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Survival Measures and Interacting Intensity Model: with Applications in Guaranteed Debt Pricing *

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Abstract

This paper studies survival measures in credit risk models. Survival measure, which was first introduced by Schönbucher [12] in the framework of defaultable LMM, has the advantage of eliminating default indicator variable directly from the expectation by absorbing it into Randon-Nikodym density process. Survival measure approach was further extended by Collin-Duresne [4] to avoid calculating a troublesome jump in IBPR reduced-form model. This paper considers survival measure in "HBPR" model, i.e. default time is characterized by Cox construction, and studies the relevant drift changes and martingale representations. This paper also takes advantage of survival measure to solve the looping default problem in interacting intensity model with stochastic intensities. Guaranteed debt is priced under this model, as an application of survival measure and interacting intensity model. Detailed numerical analysis is performed in this paper to study influence of stochastic pre-default intensities and contagion on value of a two firms' bilateral guaranteed debt portfolio.

JEL classification: G12; G13

Keywords: Survival Measure, Interacting Intensity Model, Measure Change, Guaranteed Debt, Mitigation and Contagion.

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1 Introduction

As well known, the methodology for modeling default risk can be split into two main approaches, the structural approach and reduced form approach. Structural model treats the default time as first passage time of firm value process over a default barrier. This approach is intuitive in the way that it models default as endogenous event that is determined by the structure of balance sheet of the company. Nonetheless, many significant drawbacks of structural approach constrict the application of this model. Firstly, the firm value process and default barrier can hardly be observed, that is only partial information is available in the market, for the reason that firm asset is not tradeable and structure of balance sheet is complex to identify. Moreover, first passage time characterization of default time of a continuous process over

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default barrier implies that default time is predictable time, leading to unnatural features such as null spreads for short maturities.

Reduced form approach, on the other hand, lies on the assumption that default time is a totally inaccessible time. There are mainly two categories of reduced form models, "Intensity Based Pricing Rule" (IBPR) and "Hazard Based Pricing Rule" (HBPR), termed by Jeanblanc [7]. In IBPR model, see Duffie [5], default time is a stopping time of the whole market filtration. The model is based on the existence of an "intensity rate process": a non-negative process satisfying a compensation property. The main problem in this methodology is that the pricing rule leads to a non tractable formula, involving computations complex to handle. HBPR model, see Lando [9], is based on the computation of the "Hazard process" and lies on the assumption of decomposing market information into two filtration: a reference filtration expanded by information of default-free assets and a filtration expanded by the progressive knowledge of credit event. The decomposition of market information in HBPR model concludes a pricing formula much more convenient to use. However, it depends on the assumption of the existence of decomposition of all available information into "default-free information" and "default event information", as well as some technical requirements about the hazard process, see Bielecki [3] for detailed discussion.

Survival measure is first introduced by Schönbucher [12] in a so called LIBOR Market Model with Default Risk ("defaultable LMM" for short), where defaultable effective forward rates in discrete tenor are modeled in the way similar to default free LIBOR Market Model. Survival measure performs the role to eliminate default indicator in expectation when pricing credit derivatives, just like the effect reduced form model has by replacing survival indicator variable with negative exponential of integrating intensity process or hazard process. Survival measure is motivated by the attempt to eliminate default indicator variable in expectation of defautable LMM without using intensity models. This idea is extended by Collin-Duresne [4] to a general formula which avoids the problem of calculating a non-tractable discounted expected jump term in IBPR reduced-form model. Further application of survival measure approach is explored by Leung [8] to overcome the difficulty solving looping default problem in interacting intensity model, which is an intuitive and direct approach of characterizing default contagion. However, default intensities are assumed to be constant parameters in Leung [8]'s work. This paper considers survival measure in "HBPR" model, i.e. default time is characterized by Cox construction, and studies the relevant drift changes and martingale representations. Based on these analysis, interacting intensity model is extended to allow stochastic default intensities which are driven by Brownian motions.

Guaranty, as one of the three major means of mitigation in commercial banks, has both the effect of mitigation and contagion. If the guarantee does default before maturity, but the guarantor does not, even after he takes over the guarantee's obligation, then less loss would incurred for commercial banks. On the other hand, the guarantor would increase its own probability of default by taking over guarantee's obligation, which is the side-effect of his participation in a guaranty relationship. Mitigation has the effect of reducing total loss whilst contagion induces higher probability of large loss. Therefore, a natural question would be to exactly evaluate both of these two effects so as to tell whether banks should use guaranty to mitigate in those particular loans. Li and Bao [11] establishes a framework for analysis of mitigation and contagion effect of guaranteed debt where contagion is modeled by interacting intensities with constant parameters. Analytical solutions are attained through the approach of survival measure. A term Conditional Odds Ratio is defined to set up a criterion for gauging the difference between mitigation effect and contagion risk in a pair of guaranteed debt. This paper extends Li and Bao [11]'s work to allow stochastic default intensities which are driven by

Brownian motions..

The remaining sections are organized as follows. IBPR and HBPR are briefly introduced in Section 2. Section 3 considers extensions of IBPR and HBPR via survival measure changes. Two alternative measure changes are compared and summarized in this section. Section 4 explores the applications of survival measure approach in interacting intensity model with stochastic default intensities. Mitigation and contagion effect in guaranteed debt is priced in Section 5, as an application of survival measure approach and interacting intensity model. Section 6 performs a series of numerical experiments to analyze impact of contagion and stochastic pre-default intensities on value of a two firms' bilateral guaranteed debt portfolio. This paper is concluded in Section 7.

2 Reduced Form Models

This section presents the "Intensity Based Pricing Rule" and "Hazard Based Pricing Rule", the two main approaches in reduced form modeling. In intensity based framework, default time τ is a stopping time in a given filtration $\mathbb{G}$, which represents the full information of market. The default indicator process H_t is defined as the $\mathbb{G}$-adapted increasing càdlàg process $1_{\{\tau\leq t\}}$, which is obviously a $\mathbb{G}$ -submartingale thus assures the existence of unique $\mathbb{G}$-predictable increasing process $\Lambda_t^{\mathbb{G}}$, called the compensator of H_t, such that the following process

$$M_t = H_t - \Lambda_t^{\mathbb{G}} \tag{2.1}$$

is a $\mathbb{G}$-martingale. As the default indicator process H_t vanishes after default, $\Lambda_t^{\mathbb{G}}$ have to be constant after default so as to ensure that M_t is martingale. This means $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$. In the light of definition of totally inaccessible time, it is not hard to check that $\Lambda_t^{\mathbb{G}}$ is continuous if and only if τ is a $\mathbb{G}$-totally inaccessible stopping time. Derivative of $\Lambda_t^{\mathbb{G}}$ with respect to Lebesgue measure is denoted by $\lambda_t^{\mathbb{G}}$, if exists, such that

$$\Lambda_t^{\mathbb{G}} = \int_0^t \lambda_s^{\mathbb{G}} ds, \qquad \forall t \geq 0$$

.

$\lambda_t^{\mathbb{G}}$ is called the intensity rate process of τ and vanishes after default.

Duffie [5] proposes the pricing formula in IBPR for defaultable contingent claim (X, D_t), with survival contingent claim $X \in \mathcal{G}_T$ and cumulative dividend process D_t being $\mathcal{G}_t$ predictable. IBPR pricing formula is expressed as

$$\begin{aligned} S_t &= E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s r_u du\right\} 1_{\{\tau>s\}} dD_s + \exp\left\{-\int_t^T r_u du\right\} X 1_{\{\tau>T\}} \middle| \mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}} \cdot \left\{V_t - E^{\mathbb{Q}}\left[\exp\left\{-\int_t^\tau r_u du\right\} \Delta V_\tau \middle| \mathcal{G}_t\right]\right\} \end{aligned} \tag{2.2}$$

where the expectation is computed under martingale measure $\mathbb{Q}$, which is assumed to exist, and pre-default value V_t is defined as

$$V_t = E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s \left[r_u + \lambda_u^{\mathbb{G}}\right] du\right\} dD_s + \exp\left\{-\int_t^T \left[r_u + \lambda_u^{\mathbb{G}}\right] du\right\} X \middle| \mathcal{G}_t\right] \tag{2.3}$$

Detailed proof of formulas (2.2) and (2.3) are referred to Duffie [5].

The main difficulty of IBPR pricing formula (2.2) is the computation of the jump of pre-default process V_t at default time τ. Generally, V_t is in no way to be continuous but in some special cases. For example, if market filtration $\mathcal{G}_t$ can be decomposed as $\mathcal{G}_t = \mathcal{F}_t^c \vee \mathcal{F}_t^I$, with $\mathcal{F}_t^c$ being continuous sub-filtration and (X, D_t) being independent from $\mathcal{F}_t^I$ conditional on $\mathcal{F}_t^c$.

In hazard based framework, the default time τ is still a stopping time in market filtration $\mathbb{G}$, but with additional assumption that $\mathcal{G}_t$ is decomposed as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, with $\mathbb{F} = \{\mathcal{F}_t, t > 0\}$ being default-free filtration that is expanded by information from default-free assets, and $\mathbb{H} = \{\mathcal{H}_t, t > 0\}$ being default filtration expanded by the progressive knowledge of credit event, i.e. $\mathcal{H}_t = \sigma(\{H_s\}_{s\leq t})$. τ is assumed not to be $\mathbb{F}$-stopping time. Thus, it is reasonable to define the following conditional default probability

$$F_t = \mathbb{Q}\{\tau \leq t | \mathcal{F}_t\}$$

which is assumed to satisfy $F_t < 1, \forall t > 0$.

Given F_t, define hazard process Γ_t as

$$\Gamma_t = -\ln(1 - \mathrm{F_t})$$

Under the assumption of F_t being continuous and monotonically increasing, it can be shown that

$$M_t = H_t - \Gamma_{t\wedge\tau}$$

is $\mathbb{G}$-martingale. Uniqueness of Doob-Meyer decomposition of H_t asserts that Γ_t is compensator of H_t with respect to filtration $\mathbb{G}$, that is, $\Gamma_t = \Lambda_t^{\mathbb{F}}$. However, this equality does not hold when the assumption of F_t's continuity and monotonicity is not true.

Based on the assumption of F_t being continuous and monotonically increasing, Lando [9] proposes a pricing formula under HBPR which is similar with formulas (2.2) and (2.3), but involves no jump of any processes. Assume Γ_t is absolutely continuous with respect to Lebesgue measure, and have derivative γ_t. Then Bielecki [3] proves $\lambda_t^{\mathbb{F}} = \gamma_t$, and

$$\begin{aligned} S_t &= E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s r_u du\right\} 1_{\{\tau>s\}} dD_s + \exp\left\{-\int_t^T r_u du\right\} X 1_{\{\tau>T\}} \middle| \mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}} \cdot E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s \left[r_u + \lambda_u^{\mathbb{F}}\right] du\right\} dD_s + \exp\left\{-\int_t^T \left[r_u + \lambda_u^{\mathbb{F}}\right] du\right\} X \middle| \mathcal{F}_t\right] \end{aligned} \tag{2.4}$$

Apart from the difficulty of computing jump of pre-default value process in IBPR, pricing formulas (2.2), (2.3) and (2.4) in reduced form models are similar to the pricing formula of default-free contingent claims, with default-free interest rate r_t replaced by $\tilde{r}_t = r_t + \lambda_t^{\mathbb{G}}$ or $\hat{r}_t = r_t + \lambda_t^{\mathbb{F}}$ in IBPR and HBPR, respectively. One can model $\tilde{r}_t$ or $\hat{r}_t$ using short interest rate models, such as CIR and JCIR, and price credit derivatives explicitly or numerically, with parameters calibrated from credit market and interest rate market. This is one of the major differences of reduced form model with structural model.

3 Extensions via Measure Changes

Application of IBPR pricing formulas (2.2) and (2.3) is restricted by the necessity of computing jump of V_t. Collin-Dufresne [4] proposes a so called general framework where an inequivalent measure change is performed and jump component is removed from IBPR formula. The result of Collin-Dufresne [4] is summarized in the following theorem.

Theorem 1. *Assume the compensator $\Lambda_t^{\mathbb{G}}$ of H_t with respect to $\mathbb{G}$ and its derivative $\lambda_t^{\mathbb{G}}$ exist.*

Define a probability measure change by

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = L_t|_{\mathcal{G}_t} = 1_{\{\tau>t\}} e^{\Lambda_t^{\mathbb{G}}}\Big|_{\mathcal{G}_t} \tag{3.5}$$

Then price of defaultable contingent claim (X, D_t) is uniquely determined by

$$\begin{aligned} S_t &= E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s r_u du\right\} 1_{\{\tau>s\}} dD_s + \exp\left\{-\int_t^T r_u du\right\} X 1_{\{\tau>T\}} \middle| \mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}} \cdot E^{\bar{\mathbb{P}}}\left[\int_{]t,T]} \exp\left\{-\int_t^s \left[r_u + \lambda_u^{\mathbb{G}}\right] du\right\} dD_s + \exp\left\{-\int_t^T \left[r_u + \lambda_u^{\mathbb{G}}\right] du\right\} X \middle| \bar{\mathcal{G}}_t\right] \end{aligned} \tag{3.6}$$

where expectation $E^{\bar{\mathbb{P}}}$ is performed under the new probability $\bar{\mathbb{P}}$, with respect to the new filtration $\bar{\mathbb{G}}$, which is defined as the augmentation of original filtration $\mathbb{G}$ by the null sets of the probability measure $\bar{\mathbb{P}}$. □

Proof of this theorem is straightforward. One can first represent S_t as

$$\begin{aligned} S_t &= 1_{\{\tau>t\}} \cdot E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s [r_u + \lambda_u^{\mathbb{G}}] du\right\} L_s dD_s + \exp\left\{-\int_t^T [r_u + \lambda_u^{\mathbb{G}}] du\right\} X L_T \middle| \mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}} \cdot E^{\mathbb{Q}}\left[\int_{]t,T]} \exp\left\{-\int_t^s [r_u + \lambda_u^{\mathbb{G}}] du\right\} dD_s L_T + \exp\left\{-\int_t^T [r_u + \lambda_u^{\mathbb{G}}] du\right\} X L_T \middle| \mathcal{G}_t\right] \end{aligned}$$

Then the proof is a simple application of Bayesian formula in the situation of absolutely continuous change of probability measure. This Bayesian formula is referred to Appendix A.

Pricing formula (3.6) is similar to HBPR formula (2.4) in the way that it mimics default-free pricing rule with slight difference of changing risk-free rate r_t to risk-adjusted rate $\tilde{r}_t = r_t + \lambda_t^{\mathbb{G}}$. However, formula (3.6) is significantly different from HBPR formula (2.4) by replacing martingale measure $\mathbb{Q}$ with an inequivalent measure $\bar{\mathbb{P}}$ which is absolutely continuous with respect to $\mathbb{Q}$, with the filtration $\mathbb{G}$ changed into $\bar{\mathbb{G}}$ accordingly. The measure $\bar{\mathbb{P}}$ is called CGH **Survival Measure** for the reason that its quality concentrates on the event of survive until maturity. That is,

$$\bar{\mathbb{P}}\{A\} = E^{\mathbb{Q}}\left\{1_A \cdot \left(\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}}\right|_{\mathcal{G}_T}\right)\right\} = E^{\mathbb{Q}}\left\{1_A \cdot 1_{\{\tau>T\}} e^{\Lambda_T^{\mathbb{G}}}\right\} = 0, \quad \forall A \subset \{\tau \le T\}$$

and

$$\bar{\mathbb{P}}\{\tau > T\} = E^{\mathbb{Q}}\left\{1_{\{\tau>T\}} \cdot \left(\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}}\right|_{\mathcal{G}_T}\right)\right\} = E^{\mathbb{Q}}\left\{1_{\{\tau>T\}} \cdot 1_{\{\tau>T\}} e^{\Lambda_T^{\mathbb{G}}}\right\} = 1$$

$\bar{\mathbb{G}}$ is constructed as augmentation of original filtration $\mathbb{G}$ by the null sets of the probability measure $\bar{\mathbb{P}}$ means that $\bar{\mathbb{G}}$ is obtained by adding to the original filtration the knowledge that default will not occur before the maturity date of the security under consideration. Other than using martingale property of compensated martingale $M_t = H_t - \Lambda_{t\wedge\tau}$ in IBPR and HBPR to eliminate survival indicator $1_{\{\tau>T\}}$ from pricing formula, survival measure approach in Collin-Dufresne [4] absorbs default indicator $1_{\{\tau>T\}}$ into Randon-Nikodym density process directly, and eliminate it through measure change.

Schönbucher [12] proposes a survival measure approach in the framework of LIBOR market model with default risk, paralleling with the famous LIBOR Market Model (LMM) in interest rate market. Motivation of using survival measure in his paper is based on the direct effect of absorbing default indicator into Randon-Nikodym density process when measure is changed. Parallelism of reduced form model with short interest rate model is extended by Schönbucher

[12] to LIBOR Market Model, where effective (simply compounded) forward rate is fundamental model quantities, other than short (continuously compounded) interest rate. Schönbucher [12] treats defaultable effective forward rate as fundamental quantity in defaultable LMM, where intensity rate and short rate is not modeled. Therefore, reduced form approach can not be applied to attain pricing formula in this framework, making survival measure approach as suitable alternative.

Survival measure in Schönbucher's model is defined as

$$\left.\frac{d\bar{\mathbb{P}}_T}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = Z_t|_{\mathcal{G}_t} = 1_{\{\tau>t\}} \cdot \left.\frac{\bar{B}(t,T)}{B(t)\bar{B}(0,T)}\right|_{\mathcal{G}_t} \tag{3.7}$$

where $\bar{B}(t,T)$ is denoted as pre-default value of defaultable zero coupon bond, which is fundamental model quantity in Schönbucher's model as well, and $B(t) = B_t$ is bank account. It is not hard to check that $\bar{\mathbb{P}}_T$ is also a survival measure, meaning

$$\bar{\mathbb{P}}_T\{A\} = 0,\ \forall A \subset \{\tau \le T\} \text{ and } \bar{\mathbb{P}}_T\{\tau > T\} = 1$$

Using Bayesian formula for absolutely continuous measure change in Appendix A, one can easily attain the following pricing formula in Schönbucher's defaultable LMM.

$$\begin{aligned}
S_t &= E^{\mathbb{Q}}\left[\int_{]t,T]} B_t B_s^{-1} 1_{\{\tau>s\}} dD_s + B_t B_T^{-1} X 1_{\{\tau>T\}} \middle| \mathcal{G}_t\right] \\
&= B_t \bar{B}(0,T) E^{\mathbb{Q}}\left[\int_{]t,T]} \bar{B}^{-1}(s,T) Z_s dD_s + X Z_T \middle| \mathcal{G}_t\right] \\
&= B_t \bar{B}(0,T) E^{\mathbb{Q}}\left[\int_{]t,T]} \bar{B}^{-1}(s,T) Z_T dD_s + X Z_T \middle| \mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}} \cdot \bar{B}(t,T) E^{\bar{\mathbb{P}}_T}\left[\int_{]t,T]} \bar{B}^{-1}(s,T) dD_s + X \middle| \tilde{\mathcal{G}}_t\right]
\end{aligned} \tag{3.8}$$

where expectation $E^{\bar{\mathbb{P}}_T}$ is performed under the new probability measure $\bar{\mathbb{P}}_T$, with respect to the new filtration $\tilde{\mathbb{G}}$, which is defined as the augmentation of original filtration $\mathbb{G}$ by the null sets of the probability measure $\bar{\mathbb{P}}_T$.

One interesting property of Schönbucher's survival measure is that it can be represented as conditional forward probability measure on the survival event $\{\tau > T\}$. That is,

$$\begin{aligned}
\bar{\mathbb{P}}_T\{A\} &= \frac{E^{\mathbb{Q}}\left[1_A \cdot \frac{d\bar{\mathbb{P}}_T}{d\mathbb{Q}}\right]}{E^{\mathbb{Q}}\left[\frac{d\bar{\mathbb{P}}_T}{d\mathbb{Q}}\right]} = \frac{E^{\mathbb{Q}}\left[1_A \cdot 1_{\{\tau>T\}} \frac{1}{B(T)\bar{B}(0,T)}\right]}{E^{\mathbb{Q}}\left[1_{\{\tau>T\}} \frac{1}{B(T)\bar{B}(0,T)}\right]} \\
&= \frac{E^{\mathbb{Q}}\left[1_A \cdot 1_{\{\tau>T\}} \frac{1}{B(T)B(0,T)}\right]}{E^{\mathbb{Q}}\left[1_{\{\tau>T\}} \frac{1}{B(T)B(0,T)}\right]} = \frac{E^{\mathbb{Q}}\left[1_A 1_{\{\tau>T\}} \cdot \frac{d\mathbb{P}_T}{d\mathbb{Q}}\right]}{E^{\mathbb{Q}}\left[1_{\{\tau>T\}} \cdot \frac{d\mathbb{P}_T}{d\mathbb{Q}}\right]} = \mathbb{P}_T\{A|\,\tau > T\}
\end{aligned}$$

where forward measure $\mathbb{P}_T$ is defined by

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = \frac{B(t,T)}{B(t)B(0,T)}$$

with $B(t,T)$ being the value of default-free zero coupon bond. Survival measure $\bar{\mathbb{P}}$ can not be represented as conditional forward probability measure, because the third equality in the above equation does not hold if Randon-Nikodym density is replaced by $\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}}$.

The fundamental problem one has to consider when operating measure changes is the semi-martingale representation of original martingale under the new measure. The consequence of absolutely continuous measure change on martingale property is given in Appendix A as Girsanov's Theorem. Girsanov's Theorem in Appendix A shows that $\bar{m}_t = m_t - \int_0^t \frac{1}{L_{s-}} d\langle L, m\rangle_s$ and $\tilde{m}_t = m_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, m\rangle_s$ are $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$-martingale and $(\bar{\mathbb{P}}_T, \tilde{\mathbb{G}})$-martingale, respectively. However, computing quadratic covariation processes $\langle L, m\rangle_t$ and $\langle Z, m\rangle_t$ is not so simple for an arbitrary martingale m_t. As indicated in Bielecki [3], non-negative martingale L_t can be represented as Itô's integral with respect to the fundamental martingale M_t defined in equation (2.1), i.e. $dL_t = -L_{t-} dM_t$. Therefore, $\langle L, m\rangle_t$ can be simplified and $\bar{m}_t$ can be represented as m_t plus quadratic covariation $\langle M, m\rangle_t$. Thus $\bar{m}_t$ can be better understood under new measure $\bar{\mathbb{P}}$. The results are given in the following theorem, whose proof are referred to Collin-Dufresne [4].

Theorem 2. *Suppose the survival measure $\bar{\mathbb{P}}$ defined in equation (3.5) exists, and m_t is an arbitrary $(\mathbb{Q}, \mathbb{G})$-martingale. Then the process defined by*

$$\bar{m}_t = m_t - \int_0^t \frac{1}{L_{s-}} d\langle L, m\rangle_s = m_t + \langle M, m\rangle_t \tag{3.9}$$

is a martingale with respect to $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$. In particular,

(i) If the process m_t does not jump at the default time τ, i.e. $\Delta m_\tau = 0$, then m_t itself is $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$-martingale as well.

(ii) The default intensity $\lambda_t^{\mathbb{G}}$ and the default indicator process H_t are both equal to zero almost surely under $\bar{\mathbb{P}}$ on the interval $[0, T]$. □

Interpretation of (i) in Theorem 2 is straightforward. $\Delta m_\tau = 0$ implies that the original martingale m_t is "independent" from default time. Default of reference entity does not have sudden impact on dynamics of m_t. Then martingale property of m_t under the new measure $\bar{\mathbb{P}}$, which concentrates all quality on the event of survival until maturity, remains unchanged. Similarly, because $\bar{\mathbb{P}}$ put all weight on survival event, default intensity and default indicator process are reasonably to vanish under this measure.

In particular, if m_t is a $(\mathbb{Q}, \mathbb{G})$ Brownian motion, then m_t is still continuous martingale under $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$. It is obvious that deterministic quadratic variation remains the same under absolutely continuous probability measure change. Therefore, quadratic variation of m_t is still $\langle m\rangle_t = t$ under $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$. Lévy's characterization of Brownian motion shows that m_t is still Brownian motion under $(\bar{\mathbb{P}}, \bar{\mathbb{G}})$.

In general, $\bar{\mathbb{P}}$ does not coincide with $\bar{\mathbb{P}}_T$. Particularly, consider the situation when jump of pre-default value process V_t vanishes in IBPR. Then one can easily attain the following equation from formulas (2.2) and (2.3).

$$\left.\frac{d\bar{\mathbb{P}}_T}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = 1_{\{\tau>t\}} \cdot \left.\frac{\bar{B}(t,T)}{B(t)\bar{B}(0,T)}\right|_{\mathcal{G}_t} = 1_{\{\tau>t\}} \cdot e^{\Lambda_t^{\mathbb{G}}} \left.\frac{E^{\mathbb{Q}}\left[B_T^{-1} e^{-\Lambda_T^{\mathbb{G}}} | \mathcal{G}_t\right]}{E^{\mathbb{Q}}\left[B_T^{-1} e^{-\Lambda_T^{\mathbb{G}}}\right]}\right|_{\mathcal{G}_t}$$

Therefore, $\bar{\mathbb{P}}$ coincides with $\bar{\mathbb{P}}_T$ if and only if $B_T^{-1} e^{-\Lambda_T^{\mathbb{G}}} = Constant$, which does not holds in generic environment. Because of the good qualities $\bar{\mathbb{P}}$ has in Theorem 2, this paper will only consider applications of CGH survival measure approach in interacting intensity model with stochastic intensities and in guaranteed debt pricing problem.

CGH survival measure is defined in the framework of IBPR because $\mathbb{G}$-adapted process $\Lambda_t^{\mathbb{G}}$ is just compensator of H_t, without any specification under any (default-free) sub-filtration.

Survival measure used in next section is supposed to be defined as, for example

$$\frac{d\bar{\mathbb{P}}^i}{d\mathbb{Q}}\bigg|_{\mathcal{G}_t} = 1_{\{\tau_i > t\}} \cdot e^{\Lambda_t^{i,\bar{\mathbb{F}}^i}}\bigg|_{\mathcal{G}_t} \tag{3.10}$$

where $\Lambda_t^{i,\bar{\mathbb{F}}^i}$ is $\bar{\mathbb{F}}^i$-adapted hazard process defined in HBPR framework. Reference filtration $\bar{\mathbb{F}}^i$ is the smallest sub-filtration of $\mathbb{G}$ satisfying $\mathbb{G} = \bar{\mathbb{F}}^i \vee \mathbb{H}^i$. Usually, $\bar{\mathbb{F}}^i$ is designed to be union of default-free filtration $\mathbb{F}$ and default filtration of some other firms, say $\mathbb{H}^{-i} = \bigvee_{j \neq i} \mathbb{H}^j$, where $\mathbb{H}^j$ is default filtration of firm j. Moreover, $\Lambda_t^{i,\bar{\mathbb{F}}^i}$ is assumed to be purely $\mathbb{F}$-adapted once $\mathbb{H}^{-i}$ information is given.

In fact, the survival measure in HBPR as defined in equation (3.10) is special case of CGH survival measure if hazard process $\Lambda_t^{i,\bar{\mathbb{F}}^i}$ is assumed to be martingale hazard process, i.e. $H_t^i - \Lambda_{t\wedge\tau_i}^{i,\bar{\mathbb{F}}^i}$ is $\mathbb{G}$-martingale. Thus the above results of CGH survival measure can be applied to HBPR survival measure. The above assumption is supposed to always hold in subsequent sections.

4 Interacting Intensity Model

This section considers the application of our survival measure under HBPR framework in interacting intensity model. **Interacting intensity model** or **contagion model**, introduced by Jarrow et al. [6] is the only default dependence model that can explicitly characterize contagion among reference firms, comparing to the popular copula model, see Bao et al. [1] and Bao et al. [2] for example. The model is built upon the fundamental single-name reduced-form model, via constructing direct interacting effect among default intensities of reference firms. For instance, the model with two reference firms, say firm A and firm B, can be expressed as

$$\begin{cases} \lambda_t^A \doteq a_t^0 + a_t^1 \cdot 1_{\{\tau_B \le t\}} \\ \lambda_t^B \doteq b_t^0 + b_t^1 \cdot 1_{\{\tau_A \le t\}} \end{cases} \tag{4.11}$$

where τ_A and τ_B are default times of firm A and firm B, with stochastic hazard processes λ_t^A and λ_t^B, respectively. As indicated at the end of last section, λ_t^A and λ_t^B are supposed to be stochastic hazard process in HBPR framework, i.e. λ_t^A is $\mathbb{G}^{-A} = \mathbb{F} \vee \mathbb{H}^B$-adapted, λ_t^B is $\mathbb{G}^{-B} = \mathbb{F} \vee \mathbb{H}^A$-adapted. Furthermore, τ_A and τ_B are characterized by the following Cox construction

$$\tau_i = \inf\left\{ t > 0 \,\middle|\, \int_0^t \lambda_s^i ds \ge E_i \right\}, \quad i = A, B \tag{4.12}$$

given realization of $\mathcal{G}_\infty^{-i}$. E_A and E_B are independent unit mean exponential variables, i.e., $Exp(1)$-random variable. Intensities a_t^k and b_t^k, $k = 0, 1$ are assumed to be $\mathbb{F}$-adapted non-negative processes.

Interacting intensity model with stochastic intensities (4.11-4.12) can characterize default contagion among firms while allowing default intensities to be $\mathbb{F}$-stochastic, i.e. driven by default free information. Specifically, firm A has default hazard rate $\lambda_t^A = a_t^0$ before the default of any firm. The intensity of A will immediately jump to a higher level $\lambda_t^A = a_t^0 + a_t^1$ upon the occurrence of B's default, thus direct contagion of firm B's default information to firm A is characterized. Contagion from firm A to firm B can be explained in the same way.

The major problem of interacting intensity model is computing unconditional joint and marginal default probabilities, facing the so called Looping Default Problem as main obstacle. Pricing formula of HBPR reduced form model, as presented in section 2, implies that marginal default probability can be attained once hazard rate is given as inputs. However, hazard rates in (4.11) are still determined by default status of another firm, whose hazard rate is recursively determined by the former firm. This looping dependence while computing joint and marginal default probabilities is hard to deal with. Three alternative approaches are proposed in literatures, total hazard approach in Yu [14] and Yu [15], Markov chain approach in Leung [10] and Walker [13], CGH survival measure approach in Leung [8]. CGH survival measure is originally proposed by Collin-Dufresene [4] to extend IBPR, whose essence of absorbing default indicator into Randon-Nikodym density process is further applied by Leung [8] to tackle looping default problem in interacting intensity model where default indicator is explicitly present. However, only constant parameters a^0, a^1, b^0, b^1 are considered in Leung [8]. This paper extends interacting tensity model to allow stochastic intensities while still remains tractable. Girsanov's theorem relating to CGH survival measure presented in Section 3 can be used for our HBPR survival measure so as to attain analytical solutions.

Application of HBPR survival measure approach in interacting intensity model (4.11-4.12) is expressed in the following lemma.

Lemma 1. *For interacting intensity model with stochastic intensities (4.11-4.12), define the following two survival measures*

$$\left.\frac{d\mathbb{Q}_A}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = 1_{\{\tau_A > t\}} \cdot \exp\left\{\int_0^t \lambda_s^A ds\right\}, \quad \forall t \leq T \tag{4.13}$$

and

$$\left.\frac{d\mathbb{Q}_B}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = 1_{\{\tau_B > t\}} \cdot \exp\left\{\int_0^t \lambda_s^B ds\right\}, \quad \forall t \leq T \tag{4.14}$$

then stochastic hazard processes λ_t^A and λ_t^B can be simplified under the two survival measures as

$$\begin{cases} \lambda_t^B = b_t^0 \sim \mathbb{Q}_A - a.s. \\ \lambda_t^A = a_t^0 \sim \mathbb{Q}_B - a.s. \end{cases} \quad and \quad \begin{cases} \lambda_t^B = 0 \sim \mathbb{Q}_B - a.s. \\ \lambda_t^A = 0 \sim \mathbb{Q}_A - a.s. \end{cases} \tag{4.15}$$

Moreover, if a_t^k and b_t^k, $k = 0, 1$ are assumed to be $\mathbb{F}$-adapted non-negative Itô diffusion processes, and $\mathbb{F}$ is assumed to be expanded by Brownian motion W_t, then distributions of a_t^k and b_t^k, $k = 0, 1$ under $(\mathbb{Q}_i, \mathbb{F})$, $i = A, B$, are the same as under $(\mathbb{Q}, \mathbb{F})$. $\square$.

Proof: Equations (4.15) are straightforward because $\mathbb{Q}_A$ and $\mathbb{Q}_B$ are survival measures. Because our HBPR survival measure is compatible with CGH survival measure, as indicated at the end of last section, results in Theorem 2 concludes proof of equations (4.15).

Results in Section 3 asserts that any $\mathbb{Q}$ Brownian motion is still Brownian motion under survival measures $\mathbb{Q}_A$ and $\mathbb{Q}_B$. Because a_t^0 is $\mathcal{F}_t = \mathcal{F}_t^W$-measurable, it can be represented as some measurable functional of Brownian motion W_t, i.e. $a_t^0 = f_t\left(\{W_s\}_{s\leq t}\right)$. Because W_t is still Brownian motion under $\mathbb{Q}_B$ and functional relationship $a_t^0 = f_t\left(\{W_s\}_{s\leq t}\right)$ does not change under measure changes, a_t^0 has the same distribution under $\mathbb{Q}_B$ as $\mathbb{Q}$. Similarly, one can prove that distributions of a_t^1, b_t^0 and b_t^1 remains the same under survival measure changes. $\square$

5 Applications in Guaranteed Debt

This section considers the problem of pricing contagion and mitigation effect in guaranteed debt, as application of our HBPR survival measure approach and interacting intensity model with stochastic intensities. Two firms are considered here, denoted as firm A and firm B. By saying the two firms form a guaranty relationship we mean that A promises to take over the loss given default (LGD) of B upon the default of B, and B promises to do the same for A during the life time of this guaranty provision. Assume the bond of two firms has maturity T and of unit face value. Once default, they have recovery rates of R_A and R_B, or loss given default LGD_A and LGD_B respectively. In our model, mitigation effect is modeled in the payoff function of the firms, while contagion is modeled using interacting intensity model with stochastic intensities. To extract mitigation and contagion effects from the bond values, we need to consider two cases, with and without guaranty relationship, respectively. The two cases are assumed to have the same recovery rates and maturity.

Firstly, consider the case without guaranty. Default times are denoted by $\bar{\tau}_A$ and $\bar{\tau}_B$, with stochastic hazard procsses $\bar{\lambda}_t^A$ and $\bar{\lambda}_t^B$, which are both $\mathbb{F}$ adapted non-negative processes. To compare the difference between debt portfolios with and without guaranty, $\bar{\lambda}_t^A$ and $\bar{\lambda}_t^B$ are designed as

$$\bar{\lambda}_t^A = a_t^0 \qquad \bar{\lambda}_t^B = b_t^0 \tag{5.16}$$

Default times $\bar{\tau}_A$ and $\bar{\tau}_B$ are characterized by the following Cox construction

$$\bar{\tau}_i = \inf \left\{ t > 0 \left| \int_0^t \bar{\lambda}_s^i ds \geq \bar{E}_i \right. \right\} \quad i = A, B \tag{5.17}$$

given realization of $\mathcal{F}_\infty$. $\bar{E}_A$ and $\bar{E}_B$ are independent unit mean exponential variables, i.e., $Exp(1)$-random variable. Moreover, $\bar{E}_A$ and $\bar{E}_B$ are assumed to be independent from E_A and E_B in equations (4.12).

Payoffs of the two firms are

$$\begin{cases} \bar{H}_T^A = 1_{\{\bar{\tau}_A > T\}} + 1_{\{\bar{\tau}_A \leq T\}} \cdot R_A \\ \bar{H}_T^B = 1_{\{\bar{\tau}_B > T\}} + 1_{\{\bar{\tau}_B \leq T\}} \cdot R_B \end{cases} \tag{5.18}$$

Therefore, in the event of no firm default, the two firm portfolio's total payoff is 2, while in the event of both default the porfolio recover $(R_A + R_B)$. When only one firm defaults, the total payoff is either $(1 + R_A)$ (if A defaults) or $(1 + R_B)$ (if B defaults).

In the presence of two-way guaranty, i.e. firm A provides guaranty for firm B and firm B does the same for firm A, mitigation effect can be represented in the following payoff function

$$\begin{cases} H_T^A = 1_{\{\tau_A > T\}} + 1_{\{\tau_A \leq T\}} \left[1_{\{\tau_B > T\}} + 1_{\{\tau_B \leq T\}} R_A \right] \\ H_T^B = 1_{\{\tau_B > T\}} + 1_{\{\tau_B \leq T\}} \left[1_{\{\tau_A > T\}} + 1_{\{\tau_A \leq T\}} R_B \right] \end{cases} \tag{5.19}$$

For this case, payoff is still the same in the event of no default and both default, while in the event of only one default no virtual loss is incurred for the portfolio, which is contributed to the mitigation of guaranty. However, the survival firm will be burdened with more obligation after his counterparty defaults, thus with higher PD. Even in the events where two cases have same payoff, the relevant probabilities are not ensured to be the same. Therefore, to explicitly tell the difference of two cases, joint probabilities of default/survival must be derived first.

Lemma 2. *In the case of no guaranty, joint probabilities can be evaluated as:*

$$\begin{cases} \mathbb{Q}\{\bar{\tau}_A > T, \bar{\tau}_B > T\} = \Lambda_T^{AB} \\ \mathbb{Q}\{\bar{\tau}_A > T, \bar{\tau}_B \leq T\} = \Lambda_T^{AB} \cdot \Lambda_T^B \\ \mathbb{Q}\{\bar{\tau}_A \leq T, \bar{\tau}_B > T\} = \Lambda_T^{AB} \cdot \Lambda_T^A \\ \mathbb{Q}\{\bar{\tau}_A \leq T, \bar{\tau}_B \leq T\} = 1 - \Lambda_T^{AB} \left[1 + \Lambda_T^A + \Lambda_T^B \right] \end{cases} \tag{5.20}$$

with

$$\begin{cases} \Lambda_T^{AB} \doteq E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T [a_s^0 + b_s^0] ds\right\}\right] \\ \Lambda_T^{AB} \cdot \Lambda_T^{A} \doteq E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T b_s^0 ds\right\} - \exp\left\{-\int_0^T [a_s^0 + b_s^0] ds\right\}\right] \\ \Lambda_T^{AB} \cdot \Lambda_T^{B} \doteq E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T a_s^0 ds\right\} - \exp\left\{-\int_0^T [a_s^0 + b_s^0] ds\right\}\right] \end{cases} \tag{5.21}$$

□

Proof: The above joint probabilities can be easily checked because of Cox construction of $\bar{\tau}_A$ and $\bar{\tau}_B$ and independence of $\bar{E}_A$ and $\bar{E}_B$. □

Theorem 3. *Suppose default times τ_A and τ_B are modeled as interacting intensity model with stochastic intensities (4.11-4.12), and a_t^i, b_t^i, $i = 0, 1$, are assumed to be $\mathbb{F}$-adapted non-negative Itô diffusion processes, which are obviously continuous $\mathbb{G}$-semimartingales. Particularly, interacting components a_t^1 and b_t^1 are designed as*

$$\begin{cases} a_t^1 = \eta_B \cdot b_t^0 \\ b_t^1 = \eta_A \cdot a_t^0 \end{cases} \tag{5.22}$$

Then joint probabilities of τ_A and τ_B can be expressed as

$$\begin{cases} \mathbb{Q}\{\tau_A > T, \tau_B > T\} = \Lambda_T^{AB} \\ \mathbb{Q}\{\tau_A \leq T, \tau_B > T\} = \Lambda_T^{AB} \cdot \Gamma_T^{A} \\ \mathbb{Q}\{\tau_A > T, \tau_B \leq T\} = \Lambda_T^{AB} \cdot \Gamma_T^{B} \\ \mathbb{Q}\{\tau_A \leq T, \tau_B \leq T\} = 1 - \Lambda_T^{AB}\left[1 + \Gamma_T^{A} + \Gamma_T^{B}\right] \end{cases} \tag{5.23}$$

with

$$\begin{cases} \Lambda_T^{AB} \cdot \Gamma_T^{A} \doteq \frac{1}{1-\eta_A} E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left[b_s^0 + \eta_A \cdot a_s^0\right] ds\right\} - \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \\ \Lambda_T^{AB} \cdot \Gamma_T^{B} \doteq \frac{1}{1-\eta_B} E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left[a_s^0 + \eta_B \cdot b_s^0\right] ds\right\} - \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \end{cases} \tag{5.24}$$

for $\eta_A \neq 1$ and $\eta_B \neq 1$. When $\eta_A = 1$ and $\eta_B = 1$, we have

$$\begin{cases} \Lambda_T^{AB} \cdot \Gamma_T^{A} \doteq E^{\mathbb{Q}}\left[\int_0^T a_s^0 ds \cdot \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \\ \Lambda_T^{AB} \cdot \Gamma_T^{B} \doteq E^{\mathbb{Q}}\left[\int_0^T b_s^0 ds \cdot \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \end{cases} \tag{5.25}$$

which can be seen as limits of $\Lambda_T^{AB} \cdot \Gamma_T^{A}$ and $\Lambda_T^{AB} \cdot \Gamma_T^{B}$ in equation (5.24) as $\eta_A \to 1$ and $\eta_B \to 1$. □

Proof: Define survival measures $\mathbb{Q}_A$ and $\mathbb{Q}_B$ as equations (4.13) and (4.14), then results in Lemma 1 holds. Change measure from $\mathbb{Q}$ to $\mathbb{Q}_A$, then

$$\begin{aligned} \mathbb{Q}\{\tau_A > T, \tau_B > T\} &= E^{\mathbb{Q}}\left[1_{\{\tau_A > T\}} 1_{\{\tau_B > T\}}\right] \\ &= E^{\mathbb{Q}_A}\left[1_{\{\tau_B > T\}} \exp\left\{-\int_0^T [a_s^0 + a_s^1 1_{\{\tau_B \leq s\}}] ds\right\}\right] \\ &= E^{\mathbb{Q}_A}\left[1_{\{\tau_B > T\}} \exp\left\{-\int_0^T a_s^0 ds\right\}\right] \\ &= E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T a_s^0 ds\right\} E^{\mathbb{Q}_A}\left[1_{\{\tau_B > T\}} | \mathcal{F}_T\right]\right] \\ &= E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \end{aligned}$$

$$= E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right]$$

The last equality holds for the reason that distributions of a_t^0 and b_t^0 under new measures $\mathbb{Q}_A$ is the same as $\mathbb{Q}$.

Using the same survival measure, we get

$$\mathbb{Q}\{\tau_A > T, \tau_B \leq T\} = E^{\mathbb{Q}}\left[1_{\{\tau_A > T\}} 1_{\{\tau_B \leq T\}}\right]$$

$$= E^{\mathbb{Q}_A}\left[1_{\{\tau_B \leq T\}} \exp\left\{-\int_0^T [a_s^0 + a_s^1 1_{\{\tau_B \leq s\}}] ds\right\}\right]$$

$$= E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T a_s^0 ds\right\} E^{\mathbb{Q}_A}\left[1_{\{\tau_B \leq T\}} \exp\left\{-\int_0^T a_s^1 1_{\{\tau_B \leq s\}} ds\right\} |\mathcal{F}_T\right]\right]$$

$$= E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T \left[a_s^0 + a_s^1\right] ds\right\} \int_0^T \exp\left\{-\int_0^t \left[b_s^0 - a_s^1\right] ds\right\} b_t^0 dt\right]$$

$$= E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T \left[a_s^0 + \eta_B \cdot b_s^0\right] ds\right\} \int_0^T \exp\left\{-\int_0^t (1 - \eta_B) \cdot b_s^0 ds\right\} b_t^0 dt\right] \tag{5.26}$$

The 4^{th} equality holds for the reason that τ_B's conditional p.d.f. on $\mathcal{F}_T$ can be easily attained by its $\mathbb{Q}_A$−intensity in equation (4.15).

For $\eta_B \neq 1$, the above equation can be calculated as

$$\mathbb{Q}\{\tau_A > T, \tau_B \leq T\}$$

$$= \frac{1}{1 - \eta_B} E^{\mathbb{Q}_A}\left[\exp\left\{-\int_0^T \left[a_s^0 + \eta_B \cdot b_s^0\right] ds\right\}\left[1 - \exp\left\{-\int_0^T (1 - \eta_B) b_s^0 ds\right\}\right]\right]$$

$$= \frac{1}{1 - \eta_B} E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left[a_s^0 + \eta_B \cdot b_s^0\right] ds\right\} - \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right] \tag{5.27}$$

The last equality is derived because distributions of a_t^0, a_t^1 and b_t^0 under new measures $\mathbb{Q}_A$ is the same as $\mathbb{Q}$.

For $\eta_B = 1$, equation (5.26) can be simplified as

$$\mathbb{Q}\{\tau_A > T, \tau_B \leq T\} = E^{\mathbb{Q}}\left[\int_0^T b_s^0 ds \cdot \exp\left\{-\int_0^T \left[a_s^0 + b_s^0\right] ds\right\}\right]$$

It is not so hard to check that this equation can be seen as limit of equation (5.27) as $\eta_B \to 1$ by *L'Hospital's Rule*. Finally, the second formula in (5.23) is similarly derived, and the fourth one is direct conclusion of the first three formulas. □

Notice that $\Gamma_T^A = \frac{\mathbb{Q}\{\tau_A \leq T, \tau_B > T\}}{\mathbb{Q}\{\tau_A > T, \tau_B > T\}} = \frac{\mathbb{Q}\{\tau_A \leq T | \tau_B > T\}}{\mathbb{Q}\{\tau_A > T | \tau_B > T\}}$, which is the odds ratio of A's default probability versus its survival probability conditional on B's survival until maturity. Thus, **Conditional Odds Ratio** Γ_T^A represents the comparative possibility of A's default to its survival. Γ_T^B, Λ_T^A and Λ_T^B can be explained as the similar meaning.

Moreover, notice that unlike traditional design of jump that is proportional to a firm's own pre-default intensity, such as Leung et al. [10], we assume jump of one firm's intensity is proportional to the other firm's pre-default intensity. This implies that contagion from one firm to another is represented not only by a sudden jump in its intensity, but also by transferring defaulted firm's pre-default intensity to the survival firm. The major advantage of this design is that explicit formulas for marginal survival probability of τ_2 and joint survival probability of τ_1

and τ_2 in the situation of stochastic pre-default intensities are available. Through our numerical analysis we find that we can chose contagion parameter η to effectively reflect the actual level of contagion based on firms' credit worthiness and amount of guaranteed debt. Actually, due to the relative significant sensitivity of portfolio value to contagion parameter η, it is one of the key parameters to care more about in practical application.

The above results show that although payoffs of the two cases are the same in the events of no default and both firms default, probabilities of two firm default are not identical. Therefore, their present values would differentiate from each other when restricted to this extreme bad circumstance. Meanwhile, our analysis can be released slightly because both payoffs and probabilities are identical for two cases in the event of no default, for the present value difference is zero in this situation. All in all, we need to compare the two cases in the situation of at least one default happens.

Denote present value differences of the two cases in the events of only B defaults, only A defaults and both firms default by V_1, V_2 and V_3 respectively. More specifically, when bank account B_t is assumed to be deterministic function of time, then

$$\begin{cases} V_1 = B_T^{-1}E\left\{2\cdot 1_{\{\tau_A>T,\tau_B\le T\}} - (1+R_B)1_{\{\bar\tau_A>T,\bar\tau_B\le T\}}\right\} \\ V_2 = B_T^{-1}E\left\{2\cdot 1_{\{\tau_A\le T,\tau_B> T\}} - (1+R_A)1_{\{\bar\tau_A\le T,\bar\tau_B> T\}}\right\} \\ V_3 = B_T^{-1}E\left\{(R_A+R_B)[1_{\{\tau_A\le T,\tau_B\le T\}} - 1_{\{\bar\tau_A\le T,\bar\tau_B\le T\}}]\right\} \end{cases} \tag{5.28}$$

The following theorem summarizes the above analysis and gives the exact pricing of mitigation and contagion effects incurred by the guaranty relationship in the pair of firms.

Theorem 4. *Based on the above assumption and analysis, the value V of two-way guaranty is proportional to the difference of (weighted) conditional odds ratios:*

$$\begin{aligned} V &= (LGD_A + LGD_B)B_T^{-1}\Lambda_T^{AB} \\ &\quad \left\{\left(\Gamma_T^A+\Gamma_T^B\right) - \left(\tilde\Lambda_T^A+\tilde\Lambda_T^B\right)\right\} \end{aligned} \tag{5.29}$$

where

$$\begin{cases} \tilde\Lambda_T^A = \frac{LGD_B}{LGD_A+LGD_B}\Lambda_T^A \\ \tilde\Lambda_T^B = \frac{LGD_A}{LGD_A+LGD_B}\Lambda_T^B \end{cases}$$

are weighted conditional odds ratios by their courterparty's proportional LGD. □

Proof: Firstly, V_1 can be evaluated as

$$\begin{aligned} V_1 &= 2B_T^{-1}\mathbb{Q}\{\tau_A>T,\tau_B\le T\} \\ &\quad -(1+R_B)B_T^{-1}\mathbb{Q}\{\bar\tau_A>T,\bar\tau_B\le T\} \\ &= B_T^{-1}\Lambda_T^{AB}\left[2\cdot\Gamma_T^B - (1+R_B)\Lambda_T^B\right] \end{aligned}$$

Similarly, we get

$$V_2 = B_T^{-1}\Lambda_T^{AB}\left[2\cdot\Gamma_T^A - (1+R_A)\Lambda_T^A\right]$$

and

$$V_3 = (R_A+R_B)B_T^{-1}\Lambda_T^{AB}\left[\left(\Lambda_T^A+\Lambda_T^B\right) - \left(\Gamma_T^A+\Gamma_T^B\right)\right]$$

Combine them together, we finally derive (5.29). □

One direct conclusion of equation (5.29) is that mitigation value is great than contagion risk value in the guaranteed debt portfolio of two firms if and only if the following is true

$$\left(\Gamma_T^A+\Gamma_T^B\right) > \left(\tilde\Lambda_T^A+\tilde\Lambda_T^B\right) \tag{5.30}$$

The $\mathbb{F}$ adapted pre-default intensities a_t^0 and b_t^0 can be modeled by affine short rate models such as CIR and CIR++, which are driven by $(\mathbb{Q},\mathbb{F})$ Brownian motions. Hence, as well known, the conditional odds ratios can be explicitly evaluated in affine short rate models.

For instance, suppose a_t^0 and b_t^0 are driven by state variables x_t and z_t which are assumed to be $(\mathbb{F},\mathbb{Q})$-affine processes. As an illustration, x_t and z_t are supposed to be independent CIR processes, i.e.

$$\begin{cases} dx_t = k_x\left[\theta_x - x_t\right]dt + \sigma_x\sqrt{x_t}dW_t^x \\ dz_t = k_z\left[\theta_z - z_t\right]dt + \sigma_z\sqrt{z_t}dW_t^z \end{cases}, \text{ with } dW_t^x \perp dW_t^z \tag{5.31}$$

under martingale measure $\mathbb{Q}$. Parameters $\kappa_x, \kappa_z, \theta_x, \theta_z, \sigma_x$ and σ_z are supposed to satisfy $2k_x\theta_x > \sigma_x^2$ and $2k_z\theta_z > \sigma_z^2$ so that 0 is unattainable for x_t and z_t. a_t^0 and b_t^0 are assumed to be affine functions of state variable x_t and z_t, i.e.

$$\begin{cases} a_t^0 = \alpha^A x_t + \beta^A z_t \\ b_t^0 = \alpha^B x_t + \beta^B z_t \end{cases} \tag{5.32}$$

where α^i and β^i, $i = A, B$ are non-negative constant parameters. Dependence of a_t^0 and b_t^0 are fully characterized by commonly reliance on two independent CIR factors. The following lemma gives some results about pricing in CIR model that will be needed in the sequel of this paper.

Lemma 3. *Suppose factor X_t is an $(\mathbb{F},\mathbb{Q})$-CIR process with parameters (k,θ,σ), i.e.*

$$dX_t = k\left[\theta - X_t\right]dt + \sigma\sqrt{X_t}dW_t \tag{5.33}$$

Then for $\alpha > 0$, αX_t is also an $(\mathbb{F},\mathbb{Q})$-CIR process, with parameters $(k, \alpha\theta, \sqrt{\alpha}\sigma)$. Moreover, pricing formulas of $P_X(t,T;\alpha)$ and $N_X(t,T;\alpha)$ can be represented as

$$P_X(t,T;\alpha) = E\left[\exp\left\{-\int_t^T \alpha X_s ds\right\}\middle|\mathcal{F}_t\right] = A_X(t,T;\alpha)e^{-B_X(t,T;\alpha)\cdot\alpha X_t} \tag{5.34}$$

where $A_X(t,T;\alpha)$ and $B_X(t,T;\alpha)$ are deterministic functions of t and T, given as

$$\begin{cases} A_X(t,T;\alpha) = \left[\dfrac{2he^{(T-t)(k+h)/2}}{2h + (k+h)\left[e^{(T-t)h} - 1\right]}\right]^{\frac{2k\theta}{\sigma^2}} \\ B_X(t,T;\alpha) = \dfrac{2\left[e^{(T-t)h} - 1\right]}{2h + (k+h)\left[e^{(T-t)h} - 1\right]} \end{cases} \tag{5.35}$$

with h denoted as $h = \sqrt{k^2 + 2\alpha\sigma^2}$.

$$N_X(t,T;\alpha) = E\left[\int_t^T x_s ds \cdot \exp\left\{-\int_t^T \alpha\cdot x_s ds\right\}\middle|\mathcal{F}_t\right] = \tilde{A}_X(t,T,x_t;\alpha)P_X(t,T;\alpha) \tag{5.36}$$

with

$$\tilde{A}_X(t,T,X_t;\alpha) = E_X(t,T;\alpha) + F_X(t,T;\alpha)X_t \tag{5.37}$$

and

$$\begin{cases} E_X(t,T;\alpha) = \dfrac{\kappa\theta}{h^2}\dfrac{\left\{-2\kappa\left[e^{(T-t)h} - 1\right] + h(T-t)\left[2\kappa + (\kappa+h)\left[e^{(T-t)h} - 1\right]\right]\right\}}{2h + (\kappa+h)\left[e^{(T-t)h} - 1\right]} \\ F_X(t,T;\alpha) = \dfrac{2\alpha\sigma^2}{h}\dfrac{\left[2he^{(T-t)h}(T-t) - e^{2(T-t)h} + 1\right]}{\left(2h + (\kappa+h)\left[e^{(T-t)h} - 1\right]\right)^2} + \dfrac{2\left[e^{(T-t)h} - 1\right]}{2h + (\kappa+h)\left[e^{(T-t)h} - 1\right]} \end{cases} \tag{5.38}$$

$\square$.

Proof of $A_X(t,T;\alpha)$ and $B_X(t,T;\alpha)$, which are well known formulas, is standard and is omitted in this paper. Derivation of $E_X(t,T;\alpha)$ and $F_X(t,T;\alpha)$ is deferred to Appendix B.

Particularly, we denote $P_X(t,T) = P_X(t,T;1)$, i.e.

$$P_X(t,T) = E\left[\exp\left\{-\int_t^T X_s ds\right\}\middle|\mathcal{F}_t\right] = A_X(t,T)e^{-B_X(t,T)X_t} \tag{5.39}$$

where $A_X(t,T) = A_X(t,T;1)$, $B_X(t,T) = B_X(t,T;1)$ and h is simplified as $h = \sqrt{\kappa^2 + 2\sigma^2}$. For notational convenience, we define the following function of α and β

$$P(\alpha,\beta) = E\left[\exp\left\{-\int_0^T [\alpha x_s + \beta z_s]\, ds\right\}\middle|\mathcal{F}_t\right] = P_x(0,T;\alpha)\cdot P_z(0,T;\beta) \tag{5.40}$$

with dependence of $P(\alpha,\beta)$ on $(x_0,k_x,\theta_x,\sigma_x)$ and $(z_0,k_z,\theta_z,\sigma_z)$ implicitly represented. Consequently, joint survival probability Λ_T^{AB} and conditional odds ratios Λ_T^i, Γ_T^i, $i = A, B$ in Lemma 2 and Theorem 3 can be represented in the sequel theorem.

Theorem 5. *Assume default times τ_A and τ_B are modeled as interacting intensity model with stochastic intensities (4.11,4.12, 5.22), and a_t^0, b_t^0 are assumed to be dependent on $(\mathbb{F},\mathbb{Q})$-CIR processes (5.31) as equation (5.32), then the joint survival probability Λ_T^{AB} and conditional odds ratios Λ_T^i, Γ_T^i, $i = A, B$ in Lemma 2 and Theorem 3 can be expressed as*

$$\begin{cases} \Lambda_T^{AB} = P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right) \\ \Lambda_T^A = \dfrac{P\left(\alpha^B, \beta^B\right)}{P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)} - 1 \\ \Lambda_T^B = \dfrac{P\left(\alpha^A, \beta^A\right)}{P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)} - 1 \end{cases} \tag{5.41}$$

and

$$\Gamma_T^A = \begin{cases} \dfrac{1}{1-\eta_A}\left[\dfrac{P\left(\alpha^B + \eta_A\cdot\alpha^A, \beta^B + \eta_A\cdot\beta^A\right)}{P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)} - 1\right] & for\, \eta_A \neq 1 \\ \alpha^A \tilde{A}_x(0,T,x_0;\alpha^A + \alpha^B) + \beta^A \tilde{A}_z(0,T,x_0;\alpha^A + \alpha^B) & for\, \eta_A = 1 \end{cases} \tag{5.42}$$

$$\Gamma_T^B = \begin{cases} \dfrac{1}{1-\eta_B}\left[\dfrac{P\left(\alpha^A + \eta_B\cdot\alpha^B, \beta^A + \eta_B\cdot\beta^B\right)}{P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)} - 1\right] & for\, \eta_B \neq 1 \\ \alpha^B \tilde{A}_x(0,T,x_0;\alpha^A + \alpha^B) + \beta^B \tilde{A}_z(0,T,x_0;\alpha^A + \alpha^B) & for\, \eta_B = 1 \end{cases} \tag{5.43}$$

□

Proof: Using the notation in equation (5.40), equation (5.41) is straightforward. For $\eta_A \neq 1$, we get

$$\begin{aligned} \Lambda_T^{AB}\cdot\Gamma_T^A &= \frac{1}{1-\eta_A}E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left[\left(\alpha^B + \eta_A\cdot\alpha^A\right)x_s + \left(\beta^B + \eta_A\cdot\beta^A\right)z_s\right]ds\right\} - \Lambda_T^{AB}\right] \\ &= \frac{1}{1-\eta_A}\left[P\left(\alpha^B + \eta_A\cdot\alpha^A, \beta^B + \eta_A\cdot\beta^A\right) - P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)\right] \end{aligned}$$

Therefore, in the case of $\eta_A \neq 1$, conditional odds ratio Γ_T^A in the interacting intensity model can be given as

$$\Gamma_T^A = \frac{1}{1-\eta_A}\left[\frac{P\left(\alpha^B + \eta_A\cdot\alpha^A, \beta^B + \eta_A\cdot\beta^A\right)}{P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)} - 1\right]$$

When $\eta_A = 1$, conditional odds ratio Γ_T^A can be attained by letting $\eta_A \to 1$ in the above equation. Alternatively, we use Lemma B to calculate Γ_T^A for the case of $\eta_A = 1$.

$$\Lambda_T^{AB} \cdot \Gamma_T^A = E^{\mathbb{Q}}\left[\int_0^T a_s^0 ds \cdot \exp\left\{-\int_0^T \left[b_s^0 + a_s^0\right] ds\right\}\right]$$

$$= E^{\mathbb{Q}}\left[\int_0^T \left[\alpha^A x_s + \beta^A z_s\right] ds \cdot \exp\left\{-\int_0^T \left[\left(\alpha^A + \alpha^B\right) x_s + \left(\beta^A + \beta^B\right) z_s\right] ds\right\}\right]$$

$$= \alpha^A E^{\mathbb{Q}}\left[\int_0^T x_s ds \cdot \exp\left\{-\int_0^T \left(\alpha^A + \alpha^B\right) x_s ds\right\}\right] \cdot E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left(\beta^A + \beta^B\right) z_s ds\right\}\right]$$

$$+\beta^A E^{\mathbb{Q}}\left[\int_0^T z_s ds \cdot \exp\left\{-\int_0^T \left(\beta^A + \beta^B\right) z_s ds\right\}\right] \cdot E^{\mathbb{Q}}\left[\exp\left\{-\int_0^T \left(\alpha^A + \alpha^B\right) x_s ds\right\}\right]$$

$$= \left[\alpha^A \tilde{A}_x(0,T,x_0;\alpha^A + \alpha^B) + \beta^A \tilde{A}_z(0,T,x_0;\alpha^A + \alpha^B)\right] P\left(\alpha^A + \alpha^B, \beta^A + \beta^B\right)$$

Thus the expression of Γ_T^A for the case of $\eta_A = 1$ is exactly given as equation (5.42). Expression of Γ_T^B can be attained similarly. □

	Initial values	k	θ	σ
Factor x_t	0.03	0.50	0.05	0.50
Factor z_t	0.01	0.80	0.02	0.20

Table 1: Parameters of latent CIR factors.

6 Numerical Analysis for Guaranteed Debt Portfolio

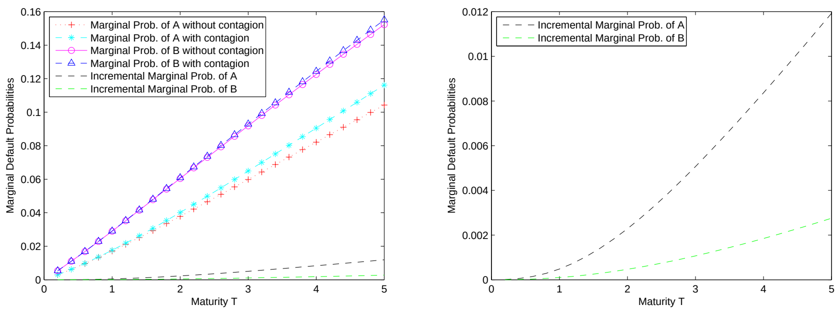


Figure 1: Marginal probabilities of A and B with/without contagion.

This section performs some numerical analysis of the analytical pricing formula of guaranteed debt portfolio in equation (5.29), especially tests the guaranty effect and contagion risk (loss) in our interacting intensity model. We adopt a group of parameters with reasonable sense for two firms with modest credit risk. Parameters of the two latent CIR factors underlying pre-default intensities are illustrated in Table 1. Parameters α^i, β^i, $i = A, B$ determine the mutual

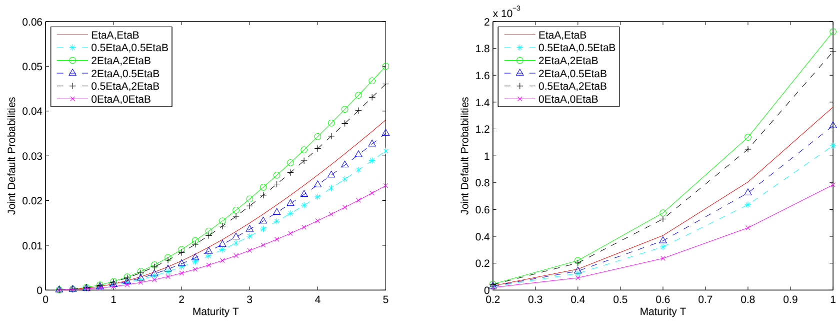


Figure 2: Impact of magnitude of contagion on joint default probabilities.

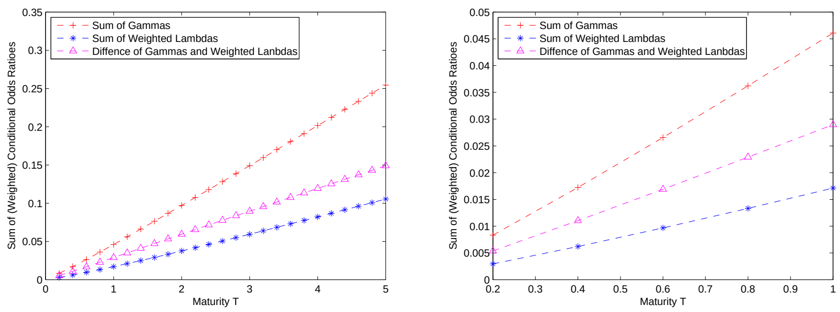
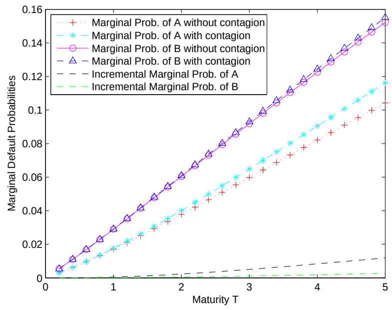


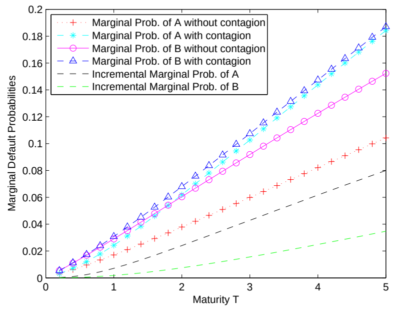
Figure 3: Mitigation v.s. Contagion: in the cases of small and modest contagion.

dependence of pre-default intensities a_t^0 and b_t^0. We take $\alpha^A = \beta^B = 0.2$, $\alpha^B = \beta^A = 0.8$ as the benchmark parameters to get a modest five year mean correlation of 0.7922. Because pre-default intensity b_t^0 has larger weight on factor x_t, which has larger initial value and long term mean value θ, firm B is pre-assumed to have greater credit risk than firm A. Due to the common sense of negative relationship of default risk and recovery rate, loss given default (LGD) of firm B is assumed to be greater than firm A. We take $LGDA = 0.6$ and $LGDB = 0.7$ as an illustration. Contagion parameters η_A and η_B characterize the magnitude of contagion of default risk from firm A to firm B, and vise versa. η_A and η_B represent the units of a_t^0 and b_t^0 as increment to intensities of firm B and firm A upon default of counterparty firm. We take $\eta_A = \eta_B = 0.5$ as benchmark case, and study influence of various value of η on the guaranteed debt portfolio.

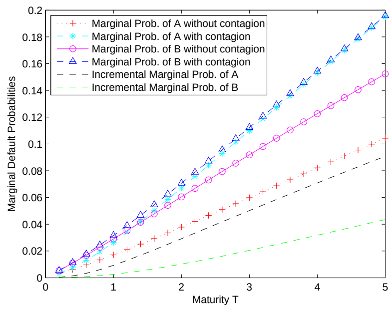
Under the specification of benchmark parameters, we first study the influence of contagion between the two firms on marginal probabilities of individual firms. Figure 1 presents the results of marginal probabilities of firm A and firm B. Firm A has smaller pre-default intensity implies that firm A is less likely to default before firm B. Thus impact of contagion on firm B (from firm A) is smaller than firm A (from firm B). Five year PD of firm A in the case of no contagion is 0.1042, with an increase of 0.0119(by 11.44%) by introducing contagion to the mutual dependence structure of two firms. Firm B has a five year PD of 0.1523 in absence of contagion, with a small increase 0.0028 (by 1.81%) through contagion from firm A. This figure shows an modest impact of contagion on firm A's marginal PD, while shows a quite small impact of contagion on firm B's marginal PD. This primarily results from the fact that firm B has greater pre-default intensity than firm A, implying that firm B is more possible to default before firm A and, moreover, transforms a greater amount of intensity to firm A (note that $\eta_A = \eta_B = 0.5$ for our benchmark case) once firm B actually defaults.



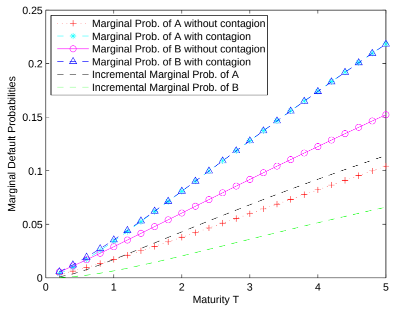
(a) Marginal prob. with $\eta_A = \eta_B = 0.5$



(b) Marginal prob. with $\eta_A = \eta_B = 10$



(c) Marginal prob. with $\eta_A = \eta_B = 15$



(d) Marginal prob. with $\eta_A = \eta_B = 50$

Figure 4: Marginal probabilities of A and B in the cases of large contagion

Subsequently, we study the influence of contagion between the two firms on joint default probabilities, mainly in the benchmark case and present various comparative cases in the same figure. Figure 2 reports our results. For the case of no contagion, five year joint PD is 0.0233, which is greater than $0.0159 = 0.1042 \times 0.1523$ that is joint PD should no contagion and pre-default intensities correlation exist. The increase by percentage of 46.54% is solely due to correlations between a_t^0 and b_t^0, which reflects the fact that stochastic pre-default intensities accounts for significant part of firms' default correlation, especially for the circumstance of no firm has defaulted. This is primarily attributed to our extended interacting intensity model that includes randomness into pre-default intensities, contrasting to traditional models such as Leung [8] where pre-default intensities are deterministic functions of time, implying independence of creditworthiness before any firm defaults. Furthermore, Figure 2 shows that introducing contagion into the dependence structure with benchmark value of η_A and η_B increases joint PD from 0.0233 to 0.0380 by a percentage of 62.93%, which would be a 139% increase with respect to 0.0159. This result gives a preliminary picture showing that default dependence due to contagion is greater than default dependence resulting from pre-default intensities correlation, but still not so convincing because our analysis here depends on choice of parameters η_A and η_B. Detailed analysis of contagion and pre-intensities correlation would be performed later in this section to confirm this assertion. Actually, by decreasing η_A and η_B to one half of benchmark value, we get smaller impact of contagion. Figure 2 shows that for the case of (0.5EtaA, 0.5EtaB) five year joint PD increases 0.0078, only by a percentage of 33.22%. In the case of (2EtaA, 2EtaB), five year joint PD increases 0.0267 by the percentage of 114.3%. Finally, we study the contribution of contagion from firm A and firm B via increasing η_A by 2 times while decreasing η_B by 0.5 times and doing the opposite again. Figure 2 shows that joint PD curve for the former is entirely below benchmark case, while joint PD curve for the later is entirely above benchmark case. This result confirms the analysis earlier that greater pre-default intensity of firm B concludes a larger possibility that firm B will default before firm A, thus implies larger effect of contagion from firm B. Furthermore, in our model default of firm B will transform its pre-default intensity to firm A which will increase contagion effect to a larger extent. Fortunately, η_B is still a free parameter that will dominate transform of absolute magnitude of intensity from firm B to firm A. By choosing an appropriate value of η_B would control the intensity transform to a level reflecting actual circumstance.

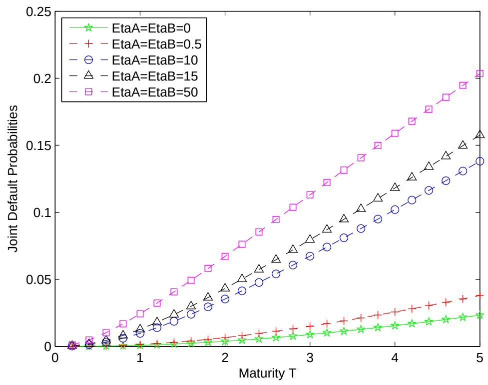


Figure 5: Impact large contagion on joint default probabilities.

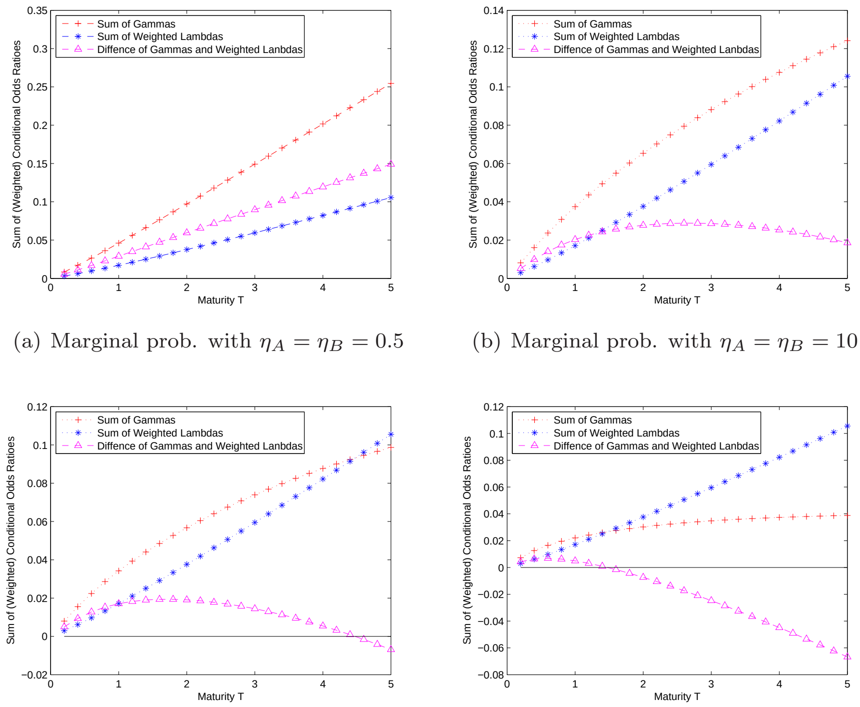
After checking influence of contagion on marginal PD and joint PD, we now analyze its impact on the value of guaranteed debt portfolio, that is analyzing mitigation effect of guaranty. Formula (5.29) shows that value of guaranteed debt portfolio is proportional to difference between sum of conditional odds ratios with contagion (Gammas) and sum of weighted conditional odds ratios without contagion (Weighted Lambdas). Positive difference implies that mitigation value is greater than contagion risk (greater probability of larger loss). Figure 3 presents results of our analysis. For the benchmark case, with small and modest value of contagion parameter, we find the "Gammas" line is consistently above "Weighted Lambdas" line, which implies that value of guaranteed debt portfolio is consistently positive for the five year period. The "Difference" line in Figure 3 lies between "Gammas" line and "Weighted Lambdas" line shows that mitigation benefit is greater than contagion risk in a reasonably distance when contagion parameters are assumed to be modest as our benchmark case. However, when contagion parameters increase, impact of contagion on joint default probability of two firms will increase and offsets mitigation effect of deferring first firm's loss given default. This assertion will be detailed analyzed later in this section.

Beyond the benchmark case, we assume contagion parameter η tend to large value and analyze the changes of marginal default probabilities, joint default probability and guaranteed debt portfolio value. We find in the previous analysis that for modest contagion parameter $\eta_A = \eta_B = 0.5$, mitigation benefit is consistently greater than contagion risk in a reasonable distance. Nonetheless, some counterparty firms may have larger contagion parameters, for instance, due to the considerably large guaranteed debt for guarantor that would demand an immediate large amount of cash to pay off debt subject to default of guarantee. In other cases, guarantor and guarantee might be related parties such that default of guarantee will deduce a sudden decrease of guarantor's credit worthiness which obvious resulting from factors other than guaranty relationship. We consider three cases here with $\eta_A = \eta_B$ =10, 15, 50, respectively. Figure 4 reports our results of impact on marginal default probabilities. As indicated before, marginal probabilities of firm A and B in the case of no contagion are 0.1042 and 0.1523 respectively. By increasing contagion parameter to 10, five year marginal PD of firm A increases 0.08 (by 76.73%), and five year PD of firm B increases 0.0347 (by 22.81%). For the cases $\eta_A = \eta_B$ =15 and 50, firm A's five year PD increases 0.0908 (by 87.06%) and 0.1142 (by 109.53%), while firm B's five year PD increases 0.0435 (by 28.55%) and 0.0659 (by 43.30%). Comparing these results to the benchmark case we find that marginal PD's are subject to considerably larger increases in the case of greater contagion parameters, especially for firm A who has a riskier counterparty than itself.

For joint default probability of the two firms subject to large contagion, we present our results in Figure 5. For the cases of $\eta_A = \eta_B$ =10, 15, 50, percentage changes of joint default probabilities are 491.70%, 575.35% and 771.97% respectively. This confirms the assertion that contagion has larger impact on joint default probability than marginal default probabilities especially in the situation of large η's.

Guaranty has mitigation benefit for bankers in the view of portfolio, by deferring loss given default of the earlier default firm to the default of second firm or never if the guarantor does not default before maturity. In presence of large contagion, joint default probability increases significantly, inducing a larger possibility of loss of both firms LGD's. If two firms are more likely to default simultaneously, protection of the first default firm's LGD is weaker, which not only fails at mitigating guarantee's debt, but also increases the survival firm's credit risk concurring a default risk that would not exist should absence of guaranty. Therefore, when contagion parameter is greater than some threshold value, mitigation benefit disappears due to increasing contagion risk. Figure 6 reports our results on η's influence on value of guaranteed debt portfolio, representing in the form of difference between sum of Gammas and sum of weighted Lambdas. Figure 6(a) shows the result in benchmark case where $\eta_A = \eta_B = 0.5$. In this modest situation mitigation benefit is significantly greater than contagion risk (potential loss) with a reasonable distance. When $\eta_A = \eta_B$ increases to 10, Figure 6(b) shows that conditional odds ratios difference decreases significantly to the extent that five year value almost touches zero. When $\eta_A = \eta_B = 15$ as in Figure 6(c), five year value of odds ratios difference is negative, implying that contagion risk dominates mitigation benefit in the view of portfolio for a five year period. If $\eta_A = \eta_B$ surges to 50 as Figure 6(d), 2 year value of odds ratios difference is negative, while the shorter term value only lies above zero by small negligible amount. Therefore, for the situation of two firms with large contagion, guaranty would have negative impact on debt portfolio.

Sum of weighted Lambdas is independent from contagion parameter η, thus negative value of conditional odds ratios difference results mainly from decrease of sum of Gammas. Actually, conditional odds ratios of two firms in the presence contagion, i.e. Gammas, plunges significantly as for the early stage of η tending to large value, especially for the longer 5 year period as show in Figure 7. This result might sound counterintuitive at the first glance because increasing of



(a) Marginal prob. with $\eta_A = \eta_B = 0.5$ (b) Marginal prob. with $\eta_A = \eta_B = 10$

(c) Marginal prob. with $\eta_A = \eta_B = 15$ (d) Marginal prob. with $\eta_A = \eta_B = 50$

Figure 6: Mitigation v.s. Contagion: in the cases of large contagion.

contagion parameter would definitely increases marginal default probability, as shown in Figure 4, thus results in increasing of "odds ratio for default". However, Γ_T^A and Γ_T^B are defined as "conditional odds ratios for default" given survival of the other firm. For instance, Γ_T^A is defined as $\Gamma_T^A = \frac{\mathbb{Q}\{\tau_A \leq T | \tau_B > T\}}{\mathbb{Q}\{\tau_A > T | \tau_B > T\}} = \frac{\mathbb{Q}\{\tau_A \leq T, \tau_B > T\}}{\mathbb{Q}\{\tau_A > T, \tau_B > T\}}$. The denominator in second equality is irrelevant with contagion as shown in equation (5.23). The numerator in this equality is counter value of joint default probability $\mathbb{Q}\{\tau_A \leq T, \tau_B \leq T\}$, i.e. it equals to $1 - \mathbb{Q}\{\tau_A \leq T, \tau_B \leq T\}$. Our previous analysis shows that $\mathbb{Q}\{\tau_A \leq T, \tau_B \leq T\}$ increases significantly due to surge of contagion parameter, thus deducing significant plunge of $\mathbb{Q}\{\tau_A \leq T, \tau_B > T\}$ resulting the pattern of Γ_T^A with respect to η as in Figure 7. Alternatively, there is a straightforward interpretation of this phenomenon. Because default of firm A would induce a large possibility of default of firm B, thus a small chance for firm B to survive. Given the survival of firm B in the circumstance of large contagion, one would more likely to believe that firm A has not default, meaning that firm A has small probability of default before maturity.

More specifically, we draw the curve of joint default probability and conditional odds ratio difference with respect to η in Figure 8 and Figure 9. The pattern is consistent with our previous analysis about surge of joint default probability and plunge of conditional odds ratio difference as η tends to large value, as well as the result that firm B's contagion effect is more significant than firm A due to their difference of pre-default intensities.

Finally, we perform some experiments for various level of correlation between pre-default intensities ranging from 0 to 1, including the benchmark case. Besides assuming $\alpha^A = \beta^B = 0.2$,

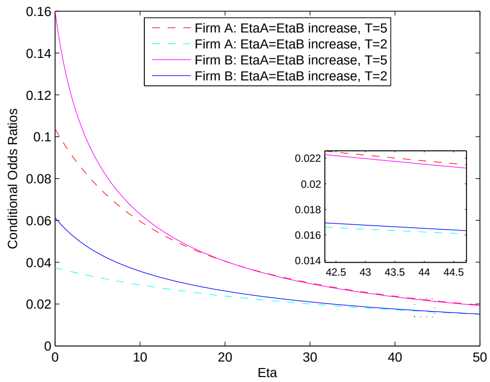


Figure 7: Impact of contagion on conditional odds ratios: illustration of joint prob. as function of η.

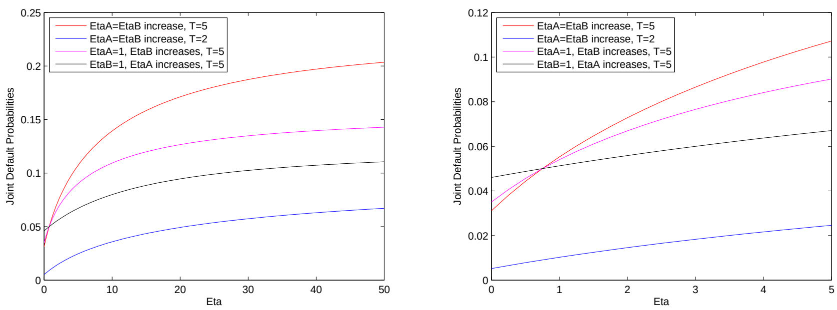


Figure 8: Impact of contagion on joint default probability of A and B: illustration of joint prob. as function of η.

$\alpha^B = \beta^A = 0.8$ as in the benchmark situation, which results in a five year mean correlation coefficient of 0.7922, we consider an alternative situation where $\alpha^A = \beta^B = 0.1$, $\alpha^B = \beta^A = 0.9$ implying a five year mean correlation coefficient 0.4805. Moreover, we include two extreme circumstance where pre-default intensities are perfectly correlated, i.e. $\alpha^A = \beta^B = \alpha^B = \beta^A = 0.5$, and are mutually independent with $\alpha^A = \beta^B = 1$, $\alpha^B = \beta^A = 0$. Throughout this experiment, we assume contagion is present as benchmark case. Figure 10 presents our experiments results. It is obvious from Figure 10(a) that joint PD curves are bounded by the perfectly correlated case and one of the completely independent case. The five year PD of independent case is 0.0344, while the value of perfectly correlated case is 0.0400, increasing by percentage 16.28%. The above four pieces of curve in Figure 10(b) represent sum of Gammas, while the below four pieces of curve represent conditional odds ratio differences for the four situation. Influence of correlation on sum of Gammas is straightforward from this figure that smaller correlation coefficient implies larger sum. This can be explained similarly as the experiment result of analyzing impact of large η's on Gammas. Nonetheless, weighted Lambdas are no longer independent from correlation coefficient, and this results in a humped pattern of conditional

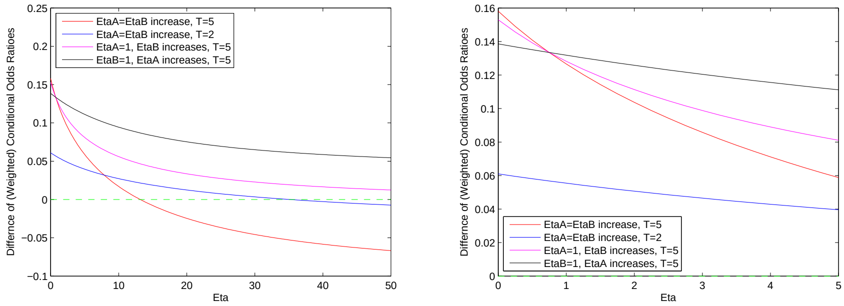
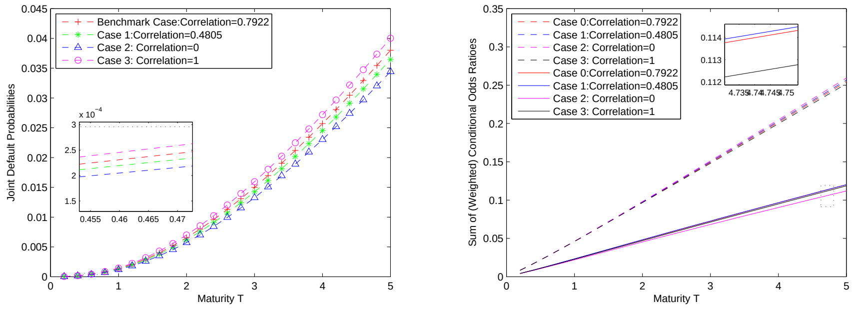


Figure 9: Impact of contagion on joint default probability of A and B: illustration of difference between sums of (weighted) conditional odds ratios as function of η.



(a) For joint PD.

(b) For conditional odds ratios difference.

Figure 10: Impact of correlation between pre-default intensities a_t^0 and b_t^0 on joint default probability and conditional odds ratios difference.

odds ratio difference on pre-default intensities' correlation. For the independent case, smallest difference is detected. When increasing correlation from 0 to 0.4805, the difference increases to a maximum value. When correlation continue growing to 0.7922 and to 1, the difference decreases accordingly. However, the overall influence on the difference conditional odds ratio is subtle.

7 Conclusion

This paper studies survival measures in credit risk models. Unlike survival measures in literature, we consider survival measure in "HBPR" model, which means that default time is characterized by Cox construction, and studies the relevant drift changes and martingale representations. This paper also takes advantage of survival measure to solve the looping default problem in interacting intensity model with stochastic intensities. Guaranteed debt is priced under this model, as an application of survival measure and interacting intensity model. Default intensities are modeled as affine function of CIR state variables, and analytical formula for value of two-way guaranty for a two firm portfolio is attained. This paper also performs a series of numerical experiments to study influence of stochastic pre-default intensities and contagion from interacting intensities model on marginal/joint probabilities and value of two-way guaranty. Our results show that correlation from pre-default intensities, through commonly dependence of two independent CIR latent variables, accounts for significant part of joint default probability, especially for the circumstance of no firm has defaulted yet. This is more realistic in the sense that it considers default dependence other than direct contagion. However, influence of different level of correlation coefficient on value of two-way guaranty is subtle. Our tests about contagion parameter show that contagion increases marginal probabilities significantly, while increases joint default probability in a larger percentage. In the circumstance of modest contagion parameter, mitigation benefit of guaranty is greater than contagion risk (loss) in a reasonable distance. As contagion parameter increases, joint default probabilities grows accordingly, and deduces mitigation benefit gradually. When contagion parameter is large enough, mitigation benefit is completely offset, and even incurs a net loss to guaranteed debt portfolio. Relatively high sensitivity of our model with respect to contagion parameter η emphasizes the key role of η for practical application. By carefully estimating η from firm's data, banks can use our formula to determine whether a two-way guaranty is appropriate. Estimation and analysis of factors that affect η will be performed in the subsequent work.

Appendix

A Bayesian Formula and Girsanov's Theorem

Suppose Z_t is a non-negative $(\mathbb{G}, \mathbb{Q})$-martingale. Define an absolutely continuous measure change through the following Randon-Nikodym density process as

$$\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{G}_t} = Z_t \geq 0, \quad \forall t > 0$$

Then measure changed conditional expectation and semimartingale representation can be given as

Bayesian Formula: Denote $\bar{\mathbb{G}} = \{\bar{\mathcal{G}}_t\}$ as augmentation of original filtration $\mathbb{G}$ by the null sets of the probability measure $\mathbb{P}$. Then

$$E^{\mathbb{Q}}\left[Z_\infty \cdot H|\mathcal{G}_t\right] = Z_t \cdot E^{\mathbb{P}}\left[H|\bar{\mathcal{G}}_t\right], \quad \forall H \in \mathcal{G}_\infty$$

Girsanov's Theorem: For any $(\mathbb{G}, \mathbb{Q})$-martingale m_t, the following defined $\bar{m}_t$ is $(\bar{\mathbb{G}}, \mathbb{P})$-martingale

$$\bar{m}_t = m_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, m\rangle_s, \quad \forall t \geq 0$$

This is equivalent to saying that any $(\mathbb{G}, \mathbb{Q})$-martingale m_t can be represented as a $(\bar{\mathbb{G}}, \mathbb{P})$-semimartingale

$$m_t = \bar{m}_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, m\rangle_s, \quad \forall t \geq 0$$

B Analytical Solution of Affine Model

Proof of Lemma 3: We need to prove the expression of $N_X(t,T;\alpha)$, i.e. we have to derive the expression of $E_X(t,T;\alpha)$ and $F_X(t,T;\alpha)$. First, we note that

$$P_X(t,T;\alpha) = E\left[\exp\left\{-\int_t^T \alpha X_s ds\right\}\middle|\mathcal{F}_t\right] = A_X(t,T;\alpha)e^{-B_X(t,T;\alpha)\cdot\alpha X_t}$$

Therefore, we have

$$\begin{aligned} N_X(t,T;\alpha) &= E\left[\int_t^T X_s ds\cdot\exp\left\{-\int_t^T \alpha X_s ds\right\}\middle|\mathcal{F}_t\right] = -\partial_\alpha P_X(t,T;\alpha) \\ &= \left[\frac{-\partial_\alpha A_X(t,T;\alpha)}{A_X(t,T;\alpha)} + \left[\partial_\alpha\left[B_X(t,T;\alpha)\right]\alpha + B_X(t,T;\alpha)\right]X_t\right]P_X(t,T;\alpha) \end{aligned}$$

Due to some complex calculation, we attain derivatives of $A_X(t,T;\alpha)$ and $B_X(t,T;\alpha)$ with respect to α as

$$\frac{\partial_\alpha A_X(t,T;\alpha)}{A_X(t,T;\alpha)} = \frac{\kappa\theta}{h^2}\frac{\left\{2\kappa\left[e^{(T-t)h}-1\right] - h(T-t)\left[2\kappa + (\kappa+h)\left[e^{(T-t)h}-1\right]\right]\right\}}{2h+(\kappa+h)\left[e^{(T-t)h}-1\right]}$$

and

$$\partial_\alpha B_X(t,T;\alpha) = \frac{2\sigma^2}{h}\frac{\left[2he^{(T-t)h}(T-t) - e^{2(T-t)h}+1\right]}{\left(2h+(\kappa+h)\left[e^{(T-t)h}-1\right]\right)^2}$$

Put these two equations back into $-\partial_\alpha P_X(t,T;\alpha)$, we conclude the analytical expression of $N_X(t,T;\alpha)$ as in equation (5.36). □

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