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Abstract

For any transferable utility game in coalitional form with nonempty core, we show that that the number of blocks required to switch from an imputation out of the core to an imputation in the core is less than or equal to $n(n-1)/2$, where n is the cardinality of the player set. This number considerably improves the upper bounds found so far by Kóczy [8] and Yang [17]. Our result relies on an altered version of the procedure proposed by Sengupta and Sengupta [13]. The use of the Davis-Maschler reduced-games is also pointed out.

Keywords: Core, excess function, dominance path, Davis-Maschler reduced-game.
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1 Introduction

The core (Gillies [5]) is the solution concept for transferable utility games in coalitional form (henceforth, simply called a TU-game) that collects the imputations which cannot be blocked by any coalition of players. As such, it satisfies the internal stability property. Nevertheless, the core is often criticized on two aspects. Firstly, it does not account for every imputation it excludes, since it does not always satisfies the external stability property. Secondly, Harsanyi [6] and Chwe [3] consider that this solution concept is too myopic because it neglects the effect of indirect dominance relations.

Harsanyi [6] introduces two new indirect dominance relations in order to cope with these lacks. In the first one, there exists a direct domination between any two consecutive imputations in the chain of blocks. Even though Harsanyi originally applies this indirect dominance relation to study the von Neumann-Morgenstern stable sets (von Neumann and Morgenstern [16]), Sengupta and Sengupta [13] employ it to show that the core is indirectly externally stable: starting from any imputation that stands outside the core, there always exists a chain of blocks which terminates in the core. In other words, the core can be considered as a von Neumann-Morgenstern stable set of the indirect dominance relation. As such, the above-mentioned chain of blocks is finite. This result has initiated the literature on the accessibility of the core. The central question that has appeared is to determine a upper bound on the number of elements of the chain of blocks needed to access the core.

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Several recent articles try to answer this question and this article improves on the existing answers. Sengupta and Sengupta [13] introduce a procedure which ensures that the current allocation is an imputation and that the core is reached after a bounded number of blocks. However, the upper bound is not studied. Kóczy [8] provides the first description of an upper bound, but it is overwhelmingly difficult to compute it and relies on the game under consideration. Kóczy [8] is also aware that his bound should be lowered. This is precisely what Yang [17] achieves by mean a more restrictive dominance relation, which never makes the past blocking coalitions worse off. Yang's construction is very instructive but his new upper bound is also sensitive to the game under consideration since it is equal to the number of active coalitions in the game, *i.e.* those coalitions different from the grand coalition for which the worth is greater than the sum of the stand-alone worths of its members. Even if this number can be explicitly computed, it can be exponentially large compared to the number of players.

In this article, we show that the core of any TU-game with n players can be accessed with at most $n(n-1)/2$ blocks.[1] This result gathers many of the advantages of the above-mentioned literature while it does not bear any of its main drawbacks. Firstly, it improves dramatically the bound found by Kóczy [8], proving that the procedure which leads to the core is polynomial in the number of player and not exponential as it was suspected until now. Secondly, our upper bound is very easy to compute, while it is not the case in Kóczy [8]. Thirdly, our bound relies solely on the number of players in the game, but not on the structure of the coalitional function, while the bounds in Kóczy [8] and Yang [17] vary across TU-games with a same number of players. For instance, the bound found by Yang [17] raises from 0 for a game with no active coalition to $2^n - (n+2)$ for a game with the maximal number of active coalitions.

The procedure which is used to prove our result is similar to the one introduced by Sengupta and Sengupta [13]. In particular, both procedures share the idea of using a core element as a reference point. Nevertheless, we introduce two major differences. The first difference is related to the choice, at each step of the procedure, of the coalition with respect to which the block is constructed. While Sengupta and Sengupta [13] choose a coalition whose excess is maximal with respect to the current imputation, we select a coalition among the smallest coalitions with positive excess. The second difference is related to the way the excess of this chosen coalition is redistributed at each step. While Sengupta and Sengupta [13] split this excess equally among the players of the chosen blocking coalition whose payoff is less than their core reference payoff, we try to fill in the gap between their current payoff and their core reference payoff for as many players of the chosen coalition as possible. The objective of these two changes is to give to as many new players as possible their core reference payoff at each step of the procedure.

Another feature of our procedure is that at each step of the procedure, we make use of the Davis-Maschler reduced-games with respect to the core reference point and to the current coalition of players who do not get their core reference payoff. Such Davis-Maschler reduced-games describe situations in which all the players agree that the left players get their core reference payoffs but continue to cooperate with the remaining players, subject to the foregoing agreement. The Davis-Maschler reduced-games are well known for being the basis of the so-called reduced-game property, which states that if an allocation is prescribed by some solution concept in a TU-game, then the restriction of this allocation to any coalition of players is also prescribed by the solution concept in the reduced-game associated with these coalition and allocation.

The literature on the accessibility of the core belongs to a broader literature on the accessibility

[1] It is a pure coincidence that this bound is equal to the number of pairs of players in any n-person game.

of cooperative solution concepts. Manea [9] considers a tatônnement procedure that converges to the core. This procedure is very simple and easy to implement but suffers from two lacks: sometimes it never reaches the core and at each step the current allocation is not necessary an imputation. Stearns [15] and Justman [7] provide procedures that converge to the bargaining set (Aumann and Maschler [1]) and the Kernel (Davis and Maschler [4]), and to the Nucleolus (Schmeidler [12]), respectively. These procedures are similar in spirit to the procedures used in Manea [9]. Another contribution to this literature relies on the second type of indirect dominance relation proposed by Harsanyi [6]. This indirect dominance relation assumes that the members of the current blocking coalition do not compare their current payoff to their payoff in the next imputation of the chain, but to their payoff in the last imputation of the chain. This is the reason why the behavior of these coalitions of players is said to incorporate farsightedness. Chwe [3] uses this indirect dominance relation in the general context of social situations, and introduces and studies three new solution concepts: the von Neumann-Morgenstern farsighted stable set, the farsighted core and the largest consistent set. Béal, Durieu and Solal [2] examine these solution concepts for the class of all TU-games. They characterize the von Neumann-Morgenstern farsighted stable sets and it is implicit in their proof that the number of imputations required to access any von Neumann-Morgenstern farsighted stable set is at most three (Lemma 2 on page 308). Since any von Neumann-Morgenstern farsighted stable sets belongs to the largest consistent set (Chwe [3], Proposition 3), the same bound is obtained for the accessibility of the largest consistent set. Finally, the accessibility of the farsighted core is trivial since Béal, Durieu and Solal [2] show that it is either empty or coincides with the set of all imputations.

The rest of the article is organized as follows. Section 2 gives the definitions that are necessary to state the main theorem. Section 3 introduces the Davis-Maschler reduced-games and proves two lemmas that will be used in the proof of the main theorem. Section 4 completes the proof of the main theorem. In this section, we first describe the procedure that will be the corner stone of the proof of the main theorem. Then we demonstrate numerous claims on which the proof of the main theorem relies, and the proof of the theorem itself. Section 5 provides examples in which our procedure is compared to the alternative approaches. Section 6 concludes with an open question.

2 Preliminaries and the result

Let $\subseteq$ denote weak set inclusion and $\subset$ denote proper set inclusion. We use the notation $|S|$ to denote the number of elements in a finite set S.

A *TU-game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is a nonempty finite *player set* and $v : 2^N \longrightarrow \mathbb{R}$ is a real-valued function such that $v(\emptyset) = 0$. A nonempty subset S of N is called a *coalition* and $v(S)$ is interpreted as the *worth* of coalition S.

An *allocation* $x \in \mathbb{R}^n$ on N is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. Given an allocation $x \in \mathbb{R}^n$ and a coalition S, x_S will denote the projection of x on S and $x(S)$ will stand for $\sum_{i \in S} x_i$. For two allocations $x, y \in \mathbb{R}^n$, we write $x_S > y_S$ if $x_i \geq y_i$ for each $i \in S$ but $x_S \neq y_S$.

An allocation $x \in \mathbb{R}^n$ is *efficient* if $x(N) = v(N)$ and *individually rational* if $x_i \geq v(\{i\})$ for each $i \in N$. An individually rational and efficient allocation is referred to as an *imputation* and the set of imputations of the TU-game (N, v) is denoted by $I(N, v)$.

Let Γ be the set of all finite TU-games. A *solution* on Γ is a function F which assigns to each $(N, v) \in \Gamma$ a set of allocations $F(N, v)$.

Given an allocation $x \in \mathbb{R}^n$, the *excess* $e(x, S)$ of a coalition S is given by

$$e(x, S) = v(S) - x(S).$$

The *core* is the solution C on Γ that assigns to each TU-game $(N, v) \in \Gamma$ the set $C(N, v)$ of imputations for which the excess of each coalition is not positive, *i.e.*

$$C(N, v) = \{x \in I(N, v) : e(x, S) \leq 0, \forall S \in 2^N \backslash \{\emptyset\}\}.$$

Denote by Γ^c the set of TU-games in Γ with a nonempty core.

Let $x, y \in \mathbb{R}^n$ be two efficient allocations. We say that x *directly dominates* y via coalition S, which we denote by $x >_S y$, if $x_S > y_S$ and $x(S) \leq v(S)$. We say that x *indirectly dominates* y if there exists a finite sequence of efficient allocations $(x^0, \ldots, x^m)$ and a finite collection of coalitions $(S_0, \ldots, S_{m-1})$ such that $x^0 = x$, $x^m = y$ and for each $k = 0, \ldots, m-1$,

$$x^{k+1} >_{S_k} x^k.$$

We call the sequence $(x^0, S_0, x^1, \ldots, x^{m-1}, S_{m-1}, x^m)$ a *dominance path*, and m its length.

The main result of the article can be stated as follows.

Theorem 1 *Let $(N, v) \in \Gamma^c$ be a TU-game. Then for each imputation $x \in I(N, v) \backslash C(N, v)$, there exists a core element $c \in C(v)$ and a dominance path from x to c with length smaller than or equal to $n(n-1)/2$ and such that each allocation along the path is an imputation.*

3 Reduced-games equivalences

This section establishes connections between the cores of Davis-Maschler reduced-games and of the original game. These intermediary results will be used later on to prove the main theorem. Let $S \subset N$ be any coalition different from N and $x \in \mathbb{R}^n$ any efficient allocation. Davis and Maschler [4] propose the *reduced-game* with respect to S and x, denoted by $(S, v_{S,x})$ and defined as

$$v_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N) - x(N \backslash S) & \text{if } T = S, \\ \max_{R \in 2^{N \backslash S}} \big(v(T \cup R) - x(R)\big) & \text{otherwise.} \end{cases}$$

In this article, the direct dominance relation $>_S$ and the excess function $e(\cdot)$ will be used in the Davis-Maschler reduced-games as well without any risk of confusion.

A solution F on Γ satisfies the *reduced-game property* if for each $(N, v) \in \Gamma$, each coalition $S \subset N$ and each $x \in F(N, v)$, it holds that $x_S \in F(S, v_{S,x})$. It is well-known that the core satisfies the reduced-game property. In fact, the reduced-game property is one of the axioms used by Peleg [11] in order to characterize the core.

In the remainder of the article, we will focus on the set Γ^c of TU-games with a nonempty core. For TU-games in this class, we will construct Davis-Maschler reduced-games with respect to core elements only. This section establishes two interesting properties of such reduced-games. To begin with, we formulate some remarks. So let $(N, v) \in \Gamma^c$ be any TU-game with a nonempty core, $S \subset N$ be any coalition and $c \in C(N, v)$ be any core element of (N, v). Note that the Davis-Maschler reduced-game $(S, v_{S,c})$ can be reformulated by the following simpler expression:

$$v_{S,c}(T) = \max_{R \in 2^{N \backslash S}} \big(v(T \cup R) - c(R)\big)$$

for each $T \in 2^S$. In fact, $c \in C(N,v)$ implies $e(c,R) \leq 0$ for each $R \in 2^{N\backslash S}$, which in turn leads to $v_{S,c}(\emptyset) = 0$ by choosing $R = \emptyset$, and Claim 1 below proves that the above-mentioned reformulation of $(S, v_{S,c})$ implies $v_{S,c}(S) = c(S)$.

Claim 1 *Let $(N,v) \in \Gamma^c$ be any TU-game with a nonempty core, $S \subset N$ be any coalition and $c \in C(N,v)$ be any core element of (N,v). Then $\max_{R\in 2^{N\backslash S}} \big(v(S\cup R) - c(R)\big) = c(S)$.*

Proof. On the one hand, $\max_{R\in 2^{N\backslash S}} \big(v(S\cup R) - c(R)\big) \geq v(S \cup (N\backslash S)) - c(N\backslash S) = c(S)$. On the other hand, $\max_{R\in 2^{N\backslash S}} \big(v(S\cup R) - c(R)\big) \leq \max_{R\in 2^{N\backslash S}} \big(c(S\cup R) - c(R)\big) = c(S)$. ■

For simplicity, for each $T \in 2^S$, we denote by $\overline{T}$ a coalition in $2^{N\backslash S}$ such that $v_{S,c}(T) = v(T\cup\overline{T}) - c(\overline{T})$.

Lemma 1 *Consider any $(N,v) \in \Gamma^c$, any coalition $S \subset N$ and any $c \in C(N,v)$. Pick any allocation $x \in \mathbb{R}^n$ such that $x_{N\backslash S} = c_{N\backslash S}$. Then, $x \in C(N,v)$ if and only if $x_S \in C(S, v_{S,c})$.*

Proof. Firstly, suppose that $x \in C(N,v)$. Choose any coalition $T \in 2^S$. By definition of $v_{S,c}$, it holds that $v_{S,c}(T) = v(T\cup\overline{T}) - c(\overline{T})$. Since $x \in C(N,v)$, we have $v(T\cup\overline{T}) \leq x(T\cup\overline{T})$. Therefore, $v_{S,c}(T) \leq x(T\cup\overline{T}) - c(\overline{T}) = x_S(T)$ since $x_{\overline{T}} = c_{\overline{T}}$. Regarding efficiency of x_S in $(S, v_{S,c})$, we have

$$x_S(S) = x(N) - x(N\backslash S) = c(N) - c(N\backslash S) = c(S) = v_{S,c}(S).$$

where the last equality follows from Claim 1.

Secondly, suppose that $x_S \in C(S, v_{S,c})$. Since $x_{N\backslash S} = c_{N\backslash S}$ and $x_S(S) = v_{S,c}(S) = c(S)$, we get $x(N) = c(N) = v(N)$, proving that x is an efficient allocation in (N,v). Next, choose any coalition $T \in 2^N$. The definition of $v_{S,c}$ and $x_S \in C(S, v_{S,c})$ imply that

$$\begin{aligned} v(T) &= v((T\cap S)\cup(T\backslash S)) \\ &= v((T\cap S)\cup(T\backslash S)) - c(T\backslash S) + c(T\backslash S) \\ &\leq v_{S,c}(T\cap S) + c(T\backslash S) \\ &\leq x_S(T\cap S) + c(T\backslash S) \\ &= x(T), \end{aligned}$$

which proves the result. ■

Note that the proof of the first implication does not follow from the reduced-game property since the imputation x that is proved to be in the core can be different from the core element c from which the Davis-Maschler reduced-game is constructed.

Lemma 2 *Consider any $(N,v) \in \Gamma^c$, any coalition $S \subset N$ and any $c \in C(N,v)$. Pick any two efficient allocations $x, y \in \mathbb{R}^n$ such that $x_{N\backslash S} = y_{N\backslash S} = c_{N\backslash S}$ and a coalition $T \subset S$. Then $x >_{T\cup\overline{T}} y$ in (N,v) if and only if $x_S >_T y_S$ in $(S, v_{S,c})$.*

Proof. Firstly, assume that $x >_{T\cup\overline{T}} y$ in (N,v). Then $x_{T\cup\overline{T}} > y_{T\cup\overline{T}}$. In addition, $\overline{T} \in 2^{N\backslash S}$ implies $x_{\overline{T}} = y_{\overline{T}}$. It follows that $x_T > y_T$. Next, $x >_{T\cup\overline{T}} y$ in (N,v) also means that $x(T\cup\overline{T}) \leq v(T\cup\overline{T})$. Therefore, by definition of x, we get $v_{S,c}(T) = v(T\cup\overline{T}) - c(\overline{T}) \geq x(T\cup\overline{T}) - c(\overline{T}) = x(T)$. We conclude that $x_S >_T y_S$ in $(S, v_{S,c})$.

Secondly, assume that $x_S >_T y_S$ in $(S, v_{S,c})$. Then $x_T > y_T$. Since $x_{\overline{T}} = y_{\overline{T}}$ by assumption, this implies that $x_{T\cup\overline{T}} > y_{T\cup\overline{T}}$. Moreover, $x_S >_T y_S$ in $(S, V_{S,c})$ also means that $x(T) \leq v_{S,c}(T) = v(T\cup\overline{T}) - c(\overline{T})$. Thus, $v(T\cup\overline{T}) \geq x(T) + c(\overline{T}) = x(T\cup\overline{T})$. Therefore $x >_{T\cup\overline{T}} y$ in (N,v). ■

4 Proof of Theorem 1

4.1 The procedure

We follow the approach developed Sengupta and Sengupta [13] but introduce two differences between their construction and ours, which will be highlighted throughout this section.

Consider a TU-game $(N, v) \in \Gamma^c$. From now on, we fix an imputation $x^0 \in I(N, v)$ of (N, v) and an arbitrary core element $c \in C(N, v)$. We would like to construct a sequence of imputations $(x^0, \dots, x^m, \dots)$, a sequence of coalitions $(S_0, \dots, S_m, \dots,)$ and a sequence of TU-games $((N_0, v_{N_0,c}), \dots, (N_m, v_{N_m,c}), \dots)$ with the following properties:

1. there is a $m \leq n(n-1)/2$ such that $x^m \in C(N_m, v_{N_m,c})$;
2. for each $k = 0, \dots, m-1$, $x^{k+1}_{N_k} >_{S_k} x^k_{N_k}$ in the TU-game $(N_k, v_{N_k,c})$;
3. for each $k = 0, \dots, m-1$, $x^k_{N \setminus N_k} = c_{N \setminus N_k}$.

Therefore, we will be able to use Lemmas 1 and 2 to state that $x^m_{N_m} \in C(N_m, v_{N_m,c})$ implies $x^m \in C(N, v)$.

Before proceeding to the construction of the sequence of imputations $(x^0, \dots, x^m, \dots)$, we give, for each $k = 0, 1, \dots,$ the definitions of the TU-game $(N_k, v_{N_k,c})$ and coalition S_k assuming that the imputation x^k has already been defined. So let N_k be the set of players in N_{k-1} who are assigned in x^k a different payoff than in c, *i.e.*

$$N_k = \{i \in N_{k-1} : x^k_i \neq c_i\},$$

if $k \geq 1$, and $N_0 = \{i \in N : x^0_i \neq c_i\}$. The TU-game $(N_k, v_{N_k,c})$ is the Davis-Maschler reduced-game of (N, v) with respect to N_k and c, *i.e.* for each $S \in 2^{N_k}$,

$$v_{N_k,c}(S) = \max_{R \in 2^{N \setminus N_k}} \big(v(S \cup R) - c(R)\big).$$

One can remark that the TU-game "reduction" is a transitive operation in the sense that for each $k \geq 1$, the TU-game $(N_k, v_{N_k,c})$ is also the Davis-Maschler reduced-game of $(N_{k-1}, v_{N_{k-1},c})$ with respect to N_k and c. Next, in the Davis-Maschler reduced-game $(N_k, v_{N_k,c})$, we will denote $E_k(x)$ the set of coalitions with positive excess with respect to the efficient allocation $x \in \mathbb{R}^{|N_k|}$. We choose coalition S_k among the set of smallest coalitions with positive excess with respect to $x^k_{N_k}$ in $(N_k, v_{N_k,c})$, *i.e.*

$$S_k \in \arg \min_{S \in E_k(x^k_{N_k})} |S|.$$

We will stop the construction of the sequence as soon as choosing such a coalition will become impossible. Note that S_k exists and is nonempty if and only if $x^k_{N_k} \notin C(N_k, v_{N_k,c})$. Here is a first difference between our procedure and the one in Sengupta and Sengupta [13], since they choose a coalition with the maximal excess. Kóczy [8] also chooses a coalition with the maximal excess. In addition to coalition S_k, we define the coalition T_k of players in S_k who are assigned less in x^k than in c, *i.e.*

$$T_k = \{i \in S_k : x^k_i < c_i\}.$$

Without any loss of generality, the players in T_k are labeled $i_1, \ldots, i_{|T_k|}$ in such a way that for each $q = 1, \ldots, |T_k| - 1$, it holds that

$$i_q < i_{q+1} \Longleftrightarrow c_{i_q} - x^k_{i_q} \leq c_{i_{q+1}} - x^k_{i_{q+1}}.$$

In other words, we order the players in T_k according to the difference between their payoffs in c and in x^k, from the smallest to the largest. Among the first collection of claims, Claims 2, 3, 4 and 6 are quite similar to the ones obtained by Sengupta and Sengupta [13]. We give the proofs for completeness.

Claim 2 *If $x^k_{N_k} \notin C(N_k, v_{N_k,c})$ then $T_k \neq \emptyset$.*

Proof. By contradiction, suppose $x^k_{N_k} \notin C(N_k, v_{N_k,c})$ but $T_k = \emptyset$. Then the reduced-game property implies that $x^k(S_k) \geq c(S_k) \geq v_{N_k,c}(S_k)$, which in turn implies that $e(x^k_{N_k}, S_k) \leq 0$, a contradiction with the definition of S_k. ■

Claim 3 *If $x^k_{N_k} \notin C(N_k, v_{N_k,c})$ and $x^k(N_k) = v_{N_k,c}(N_k)$, then $N_k \neq S_k$.*

Proof. If $x^k_{N_k} \notin C(N_k, v_{N_k,c})$, then S_k is nonempty. By contradiction, assume $N_k = S_k$. By definition of S_k, we have $x^k(S_k) < v_{N_k,c}(S_k)$. Therefore we obtain $x^k(N_k) < v_{N_k,c}(N_k)$, a contradiction with the assumption $x^k(N_k) = v_{N_k,c}(N_k)$. ■

Note that it is implicit from the statement of almost all claims that the restriction of the current allocation x^k to N_k does not belong to $C(N_k, v_{N_k,c})$. In order to construct x^{k+1} from x^k, choose one of the players in $N_k \backslash S_k$ depicted in Claim 3, say player j_k. In x^{k+1}, we use the excess $e(x^k_{N_k}, S_k)$ so as to rise x^k_i up to c_i for as many players i in T_k as possible. This is the reason why the players $i \in T_k$ are ordered by the smallest amount they need in order to fill in the gap between x^k_i and c_i. Any other player $i \in S_k$ will keep his payoff x^k_i. Lastly, each player $i \in N \backslash S_k$ will receive c_i except the chosen player j_k who will get the remainder of $v_{N_k,c}(N_k)$, which cannot be less than c_{j_k}. Formally,

$$x^{k+1}_i = \begin{cases} c_i & \text{if } i = i_t \in T_k \text{ and } \displaystyle\sum_{r=1}^{t} \left(c_{i_r} - x^k_{i_r}\right) \leq e(x^k_{N_k}, S_k) \\ & \text{or } i \in N \backslash (S_k \cup \{j_k\}), \\ x^k_i + e(x^k_{N_k}, S_k) - \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x^k_{i_r}\right) & \text{if } i = i_t \in T_k \text{ and} \\ & \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x^k_{i_r}\right) \leq e(x^k_{N_k}, S_k) < \sum_{r=1}^{t} \left(c_{i_r} - x^k_{i_r}\right), \\ x^k_i & \text{if } i = i_t \in T_k \text{ and} \\ & \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x^k_{i_r}\right) > e(x^k_{N_k}, S_k) \text{ or } i \in S_k \backslash T_k, \\ c_i + v_{N_k,c}(N_k) - v_{N_k,c}(S_k) - c(N_k \backslash S_k) & \text{if } i = j_k. \end{cases}$$

Here is a second difference with Sengupta and Sengupta [13]. In order to satisfy the coalition S_k, these authors split equally the excess $e(x^k_{N_k}, S_k)$ among the players in T_k. By contrast, we order the players in T_k so as to give as many players in T_k the difference between their current payoff in x^k and their targeted core payoff in c.

In the rest of this section, we show that for each $k = 0, 1, \ldots,$ $x^{k+1}_{N_k} >_{S_k} x^k_{N_k}$. the proof that there is an integer $m \leq n(n-1)/2$ such that $x^m_{N_m} \in (N_m, v_{N_m,c})$ is delegated to section 4.2.

Claim 4 *If $x^k_{N_k} \notin C(N_k, v_{N_k,c})$, then for each $i \in N_k \backslash T_k$, it holds that $x^{k+1}_i \geq c_i$.*

Proof. By definition of S_k, $x^{k+1}_i \geq c_i$ for each $i \in S_k \backslash T_k$. By construction of x^{k+1}, the players in $N_k \backslash (S_k \cup \{j_k\})$ get $x^{k+1}_i = c_i$. So it remains to show that $x^{k+1}_{j_k} \geq c_{j_k}$. By construction of $x^{k+1}_{j_k}$, it is sufficient to prove that $v_{N_k,c}(N_k) - v_{N_k,c}(S_k) - c(N_k \backslash S_k) \geq 0$. By Claim 1, we know that $c(N_k) = v_{N_k,c}(N_k)$. Therefore, since $c_{N_k} \in C(N_k, v_{N_k,c})$, we can write

$$\begin{aligned} v_{N_k,c}(N_k) - v_{N_k,c}(S_k) - c(N_k \backslash S_k) &= c(N_k) - v_{N_k,c}(S_k) - c(N_k \backslash S_k) \\ &= c(S_k) - v_{N_k,c}(S_k) \\ &\geq 0, \end{aligned}$$

as desired. ■

Claim 5 *For each $k = 0, 1, \dots,$ it holds that $x^k \in I(N, v)$ and $x^k(N_k) = v_{N_k,c}(N_k)$.*

Proof. For the first part of the claim, we proceed by induction on $k = 0, 1, \dots$.
Initial step: By assumption, the starting allocation x^0 is an imputation of (N, v).
Induction hypothesis: let $k \in \mathbb{N}$ and assume that $x^q \in I(N, v)$ for each $q = 0, 1, \dots, k$.
Induction step: Consider the allocation x^{k+1}. By the induction hypothesis, we know that $x^k \in I(N, v)$. By construction of x^{k+1} and $c \in C(N, v)$, we have $x^{k+1}_i = c_i \geq v(\{i\})$ for each $i \in N \backslash N_k$. Claim 4 implies that $x^{k+1}_i \geq v(\{i\})$ for each $i \in N_k \backslash T_k$. Moreover, for each $i \in T_k$ we have $x^{k+1}_i \geq x^k_i$ by construction, which implies $x^{k+1}_i \geq v(\{i\})$ since $x^k \in I(N, v)$. Thus x^{k+1} is individually rational.

Next, $x^{k+1}(N)$ can be decomposed as follows:

$$\begin{aligned} x^{k+1}(N) &= x^{k+1}(S_k) + x^{k+1}(N \backslash (S_k \cup \{j_k\})) + x^{k+1}_{j_k} \\ &= v_{N_k,c}(S_k) + c(N \backslash (S_k \cup \{j_k\})) + c_{j_k} + v_{N_k,c}(N_k) - v_{N_k,c}(S_k) - c(N_k \backslash S_k) \\ &= c(N \backslash N_k) + v_{N_k,c}(N_k). \end{aligned}$$

Since $v_{N_k,c}(N_k) = c(N_k)$ and c is an efficient allocation, we obtain $x^{k+1}(N) = c(N) = v(N)$, which proves that x^{k+1} is also efficient.

For the second part of the claim, fix any $k = 0, 1, \dots$. Recall that $v_{N_k,c}(N_k) = c(N_k)$ and $x^k_i = c_i$ for each $i \in N \backslash N_k$. Thus, $x^k(N_k) = v(N) - x^k(N \backslash N_k) = c(N) - c(N \backslash N_k) = c(N_k) = v_{N_k,c}(N_k)$. ■

Claim 6 *If $x^k_{N_k} \notin C(N_k, v_{N_k,c})$, then $x^{k+1}_{N_k} >_{S_k} x^k_{N_k}$.*

Proof. By construction, we have $x^{k+1}(S_k) = v_{N_k,c}(S_k)$. Furthermore, $T_k \neq \emptyset$ by Claim 2, which implies that there is $i \in T_k$ such that $x^{k+1}_i > x^k_i$. In addition, it is clear that the construction of x^{k+1} ensures that $x^{k+1}_i \geq x^k_i$ for each $i \in S_k$. Thus $x^{k+1}_{N_k} >_{S_k} x^k_{N_k}$. ■

4.2 Analysis

In order to prove that our procedure ends up in the core after at most $n(n-1)/2$ blocks, we prove several extra claims. The two key variables are the size of N_k and the structure of $E_k(x^k_{N_k})$ as explained in the following. For any Davis-Maschler reduced-game $(N_k, V_{N_k,c})$, consider the following three conditions on the set $E_k(x^k_{N_k})$:

1. there exists $S \in E_k(x^k_{N_k})$ such that $|S| < |N_k| - 1$,

2. $|S| = |N_k| - 1$ for each $S \in E_k(x^k_{N_k})$, and there exists a coalition $S \in E_k(x^k_{N_k})$ such that $c(S_k) = v_{N_k,c}(S_k)$ or there exist a coalition $S \in E_k(x^k_{N_k})$ and a player $i \in S$ such that $0 < c_i - x^k_i \leq e(x^k_{N_k}, S)$,

3. $|S| = |N_k| - 1$ for each $S \in E_k(x^k_{N_k})$, and there neither exist a coalition $S \in E_k(x^k_{N_k})$ such that $c(S_k) = v_{N_k,c}(S_k)$ nor both a coalition $S \in E_k(x^k_{N_k})$ and a player $i \in S$ such that $0 < c_i - x^k_i \leq e(x^k_{N_k}, S)$.

Obviously, any set $E_k(x^k_{N_k})$ satisfies one and only one of these conditions, and if $|N_k| \leq 2$, then $E_k(x^k_{N_k})$ cannot satisfy condition 1. It is easy to check that once a player $i \in N_k$ is allocated a payoff $x^{k+1}_i = c_i$, then his payoff will not be altered in the subsequent steps. As a consequence, for each $k = 0, 1, \ldots$, it holds that $N_{k+1} \subseteq N_k$. Claim 7 below identifies the conditions on the set $E_k(x^k_{N_k})$ for which this inclusion is strict and Claim 8 explains that if $N_{k+1} = N_k$, then the number of coalitions of size $|N_k| - 1$ in $E_{k+1}(x^{k+1}_{N_{k+1}})$ must be less than in $E_k(x^k_{N_k})$.

Claim 7 *If either condition 1 or 2 is satisfied, then $N_{k+1} \subset N_k$.*

Proof. Firstly, suppose that condition 1 is satisfied. Then S_k is chosen such that $|S_k| < |N_k| - 1$. This means that there exists at least one player $i \in N_k \backslash (S_k \cup \{j_k\})$. Any such player is allocated a payoff $x^{k+1}_i = c_i$. Since N_{k+1} contains those players in N_k that are allocated a payoff different from the one they receive in c, $i \notin N_{k+1}$. We conclude that $N_{k+1} \subset N_k$.

Secondly, suppose that condition 2 is satisfied. If there exists a coalition $S \in E_k(x^k_{N_k})$ such that $c(S_k) = v_{N_k,c}(S_k)$, then from $x^{k+1}(S_k) = v_{N_k,c}(S_k)$, $|S_k| = |N_k| - 1$ and Claim 5, we get $x^{k+1}_{j_k} = c_{j_k}$ for the unique player j_k in $N_k \backslash S_k$. Thus $j_k \notin N_{k+1}$ and we conclude that $N_{k+1} \subset N_k$. If there exist a coalition $S \in E_k(x^k_{N_k})$ and a player $i \in S$ such that $0 < c_i - x^k_i \leq e(x^k_{N_k}, S)$, then consider player $i_1 \in T_k$. We know that $c_{i_1} - x^k_{i_1} \leq e(x^k_{N_k}, S_k)$. Therefore, $x^{k+1}_{i_1} = c_{i_1}$. For the same reason as above, $i_1 \notin N_{k+1}$ and we conclude once again that $N_{k+1} \subset N_k$. ■

Claim 8 *If condition 3 is satisfied, then $N_{k+1} = N_k$ and $\{S \in E_{k+1}(x^{k+1}_{N_{k+1}}) : |S| = |N_{k+1}| - 1\} \subseteq E_k(x^k_{N_k}) \backslash \{S_k\}$.*

Proof. Suppose condition 3 is satisfied. By construction of x^{k+1}, we know that each player $i \in S_k \backslash T_k$ will get $x^{k+1}_i = x^k_i > c_i$. By condition 3, we also know that the excess of S_k is too small to increase the payoff of some player $i \in T_k$ up to c_i. Finally condition 3 tells us that $c(S_k) \neq v_{N_k,c}(S_k)$. Because $c_{N_k} \in C(N_k, v_{N_k,c})$, this means $c(S_k) > v_{N_k,c}(S_k)$. Since $x^{k+1}_{N_k}$ is efficient in $(N_{k+1}, v_{N_{k+1},c})$ by Claim 5 and $x^{k+1}(S_k) = v_{N_k,c}(S_k)$ by construction, we get $x^{k+1}_{j_k} = v_{N_k,c}(N_k) - v_{N_k,c}(S_k) > c(N_k) - c(S_k) = c_{j_k}$ for the unique player j_k in $N_k \backslash S_k$. As a consequence, we obtain $N_{k+1} = N_k$. Note that this implies that $(N_{k+1}, v_{N_{k+1},c}) = (N_k, v_{N_k,c})$ and thus that $E_{k+1}(\cdot) = E_k(\cdot)$. Recall these equivalences throughout this proof.

In order to prove the second part of the claim, we start by the following observations. By condition 3, player $i_1 \in T_k$ gets $x^{k+1}_{i_1} = x^k_{i_1} + e(x^k_{N_k}, S_k) > x^k_{i_1}$, and for each player $i \in S_k \backslash \{i_1\}$, it holds that $x^{k+1}_i = x^k_i$. Thus, i_1 is the unique player in S_k who increases his payoff from x^k to x^{k+1}. Claim 5 proves that $x^{k+1}_{N_{k+1}}$ is an efficient allocation in $(N_{k+1}, v_{N_{k+1},c})$, so that we have $x^{k+1}_{j_k} = x^k_{j_k} - e(x^k_{N_k}, S_k)$ for the unique player j_k in $N_k \backslash S_k$. With these observations in mind, we can consider successively all coalitions of size $|N_k| - 1$ in 2^{N_k}.

Firstly, consider the chosen coalition S_k or equivalently $N_k\backslash\{j_k\}$. By construction, we know that $x^{k+1}(S_k) = v_{N_k,c}(S_k)$, which means that $S_k \notin E_{k+1}(x^{k+1}_{N_{k+1}})$.

Secondly, pick any player $i \in N_k\backslash\{i_1, j_k\}$ and consider the coalition $N_k\backslash\{i\}$. Because $N_k\backslash\{i\}$ contains both players i_1 and j_k, it follows that $x^{k+1}(N_k\backslash\{i\}) = x^k(N_k\backslash\{i\})$. Therefore, $N_k\backslash\{i\} \in E_k(x^k_{N_k})$ if and only if $N_k\backslash\{i\} \in E_{k+1}(x^{k+1}_{N_{k+1}})$.

Thirdly, it remains to consider one last coalition of size $|N_k| - 1$, namely coalition $N\backslash\{i_1\}$. By condition 3, we know that $x^{k+1}_{i_1} < c_{i_1}$. Together with the efficiency of allocations $x^{k+1}_{N_{k+1}}$ and $c_{N_{k+1}}$ in $(N_{k+1}, v_{N_{k+1},c})$, this implies that

$$x^{k+1}(N_k\backslash\{i_1\}) = x^{k+1}(N_{k+1}\backslash\{i_1\}) > c(N_{k+1}\backslash\{i_1\}) \geq v_{N_{k+1},c}(N_{k+1}\backslash\{i_1\}).$$

where the last equality follows from the fact that $c_{N_{k+1}} \in C(N_{k+1}, v_{N_{k+1},c})$. Thus, $N_k\backslash\{i_1\} \notin E_{k+1}(x^{k+1}_{N_{k+1}})$. We conclude that $\{S \in E_{k+1}(x^{k+1}_{N_{k+1}}) : |S| = |N_{k+1}| - 1\} \subseteq E_k(x^k_{N_k})\backslash\{S_k\}$. ∎

Claim 10 below is essential for the proof of Theorem 1. It determines how many steps are necessary to ensure a reduction of the population of a Davis-Maschler reduced-game. Its proof relies on Claims 7 and 8, and on the maximal size that the set $E_k(x^k_{N_k})$ can have if condition 3 is satisfied. Claim 9 specifies how an upper bound on this size can be obtained.

Claim 9 *Consider a TU-game $(N, v) \in \Gamma^c$, c an element of $C(N, v)$ and an efficient allocation $x \in \mathbb{R}^n$. For each $i \in N$ such that $x_i \leq c_i$, we have $e(x, N\backslash\{i\}) \leq 0$.*

Proof. The inequality

$$v(N\backslash\{i\}) \leq c(N\backslash\{i\}) = c(N) - c_i = x(N) - c_i \leq x(N) - x_i = x(N\backslash\{i\})$$

proves the statement. ∎

Claim 10 *For any $k \in \mathbb{N}$, at least one of the following cases holds:*

- *there exists $q \in \{1, \ldots, |N_k| - 1\}$ such that $x^{k+q}_{N_{k+q}} \in C(N_{k+q}, v_{N_{k+q},c})$,*
- *$|N_{k+|N_k|-1}| \leq |N_k| - 1$.*

Proof. For the sake of contradiction, assume that none of the two cases holds. In particular, $|N_{k+|N_k|-1}| = |N_k|$ since for each $q \in \{1, \ldots, |N_k| - 1\}$, we have $N_{k+q} \subseteq N_{k+q-1}$ by construction of x^{k+q}. We have seen that, when either condition 1 or 2 is satisfied, $|N_{k+1}| \leq |N_k| - 1$, which means that $|N_{k+|N_k|-1}| \leq |N_k| - 1$. So it must be the case that condition 3 holds for any N_{k+q}, $q \in \{0, \ldots, |N_k| - 2\}$. In other words, for each $q \in \{0, \ldots, |N_k| - 2\}$, the set $E_{k+q}(x^{k+q}_{N_{k+q}})$ only contains coalitions of size $|N_k| - 1$.

Now, let p^k_q denote the number of coalitions in $E_{k+q}(x^{k+q}_{N_{k+q}})$, $q \in \{0 \ldots, |N_k| - 1\}$. On the one hand, there exists an agent $i \in N_k$ such that $x^k_i < c_i$ by Claim 2. From Claim 9, this implies that $p^k_0 \leq |N_k| - 1$. On the other hand, Claim 8 proves that $p^k_q > p^k_{q+1}$ for each $q \in \{0, \ldots, |N_k| - 2\}$. As a consequence, we get $|N_k| - 1 \geq p^k_0 > p^k_1 > \cdots > p^k_{|N_k|-2} \geq 1$, which enforces that $p^k_q = |N_k| - 1 - q$ for each $q \in \{0, \ldots, |N_k| - 2\}$. In particular, it holds that $p^k_0 = |N_k| - 1$, which in turn implies that there exists a unique player in N_k, denoted by i for the rest of the proof, such that $x^k_i < c_i$. Thus, for any agent $j \in N_k\backslash\{i\}$, we have $x^k_j \geq c_j$, which yields $x^{k+q}_j \geq c_j$ for each integer $q \geq 0$ by construction.

Next consider the set $E_{k+|N_k|-1}(x^{k+|N_k|-1}_{N_{k+|N_k|-1}})$. From Claim 8, we know that $E_{k+|N_k|-1}(x^{k+|N_k|-1}_{N_{k+|N_k|-1}})$ does not contain any coalition of size $|N_k|-1$. So any nonempty coalition $S \in E_{k+|N_k|-1}(x^{k+|N_k|-1}_{N_{k+|N_k|-1}})$ that we can pick has a size at most $|N_k|-2$. There are two possibilities. Firstly, if the above-mentioned player $i \notin S$, then for each player $j \in S$, $x^{k+|N_k|-1}_j \geq c_j$, and thus $x^{k+|N_k|-1}(S) \geq c(S) \geq v(S)$. Secondly, suppose that $i \in S$. Recall that condition 3 holds at each step $k+q$, $q \in \{0,\dots,|N_k|-2\}$. In particular, $|S_{k+|N_k|-2}| = |N_k|-1$ and $j_{k+|N_k|-2}$ is the unique player in $N_k \backslash S_{k+|N_k|-2}$. Recall also that the satisfaction of coalition $S_{k+|N_k|-2}$ implies the equality $x^{k+|N_k|-1}_i - x^{k+|N_k|-2}_i = x^{k+|N_k|-2}_{j_{k+|N_k|-2}} - x^{k+|N_k|-1}_{j_{k+|N_k|-2}}$ since player i is the unique player such that $x^{k+|N_k|-2}_i < c_i$ at step $k+|N_k|-2$. From this equality, we obtain: $x^{k+|N_k|-1}(S) \geq x^{k+|N_k|-2}(S)$. In addition, since the size of S is at most $|N_k|-2$, condition 3 yields $S \notin E_{k+|N_k|-2}(x^{k+|N_k|-2}_{N_{k+|N_k|-2}})$, so that we also obtain $x^{k+|N_k|-2}(S) \geq v_{N_{k+|N_k|-2},c}(S)$. Therefore we get $x^{k+|N_k|-1}(S) \geq v_{N_{k+|N_k|-2},c}(S)$. We conclude that $E_{k+|N_k|-1}(x^{k+|N_k|-1}_{N_{k+|N_k|-1}}) = \emptyset$, or equivalently that $x^{k+|N_k|-1}_{N_{k+|N_k|-1}} \in C(N_{k+|N_k|-1}, v_{N_{k+|N_k|-1},c})$, a contradiction with the initial assumption. ■

We now have the material required to prove Theorem 1.

Proof. [Theorem 1] Claim 6 proves that for each $k = 0, 1, \dots$ such that $x^k_{N_k} \notin C(N_k, v_{N_k,c})$, then $x^{k+1}_{N_k} >_{S_k} x^k_{N_k}$. By Lemma 2, this implies that for each such $k = 0, 1, \dots$, we have $x^{k+1} >_{S_k \cup \overline{S_k}} x^k$. Thus, the sequence $(x^0, S_0 \cup \overline{S_0}, x^1, S_1 \cup \overline{S_1}, \dots, x^k, S_k \cup \overline{S_k}, x^{k+1})$ is a dominance path of length $k+1$ between x^0 and x^{k+1}.

By way of contradiction, assume that, for each integer $m \leq n(n-1)/2$, we have $x^m \notin C(N, v)$. For each $k \in \{0,\dots,n\}$, define the integer $\sigma(k) = \sum_{q=1}^{k}(n-q)$. Hence $\sigma(0) = 0$ and $\sigma(n) = n(n-1)/2$. We prove below by induction that, for each $k \in \{0,\dots,n\}$, it holds that $|N_{\sigma(k)}| \leq n-k$.
Initial step: It is obviously that the inequality $|N_{\sigma(0)}| = |N_0| \leq n$ follows from the definition of N_0.
Induction hypothesis: Pick any $k \in \{0,\dots,n-1\}$ and assume that $|N_{\sigma(q)}| \leq n-q$ for each integer $q \in \{0,\dots,k\}$.
Induction step: Let us show that $N_{\sigma(k+1)} \leq n-k-1$. Recall that $\sigma(k) < n(n-1)/2$ so that $x^{\sigma(k)} \notin C(N, v)$ by assumption, which is equivalent to $x^{\sigma(k)}_{N_{\sigma(k)}} \notin C(N_{\sigma(k)}, v_{N_{\sigma(k)},c})$ by Lemma 1. Together with Claim 10, this implies that

$$|N_{\sigma(k)+|N_{\sigma(k)}|-1}| \leq |N_{\sigma(k)}| - 1. \tag{1}$$

By the induction hypothesis, we have $|N_{\sigma(k)}| \leq n-k$, so that both sides of the inequality (1) can be bounded as follows:

$$|N_{\sigma(k)+n-k-1}| \leq |N_{\sigma(k)+|N_{\sigma(k)}|-1}| \leq |N_{\sigma(k)}| - 1 \leq n-k-1.$$

Since $\sigma(k)+n-k-1$ is precisely $\sigma(k+1)$, we conclude that $|N_{\sigma(k+1)}| \leq n-k-1$ as desired. In particular, $|N_{\sigma(n)}| \leq n-n$, which means that $N_{n(n-1)/2}$ is the empty set. In other words $x^{n(n-1)/2} = c$, which contradicts the initial assumption that $x^{n(n-1)/2} \notin C(N, v)$. ■

5 A comparison with the alternative procedures

In this section, we compare our procedure with the other approaches by means of examples. A first example shows that the tatônnement procedure adopted by Manea [9] may not reach the core and that the allocations created along the procedure may not be imputations. The tatônnement procedure of Manea [9] consists, at each step k, in choosing one of the coalitions with the greatest excess with respect to the current allocation x^k, say S_k, and to split equally this excess among the players in S_k, while the loss is shared equally among the players in the complementary coalition $N\backslash S_k$, *i.e.*

$$x_i^{k+1} = \begin{cases} x_i^k + e(x^k, S_k)/|S_k| & \text{if } i \in S_k, \\ x_i^k - e(x^k, S_k)/|N\backslash S_k| & \text{if } i \in N\backslash S_k. \end{cases}$$

So consider the three-player TU-game (N, v), with $N = \{1, 2, 3, 4\}$ and with non-zero worths given by the following table

S	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{1,2,3,4\}$
$v(S)$	10	10	10	8	8	8	12

Imputation $c = (9, 1, 1, 1)$ proves that the core of (N, v) is nonempty. Let imputation $x^0 = (0, 0, 0, 12) \notin C(N, v)$ be our departure point. Note that the largest excess of 10 with respect to x^0 is borne by coalitions $\{1, 2\}$ and $\{1, 3\}$. Without loss of generality, choose coalition $\{1, 2\}$. Then the constructed allocation $x^1 = (5, 5, -5, 7)$ at the next step is not an imputation in (N, v). Continuing in this fashion, it can be checked that the chain of blocks $(x^1, x^2, \dots)$ will never reach the core since at any step of the tatônnement, coalition $\{1, 2\}$ or $\{1, 3\}$ or $\{2, 3, 4\}$ will have positive excess.

A second example illustrates that our bound considerably improves the best bound found so far. Consider a n-player TU-game given with $N = \{1, \dots, n\}$ and $v(\{i\}) = 0$ for each $i \in N$, $v(N) = n$ and $v(S) = 1$ otherwise. For any $n \in \mathbb{N}$, the allocation c such that $c_i = 1$ for each $i \in N$ belongs to the core. In addition, the game has the maximal number of active coalitions, that is $2^n - (n + 2)$, and this number corresponds to the bound found by Yang [17]. For n sufficiently large, the difference between this bound and our is considerable. For instance, if $n = 20$, then Yang [17] obtains 1048554 while our bound shrinks to 190. According to Yang, the bound found by Kóczy [8] is even greater. It can be argue that the bound of Yang is well-suited for TU-games with a large number of players but few active coalitions. However, one can note that even TU-games with a very simple structure can have a large number of active coalitions. For instance, consider a unanimity TU-game (N, u_S) on coalition $S \in 2^N\backslash\{\emptyset\}$. The number of active coalitions in (N, u_S) is $2^{n-|S|} - 1$ if $|S| \geq 2$ and 0 if $|S| = 1$. Therefore, the bound of Yang is greater than our bound whenever

$$2 \leq |S| \leq n + 1 - \log_2(n + 1) - \log_2(n - 2).$$

Again, if $n = 20$, the above condition is met for each unanimity TU-game (N, u_S) such that $2 \leq |S| \leq 13$. There are other classes of games with nonempty cores which contain a large number of active coalitions, such as strictly superadditive TU-games and strictly convex TU-games. Another similar class of TU-games is the class of queueing games studied by Maniquet [10]. In such games, only coalitions of size two have a non-zero dividend, which implies that all coalitions of size at least two are active. One last class of such games is the class of market games introduced by Shapley [14]. For any TU-game in this class, any coalition containing at least one buyer and one seller has a worth of at least a unit, which is greater than the zero worth obtained from summing the stand-alone worths of the members of the coalition.

Finally, it is difficult to make a comparison with the procedure in Sengupta and Sengupta [13]. The two approaches are rather similar, but we believe that the two differences, on the choice of the active coalition and on the redistribution of its excess among its members, are crucial in establishing our result. The reader might have the same feeling after reading our proofs. Even if we did not manage to measure the maximal number of blocks required by their procedure, we believe that it is greater than $n(n-1)/2$.

6 Conclusion

The new upper bound on the number of blocks required to access the core that we provide in this article considerably improves the former existing bounds. We did not manage to prove the same result with the procedure of Sengupta and Sengupta [13]. Therefore, we suspect that the two differences between their procedure are the procedure used in this article are crucial in establishing the result. Moreover, we expect that this bound can be lowered. In fact, we did not succeed in finding examples where our bound is tight, which might suggests that the "worst-case" scenarios considered in the proofs of Claims 8, 9 and 10 and Theorem 1 are very unlikely to occur simultaneously. We therefore wonder whether our polynomial number of blocks can be sufficiently reduced to become linear. This question is left for future works.

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