbb{I}_{\bar{A}} + x_u\mathbb{I}_{\bar{D}}$ *where* $\bar{A} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > \bar{a}\right\}$, *and* $\bar{D} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < \bar{a}\right\}$. *Define the expected return of the 'Bar-System' as* $\bar{z} = E[\bar{X}] = x_dP(\bar{A}) + x_uP(\bar{D})$.

Lemma 3.3 $\bar{z}$ *is the highest expected return that can be obtained by a self-financing portfolio with initial capital* x_0 *whose value is bounded between* x_d *and* x_u*:*

$$\bar{z} = \max_{X \in \mathcal{F}} E[X] \quad s.t. \quad \tilde{E}[X] = x_r = x_0e^{rT}, \quad x_d \leq X \leq x_u \ a.s..$$

In the following lemma, we vary the 'x' value in the *Two-Line Configurations* $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ and $X = x_d\mathbb{I}_A + x\mathbb{I}_B$, while maintaining the capital constraints respectively. We observe their expected returns to vary between values x_r and $\bar{z}$ in a monotone and continuous fashion.

Lemma 3.4 *For fixed* $-\infty < x_d < x_r < x_u < \infty$.

1. *Given any* $x \in [x_d, x_r]$, *let 'b' be a solution to the capital constraint* $\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ *in* ***Degenerated Constraints 1*** *for the* ***Two-Line Configuration*** $X = x\mathbb{I}_B + x_u\mathbb{I}_D$. *Define the expected return of the resulting Two-Line Configuration as* $z(x) = E[X] = xP(B) + x_uP(D)$.[†] *Then* $z(x)$ *is a continuous function of* x *and decreases from* $\bar{z}$ *to* x_r *as* x *increases from* x_d *to* x_r.

2. *Given any* $x \in [x_r, x_u]$, *let 'a' be a solution to the capital constraint* $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) = x_r$ *in* ***Degenerated Constraints 2*** *for the* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x\mathbb{I}_B$. *Define the expected return of the resulting Two-Line Configuration as* $z(x) = E[X] = x_dP(A) + xP(B)$. *Then* $z(x)$ *is a continuous function of* x *and increases from* x_r *to* $\bar{z}$ *as* x *increases from* x_r *to* x_u.

From now on, we will concern ourselves with requirements on the expected return in the interval $z \in [x_r, \bar{z}]$. Lemma 3.3 ensures that there are no feasible solutions to the Main Problem (2) if we require a higher expected return than $\bar{z}$. We now make the argument that, when the return requirement is below x_r, the optimal solution to the **One-Constraint Problem** automatically satisfies the additional return constraint, thus is the optimal solution to the **Two-Constraint Problem**. Lemma 3.4 demonstrates that the **Two-Line Configuration** $X = x_d\mathbb{I}_A + x\mathbb{I}_B$ satisfying the capital constraint, will also satisfy the return constraint in this case. We refer to Li and Xu [20] for the general optimal solution to the **One-Constraint Problem**. For

[†]Threshold 'b' and consequently sets 'B' and 'D' are all dependent on 'x' through the capital constraint, therefore $z(x)$ is not a linear function of x.

convenience, we restate them later in Theorem 3.11 under additional Assumption 2.1. From this theorem, we see that the optimal solution to the **One-Constraint Problem**, either takes this *Two-Line Configuration* form with $x = x^*$ which we call the 'Star-System' $X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$; or coincides with the 'Bar-System'; or results from the pure money market account investment with expected return x_r. Lemma 3.3 and Lemma 3.4 then lead to the conclusion that a return constraint where $z \in (-\infty, x_r)$ is too weak to differentiate the **Two-Constraint Problem** from the **One-Constraint Problem** as their optimal solutions concur.

Definition 3.5 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$, define x_{z1} and x_{z2} to be the corresponding x value for* ***Two-Line Configurations*** *$X = x\mathbb{I}_B + x_u \mathbb{I}_D$ and $X = x_d \mathbb{I}_A + x\mathbb{I}_B$ that satisfy* ***Degenerated Constraints 1*** *and* ***Degenerated Constraints 2*** *respectively.*

Definition 3.5 implies when we fix the level of expected return z, we can find two particular feasible solutions: $X = x_{z1}\mathbb{I}_B + x_u \mathbb{I}_D$ satisfying $\tilde{E}[X] = x_{z1}\tilde{P}(B) + x_u \tilde{P}(D) = x_r$ and $E[X] = x_{z1}P(B) + x_u P(D) = z$; $X = x_d \mathbb{I}_A + x_{z2}\mathbb{I}_B$ satisfying $\tilde{E}[X] = x_d \tilde{P}(A) + x_{z2}\tilde{P}(B) = x_r$ and $E[X] = x_d P(A) + x_{z2} P(B) = z$. The values x_{z1} and x_{z2} are well-defined because Lemma 3.4 guarantees $z(x)$ to be an invertible function in both cases. We summarize in the following lemma whether the *Two-Line Configurations* satisfying the capital constraints meet or fail the return constraint as x ranges over its domain $[x_d, x_u]$ for the *Two-Line* and *Three-Line Configurations* in Definition 3.1.

Lemma 3.6 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$.*

1. *If we fix $x \in [x_d, x_{z1}]$, the Two-Line Configuration $X = x\mathbb{I}_B + x_u \mathbb{I}_D$ which satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u \tilde{P}(D) = x_r$ in Degenerated Constraints 1* ***satisfies*** *the expected return constraint: $E[X] = xP(B) + x_u P(D) \geq z$;*

2. *If we fix $x \in (x_{z1}, x_r]$, the Two-Line Configuration $X = x\mathbb{I}_B + x_u \mathbb{I}_D$ which satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u \tilde{P}(D) = x_r$ in Degenerated Constraints 1* ***fails*** *the expected return constraint: $E[X] = xP(B) + x_u P(D) < z$;*

3. *If we fix $x \in [x_r, x_{z2})$, the Two-Line Configuration $X = x_d \mathbb{I}_A + x\mathbb{I}_B$ which satisfies the capital constraint $\tilde{E}[X] = x_d \tilde{P}(A) + x\tilde{P}(B) = x_r$ in Degenerated Constraints 2* ***fails*** *the expected return constraint: $E[X] = xP(B) + x_u P(D) < z$;*

4. *If we fix $x \in [x_{z2}, x_u]$, the Two-Line Configuration $X = x_d \mathbb{I}_A + x\mathbb{I}_B$ which satisfies the capital constraint $\tilde{E}[X] = x_d \tilde{P}(A) + x\tilde{P}(B) = x_r$ in Degenerated Constraints 2* ***satisfies*** *the expected return constraint: $E[X] = xP(B) + x_u P(D) \geq z$.*

We turn our attention to solving **Step 1** of the **Two-Constraint Problem** (4):

Step 1: Minimization of Expected Shortfall

$$\begin{aligned} & v(x) = \inf_{X \in \mathcal{F}} E[(x - X)^+] \\ \text{subject to} \quad & E[X] \geq z, \quad (\textit{return constraint}) \\ & \tilde{E}[X] = x_r, \quad (\textit{capital constraint}) \\ & x_d \leq X \leq x_u \ a.s.; \end{aligned}$$

Notice that a solution is called for any given real number x, independent of the return level z or capital level x_r. From Lemma 3.6 and the fact that the *Two-Line Configurations* are optimal solutions to **Step 1** of the **One-Constraint Problem** (see Theorem 2.2 in Li and Xu [20]), we can immediately draw the following conclusion.

Proposition 3.7 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$.*

1. *If we fix $x \in [x_d, x_{z1}]$, then there exists a **Two-Line Configuration** $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ which is the optimal solution to **Step 1** of the **Two-Constraint Problem**;*

2. *If we fix $x \in [x_{z2}, x_u]$, then there exists a **Two-Line Configuration** $X = x_d\mathbb{I}_A + x\mathbb{I}_B$ which is the optimal solution to **Step 1** of the **Two-Constraint Problem**.*

When $x \in (x_{z1}, x_{z2})$, Lemma 3.6 shows that the Two-Line Configurations which satisfy the capital constraints ($\tilde{E}[X] = x_r$) do not generate high enough expected return ($E[X] < z$) to be feasible anymore. It turns out that a novel solution of *Three-Line Configuration* is the answer: it can be shown to be both feasible and optimal.

Lemma 3.8 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$. Given any $x \in (x_{z1}, x_{z2})$, let the pair of numbers $(a, b) \in \mathbb{R}^2$ ($b \leq a$) be a solution to the capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in **General Constraints** for the **Three-Line Configuration** $X = x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D$. Define the expected return of the resulting Three-Line Configuration as $z(a, b) = E[X] = x_dP(A) + xP(B) + x_uP(D)$. Then $z(a, b)$ is a continuous function which decreases from $\bar{z}$ to a number below z:*

1. *When $a = b = \bar{a}$ from Definition 3.2 of 'Bar-System', the Three-Line Configuration degenerates to $X = \bar{X}$ and $z(\bar{a}, \bar{a}) = E[\bar{X}] = \bar{z}$.*

2. *When $b < \bar{a}$ and $a > \bar{a}$, $z(a, b)$ decreases continuously as b decreases and a increases.*

3. *In the extreme case when $a = \infty$, the Three-Line configuration becomes the Two-Line Configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$; in the extreme case when $b = 0$, the Three-Line configuration becomes the Two-Line Configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B$. In either case, the expected value is below z by Lemma 3.6.*

Proposition 3.9 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$. If we fix $x \in (x_{z1}, x_{z2})$, then there exists a* ***Three-Line Configuration*** *$X = x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D$ that satisfies the* ***General Constraints*** *which is the optimal solution to* ***Step 1*** *of the* ***Two-Constraint Problem****.*

Combining Proposition 3.7 and Proposition 3.9 (a main result on the optimality of the Three-Line Configuration), we arrive to the following summary of the solutions.

Theorem 3.10 (Solution to Step 1: Minimization of Expected Shortfall)
For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in [x_r, \bar{z}]$. $X(x)$ and the corresponding value function $v(x)$ described below are optimal solutions to ***Step 1: Minimization of Expected Shortfall*** *of the* ***Two-Constraint Problem****:*

- *$x \in (-\infty, x_d]$: $X(x) =$ any random variable with values in $[x_d, x_u]$ satisfying both the capital constraint $\tilde{E}[X(x)] = x_r$ and the return constraint $E[X(x)] \geq z$. $v(x) = 0$.*
- *$x \in [x_d, x_{z1}]$: $X(x) =$ any random variable with values in $[x, x_u]$ satisfying both the capital constraint $\tilde{E}[X(x)] = x_r$ and the return constraint $E[X(x)] \geq z$. $v(x) = 0$.*
- *$x \in (x_{z1}, x_{z2})$: $X(x) = x_d\mathbb{I}_{A_x} + x\mathbb{I}_{B_x} + x_u\mathbb{I}_{D_x}$ where A_x, B_x, D_x are determined by a_x and b_x as in (18) satisfying the General Constraints: $\tilde{E}[X(x)] = x_r$ and $E[X(x)] = z$. $v(x) = (x - x_d)P(A_x)$.*
- *$x \in [x_{z2}, x_u]$: $X(x) = x_d\mathbb{I}_{A_x} + x\mathbb{I}_{B_x}$ where A_x, B_x are determined by a_x as in Definition 3.1 satisfying both the capital constraint $\tilde{E}[X(x)] = x_r$ and the return constraint $E[X(x)] \geq z$. $v(x) = (x - x_d)P(A_x)$.*
- *$x \in [x_u, \infty)$: $X(x) = x_d\mathbb{I}_{\bar{A}} + x_u\mathbb{I}_{\bar{B}} = \bar{X}$ where $\bar{A}, \bar{B}$ are associated to $\bar{a}$ as in Definition 3.2 satisfying both the capital constraint $\tilde{E}[X(x)] = x_r$ and the return constraint $E[X(x)] = \bar{z} \geq z$.*
  $v(x) = (x - x_d)P(\bar{A}) + (x - x_u)P(\bar{B})$.

To solve **Step 2** of the **Two-Constraint Problem**, and thus the Main Problem (2), we need to find

$$\frac{1}{\lambda} \inf_{x \in \mathbb{R}} (v(x) - \lambda x),$$

where $v(x)$ has been computed in Theorem 3.10. Depending on the z level in the return constraint being lenient or strict, the solution is sometimes obtained by the Two-Line Configuration which is optimal to the

One-Constraint Problem, and at other times obtained by a true Three-Line configuration. To proceed in this direction, we recall the solution to the **One-Constraint Problem** from Li and Xu [20].

Theorem 3.11 (Theorem 2.10 and Remark 2.11 in Li and Xu [20] when $x_u < \infty$)

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$. $X = x_r$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$.

   - *If* $\frac{1}{\bar{a}} \leq \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$ *(see Definition 3.2 for the 'Bar-System'), then* $\bar{X} = x_d \mathbb{I}_{\bar{A}} + x_u \mathbb{I}_{\bar{D}}$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(\bar{X}) = -x_r + \frac{1}{\lambda}(x_u - x_d)(P(\bar{A}) - \lambda \tilde{P}(\bar{A})).$$

   - *Otherwise, let* a^* *be the solution to the equation* $\frac{1}{a} = \frac{\lambda - P(A)}{1-\tilde{P}(A)}$. *Associate sets* $A^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a^*\right\}$ *and* $B^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) \leq a^*\right\}$ *to level* a^*. *Define* $x^* = \frac{x_r - x_d \tilde{P}(A^*)}{1-\tilde{P}(A^*)}$ *so that configuration*

$$X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$$

   satisfies the capital constraint $\tilde{E}[X^*] = x_d \tilde{P}(A^*) + x^* \tilde{P}(B^*) = x_r$.[‡] *Then* X^* *(we call the* **'Star-System'***) is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

Definition 3.12 *In part 2 of Theorem 3.11, define* $z^* = \bar{z}$ *in the first case when* $\frac{1}{\bar{a}} \leq \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$*; define* $z^* = E[X^*]$ *in the second case when* $\frac{1}{\bar{a}} > \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$.

We see that when z is smaller than z^*, the binary solutions X^* and $\bar{X}$ provided in Theorem 3.11 are indeed the optimal solution to **Step 2: Minimization of Conditional Value-at-Risk** of the **Two-Constraint**

[‡]Equivalently, (a^*, x^*) can be viewed as the solution to the capital constraint and the first order Euler condition in equations (11) and (12).

Problem. However, when z is greater than z^* these Two-Line Configurations are no longer feasible in the **Two-Constraint Problem**. We now show that the Three-Line Configuration is not only feasible but also optimal. First we establish the convexity of the objective function and its continuity in a Lemma.

Lemma 3.13 *$v(x)$ is a convex function for $x \in \mathbb{R}$, and thus continuous.*

Proposition 3.14 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and a fixed level $z \in (z^*, \bar{z}]$. Suppose $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$. The solution (a^{**}, b^{**}, x^{**}) (and consequently, A^{**}, B^{**} and D^{**}) to the equations*

$$\begin{aligned} x_d P(A) + xP(B) + x_u P(D) &= z, \quad (\textit{return constraint}) \\ x_d \tilde{P}(A) + x\tilde{P}(B) + x_u \tilde{P}(D) &= x_r, \quad (\textit{capital constraint}) \\ P(A) + \frac{\tilde{P}(B) - bP(B)}{a-b} - \lambda &= 0, \quad (\textit{first order Euler condition}) \end{aligned}$$

exists. $X^{**} = x_d \mathbb{I}_{A^{**}} + x^{**} \mathbb{I}_{B^{**}} + x_u \mathbb{I}_{D^{**}}$ (we call the **'Double-Star System'***) is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^{**}) = \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right).$$

Joining Proposition 3.14 with Theorem 3.11, we arrive to the Main Theorem of this paper.

Theorem 3.15 (Minimization of Conditional Value-at-Risk When $x_u < \infty$)
For fixed $-\infty < x_d < x_r < x_u < \infty$.

1. *Suppose $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ and $z = x_r$. The pure money market account investment $X = x_r$ is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ and $z \in (x_r, \bar{z}]$. The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X) = -x_r.$$

3. *Suppose $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ and $z \in [x_r, z^*]$ (see Definition 3.12 for z^*).*

- *If* $\frac{1}{a} \leq \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$ *(see Definition 3.2), then the* **'Bar-System'** $\bar{X} = x_d \mathbb{I}_{\bar{A}} + x_u \mathbb{I}_{\bar{D}}$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(\bar{X}) = -x_r + \frac{1}{\lambda}(x_u - x_d)(P(\bar{A}) - \lambda \tilde{P}(\bar{A})).$$

- *Otherwise, the* **'Star-System'** $X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$ *defined in Theorem 3.11 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

4. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in (z^*, \bar{z}]$. *the* **'Double-Star-Sytem'** $X^{**} = x_d \mathbb{I}_{A^{**}} + x^{**} \mathbb{I}_{B^{**}} + x_u \mathbb{I}_{D^{**}}$ *defined in Proposition 3.14 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^{**}) = \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right).$$

We observe that the pure money market account investment is rarely optimal. The condition that the Radon-Nikodým derivative is bounded above ($\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$) is not satisfied in typical continuous distribution model, for example the Black-Scholes model. When the return constraint is low $z \in [x_r, z^*]$, the Two-Line Configurations which are optimal to the $CVaR$ minimization problem without the return constraint is also the optimal when adding the return constraint. When the return constraint is materially high $z \in (z^*, \bar{z}]$, the optimal Three-Line-Configuration takes the value of the upper bound x_u to raise the expected return although the minimal risk will be compromised at a higher level. We have already seen this in a numerical example in Section 2.2.

3.2 Case $x_u = \infty$: No Upper Bound

We first restate the solution to the **One-Constraint Problem** from Li and Xu [20] in the current context: when $x_u = \infty$, we interpret $\bar{A} = \Omega$ and $\bar{z} = \infty$.

Theorem 3.16 (Theorem 2.10 and Remark 2.11 in Li and Xu [20] when $x_u = \infty$)

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$. *The pure money market account investment* $X = x_r$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$. *The* **'Star-System'** $X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$ *defined in Theorem 3.11 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

We observe that although there is no upper bound for the portfolio value, the optimal solution remains bounded from above, and the minimal $CVaR$ is bounded from below. The question of minimizing $CVaR$ risk of a self-financing portfolio (bounded from below by x_d to exclude arbitrage) from initial capital x_0 is meaningful in the sense that the risk will not approach $-\infty$ and the minimal risk can be achieved by an optimal portfolio. We will see in the following theorem that in the case we add substantial return constraint to the $CVaR$ minimization problem, although the minimal risk can still be calculated, they are truly infimum and not minimum, thus they can be approximated closely by a sub-optimal portfolio, but not achieved by an optimal portfolio.

Theorem 3.17 (Minimization of Conditional Value-at-Risk When $x_u = \infty$)
For fixed $-\infty < x_d < x_r < x_u = \infty$.

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ *and* $z = x_r$. *The pure money market account investment* $X = x_r$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ *and* $z \in (x_r, \infty)$. *The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X) = -x_r.$$

3. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in [x_r, z^*]$. *The* **'Star-System'** $X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$ *defined in Theorem 3.11 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

4. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in (z^*, \infty)$. *The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

From the proof of the above theorem in the **Appendix**, we note that in case *4*, we can always find a Three-Line Configuration as a sub-optimal solution, i.e., there exists for every $\epsilon > 0$, a corresponding portfolio $X_\epsilon = x_d \mathbb{I}_{A_\epsilon} + x_\epsilon \mathbb{I}_{B_\epsilon} + \alpha_\epsilon \mathbb{I}_{D_\epsilon}$ which satisfies the *General Constraints* and produces a $CVaR$ level close to the lower bound: $CVaR(X_\epsilon) \leq CVaR(X^*) + \epsilon$.

4 Future Work

In Assumption 2.1, we require the Radon-Nikodým derivative to have continuous distribution. When this assumption is weakened, the main results should still hold, albeit in a more complicated form. The outcome in its format resembles techniques employed in Föllmer and Leukert [13] and Li and Xu [20] where the point masses on the thresholds for the Radon-Nikodým derivative in Definition (18) have to be dealt with carefully. It will also be interesting to extend the closed-form solution for CVaR minimization to the minimization of Law-Invariant Risk Measures in general. Investigation of the solutions in incomplete markets is a natural broadening of curiosity: will the Third-Line Configuration remain optimal?

5 Appendix

Proof of Lemma 3.3. The problem of

$$\bar{z} = \max_{X \in \mathcal{F}} E[X] \quad s.t. \quad \tilde{E}[X] = x_r, \quad x_d \leq X \leq x_u \, a.s.$$

is equivalent to the Expected Shortfall Problem

$$\bar{z} = -\min_{X\in\mathcal{F}} E[(x_u - X)^+] \quad s.t. \quad \tilde{E}[X] = x_r, \quad X \geq x_d \, a.s.$$

Therefore, the answer is immediate. ◇

Proof of Lemma 3.4. Choose $x_d \leq x_1 < x_2 \leq x_r$. Let $X_1 = x_1\mathbb{I}_{B_1} + x_u\mathbb{I}_{D_1}$ where $B_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) \geq b_1\right\}$ and $D_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}$. Choose b_1 such that $\tilde{E}[X_1] = x_r$. This capital constraint means $x_1\tilde{P}(B_1) + x_u\tilde{P}(D_1) = x_r$. Since $\tilde{P}(B_1) + \tilde{P}(D_1) = 1$, $\tilde{P}(B_1) = \frac{x_u - x_r}{x_u - x_1}$ and $\tilde{P}(D_1) = \frac{x_r - x_1}{x_u - x_1}$. Define $z_1 = E[X_1]$. Similarly, z_2, X_2, B_2, D_2, b_2 corresponds to x_2 where $b_1 > b_2$ and $\tilde{P}(B_2) = \frac{x_u - x_r}{x_u - x_2}$ and $\tilde{P}(D_2) = \frac{x_r - x_2}{x_u - x_2}$. Note that $D_2 \subset D_1$, $B_1 \subset B_2$ and $D_1 \backslash D_2 = B_2 \backslash B_1$. We have

$$\begin{aligned}
z_1 - z_2 &= x_1 P(B_1) + x_u P(D_1) - x_2 P(B_2) - x_u P(D_2) \\
&= (x_u - x_2)P(B_2\backslash B_1) - (x_2 - x_1)P(B_1) \\
&= (x_u - x_2)P\left(b_2 < \tfrac{d\tilde{P}}{dP}(\omega) < b_1\right) - (x_2 - x_1)P\left(\tfrac{d\tilde{P}}{dP}(\omega) \geq b_1\right) \\
&= (x_u - x_2)\int_{\left\{b_2 < \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}} \tfrac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega) - (x_2 - x_1)\int_{\left\{\frac{d\tilde{P}}{dP}(\omega) \geq b_1\right\}} \tfrac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega) \\
&> (x_u - x_2)\frac{1}{b_1}\tilde{P}(B_2\backslash B_1) - (x_2 - x_1)\frac{1}{b_1}\tilde{P}(B_1) \\
&= (x_u - x_2)\frac{1}{b_1}\left(\frac{x_u - x_r}{x_u - x_2} - \frac{x_u - x_r}{x_u - x_1}\right) - (x_2 - x_1)\frac{1}{b_1}\frac{x_u - x_r}{x_u - x_1} = 0.
\end{aligned}$$

For any given $\epsilon > 0$, choose $x_2 - x_1 \leq \epsilon$, then

$$\begin{aligned}
z_1 - z_2 &= (x_u - x_1)P(B_2\backslash B_1) - (x_2 - x_1)P(B_2) \\
&\leq (x_u - x_1)P(B_2\backslash B_1) \\
&\leq (x_u - x_1)\left(\frac{x_u - x_r}{x_u - x_2} - \frac{x_u - x_r}{x_u - x_1}\right) \\
&\leq \frac{(x_2 - x_1)(x_u - x_r)}{x_u - x_2} \leq x_2 - x_1 \leq \epsilon.
\end{aligned}$$

Therefore, z decreases continuously as x increases when $x \in [x_d, x_r]$. When $x = x_d$, $z = \bar{z}$ from Definition 3.2. When $x = x_r$, $X \equiv x_r$ and $z = E[X] = x_r$. Similarly, we can show that z increases continuously from x_r to $\bar{z}$ as x increases from x_r to x_u. ◇

Lemma 3.6 is a logical consequence of Lemma 3.4 and Definition 3.5; Proposition 3.7 follows from Lemma

3.6; so their proofs will be skipped.

Proof of Lemma 3.8. Choose $-\infty < b_1 < b_2 \leq \bar{b} = \bar{a} \leq a_2 < a_1 < \infty$. Let configuration $X_1 = x_d \mathbb{I}_{A_1} + x\mathbb{I}_{B_1} + x_u \mathbb{I}_{D_1}$ correspond to the pair (a_1, b_1) where $A_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a_1\right\}, B_1 = \left\{\omega \in \Omega : b_1 \leq \frac{d\tilde{P}}{dP}(\omega) \leq a_1\right\}, D_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}$. Similarly, let configuration $X_2 = x_d \mathbb{I}_{A_2} + x\mathbb{I}_{B_2} + x_u \mathbb{I}_{D_2}$ correspond to the pair (a_2, b_2). Define $z_1 = E[X_1]$ and $z_2 = E[X_2]$. Since both X_1 and X_2 satisfy the capital constraint, we have

$$x_d \tilde{P}(A_1) + x\tilde{P}(B_1) + x_u \tilde{P}(D_1) = x_r = x_d \tilde{P}(A_2) + x\tilde{P}(B_2) + x_u \tilde{P}(D_2).$$

This simplifies to the equation

$$(x - x_d)\tilde{P}(A_2 \backslash A_1) = (x_u - x)\tilde{P}(D_2 \backslash D_1). \tag{19}$$

Then

$$\begin{aligned}
z_2 - z_1 &= x_d P(A_2) + xP(B_2) + x_u P(D_2) - x_d P(A_1) - xP(B_1) - x_u P(D_1) \\
&= (x_u - x)P(D_2 \backslash D_1) - (x - x_d)P(A_2 \backslash A_1) \\
&= (x_u - x)P(D_2 \backslash D_1) - (x_u - x)\frac{\tilde{P}(D_2 \backslash D_1)}{\tilde{P}(A_2 \backslash A_1)} P(A_2 \backslash A_1) \\
&= (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{P(D_2 \backslash D_1)}{\tilde{P}(D_2 \backslash D_1)} - \frac{P(A_2 \backslash A_1)}{\tilde{P}(A_2 \backslash A_1)}\right) \\
&= (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{\int_{\left\{b_1 \leq \frac{d\tilde{P}}{dP}(\omega) < b_2\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega)}{\tilde{P}(D_2 \backslash D_1)} - \frac{\int_{\left\{a_2 < \frac{d\tilde{P}}{dP}(\omega) \leq a_1\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega)}{\tilde{P}(A_2 \backslash A_1)}\right) \\
&\geq (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{1}{b_2} - \frac{1}{a_2}\right) > 0.
\end{aligned}$$

Suppose the pair (a_1, b_1) is chosen so that X_1 satisfies the budget constraint $\tilde{E}[X_1] = x_r$. For any given $\epsilon > 0$, choose $b_2 - b_1$ small enough such that $P(D_2 \backslash D_1) \leq \frac{\epsilon}{x_u - x}$. Now choose a_2 such that $a_2 < a_1$ and equation (19) is satisfied. Then X_2 also satisfies the budget constraint $\tilde{E}[X_2] = x_r$, and

$$z_2 - z_1 = (x_u - x)P(D_2 \backslash D_1) - (x - x_d)P(A_2 \backslash A_1) \leq (x_u - x)P(D_2 \backslash D_1) \leq \epsilon.$$

We conclude that the expected value of the Three-Line configuration decreases continuously as b decreases

and a increases. $\diamond$

In the following we provide the main proof of the paper: the optimality of the Three-Line configuration.

Proof of Proposition 3.9. Denote $\rho = \frac{d\tilde{P}}{dP}$. According to Lemma 3.8, there exists a Three-Line configuration $\hat{X} = x_d \mathbb{I}_A + x\mathbb{I}_B + x_u \mathbb{I}_D$ that satisfies the General Constraints:

$$E[X] = x_d P(A) + xP(B) + x_u P(D) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x\tilde{P}(B) + x_u \tilde{P}(D) = x_r.$$

where

$$A = \{\omega \in \Omega \,:\, \rho(\omega) > \hat{a}\}, \quad B = \left\{\omega \in \Omega \,:\, \hat{b} \le \rho(\omega) \le \hat{a}\right\}, \quad D = \left\{\omega \in \Omega \,:\, \rho(\omega) < \hat{b}\right\}.$$

As standard for convex optimization problems, if we can find a pair of Lagrange multipliers $\lambda \ge 0$ and $\mu \in \mathbb{R}$ such that $\hat{X}$ is the solution to the minimization problem

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E[(x - X)^+ - \lambda X - \mu\rho X] = E[(x - \hat{X})^+ - \lambda \hat{X} - \mu\rho \hat{X}], \tag{20}$$

then $\hat{X}$ is the solution to the constrained problem

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E[(x - X)^+], \quad s.t. \quad E[X] \ge z, \quad \tilde{E}[X] = x_r.$$

Define

$$\lambda = \frac{\hat{b}}{\hat{a} - \hat{b}}, \quad \mu = -\frac{1}{\hat{a} - \hat{b}}.$$

Then (20) becomes

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E\left[(x - X)^+ + \tfrac{\rho - \hat{b}}{\hat{a} - \hat{b}} X\right].$$

Choose any $X \in \mathcal{F}$ where $x_d \le X \le x_u$, and denote $G = \{\omega \in \Omega \,:\, X(\omega) \ge x\}$ and $L = \{\omega \in \Omega \,:\, X(\omega) < x\}$.

Note that $\frac{\rho-\hat{b}}{\hat{a}-\hat{b}} > 1$ on set A, $0 \le \frac{\rho-\hat{b}}{\hat{a}-\hat{b}} \le 1$ on set B, $\frac{\rho-\hat{b}}{\hat{a}-\hat{b}} < 0$ on set D. Then the difference

$$
\begin{aligned}
& E\left[(x-X)^+ + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}X\right] - E\left[(x-\hat{X})^+ + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\hat{X}\right] \\
&= E\left[(x-X)\mathbb{I}_L + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}X\left(\mathbb{I}_A + \mathbb{I}_B + \mathbb{I}_D\right)\right] - E\left[(x-x_d)\,\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}(x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D)\right] \\
&= E\left[(x-X)\mathbb{I}_L + \left(\tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}(X-x_d) - (x-x_d)\right)\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\,\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\,\mathbb{I}_D\right] \\
&\ge E\left[(x-X)\mathbb{I}_L + (X-x)\,\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\,\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\,\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap A} + \mathbb{I}_{L\cap B} + \mathbb{I}_{L\cap D}\right) + (X-x)\left(\mathbb{I}_{A\cap G} + \mathbb{I}_{A\cap L}\right) + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\,\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\,\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap B} + \mathbb{I}_{L\cap D}\right) + (X-x)\,\mathbb{I}_{A\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\,\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\,\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap B} + \mathbb{I}_{L\cap D}\right) + (X-x)\,\mathbb{I}_{A\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\left(\mathbb{I}_{B\cap G} + \mathbb{I}_{B\cap L}\right) + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\left(\mathbb{I}_{D\cap G} + \mathbb{I}_{D\cap L}\right)\right] \\
&= E\Big[(x-X)\left(1 - \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\right)\mathbb{I}_{B\cap L} + \left(x - X + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\right)\mathbb{I}_{D\cap L} + (X-x)\,\mathbb{I}_{A\cap G} \\
&\qquad + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x)\,\mathbb{I}_{B\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\,(X-x_u)\,\mathbb{I}_{D\cap G}\Big] \ge 0.
\end{aligned}
$$

The last inequality holds because each term inside the expectation is greater than or equal to zero. ◇

Theorem 3.10 is a direct consequence of Lemma 3.6, Proposition 3.7, and Proposition 3.9.

Proof of Lemma 3.13. The convexity of $v(x)$ is a simple consequence of its definition (4). Real-valued convex functions on $\mathbb{R}$ are continuous on its interior of the domain, so $v(x)$ is continuous on $\mathbb{R}$. ◇

Proof of Proposition 3.14. For $z \in (z^*, \bar{z}]$, **Step 2** of the **Two-Constraint Problem**

$$\frac{1}{\lambda}\inf_{x\in\mathbb{R}}(v(x) - \lambda x)$$

is the minimum of the following five sub-problems after applying Theorem 3.10:

Case 1

$$\frac{1}{\lambda}\inf_{(-\infty,x_d]}(v(x) - \lambda x) = \frac{1}{\lambda}\inf_{(-\infty,x_d]}(-\lambda x) = -x_d;$$

Case 2

$$\frac{1}{\lambda}\inf_{[x_d,x_{z1}]}(v(x) - \lambda x) = \frac{1}{\lambda}\inf_{[x_d,x_{z1}]}(-\lambda x) = -x_{z1} \le -x_d;$$

Case 3

$$\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}\left((x-x_d)P(A_x)-\lambda x\right);$$

Case 4

$$\frac{1}{\lambda}\inf_{[x_{z2},x_u]}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{[x_{z2},x_u]}\left((x-x_d)P(A_x)-\lambda x\right);$$

Case 5

$$\frac{1}{\lambda}\inf_{[x_u,\infty)}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{[x_u,\infty)}\left((x-x_d)P(\bar{A})+(x-x_u)P(\bar{B})-\lambda x\right).$$

Obviously, **Case 2** dominates **Case 1** in the sense that its minimum is lower. In **Case 3**, by the continuity of $v(x)$, we have

$$\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}\left((x-x_d)P(A_x)-\lambda x\right)\leq\frac{1}{\lambda}\left((x_{z1}-x_d)P(A_{x_{z1}})-\lambda x_{z1}\right)=-x_{z1}.$$

The last equality comes from the fact $P(A_{x_{z1}})=0$: As in Lemma 3.8, we know that when $x=x_{z1}$, the Three-Line configuration $X=x_d\mathbb{I}_A+x\mathbb{I}_B+x_u\mathbb{I}_D$ degenerates to the Two-Line configuration $X=x_{z1}\mathbb{I}_B+x_u\mathbb{I}_D$ where $a_{x_{z1}}=\infty$. Therefore, **Case 3** dominates **Case 2**. In **Case 5**,

$$\begin{aligned}\frac{1}{\lambda}\inf_{[x_u,\infty)}(v(x)-\lambda x)&=\frac{1}{\lambda}\inf_{[x_u,\infty)}\left((x-x_d)P(\bar{A})+(x-x_u)P(\bar{B})-\lambda x\right)\\&=\frac{1}{\lambda}\inf_{[x_u,\infty)}\left((1-\lambda)x-x_dP(\bar{A})-x_uP(\bar{B})\right)\\&=\frac{1}{\lambda}\left((1-\lambda)x_u-x_dP(\bar{A})-x_uP(\bar{B})\right)\\&=\frac{1}{\lambda}\left((x_u-x_d)P(\bar{A})-\lambda x_u\right)\\&\geq\frac{1}{\lambda}\inf_{[x_{z2},x_u]}\left((x-x_d)P(A_x)-\lambda x\right).\end{aligned}$$

Therefore, **Case 4** dominates **Case 5**. When $x\in[x_{z2},x_u]$ and $\operatorname{ess\,sup}\frac{d\tilde{P}}{dP}>\frac{1}{\lambda}$, Theorem 3.10 and Theorem 3.11 imply that the infimum in **Case 4** is achieved either by $\bar{X}$ or X^*. Since we restrict $z\in(z^*,\bar{z}]$ where $z^*=\bar{z}$ by Definition 3.12 in the first case, we need not consider this case in the current proposition. In the second case, Lemma 3.4 implies that $x^*<x_{z2}$ (because $z>z^*$). By the convexity of $v(x)$, and then the

continuity of $v(x)$,

$$\begin{aligned}\frac{1}{\lambda}\inf_{[x_{z2},x_u]}((x-x_d)P(A_x)-\lambda x) &= \frac{1}{\lambda}\left((x_{z2}-x_d)P(A_{x_{z2}})-\lambda x_{z2}\right)\\ &\geq \frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}((x-x_d)P(A_x)-\lambda x)\,.\end{aligned}$$

Therefore, **Case 3** dominates **Case 4**. We have shown that **Case 3** actually provides the globally infimum:

$$\frac{1}{\lambda}\inf_{x\in\mathbb{R}}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}(v(x)-\lambda x).$$

Now we focus on $x\in(x_{z1},x_{z2})$, where $X(x)=x_d\mathbb{I}_{A_x}+x\mathbb{I}_{B_x}+x_u\mathbb{I}_{D_x}$ satisfies the general constraints:

$$\begin{aligned}E[X(x)]&=x_dP(A_x)+xP(B_x)+x_uP(D_x)=z,\\ \tilde{E}[X(x)]&=x_d\tilde{P}(A_x)+x\tilde{P}(B_x)+x_u\tilde{P}(D_x)=x_r,\end{aligned}$$

and the definition for sets A_x, B_x and D_x are

$$A_x=\left\{\omega\in\Omega\,:\,\tfrac{d\tilde{P}}{dP}(\omega)>a_x\right\},\quad B_x=\left\{\omega\in\Omega\,:\,b_x\leq\tfrac{d\tilde{P}}{dP}(\omega)\leq a_x\right\},\quad D_x=\left\{\omega\in\Omega\,:\,\tfrac{d\tilde{P}}{dP}(\omega)<b_x\right\}.$$

Note that $v(x)=(x-x_d)P(A_x)$ (see Theorem 3.10). Since $P(A_x)+P(B_x)+P(D_x)=1$ and $\tilde{P}(A_x)+\tilde{P}(B_x)+\tilde{P}(D_x)=1$, we rewrite the capital and return constraints as

$$\begin{aligned}x-z&=(x-x_d)P(A_x)+(x-x_u)P(D_x),\\ x-x_r&=(x-x_d)\tilde{P}(A_x)+(x-x_u)\tilde{P}(D_x).\end{aligned}$$

Differentiating both sides with respect to x, we get

$$\begin{aligned}P(B_x)&=(x-x_d)\frac{dP(A_x)}{dx}+(x-x_u)\frac{dP(D_x)}{dx},\\ \tilde{P}(B_x)&=(x-x_d)\frac{d\tilde{P}(A_x)}{dx}+(x-x_u)\frac{d\tilde{P}(D_x)}{dx}.\end{aligned}$$

Since

$$\frac{d\tilde{P}(A_x)}{dx}=a_x\frac{dP(A_x)}{dx},\quad \frac{d\tilde{P}(D_x)}{dx}=b_x\frac{dP(D_x)}{dx},$$

we get

$$\frac{dP(A_x)}{dx} = \frac{\tilde{P}(B_x) - bP(B_x)}{(x - x_d)(a - b)}.$$

Therefore,

$$\begin{aligned}(v(x) - \lambda x)' &= P(A_x) + (x - x_d)\frac{dP(A_x)}{dx} - \lambda \\ &= P(A_x) + \frac{\tilde{P}(B_x) - bP(B_x)}{a - b} - \lambda.\end{aligned}$$

When the above derivative is zero, we arrive to the first order Euler condition

$$P(A_x) + \frac{\tilde{P}(B_x) - bP(B_x)}{a - b} - \lambda = 0.$$

To be precise, the above differentiation should be replaced by left-hand and right-hand derivatives as detailed in the Proof for Corollary 2.8 in Li and Xu [20]. But the first order Euler condition will turn out to be the same because we have assumed that the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ has continuous distribution.

To finish this proof, we need to show that there exists an $x \in (x_{z1}, x_{z2})$ where the first order Euler condition is satisfied. From Lemma 3.8, we know that as $x \searrow x_{z1}$, $a_x \nearrow \infty$, and $P(A_x) \searrow 0$. Therefore,

$$\lim_{x \searrow x_{z1}} (v(x) - \lambda x)' = -\lambda < 0.$$

As $x \nearrow x_{z2}$, $b_x \searrow 0$, and $P(D_x) \searrow 0$. Therefore,

$$\lim_{x \nearrow x_{z2}} (v(x) - \lambda x)' = P(A_{x_{z2}}) - \frac{\tilde{P}(A^c_{x_{z2}})}{a_{x_{z2}}} - \lambda.$$

This derivative coincides with the derivative of the value function of the Two-Line configuration that is optimal on the interval $x \in [x_{z2}, x_u]$ provided in Theorem 3.10 (see Proof for Corollary 2.8 in Li and Xu [20]). Again when $x \in [x_{z2}, x_u]$ and $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$, Theorem 3.10 and Theorem 3.11 imply that the infimum of $v(x) - \lambda x$ is achieved either by $\bar{X}$ or X^*. Since we restrict $z \in (z^*, \bar{z}]$ where $z^* = \bar{z}$ by Definition 3.12 in the first case, we need not consider this case in the current proposition. In the second case, Lemma 3.4 implies that $x^* < x_{z2}$ (because $z > z^*$). This in turn implies

$$P(A_{x_{z2}}) - \frac{\tilde{P}(A^c_{x_{z2}})}{a_{x_{z2}}} - \lambda < 0.$$

We have just shown that there exist some $x^{**} \in (x_{z1}, x_{z2})$ such that $(v(x) - \lambda x)'|_{x=x^{**}} = 0$. By the convexity of $v(x) - \lambda x$, this is the point where it obtains the minimum value. Now

$$\begin{aligned} CVaR(X^{**}) &= \frac{1}{\lambda}\left(v(x^{**}) - \lambda x^{**}\right) \\ &= \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right). \end{aligned}$$

◇

Proof of Theorem 3.15. Case 3 and 4 are already proved in Theorem 3.11 and Proposition 3.14. In Case 1 where $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ and $z = x_r$, $X = x_r$ is both feasible and optimal by Theorem 3.11. In Case 2, fix arbitrary $\epsilon > 0$. We will look for a Two-Line solution $X_\epsilon = x_\epsilon \mathbb{I}_{A_\epsilon} + \alpha_\epsilon \mathbb{I}_{B_\epsilon}$ with the right parameters $a_\epsilon, x_\epsilon, \alpha_\epsilon$ which satisfies both the capital constraint and return constraint:

$$E[X_\epsilon] = x_\epsilon P(A_\epsilon) + \alpha_\epsilon P(B_\epsilon) = z, \tag{21}$$

$$\tilde{E}[X_\epsilon] = x_\epsilon \tilde{P}(A_\epsilon) + \alpha_\epsilon \tilde{P}(B_\epsilon) = x_r, \tag{22}$$

where

$$A_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a_\epsilon\right\}, \quad B_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) \leq a_\epsilon\right\},$$

and produces a CVaR level close to the lower bound:

$$CVaR(X_\epsilon) \leq CVaR(x_r) + \epsilon = -x_r + \epsilon.$$

First, we choose $x_\epsilon = x_r - \epsilon$. To find the remaining two parameters a_ϵ and α_ϵ so that equations (21) and (22) are satisfies, we note

$$\begin{aligned} x_r P(A_\epsilon) + x_r P(B_\epsilon) &= x_r, \\ x_r \tilde{P}(A_\epsilon) + x_r \tilde{P}(B_\epsilon) &= x_r, \end{aligned}$$

and conclude that it is equivalent to find a pair of a_ϵ and α_ϵ such that the following two equalities are

satisfied:

$$-\epsilon P(A_\epsilon) + (\alpha_\epsilon - x_r)P(B_\epsilon) = \gamma,$$
$$-\epsilon \tilde{P}(A_\epsilon) + (\alpha_\epsilon - x_r)\tilde{P}(B_\epsilon) = 0,$$

where we denote $\gamma = z - x_r$. If we can find a solution a_ϵ to the equation

$$\frac{\tilde{P}(B_\epsilon)}{P(B_\epsilon)} = \frac{\epsilon}{\gamma + \epsilon}, \tag{23}$$

then

$$\alpha_\epsilon = x_r + \frac{\tilde{P}(A_\epsilon)}{\tilde{P}(B_\epsilon)}\epsilon,$$

and we have the solutions for equations (21) and (22). It is not difficult to prove that the fraction $\frac{\tilde{P}(B)}{P(B)}$ increases continuously from 0 to 1 as a increases from 0 to $\frac{1}{\lambda}$. Therefore, we can find a solution $a_\epsilon \in (0, \frac{1}{\lambda})$ where (23) is satisfied. By definition (3),

$$CVaR_\lambda(X_\epsilon) = \frac{1}{\lambda}\inf_{x\in\mathbb{R}}\left(E[(x - X_\epsilon)^+] - \lambda x\right) \leq \frac{1}{\lambda}\left(E[(x_\epsilon - X_\epsilon)^+] - \lambda x_\epsilon\right) = -x_\epsilon.$$

The difference

$$CVaR_\lambda(X_\epsilon) - CVaR(x_r) \leq -x_\epsilon + x_r = \epsilon.$$

Under Assumption 2.1, the solution in Case 2 is almost surely unique, the result is proved. ◇

Proof of Theorem 3.17. Case 1 and 3 are obviously true in light of Theorem 3.16. The proof for Case 2 is similar to that in the Proof of Theorem 3.15, so we will not repeat it here. Since $E[X^*] = z^* < z$ in case 4, $CVaR(X^*)$ is only a lower bound in this case. We first show that it is the true infimum obtained in Case 4. Fix arbitrary $\epsilon > 0$. We will look for a Three-Line solution $X_\epsilon = x_d\mathbb{I}_{A_\epsilon} + x_\epsilon\mathbb{I}_{B_\epsilon} + \alpha_\epsilon\mathbb{I}_{D_\epsilon}$ with the right parameters $a_\epsilon, b_\epsilon, x_\epsilon, \alpha_\epsilon$ which satisfies the general constraints:

$$E[X_\epsilon] = x_d P(A_\epsilon) + x_\epsilon P(B_\epsilon) + \alpha_\epsilon P(D_\epsilon) = z, \tag{24}$$

$$\tilde{E}[X_\epsilon] = x_d \tilde{P}(A_\epsilon) + x_\epsilon \tilde{P}(B_\epsilon) + \alpha_\epsilon \tilde{P}(D_\epsilon) = x_r, \tag{25}$$

where

$$A_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a_\epsilon\right\}, \quad B_\epsilon = \left\{\omega \in \Omega \,:\, b_\epsilon \le \tfrac{d\tilde{P}}{dP}(\omega) \le a_\epsilon\right\}, \quad D_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) < b_\epsilon\right\},$$

and produces a CVaR level close to the lower bound:

$$CVaR(X_\epsilon) \le CVaR(X^*) + \epsilon.$$

First, we choose $a_\epsilon = a^*$, $A_\epsilon = A^*$, $x_\epsilon = x^* - \delta$, where we define $\delta = \frac{\lambda}{\lambda - P(A^*)}\epsilon$. To find the remaining two parameters b_ϵ and α_ϵ so that equations (24) and (25) are satisfies, we note

$$E[X^*] = x_d P(A^*) + x^* P(B^*) = z^*,$$
$$\tilde{E}[X^*] = x_d \tilde{P}(A^*) + x^* \tilde{P}(B^*) = x_r,$$

and conclude that it is equivalent to find a pair of b_ϵ and α_ϵ such that the following two equalities are satisfied:

$$-\delta(P(B^*) - P(D_\epsilon)) + (\alpha_\epsilon - x^*)P(D_\epsilon) = \gamma,$$
$$-\delta(\tilde{P}(B^*) - \tilde{P}(D_\epsilon)) + (\alpha_\epsilon - x^*)\tilde{P}(D_\epsilon) = 0,$$

where we denote $\gamma = z - z^*$. If we can find a solution b_ϵ to the equation

$$\frac{\tilde{P}(D_\epsilon)}{P(D_\epsilon)} = \frac{\tilde{P}(B^*)}{\frac{\gamma}{\delta} + P(B^*)}, \tag{26}$$

then

$$\alpha_\epsilon = x^* + \left(\frac{\tilde{P}(B^*)}{\tilde{P}(D_\epsilon)} - 1\right)\delta,$$

and we have the solutions for equations (24) and (25). It is not difficult to prove that the fraction $\frac{\tilde{P}(D)}{P(D)}$ increases continuously from 0 to $\frac{\tilde{P}(B^*)}{P(B^*)}$ as b increases from 0 to a^*. Therefore, we can find a solution

$b_\epsilon \in (0, a^*)$ where (26) is satisfied. By definition (3),

$$\begin{aligned} CVaR_\lambda(X_\epsilon) &= \frac{1}{\lambda} \inf_{x\in\mathbb{R}} \left(E[(x - X_\epsilon)^+] - \lambda x\right) \\ &\leq \frac{1}{\lambda} \left(E[(x_\epsilon - X_\epsilon)^+] - \lambda x_\epsilon\right) \\ &= \frac{1}{\lambda}(x_\epsilon - x_d)P(A_\epsilon) - x_\epsilon. \end{aligned}$$

The difference

$$\begin{aligned} CVaR_\lambda(X_\epsilon) - CVaR(X^*) &\leq \frac{1}{\lambda}(x_\epsilon - x_d)P(A_\epsilon) - x_\epsilon - \frac{1}{\lambda}(x^* - x_d)P(A^*) + x^* \\ &= \frac{1}{\lambda}(x^* - x_d)(P(A_\epsilon) - P(A^*)) + \left(1 - \frac{P(A_\epsilon)}{\lambda}\right)(x^* - x_\epsilon) = \epsilon. \end{aligned}$$

Under Assumption 2.1, the solution in Case 4 is almost surely unique, the result is proved. ◇

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