Public Debt Dynamics and Debt Feedback

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Abstract

We study the dynamics of U.S. public debt in a parsimonious VAR. We find that including debt feedback ensures the stationarity of debt while standard VARs excluding debt may imply an explosive debt path. We also find that the response of debt to inflation or interest shocks is not robust and depends on the policy regime. The recent past suggests that a positive shock to inflation increases debt while the same to interest rate decreases it. Positive shocks to growth and primary surplus unambiguously reduce debt.

JEL codes: H60, E31, E62, C32

Keywords: debt, fiscal policy, growth, VAR, impulse responses

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I. Introduction

The 2008 global financial crisis has resulted in large deficits and public debt burdens across many countries. IMF (2009) projections indicate that the level of public debt for advanced countries would reach over 100 percent of GDP by 2014, a level unseen since the Second World War. For the United States, the Congressional Budget Office (CBO) estimates that although the federal deficit would decline from 9.9 percent of GDP in 2009 to 2.6 percent in 2014, federal public debt would rise to 66 percent of GDP from 53 percent in 2009.[1]

As the US emerges from the crisis and returns to “normal” times, there would be a growing need to scale down large budget deficit and debt burden. Would debt start declining in the future? How do major macroeconomic aggregates such as growth, inflation, interest rate, and fiscal deficit affect debt dynamics? How should debt be reduced? To study the relationship between public debt and macroeconomic variables, we suggest a modified VAR framework in the tradition of Sims (1980) that includes a debt feedback equation as in Favero and Giavazzi (2007, 2009). Basically, the VAR model includes the debt to GDP ratio (and its lags) that is governed by a separate deterministic debt equation. We use the generalized impulse responses method of Koop, Pesaran, and Potter (1996) to analyze variable dynamics in reaction to shocks.

The framework outlined in this paper is parsimonious, yet important to use in the debt and fiscal policy analysis. Favero and Giavazzi (2007, 2009) argued for the importance of

[1] The CBO projections are April 2010 forecast.

using the debt feedback equation since excluding debt in the VAR had resulted in misspecification. They analyzed the effects of expenditure and revenue shocks on growth using a narrative approach of Romer and Romer (2010) and a structural approach of Blanchard and Perotti (2002). Our emphasis is, however, on public debt dynamics.[2] In general, impulse responses of main macroeconomic aggregates are not substantially altered by excluding debt feedback as shown by Favero and Giavazzi (2007, 2009). However, we show that both out-of-sample debt forecasts and debt impulse responses can substantially differ.

We show that the linear approximation of debt to GDP implicit in typical VARs may be misleading and may produce an explosive debt path and different impulse responses of debt to shocks. If the underlying debt dynamics are not on a stable path, the estimated effects of fiscal policy on macroeconomic aggregates may no longer be meaningful. It would be unreasonable to assume that an economy can sustain an unbounded level of debt. We thus add another angle to the misspecification problem discussed in Favero and Giavazzi (2007, 2009). Lastly, we use generalized impulse responses to deal with the history and shock dependence inherent to nonlinear models. The inclusion of debt to GDP in the VAR implies a nonlinear relationship among variables. Therefore, the use of generalized impulse responses provides a natural way to examine out-of-sample forecasts and impulse responses that are conditional expectations based on history and shocks.

[2] Fiscal revenue and expenditure do not appear explicitly in our VAR since we use primary balance.

Using the US data and different time periods, our findings are as follows. First, the post-1980 sample suggests that excluding the debt feedback in the VAR results in explosive debt dynamics and persistent debt impulse responses. Second, as compared to growth and primary deficit shocks, the response of public debt to monetary policy related variables, inflation and interest rate can be positive or negative depending upon the policy regime. The recent past suggests that a positive shock to inflation increases debt while higher growth, primary surplus, and interest rate lower debt. The empirical evidence indicates that the effect of higher inflation on debt will largely depend on the policy regime in place. If monetary and fiscal policy reacted to higher inflation as observed in post-1980, the inflation effect would actually be self-defeating as debt would rise. Lastly, the debt ratio is forecast to peak at about 70 percent of GDP by 2015 and gradually decrease thereafter.

Two recent papers by Hall and Sargent (2010) and Aizenman and Marion (2009) explore the role of inflation in reducing debt. Hall and Sargent (2010) show that about 23 percent of the debt reduction from 1945 to 1974 was due to inflation. Yet they indicate that the average maturity of public debt has shortened to about three years from seven in the aftermath of WWII, thus reducing the benefit of inflation in reducing the debt. Aizenman and Marion (2009) point out that although the maturity of debt is shorter now, a higher proportion of debt held by foreigners creates an incentive to inflate. They find that an inflation of 6 percent could reduce the debt to GDP ratio by about 20 percent over 4 years. Yet the authors caution that the result depends on model parameters, especially the parameter on the cost of inflation, and that modest inflation may result in unintended consequences of inflation acceleration. Our findings show that the response of debt to a positive inflation shock is not robust across samples, and the dynamics observed post-1980 would generate higher debt after several quarters. It is mostly driven by higher interest rate.

A few papers incorporate public debt in VAR estimations. However, for the most part they test for the sustainability of debt, examine fiscal policy effects, or study other countries than the U.S., and more importantly, they do not study debt impulse responses. Further, these papers either use one lag of debt in VAR (Afonso and Sousa, 2009) that may result in misspecification, use public debt as one of the endogenous variables (Hasko, 2007, and Corsetti, Meier, and Muller, 2009), or use long-term cointegration approach (Boisinnot, L'Angevin, and Monfort, 2004, and Polito and Wickens, 2007). Chung and Leeper (2007) use VAR with cross-equation restrictions arising from the present-value condition of debt sustainability. Barro (1980) studied the effect of US public debt shocks on output and unemployment using regressions without VAR dynamics. Bohn (1998) in a single regression, incorporating the tax smoothing model of Barro (1979), showed that US public debt was stationary as primary surplus reacted to higher levels of debt. Celasun, Debrun, and Ostry (2007) simulate debt paths for emerging countries based on an estimated fiscal reaction function in a panel and country specific VARs of other macroeconomic variables excluding debt feedback.[3] Many others have used cross-country data (e.g. Reinhart and Rogoff, 2010).

[3] See Celasun and Keim (2010) for an application to the U.S.

The paper is structured as follows. Section II presents methodology and data. Section III is the main section of the paper analyzing debt dynamics. Section IV concludes.

II. Empirical Model, Estimation, and Data

A. Empirical Model

To keep the model parsimonious, VAR is based on the following four variables in the endogenous vector Y specified in equation (1): primary balance to GDP ratio (primary expenditures minus revenues, pb), real growth rate (g), inflation rate based on the GDP deflator (π), and nominal average interest rate based on interest payments on debt (i). The variables used are those that enter equation (2) describing debt dynamics. The VAR specification also includes the debt to GDP ratio (d):

$$Y_t = \sum_{i=1}^{k} A_i Y_{t-i} + \sum_{i=1}^{l} \gamma_i d_{t-i} + \varepsilon_t , \tag{1}$$

where k/l are a number of lags used ($k = 4; l = 2$). Equation (2) defines the debt dynamics[4]:

$$d_t = \frac{(1 + i_t)}{(1+\pi_t)(1+g_t)} d_{t-1} + pb_t \tag{2}$$

Equations (1)-(2) define our system of equations.

B. Estimation and Impulse Responses

The model estimation is straightforward, but the computation of impulse responses (IRs) requires keeping track of the debt feedback in equation (1). VAR is estimated using OLS.

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[4] We ignore the debt residual, including non-deficit financing, in our specification. For the US, the debt residual has not been large historically as shown in Favero and Giavazzi (2007).

Similarly to Favero and Giavazzi (2007), we find that it is the change in debt that affects VAR dynamics as coefficients on lagged debt are close in absolute value but of the opposite signs. Since equation (2) includes all the estimated variables in (1) and has no parameters, it does not need any estimation. The impulse response is the difference between projections based on equations (1) and (2) with and without a shock (a "shock" path and baseline, respectively).[5]

In this paper, we follow Koop, Pesaran, and Potter (1996) in computing generalized impulse responses (GIRs) that allow us to bypass the problem of ordering dependence in the Cholesky normalization and to incorporate nonlinearities. We study conditional expectations of our variables to shocks generated from the observed relationships. In essence, a shock in this framework is an innovation to the variable together with innovations to other variables that one would expect given sample correlations among innovations. It amounts to ordering the variable first each time it is "shocked." Koop, Pesaran, and Potter (1996) show that nonlinearities introduce shock and history dependence, which may make the interpretation of traditional IRs difficult. The computed GIRs are defined as the difference between the expectations conditional on history (w) and a shock (v) for the response and on history (w) for the baseline:

$$GIR(Y; v_t, w_{t-1}, n) = E(Y_{t+n} \mid v_t, w_{t-1}) - E(Y_{t+n} \mid w_{t-1}) \; for \; n = 0,1,2,.... \qquad (3)$$

Koop, Pesaran, and Potter (1996) describe in detail how to compute GIRs. We use a simple bootstrapping procedure and an estimated variance-covariance matrix of residuals

[5] See Favero and Giavazzi (2007, 2009) for a similar treatment.

in equation (1) to generate shocks and derive GIRs based on (3).[6] We condition the calculation of GIRs on the last observations in the sample, w_{t-1}.

C. Data and Descriptive Statistics

The data used are quarterly series and are available from several sources. Total revenues, expenditures, and interest payments (seasonally adjusted) are taken from the Bureau of Economic Analysis's National Income and Product Accounts (NIPA, Table 3.2). Nominal and real GDP and GDP deflator series come from the same source. The quarterly data are available from 1947. Federal debt held by public is taken from St. Louis's Federal Reserve, FRED database. The quarterly debt series are available from 1970 while the annual data start earlier. We use equation (2) for debt dynamics to impute quarterly values between the adjacent annual figures. Our whole sample covers the period from the second quarter of 1947 to the first quarter of 2009. Given a structural break occurring at about 1980 as shown in Perotti (2004), first we present our results based on the post-1980 sample. We also perform robustness checks and discuss the implications.

The debt ratio as a percent of GDP had both downward and upward trends in the latter part of the 20th century (Figure 1). The debt level stood at about 90 percent of GDP after World War II but steadily declined afterward to the mid-20s range by the late 1970s. Debt doubled in the 1980s to about 50 percent of GDP and decreased to its mean level of

[6] We also used Monte Carlo normal sampling, and the results were similar.

about 40 percent of GDP in the 1990s (Table 1). Another debt buildup has been occurring since late 2008.

The sample after 1980 clearly shows less volatility and a switch in sign in correlations of monetary policy related variables, interest rate and inflation, with primary balance and growth (Tables 1 and 2). All variables in the model except for primary balance show lower standard deviation in the 1980-2009 period than in the earlier sample. Average inflation is lower while the average interest rate is higher after 1980 reflecting most likely tighter monetary policy. Average growth is lower. In the post-1980 sample, higher deficit is associated with lower interest rate but higher inflation, while the interest rate and growth are positively correlated. The opposite relationships are observed before 1980. Both subsamples show that higher deficit is associated with lower growth and lower debt. The interest rate and inflation also correlate negatively with debt. These correlations show some interesting patterns in the data, and we examine the dynamics of these variables in the following section.

III. Debt Dynamics

The debt feedback dynamics keep the debt level in check. The debt ratio is projected to peak at about 70 percent of GDP (starting from the second quarter of 2009) by 2014-2015 and slowly decline afterward.

To assess policies and explore the effects of macroeconomic aggregates on debt, we compute debt impulse responses. We find that the effects of primary balance and growth are large and robust while those of inflation and interest rate are smaller and depend on the sample used. Fiscal policy and growth are the main ingredients in the fiscal adjustment process.

A. Stationarity of Debt and Out-of-Sample Forecast

The debt feedback in the model ensures sustainable debt dynamics. It is evident that without the debt feedback, debt grows beyond 130-140 percent of GDP in ten years (Figure 2). The results are similar irrespective of whether only the debt feedback part is shut down or whether the debt path is computed based on VAR without debt in the model (the implicit linear approximation). In fact, even debt impulse responses largely differ, especially at longer horizons, depending upon whether the debt feedback is accounted for, shut down, or not included in VAR (Figure 3).

In addition to assessing the reaction of debt to various shocks, policymakers may also want to know how debt would evolve in the near future given the current history and/or shocks. The out-of-sample forecast renders naturally to our framework. Based on the latest observation in the sample, we project debt from the second quarter of 2009 for the next ten years (Figure 2). Although debt grows initially to about 70 percent of GDP in the next five years, it starts declining afterward as growth picks up and deficit falls. At the end of the projection period, however, it still stays well above the current level of debt. Our forecast is similar to that of CBO.

B. Debt Impulse Responses

The effects of primary balance and growth on debt dynamics are large while those of inflation and interest rates are relatively small. A positive shock to the primary balance (higher deficit in this paper's definition) of one standard deviation increases debt by about 1.5 percent of GDP in about 2.5 years (Figure 4). A one standard deviation positive shock to growth lowers debt by about 1 percent of GDP in the same timeframe. In contrast, a positive inflation shock in fact increases debt after several quarters while a shock to the interest rate does not affect debt much in the first few years (Figure 4).

To explain these results, we show a decomposition of debt IRs. The primary balance, on one hand, and the nonlinear component (comprised of the interest rate, growth, and inflation multiplied by the previous debt stock, see equation [2]), on the other hand, drive the dynamics. We approximate the nonlinear component by its linear representation (see Figure 4 and Appendix B for details).

Following a shock to the primary balance, the effect on debt is not surprising. Most of the change is driven by the primary balance and less so by the interest rate, growth, and inflation. Interestingly, a similar pattern emerges for a growth shock although initially the growth effect is relatively large. Running primary surpluses accounts for most of the decline in debt.

In the case of inflation and interest rate shocks, a change in debt is driven by much higher interest rate. Even though the primary balance is in surplus, it is not enough to compensate for a large increase in the interest rate. Growth does not contribute positively in the initial periods, either. Debt increases after the inflation shock until it starts declining as growth and primary surplus continue to improve. In the case of the interest rate shock, debt slowly declines as high primary surplus and improving growth outweigh the increasing interest rate.

C. Robustness of Debt Impulse Responses

The importance of including the debt feedback in VAR becomes apparent as sample periods vary. Similar to the 1980-2009 sample, the 1947-1979 sample produces the ever-increasing debt dynamics in VAR without the debt feedback (Figure 5). The debt impulse responses produce different dynamics as well (Figure 6). The whole sample (1947-2009) and the 1973-2009 sample (from the onset of flexible exchange rates), however, indicate that the debt path is not explosive (Appendix Figures A1 and A2). The linear approximation to debt in VAR works only in a subset of samples. In contrast, VAR with the debt feedback produces a robust result that debt dynamics are not explosive in each subsample.

Using other sample periods, we find that the interest rate and inflation effects on debt are not robust. The 1947-1979 sample similarly shows that it takes a few quarters for debt to increase following a positive inflation shock and debt does not increase following an interest rate shock (Figure 7). In contrast, the 1973-2009 sample shows that debt falls after an inflation shock while it rises after an interest rate shock (Appendix Figure A3). The whole 1947-2009 sample indicates that following an inflation or interest rate shock, debt initially falls, and after about 3 years it starts rising (Appendix Figure A4). We note that the contribution of inflation to debt dynamics was much stronger before 1980 than post-1980, indicating most likely a tighter and more independent monetary policy regime. The contribution of primary balance to debt dynamics following an inflation shock has also changed directions across samples, for instance, pushing debt levels higher in the pre-1980 period. These results indicate that higher inflation may not always be a way to lower debt. The policy regime is therefore important.

The effects of deficit and growth shocks are in the same direction across subsamples but their magnitudes differ. The 1947-1979 and 1947-2009 samples indicate a much smaller effect of deficit and growth shocks on debt. The 1973-2009 sample shows a stronger impact similar to the post-1980 period.

IV. Concluding Remarks

Using the modified VAR with the debt feedback and generalized impulse responses, we assess the dynamics of US public debt in relation to major macroeconomic aggregates. We argue that it is important to incorporate the debt feedback in VAR models as a debt path may not be stable and impulse responses become very persistent. We find that fiscal policy and growth have strong effects on debt while monetary policy related variables such as interest rate and inflation have relatively small impact and are not robust. The policy regime in place will affect the response of debt to, for instance, higher inflation. Our findings suggest that if the Fed reacts as it did in the recent past, a positive inflation shock (say, an imported inflation shock) would in fact increase debt. Finally, this time should not be any different if policymakers and economic agents respond to the debt buildup as in the past. Reaching 70 percent of GDP in 4-5 years, the debt ratio is forecast to gradually fall afterward.

Our paper hints at a potential misspecification of standard VARs that include fiscal variables. The simulation of such models could produce stationary paths for the variables explicitly included. Yet, an important but implied variable such as a stock of debt could be building up in an unreasonable fashion in the background. If it is the case, the original model without debt may not be a valid way to study the relationships among variables. Thus, one needs to keep in mind possible implications of the original model and include debt feedback as this paper illustrates.

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Table 1. Descriptive Statistics

1947:II-2009:I

	Mean	*St. Dev.*	*Min*	*Max*
Primary balance	-0.0015	0.0051	-0.0188	0.0181
Growth	0.0080	0.0100	-0.0274	0.0396
Inflation	0.0085	0.0063	-0.0084	0.0339
Interest rate	0.0158	0.0077	0.0047	0.0331
Debt	0.4125	0.1408	0.2252	0.8998

1947:II-1979:IV

	Mean	*St. Dev.*	*Min*	*Max*
Primary balance	-0.0024	0.0048	-0.0188	0.0121
Growth	0.0092	0.0116	-0.0274	0.0396
Inflation	0.0093	0.0074	-0.0084	0.0339
Interest rate	0.0107	0.0054	0.0047	0.0234
Debt	0.4368	0.1793	0.2252	0.8998

1980:I-2009:I

	Mean	*St. Dev.*	*Min*	*Max*
Primary balance	-0.0005	0.0052	-0.0126	0.0181
Growth	0.0067	0.0077	-0.0207	0.0222
Inflation	0.0076	0.0047	0.0002	0.0278
Interest rate	0.0215	0.0057	0.0091	0.0331
Debt	0.3853	0.0688	0.2449	0.4965

Table 2. Correlation Matrix

1947:II-2009:I

	Primary balance	*Growth*	*Inflation*	*Interest rate*	*Debt*
Primary balance	1				
Growth	-0.154	1			
Inflation	-0.089	-0.121	1		
Interest rate	0.260	-0.097	0.310	1	
Debt	-0.382	0.059	-0.429	-0.588	1

1947:II-1979:IV

	Primary balance	*Growth*	*Inflation*	*Interest rate*	*Debt*
Primary balance	1				
Growth	-0.093	1			
Inflation	-0.176	-0.132	1		
Interest rate	0.387	-0.080	0.651	1	
Debt	-0.507	0.024	-0.443	-0.859	1

1980:I-2009:I

	Primary balance	*Growth*	*Inflation*	*Interest rate*	*Debt*
Primary balance	1				
Growth	-0.208	1			
Inflation	0.107	-0.160	1		
Interest rate	-0.014	0.084	0.475	1	
Debt	-0.113	0.104	-0.615	-0.339	1

Figure 1. Evolution of Public Debt (1947:II-2009:I)

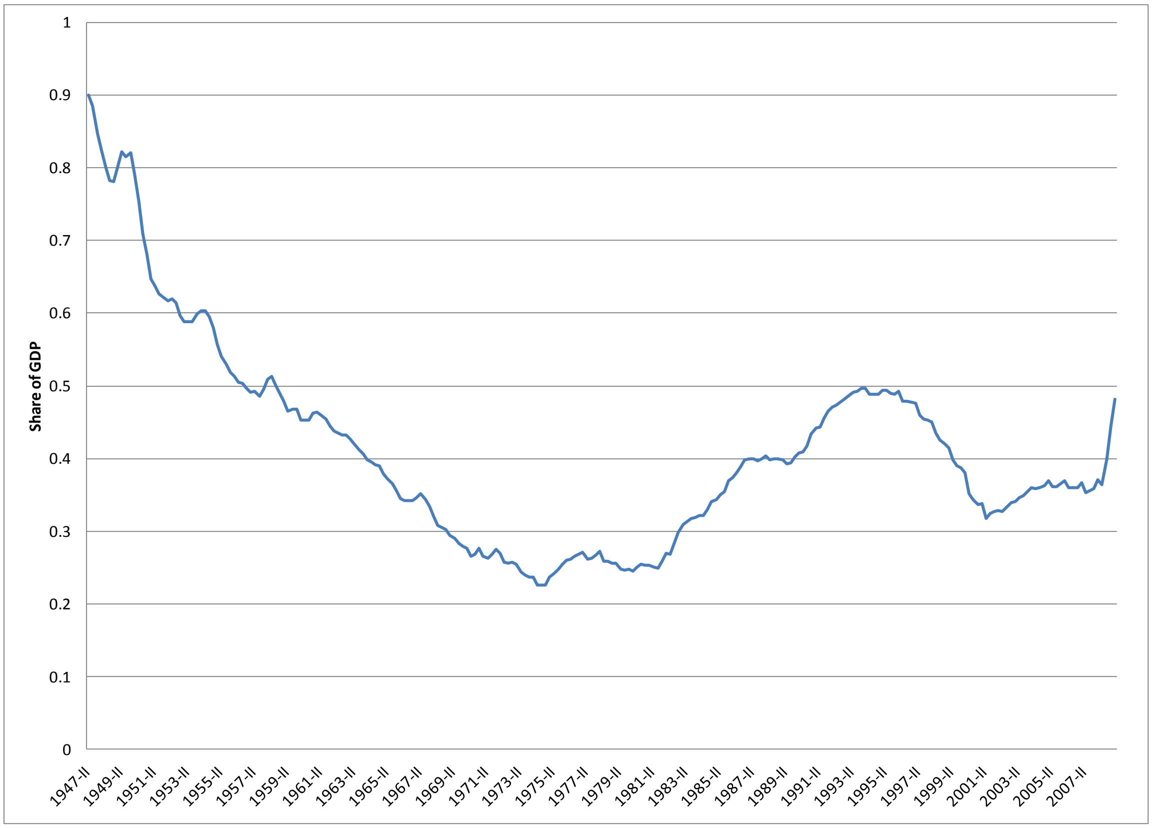


Figure 2. Debt: Out-of-Sample Forecast (1980-2009)

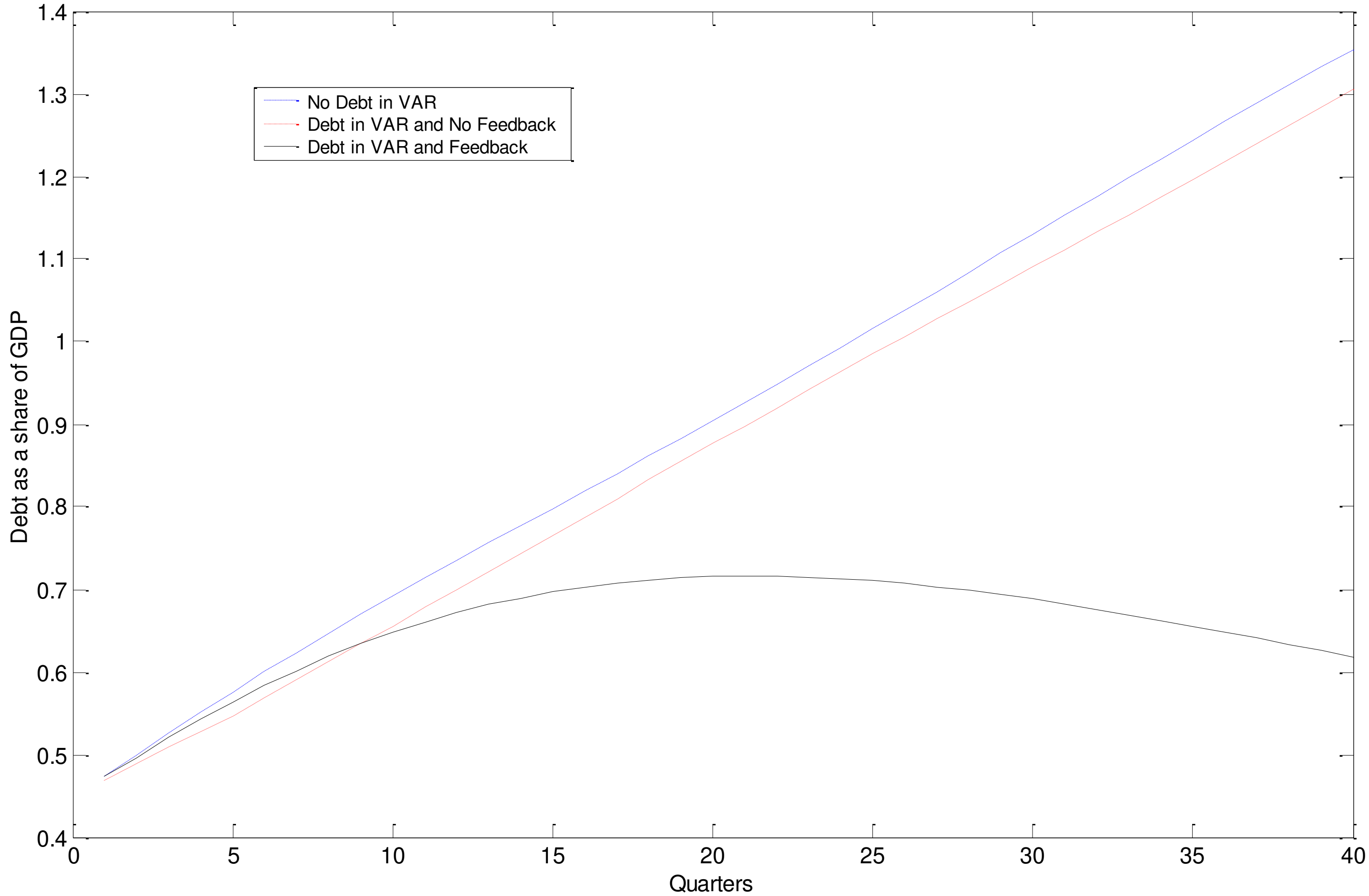


Figure 3. Debt Impulse Responses: A Comparison of VAR Models (1980-2009)

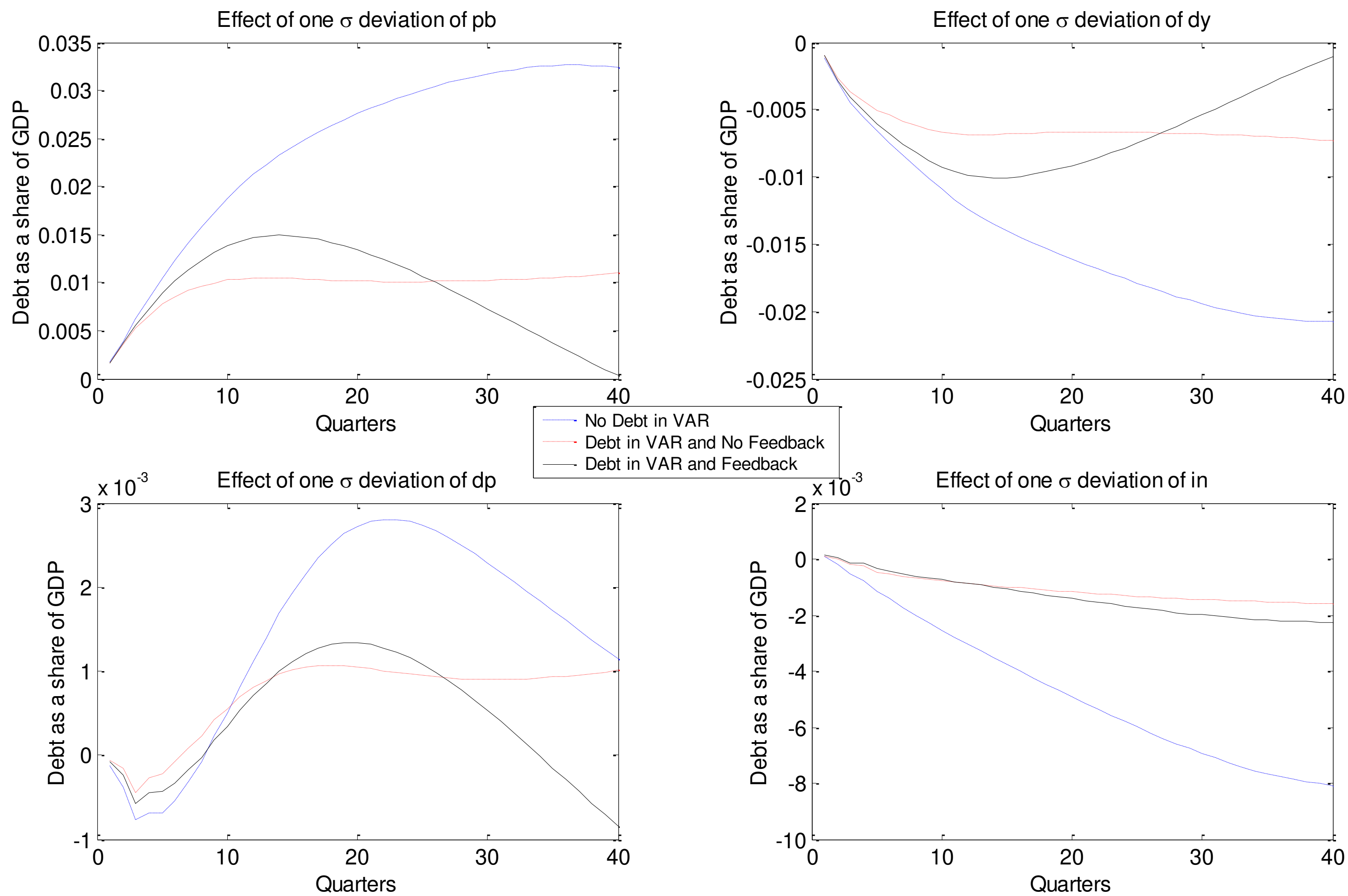


Figure 4. Debt Impulse Responses and Decomposition (1980-2009)

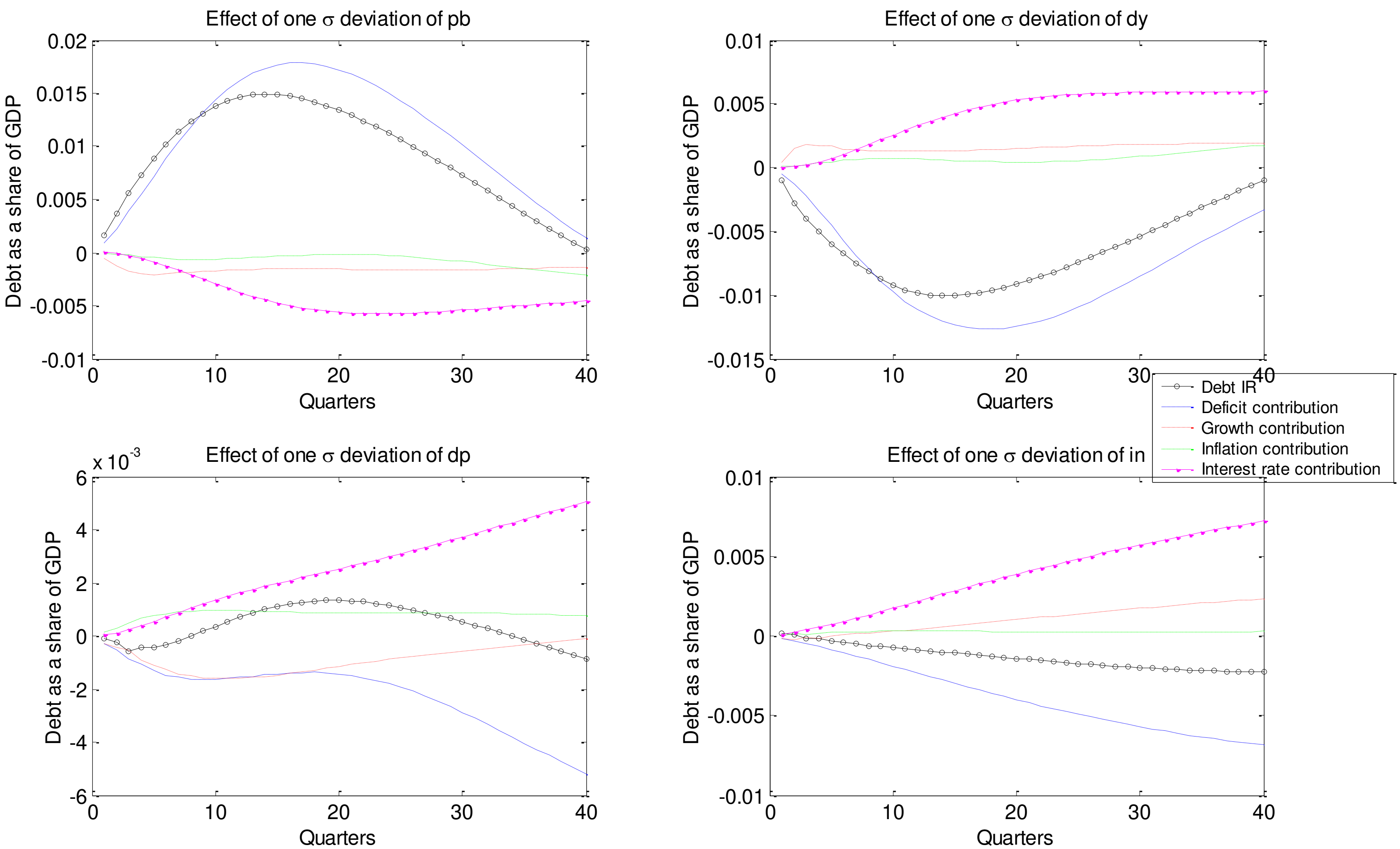


Figure 5. Debt: Out-of-Sample Forecast (1947-1979)

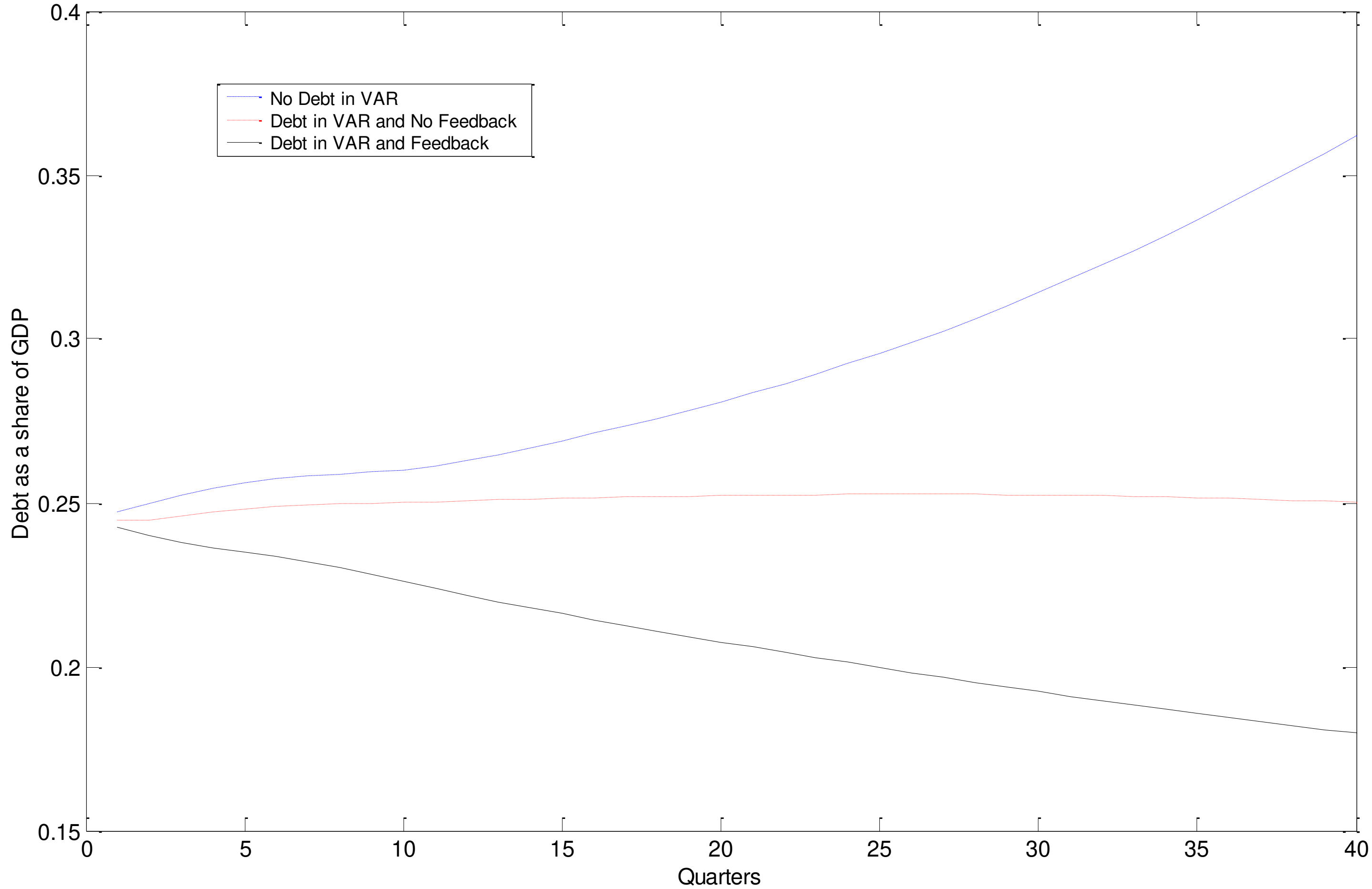


Figure 6. Debt Impulse Responses: A Comparison of VAR Models (1947-1979)

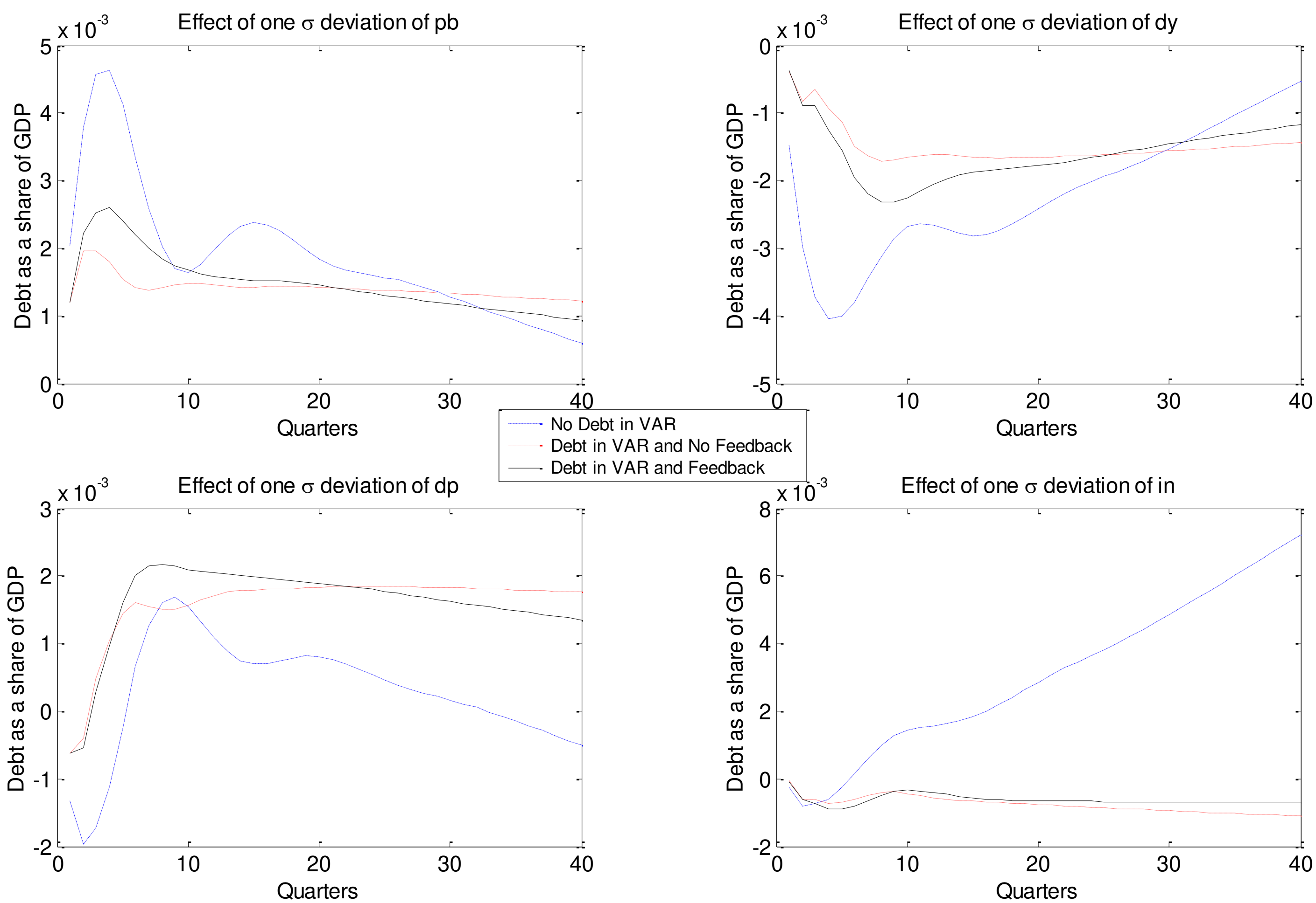
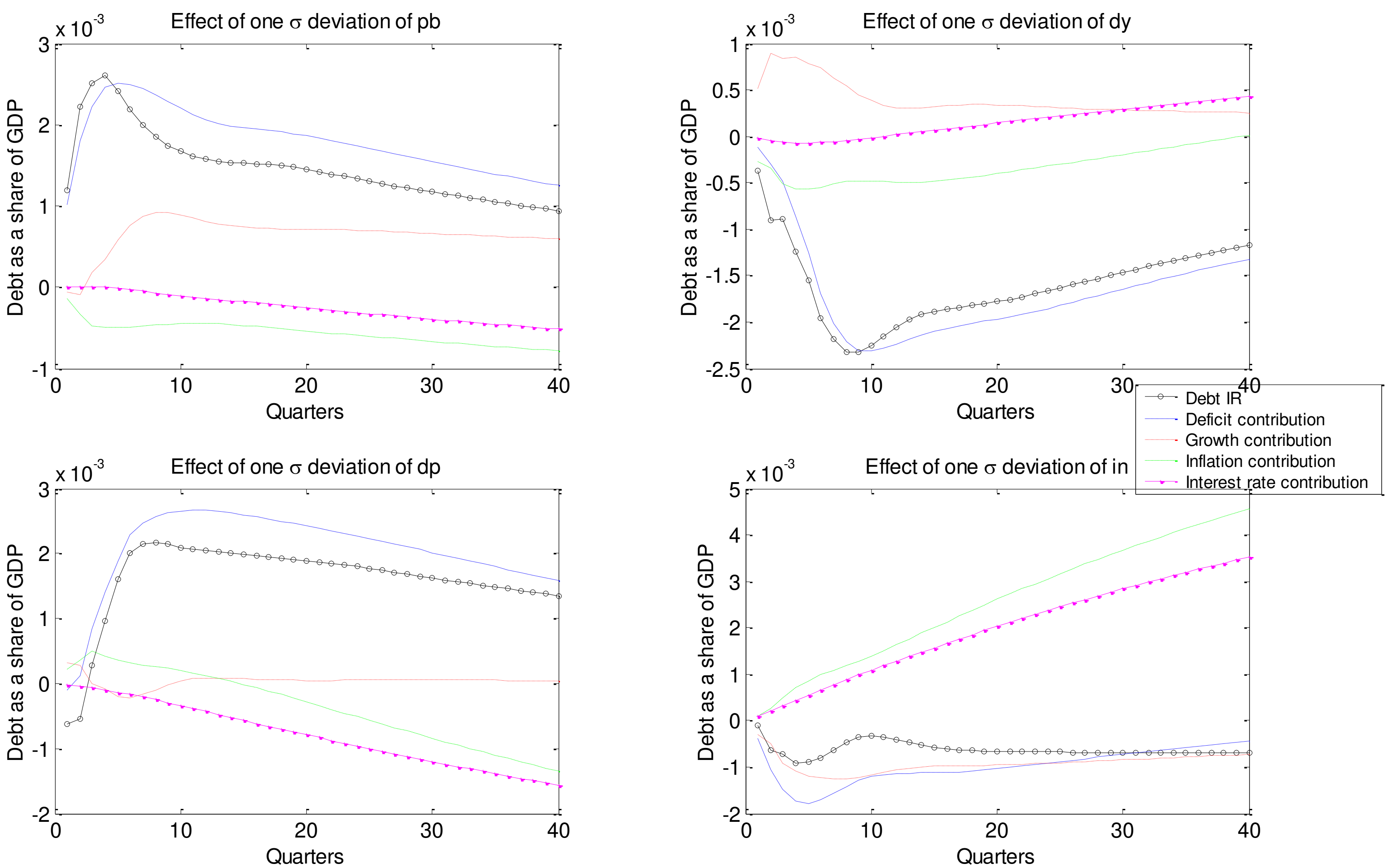


Figure 7. Debt Impulse Responses and Decomposition (1947-1979)



Appendix A

Figure A1. Debt: Out-of-Sample Forecast (1947-2009)

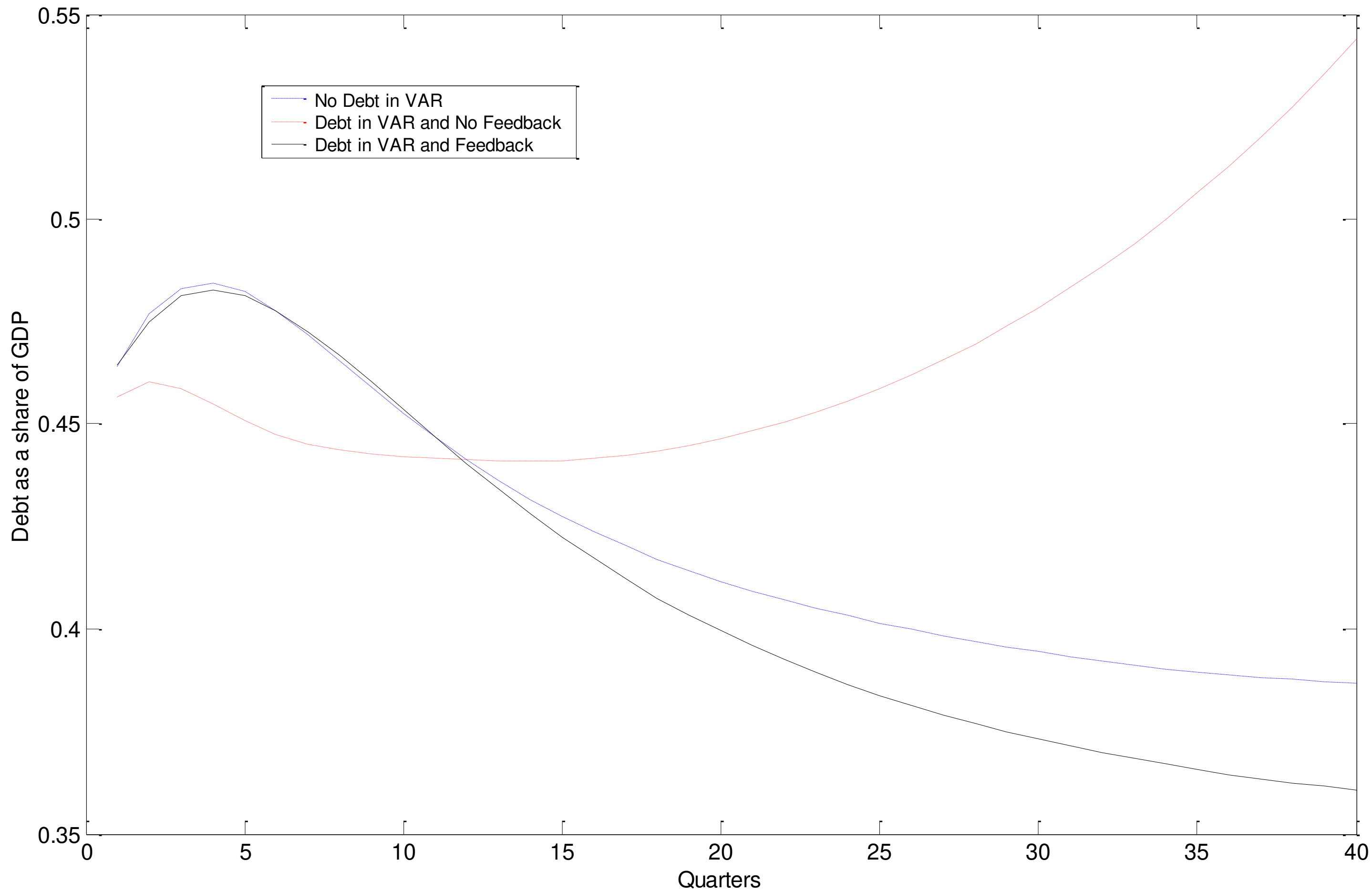


Figure A2. Debt: Out-of-Sample Forecast (1973-2009)

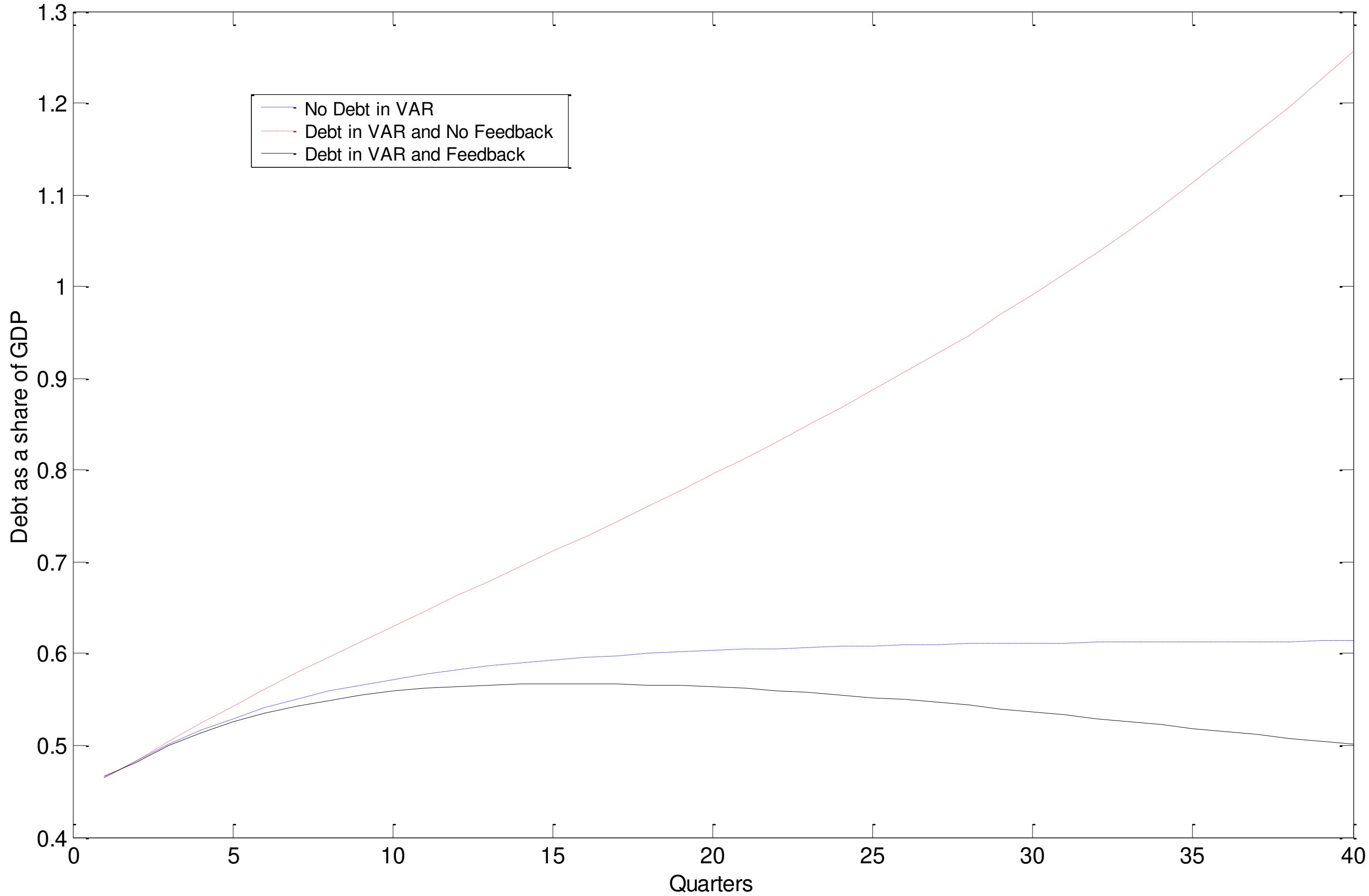


Figure A3. Debt Impulse Responses and Decomposition (1973-2009)

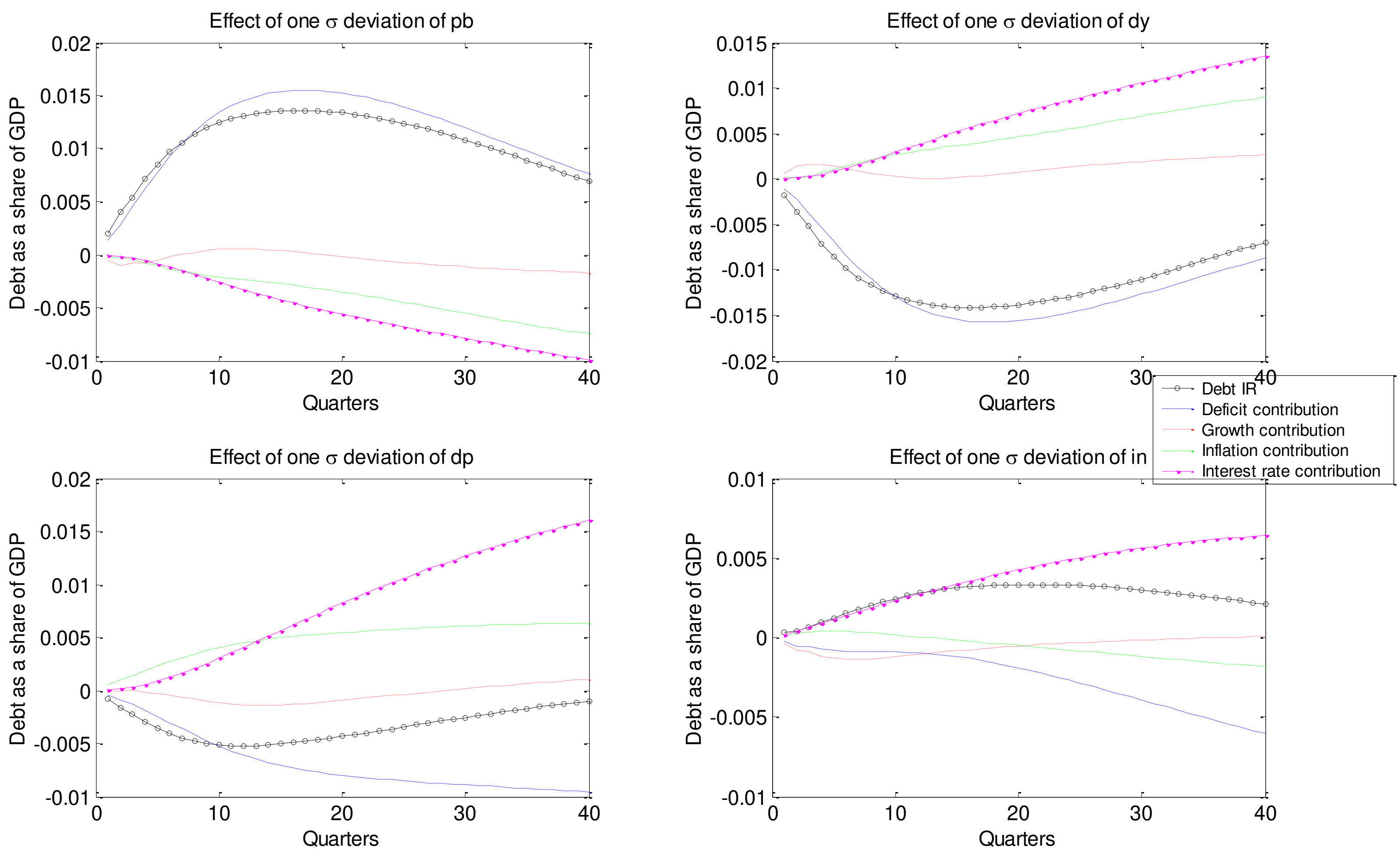
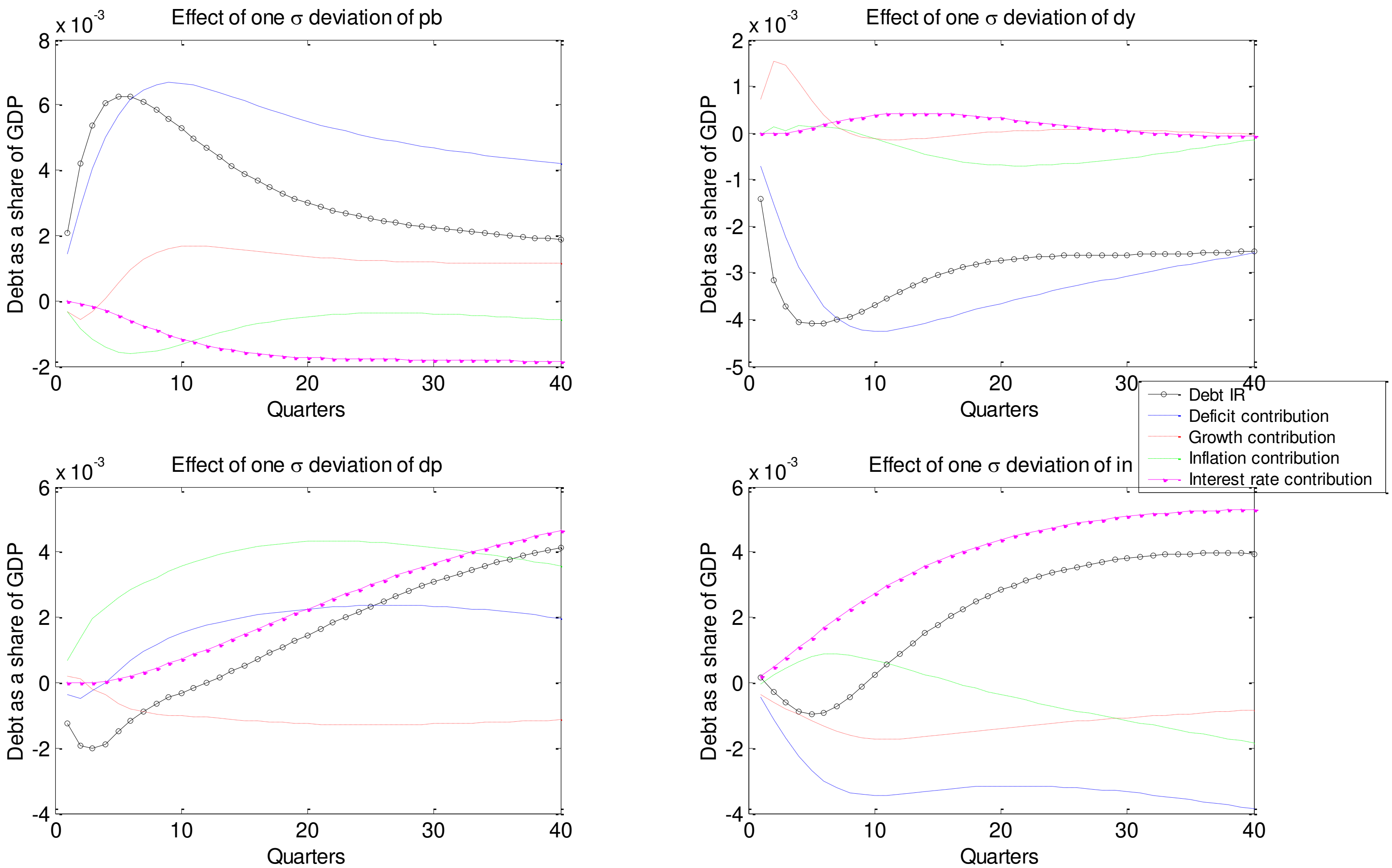


Figure A4. Debt Impulse Response and Decomposition (1947-2009)



Appendix B

We define the decomposition of the debt impulse response, d^{IR}, in terms of the contribution of each macroeconomic aggregate as follows:

$$d_t^{IR} = d_t^s - d_t^n = pb_t^* + g_t^* + \pi_t^* + i_t^*,$$

where s and n stand for "shock" and "no shock" debt paths. Using debt dynamics equation (2) in the text and approximating the nonlinear component, the components of the decomposition at time t are:

$$\begin{aligned}
pb_t^* &= \left(pb_t^s - pb_t^n\right) + \left(1 + i_t^s - \pi_t^s - g_t^s\right) pb_{t-1}^* \\
g_t^* &= \left(g_t^s - g_t^n\right) d_{t-1}^n + \left(1 + i_t^s - \pi_t^s - g_t^s\right) g_{t-1}^* \\
\pi_t^* &= \left(\pi_t^s - \pi_t^n\right) d_{t-1}^n + \left(1 + i_t^s - \pi_t^s - g_t^s\right) \pi_{t-1}^* \\
i_t^* &= \left(i_t^s - i_t^n\right) d_{t-1}^n + \left(1 + i_t^s - \pi_t^s - g_t^s\right) i_{t-1}^*
\end{aligned}$$

The first term in each equation indicates the difference between "shock" and "no shock" paths of the components scaled by the previous "no shock" debt ratio. The second term is the adjusted previous value of the component. Thus, the debt impulse response decomposition is:

$$d_t^{IR} = \Delta^{s/n} pb_t + \left(\Delta^{s/n} g_t + \Delta^{s/n} \pi_t + \Delta^{s/n} i_t\right) d_{t-1}^n + \left(1 + i_t^s - \pi_t^s - g_t^s\right) d_{t-1}^{IR},$$

where $\Delta^{s/n}$ stands for the difference between "shock" and "no shock" paths. Note also that the last term disappears in the initial period, $t = 1$, as the previous (before shock, $t = 0$), debt ratio is same.