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DISCOUNTING WHEN INCOME IS STOCHASTIC AND CLIMATE CHANGE POLICIES

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Abstract. We introduce stochastic income into the standard exponential discounting model and study dependence of effective discount rates on the type of the underlying stochastic process and agent's current income level. If the income follows a process with i.i.d. increments effective discounting is exponential. If the income follows a mean reverting process, the shape of discount rate curves depends on the margin between the agent's current income and the long-run average. The model is used to study how the willingness to pay for investments in abatement technologies depends on the current wealth of a country.

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The idea of discounting is a cornerstone in economics and finance. The traditional exponential discounting model is regarded as one of the most celebrated failures of the standard economics by behavioral economists. The latter model is based on the assumption that intertemporal choices do not depend on the decision date, which implies both stationarity of the discount function and time consistency. Economic experiments demonstrate that people behave as hyperbolic discounters: their discount functions are time-dependent and decisions may be time inconsistent or irrational.

An important implicit assumption behind the exponential discounting model is certainty of the discount function. We use the standard exponential discounting model assuming that the individual income available for consumption is a stochastic process and study dependence of the effective discount function on the type of the underlying stochastic process and on the agent's current income

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level. We derive general discount factors for gains and losses that imply several discounted utilities anomalies at once.

We introduce *the term structure of absolute risk aversion* $E[-u''(b_t)]/E[u'(b_t)]$ (where $u(\cdot)$ is the instantaneous utility function with the standard properties, b_t is the income at date t, and E is the expectations operator). We show that if the term structure of absolute risk aversion is a non-decreasing function of time (this condition is sufficient, but not necessary) then gains are discounted more than losses and the delay-speedup asymmetry follows. The gain-loss asymmetry is observed even when the discount function is exponential.

As an example, we consider the constant relative risk aversion (CRRA) utility. We show that relatively poor agents prefer to consume sooner than relatively rich agents, but relatively rich agents prefer to suffer a loss sooner than relatively poor agents. This fact, in particular, suggests an explanation of the unwillingness of poor countries to suffer costs of combat against the global warming now in order to save themselves from the losses in the future.

If the agent perceives her income as a process with i.i.d. increments, then the discounting is exponential, though it depends on the parameters of the underlying stochastic process used to model the income, on the agent's risk aversion and her current income. Rich agents discount the future gains less than poor agents, which agrees with poorer consumers' willingness to buy high priced credit products such as payday loans. On the other hand, poor agents discount future losses less than rich agents. For losses that may happen in the near future, negative discounting is observed; however, rich people do not exhibit negative discounting in this model.

In order to generate non-exponential discounting and preference reversal, we use the geometric Ornstein-Uhlenbeck (OU) (mean-reverting) process to model the income and obtain the following results[1]. If the agent is rich so that her current income is higher than the long run central tendency, then the effective discount rate increases in time (as the borrowing rate for a sound corporation), and no hyperbolic discounting is observed. This behavior of the effective discount rate corresponds to the case of so called normal yield curve in the bond markets (a pattern known as *contango* in the commodities futures markets).

If the agent is poor so that her current income is less than the long-run average by a certain non-zero margin (which depends of the risk attitude, type of uncertainty and the parameters of the income process), then the effective discount rate decreases with time, and the hyperbolic discounting is observed. This pattern is known as *backwardation* in the commodities futures markets or an inverted yield

[1]Gollier [13] shows that if the aggregate consumption follows the geometric random walk, then the socially efficient discount rate is independent of time; if the aggregate consumption follows an AR(1) process, then the socially efficient discount rate depends on time.

curve in the bond markets. Contango and backwardation patterns can be intuitively explained. The agent, whose current income is above the central tendency, values future consumption more than present consumption, because she expects her income to drop to the log-run average eventually. On the other hand, the agent, whose income is below the long-run average, expects her income to revert to the central tendency eventually, therefore she values immediate consumption more than distant consumption.

Finally, if the agent is neither too rich nor too poor, then there exists $t^* > 0$ such that the hyperbolic discounting is observed over the interval $[t^*, +\infty)$ but on $[0, t^*]$, the effective discount rate is increasing; i.e., the effective discount rate is hump-shaped[2]. The poorer the agent becomes, the higher is the probability that the hyperbolic discounting will be observed in an experiment. Hump-shaped yield curves are also observed in real markets. The traditional crude oil futures curve, for example, is typically humped: it is normal in the short-term but gives way to an inverted market for longer maturities.

Individual discounting in the uncertain world is by all means interesting and important. Social discounting has become vitally important due to the necessity of cost-benefits analysis of environmental policies. The main focus of the literature dealing with environmental policies is whether a stringent abatement policy is needed now or should the abatement begin slowly. The answer to this question, certainly, depends of the rate of discounting used to evaluate potential climate changes and costs of abatement, and the social discount rate is one of the main sources of disagreement among the economists (we relegate the review of the related literature to Section 5). To the best of our knowledge, none of the papers on environmental policies makes a distinction between rich and poor countries. Dependence of the the willingness to pay (WTP) for abatement policies on wealth is especially important to understand because rich countries are supposed to deliver about $28 bn "fast start" funding for developing countries. Our model explains why rich countries have made a promise of funding, why "fast start" funding can be expected to make progress on practical measures for tackling global warming and, why the sums so far committed are much smaller than the initial pledge.

We evaluate the fraction of consumption that a country would be willing to sacrifice at present in order to avoid big losses due to an environmental catastrophic event in the future. The aggregate consumption follows a stochastic process, the social utility function is CRRA, and the date of the catastrophe is uncertain. If the underlying stochastic process follows the GBM, WTP is independent of the country's current wealth. However, WTP is increasing in the country's rate of growth of expected utility of consumption. WTP decreases in the expected time

[2]Farmer and Geanakoplos [9] obtain a hump-shaped effective discount rate curve when they fit the experimental results obtained in [34].

until the catastrophe happens and in the expected time until abatement policies take the desirable effect. If the aggregate consumption follows the geometric OU process, WTP increases in the country's current wealth.

The rest of the paper is organized as follows. Section 1 gives an overview of the standard discounted utility and departures from this theory observed in experiments. The time preference model is specified in Section 2. In Section 3, we derive formulas for discount factors for gains and losses for a general utility function and general uncertainty and introduce the notion of the term structure of absolute risk aversion. Section 4 deals with the case of the CRRA utility function and stochastic income stream following the GBM and OU process. In Section 5 WTP for abatement policies is considered. Section 6 concludes. Technical details are relegated to the Appendix.

1. Time preference

In 1937, Samuelson [31] invented the DU theory, which compressed the influence of many factors affecting intertemporal choices into one number: the discount rate. In continuous time models, an individual with the time-separable utility u calculates the value of consumption of a stream b_t over time interval $[0, T]$ according to the formula

$$U = \int_0^T e^{-rt} u(b_t) dt, \tag{1.1}$$

where $r > 0$ is the discount rate. In discrete time models, the counterpart of equation (1.1) is

$$U = \sum_{t=0}^{T} \delta^t u(b_t), \tag{1.2}$$

where $\delta = e^{-r} \approx 1/(1 + r)$. Due to the analytical simplicity, the exponential discounted utility model was almost instantly adopted as a standard tool in intertemporal models, although [31] suggested the DU model as a convenient tool only, and explicitly disavowed an idea that individuals really optimize an integral of the form (1.1). More than 20 years later, [18] constructed an axiomatic theory of time preference which lead to the exponential discount factor in Samuelson's model. As a result, a general feeling emerged that the DU model was justified. However later, in many experimental studies, it was shown that the real behavior of individuals did not agree with the exponential discounting model. [10] present evidence that the instantaneous discount rate for gains *decreases* with time (*hyperbolic discounting*), gains are discounted more than losses (*sign effect, or gain-loss asymmetry*), greater discounting is demonstrated to avoid delay of a good than to expedite its receipt (*delay-speedup asymmetry*), an individual may prefer to expedite a payment (*negative discounting for losses*).

To account for DU anomalies, several alternative models have been developed. In the (β, δ)- model of quasi-hyperbolic discounting introduced first by [26], equation (1.2) is replaced by

$$U = u(b_0) + \sum_{t=1}^{T} \beta\delta^t u(b_t), \tag{1.3}$$

where $\beta, \delta \in (0, 1)$. Equation (1.3) is analytically simple, and captures many qualitative features of hyperbolic discounting. Thus, as in [31], the discount factors are postulated. Another strand of literature initiated by [18] deals with the axiomatic systems for time preferences, which are consistent with DU anomalies - see [24] and the bibliography therein. [11] suggested a "dual-self" model as a unified explanation for several empirical regularities. Habit formation models, reference point models and a number of other models incorporate non-standard features into the utility function. See [10], [15] and the extensive bibliography there for the list of models that depart from the DU model.

Unfortunately, it is not clear if behavioral models can be ever used as a basis of a consistent general economic theory. Note that [25] argues that it is difficult to use economic data to calibrate utility functions that depend on some variables observed in experiments on intertemporal choice. Gul and Pesendorfer [15] demonstrate that small changes to standard choice-theoretic methods suffice to analyze variables that are often ignored in standard economic models.

Noor [23] uses a standard exponential discounting model to explain the hyperbolic discounting. His main argument is that "the most likely participants in experiments may be those with the most immediate need for money;" if the participants expect a small deterministic increase in the income, then their behavior in experiments agrees with hyperbolic discounting.

The idea that time dependent discounting can be explained with an urgent need for money agrees with existence of high priced credit products such as small personal loans, pawnbroker loans, payday loans, automobile title loans, and refund anticipation loans. Prices for these products are indeed high. Finance charges are large relative to loan amounts, and annual percentage rates often exceed 100 percent (see [8] for details).

Noor [23], however, does not explain other discounted utility (DU) anomalies. For example, how the immediate need for money can account for the fact that gains are discounted more than losses?

Note that a natural model for hyperbolic discounting in discrete time would be

$$U = u(b_0) + \sum_{t=1}^{T} \prod_{t'=0}^{t-1} \beta_{t',t'+1} u(b_t), \tag{1.4}$$

where $\beta_{t,t+1}$, the discount factor between t and $t+1$, as viewed from time 0, is an *increasing* function in t. In the continuous time limit, we obtain that the hyperbolic discounting means that the instantaneous discount rate

$$r_t = -\lim_{T\to 0} \frac{\ln \beta_{t,t+T}}{T} = -\frac{\partial}{\partial T} \ln \beta_{t,t+T}|_{T=0} \tag{1.5}$$

is a *decreasing* function of t.

In finance, close analogs of the discount rates are zero-coupon bond yields. At time t, consider the bond maturing at $t+T$. Although at maturity, the payoff is deterministic (say, \$ 100), the bond price $B(t, t+T)$ is a random variable, and yield curves $t \mapsto -\ln B(t, t+T)/T$ are not flat; in fact, they can be of many shapes. The reason is that during the time period to maturity, many random events will happen in the world, and they will influence the value of the riskless zero-coupon bond. The example with the yield curves explains that however hard a researcher tries to exclude the uncertainty in an experiment on DU anomalies, the uncertainty will always remain in the background, and therefore, there is no reason to expect that the discount rate curve observed in experiments will be flat.

There is a substantial body of research in financial economics, where the behavior of bond prices and yields is derived endogenously in general equilibrium models from the exogenous stochastic dynamics of the production sector of the economy: see, e.g., [3] for one of the most popular interest rate models and [7] for the review of alternative models and further references. We conduct our analysis in the framework of a partial equilibrium model, and, to simplify the treatment of instantaneous payoffs, we consider the payoff streams in discrete time.

2. Model specification

We neither postulate the non-standard dependence of the discount factor on time as in the quasi-hyperbolic discounted utility models nor deduce it from time preference axioms. Instead, we derive general equations for the discount factors for gains and losses from several simple general assumptions.

As in [23], we define a preference relation $\succeq$ over the set of dated rewards $\mathcal{X} = \mathcal{M} \times \mathcal{T}$, where $\mathcal{M} = [0, M]$ (for some $M > 0$), and $\mathcal{T} = \mathbb{R}_+$. Let $\{b_t\}_{t\geq 0}$ be the consumer's income, then if the income is deterministic, the preference relation $\succeq$ on $\mathcal{X}$ is induced by a utility function (see [23])

$$U(m, t) = D(t)[u(b_t + m) - u(b_t)],$$

where $D(t)$ is the discount function. Using the present equivalent $\psi(m, t)$ of any dated reward (m, t), where the present equivalent is defined by $(\psi(m, t), 0) \sim (m, t)$, [23] shows that for small rewards, the money discount function $\phi^m(t) = \psi(m, t)/m$ can be matched to a hyperbolic discount function by varying the standard discount factor δ and parameters of a concave utility function u (u is assumed to be a CARA utility function in [23]).

We depart from [23] by introducing uncertainty into the standard exponential DU model. Our starting point is that an individual perceives the future – hence the utility of consumption – as uncertain. To be more specific, in this paper, we assume that the income, $\{b_t\}_{t\geq 0}$, is stochastic. In general, the uncertainty may be caused by changes both in the anticipated income and/or utility function *p*er se: obviously, the satisfaction from possession of a certain widget may change (and typically, changes) in a not completely predictable fashion. Similar ideas are used in [14] ("changing tastes") and [20] ("the perception of future events becomes increasingly "blurred" as the events are pushed further in time"), among the others. Dasgupta and Maskin [5] show that if the "average" situation entails some uncertainty about the time when payoffs are realized, the corresponding preferences may well entail hyperbolic discounting. Robson and Samuelson [30] demonstrate that aggregate uncertainty concerning survival rates can lead to non-exponential discount rates. Sou [32] derives the hyperbolic discounting from the Bayesian updating of the beliefs about the distribution of the random discount rate. Farmer and Geanakoplos [9] postulate individual stochastic discount factors to explain rationally hyperbolic discounting. Weitzman [38] argues that the discount rate should be stochastic because the future rate of return on capital is uncertain. His model also generates hyperbolic discounting.

Consider, first, the case of gains (rewards). Suppose that, at time 0, the agent is asked to compare dated payoffs (m', t) and $(m, t+T)$, where $t \geq 0$ and $T > 0$. The agent evaluates consumption streams using the standard expected discounted utility model:

$$V(b_0, \dots, b_t, \dots, b_{t+T}) = E\left[\sum_{\tau=0}^{t+T} \delta^\tau u(b_\tau)\right],$$

where $\delta \in (0,1)$ is the discount factor, and E is the expectation operator. As in [23], we assume that both rewards m and m' are small, so that the agent does not consider spreading any of the rewards over time. Then $(m, t+T) \succeq (m', t)$ iff

$$V(b_0, \dots, b_t, \dots b_{t+T} + m) - V(b_0, \dots, b_t + m', \dots b_{t+T}) \geq 0,$$

equivalently, iff

$$E\left[\sum_{\tau=0}^{t-1} \delta^\tau u(b_\tau)\right] + \delta^t E\left[u(b_t) + \sum_{\tau=t+1}^{t+T-1} \delta^{\tau-t} u(b_\tau) + \delta^T u(b_{t+T}+m)\right] -$$
$$-E\left[\sum_{\tau=0}^{t-1} \delta^\tau u(b_\tau)\right] - \delta^t E\left[u(b_t+m') + \sum_{\tau=t+1}^{t+T-1} \delta^{\tau-t} u(b_\tau) + \delta^T u(b_{t+T})\right] \geq 0,$$

equivalently, iff

$$G(T,m;m';b_t) := \delta^T E[u(b_{t+T}+m)-u(b_{t+T})] - E[u(b_t+m')-u(b_t)] \ge 0. \tag{2.1}$$

Now, let the agent be asked to compare dated losses (m',t) and $(m,t+T)$, where $T>0$. Then $(m,t+T)\succeq(m',t)$ iff

$$V(b_0,\dots,b_t,\dots b_{t+T}-m) - V(b_0,\dots,b_t-m',\dots b_{t+T}) \ge 0,$$

equivalently, iff

$$E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right] + \delta^t E\left[u(b_t)+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T}-m)\right] -$$

$$-E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right] - \delta^t E\left[u(b_t-m')+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T})\right] \ge 0,$$

equivalently, iff

$$L(T,m;m';b_t) := \delta^T E[u(b_{t+T}-m)-u(b_{t+T})] - E[u(b_t-m')-u(b_t)] \ge 0. \tag{2.2}$$

Both in case of gains and losses, we want to derive the relation between T, m, m' and b_t, which makes the agent indifferent between the dated payoffs (losses) (m',t) and $(m,t+T)$.

3. Discount factors for gains and losses.

Let $u(\cdot)\in C^2$ be increasing and strictly concave. It is impossible to derive the exact relation between T, m, m' and b_t, which makes the agent indifferent between the dated payoffs (m',t) and $(m,t+T)$, using (2.1) in a simple algebraic form. Instead, we derive an approximation assuming that both m and m' are small relative to the current consumption level.

3.1. **Discount factors for gains.** Using the Taylor expansion of order 2, we obtain the following approximation

$$\begin{aligned} G(T;m;m';b_t) &= \delta^T\left\{mE[u'(b_{t+T})]+\frac{m^2}{2}E[u''(b_{t+T})]\right\} \\ &\quad -m'E[u'(b_t)]-\frac{(m')^2}{2}E[u''(b_t)], \end{aligned} \tag{3.1}$$

whence $G(T;m;m';b_t)\ge 0$ iff

$$m'-(m')^2\frac{E[-u''(b_t)]}{2E[u'(b_t)]} \le m\delta^T\frac{E[u'(b_{t+T})]}{E[u'(b_t)]} - m^2\delta^T\frac{E[-u''(b_{t+T})]}{2E[u'(b_t)]}.$$

Letting $t=0$, and using only the linear terms,we derive that $G(T,m;m';b_0)\ge 0$ iff

$$m' \le m\delta^T\frac{E[u'(b_T)]}{u'(b_0)]}.$$

equivalently, iff

$$u'(b_0) \le \frac{m\delta^T E[u'(b_T)]}{m'}. \tag{3.2}$$

Concavity of u implies that (3.2) holds iff $b_0 \ge K$, where

$$K = \phi\left(\frac{m\delta^T E[u'(b_T)]}{m'}\right),$$

and $\phi = (u')^{-1}$. We conclude that $(m', 0) \succeq (m, T)$ if and only if $b_0 \le K$, i.e, relatively poor agents prefer immediate gratification.

To determine the present equivalent of $(m, t+T)$ at $t \ge 0$, as viewed from date 0, we need to find (m'_g, t) such that $G(T, m; m'_g; b_t) = 0$. It follows from (3.1) that m'_g is a solution of the quadratic equation

$$\begin{aligned}\delta^T \left\{ mE[u'(b_{t+T})] - \frac{m^2}{2} E[-u''(b_{t+T})] \right\} - \\ -m'_g E[u'(b_t)] + \frac{(m'_g)^2}{2} E[-u''(b_t)] = 0.\end{aligned} \tag{3.3}$$

Since both m, m'_g are small, the linear approximation to $m'_g = m'_g(m)$ is

$$m'_g = m\delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]} = mP(t, t+T), \tag{3.4}$$

where

$$P(t, t+T) = \delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]}. \tag{3.5}$$

The function $P(t, t+T)$ is the marginal rate of substitution between consumption at t and $t+T$ as perceived at time $t = 0$. We will see it later, that in fact, $P(t, t+T)$ is the discount function if the size of the rewards is small. It is natural to assume that $m'_g < m$, and, since the utility function is increasing, we also have to assume that

$$P(t, t+T) < 1. \tag{3.6}$$

Taking square of (3.4) and substituting the result for the factor $(m'_g)^2$ in the last term of (3.3), we find the second order approximation to $m'_g = m'_g(m)$: modulo $o(m^2)$ term,

$$m'_g = mP(t, t+T)\left[1 + \frac{m}{2}\left(P(t, t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right]. \tag{3.7}$$

Assuming that

$$\frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]} \ge \frac{E[-u''(b_t)]}{E[u'(b_t)]}, \tag{3.8}$$

and taking into account the standing assumption $P(t, t+T) < 1$, we obtain that the coefficient at m inside the square brackets is negative.

It follows from (3.7) that, modulo $o(m^2)$ term, the money discount factor $\mathcal{D}_g(t, T; m) = m'_g/m$ for gains is

(3.9)
$$\mathcal{D}_g(t, T; m) = P(t, t+T)\left[1 + \frac{m}{2}\left(P(t, t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right].$$

3.2. **Normal term structure of absolute risk aversion, gain-loss asymmetry and delay-speedup asymmetry.** In the Appendix, using the same argument as above, we show that relatively rich agents prefer to expedite the loss and the discount factor for losses is given by

(3.10)
$$\mathcal{D}_l(t, T; m) = P(t, t+T)\left[1 - \frac{m}{2}\left(P(t, t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right].$$

We call the mapping

$$t \mapsto E[-u''(b_t)]/E[u'(b_t)]$$

the term structure of absolute risk aversion. If (3.8) holds for all t, T, then the term structure of absolute risk aversion will be called normal (equivalently, the term structure is non-decreasing in t). If the term structure is normal and $P(t, t+T) < 1$, then it follows from (3.9) and (3.10), that gains are discounted more than losses and the delay-speedup asymmetry follows.

3.3. **Preference reversal.** Suppose that the agent is asked to compare two pairs of dated payoffs: $(m', 0)$ vs. (m, T) and (m', t) vs. $(m, t+T)$. If the agent's preferences are $(m', 0) \succ (m, T)$ and $(m, t+T) \succ (m', t)$, then we have the so-called preference reversal, or decreasing impatience. Let $(m^0_g, 0) \sim (m, T)$, and $(m^t_g, t) \sim (m, t+T)$. Then, if the preference reversal is observed, we must have $m^0_g < m' < m^t_g$. A sufficient condition is: m^t_g, the present equivalent at $t \geq 0$ as viewed from date 0, is an increasing function of t. This condition is necessary if we want to model the preference reversal between any two dates $0 \leq t' < t$, not only between 0 and t. In particular, it is clear from (3.4) that the preference reversal will be observed if $P(t, t+T)$ is an increasing function of t.

3.4. **Effective discount rates and hyperbolic discounting.** As in [23], we define the discount function as the limit of the money discount factor when the size of the reward vanishes.

It follows from (3.9) and (3.10) that

$$\lim_{m\to 0} \mathcal{D}_g(t, T; m) = \lim_{m\to 0} \mathcal{D}_l(t, T; m) = P(t, t+T),$$

therefore $P(t, t+T)$ is the discount function for gains and losses. Considering the continuous time limit of the discrete time model, and assuming that $P(t, t+T)$ is differentiable at $T = 0$, we define the effective discount rate as:

$$\rho(t) = -\frac{\partial}{\partial T} \ln P(t, t+T)\bigg|_{T=0} = -\frac{\partial}{\partial T} P(t, t+T)\bigg|_{T=0}. \tag{3.11}$$

By definition, hyperbolic discounting means that $\rho(t)$ is a decreasing function of t. On the strength of definition (3.5) of $P(t, t+T)$ and (3.11) of $\rho(t)$, the behavior of the effective discount rate depends on the agent's current income, b_0, and specification of uncertainty.

In the standard models of uncertainty, it is convenient to work with the moment-generating function of the random variable b_t:

$$\mathrm{MGF}(b, t, \gamma) = E\left[e^{\gamma b_t} | b_0 = b\right].$$

Set $\Xi(b, t, \gamma) = \ln \mathrm{MGF}(b, t, \gamma)$ (the logarithm of the moment-generating function is called the cumulant-generating function) and $r = -\ln \delta$. Then

$$P(t, t+T) = \exp[-rT + \Xi(b_0, t+T, -a) - \Xi(b_0, t, -a)],$$

and

$$\rho(a, b_0; t) = r - \frac{\partial}{\partial T}\left(\Xi(b_0, t+T, -a) - \Xi(b_0, t, -a)\right)\bigg|_{T=0}$$

Simplifying,

$$\rho(a, b_0; t) = r - \Xi_t(b_0, t, -a), \tag{3.12}$$

where $\Xi_t = \partial \Xi(\cdot, t, \cdot)/\partial t$. Hence, ρ is a decreasing function in t if Ξ_t is an increasing function in t, i.e., if Ξ is a (strictly) convex function in t. Notice that r represents the standard discount rate, and $-\Xi_t(b_0, t, -a)$ is the idiosyncratic discount rate that depends on the agent's current income, risk attitude and the underlying uncertainty. To avoid negative discounting, we need $\Xi_t(b_0, t, -a) < r$, which is satisfied automatically if Ξ is a decreasing function in t.

4. CRRA utility function

Let $u(b) = \frac{b^{1-\alpha}-1}{1-\alpha}$, where $\alpha \in (0, \infty)$. In this section, we model the stochastic income as a geometric stochastic process: $b_t = e^{X_t}$. The moment-generating function of X_t is

$$\mathrm{MGF}(x, t, \gamma) = E\left[e^{\gamma X_t} \middle| X_0 = x = \ln b\right].$$

Set $\Xi(x, t, \gamma) = \ln \mathrm{MGF}(x, t, \gamma)$. Condition (3.8) can be written in terms of $\Xi(x, t, \gamma)$ as

$$\Xi(x_0, t+T, -(\alpha+1)) - \Xi(x_0, t, -(\alpha+1)) \geq \Xi(x_0, t+T, -\alpha) - \Xi(x_0, t, -\alpha). \tag{4.1}$$

As it was shown in Section 3, the discount function is

$$P(t,t+T)=\delta^T\frac{E[u'(b_{t+T})]}{E[u'(b_t)]}=\delta^T\frac{E[b_{t+T}^{-\alpha}]}{E[b_t^{-\alpha}]}=\delta^T\frac{E[e^{-\alpha X_{t+T}}]}{E[e^{-\alpha X_t}]}.$$

Now we can use the definition (3.11) of the effective discount rate to find

$$\rho(\alpha,x_0;t)=r-\Xi_t(x_0,t,-\alpha), \tag{4.2}$$

where $x_0=\ln b_0$. We see that the hyperbolic discounting is observed iff $\Xi_t(\cdot,t,\cdot)$ is an increasing function in t, i.e. iff $\Xi(\cdot,t,\cdot)$ is a convex function in t.

4.1. **The geometric Brownian motion model.** Let $\{b_t\}$ follow the GBM, i.e., b_t is given by the following stochastic differential equation:

$$db_t=\mu b_t dt+\sigma^2 b_t dW_t,$$

where where dW_t is the increment of the standard BM with zero mean and unit variance; and μ and σ^2 are, respectively, the drift and variance of the GBM. Then $X_t=\ln b_t$ is the BM with the drift $\mu-\sigma^2/2$ and variance σ^2. We have $\Xi(x,t,\gamma)=\gamma x+t\Psi(\gamma)$, where $\Psi(\gamma)=\gamma(\mu-\sigma^2/2)+\gamma^2\sigma^2/2$, therefore $\rho(\alpha,x_0,t)=r-\Psi(-\alpha)$ is independent of t and x_0. Thus, the discounting is exponential, however, we may observe the gain-loss asymmetry. By definition,

$$P(t,t+T)=\delta^T\frac{E[e^{-\alpha X_{t+T}}]}{E[e^{-\alpha X_t}]}=\delta^T\frac{e^{-\alpha x_0+(t+T)\Psi(-\alpha)}}{e^{-\alpha x_0+t\Psi(-\alpha)}}=\delta^T e^{T\Psi(-\alpha)}=e^{-T(r-\Psi(-\alpha))}.$$

Hence, condition (3.6) becomes $r-\Psi(-\alpha)>0$. The term structure of absolute risk aversion is

$$\frac{E[-u''(b_t)]}{E[u'(b_t)]}=\alpha e^{t(\Psi(-\alpha-1)-\Psi(-\alpha))},$$

therefore, it is normal iff

$$\Psi(-\alpha-1)\ge\Psi(-\alpha). \tag{4.3}$$

We conclude that the money discount factors for gains and losses are given by

$$\begin{aligned}\mathcal{D}_g(t,T:,m) &= \delta^T e^{T\Psi(-\alpha)}\left\{1-\frac{m}{b_0}\frac{\alpha}{2}e^{(t+T)(\Psi(-\alpha-1)-\Psi(-\alpha))}\right.\\ &\quad\left.\times\left(1-\delta^T e^{T(2\Psi(-\alpha)-\Psi(-\alpha-1))}\right)\right\}, \end{aligned} \tag{4.4}$$

$$\begin{aligned}\mathcal{D}_l(t,T:,m) &= \delta^T e^{T\Psi(-\alpha)}\left\{1+\frac{m}{b_0}\frac{\alpha}{2}e^{(t+T)(\Psi(-\alpha-1)-\Psi(-\alpha))}\right.\\ &\quad\left.\times\left(1-\delta^T e^{T(2\Psi(-\alpha)-\Psi(-\alpha-1))}\right)\right\}. \end{aligned} \tag{4.5}$$

Thus, we observe that the money discount factor for gains increases in the current consumption level b_0, and the money discount factor for losses decreases in b_0. Hence, rich people discount gains less and losses more than poor people. In particular, fairly rich agents do not exhibit the negative discounting in this model.

Indeed, the negative discounting effect may be observed for small T, if $r-\Psi(-\alpha)$ is very close to 0 so that

$$-r+\Psi(-\alpha)+\frac{m}{b_0}\frac{\alpha}{2}e^{t(\Psi(-\alpha-1)-\Psi(-\alpha))}(r+\Psi(-\alpha-1)-2\Psi(-\alpha))>0. \tag{4.6}$$

For t in a finite interval $[0,\bar{t}]$, there exists b^* such that if $b_0>b^*$, then the second term in (4.6) is less than $r-\Psi(-\alpha)$. Notice that for very large t, the probability that condition $m' << b_t$ will be violated is large, hence, for large t, the quadratic approximation we started with and resulting approximate formulas are too inaccurate and should not be applied.

It remains to analyze the normality condition (4.3). Assume that the agent perceives the utility of consumption as being the same on average, i.e. as a martingale, then for any t,

$$E[u(b_t)]=u(b_0)\ \Leftrightarrow\ E\left[e^{(1-\alpha)X_t}\right]=b_0^{1-\alpha}\ \Leftrightarrow\ b_0^{1-\alpha}e^{t\Psi(1-\alpha)}=b_0^{1-\alpha}\ \Leftrightarrow\ \Psi(1-\alpha)=0.$$

Since $\Psi(0)=0$, and $\Psi(\cdot)$ is a convex function, condition $\Psi(1-\alpha)=0$ implies $\Psi(-\alpha-1)>\Psi(-\alpha)$. Notice, however, that (4.3) holds under much weaker conditions on the process. Straightforward calculations show that (4.3) is equivalent to $\mu\le\sigma^2(1+\alpha)$. One should expect that this condition holds. For instance, for stock and indices on stocks, typically, $\mu<\sigma^2$.

4.2. **Ornstein-Uhlenbeck model.** We assume that X_t is given by the stochastic differential equation

$$dX_t=\kappa(\theta-X_t)dt+\sigma dW_t, \tag{4.7}$$

where $\kappa>0$, $\theta>0$, and dW_t is the increment of the standard BM with zero mean and unit variance. The procedures for the calculations of the expectation $E[e^{\gamma X_t}]$ and resulting formula are well-known (see, e.g., [2], p. 522–523)

$$E[e^{\gamma X_t}\mid X_0=x]=\exp\left[\gamma e^{-\kappa t}x+\frac{\sigma^2\gamma^2}{4\kappa}(1-e^{-2\kappa t})+\theta\gamma(1-e^{-\kappa t})\right], \tag{4.8}$$

hence

$$\Xi(x,t,\gamma)=\gamma e^{-\kappa t}x+\frac{\sigma^2\gamma^2}{4\kappa}(1-e^{-2\kappa t})+\theta\gamma(1-e^{-\kappa t}). \tag{4.9}$$

To satisfy condition $P(t,t+T)<1$, we must have

$$Tr-(\Xi(x,t+T,-\alpha)-\Xi(x,t,-\alpha))>0,$$

equivalently,

$$r+(\theta-x)\alpha e^{-\kappa t}\frac{1-e^{-\kappa T}}{T}-\frac{\sigma^2\alpha^2}{4k}e^{-2\kappa t}\frac{1-e^{-2\kappa T}}{T}>0. \tag{4.10}$$

The LHS in (4.10) converges to $r>0$ when $T\to\infty$, hence, (4.10) is satisfied for T sufficiently large. If T is small, then, to avoid negative discounting, the effective discount rate must be positive. The effective discount rate in the OU model is

$$\rho(\alpha, x_0; t) = r - A_t(t; -\alpha)x_0 - B_t(t; -\alpha), \tag{4.11}$$

where

$$\begin{aligned} A_t(t; -\alpha) &= \alpha\kappa e^{-\kappa t}, \\ B_t(t, -\alpha) &= \frac{\sigma^2\alpha^2}{2} e^{-2\kappa t} - \alpha\theta\kappa e^{-\kappa t} \end{aligned}$$

Substituting into (4.11), we obtain

$$\rho(\alpha, x_0; t) = r + (\theta - x_0)\alpha\kappa e^{-\kappa t} - \frac{\sigma^2\alpha^2}{2} e^{-2\kappa t}, \tag{4.12}$$

and, differentiating,

$$\rho_t(\alpha, x_0; t) = \alpha\kappa e^{-2\kappa t}(-(\theta - x_0)\kappa e^{\kappa t} + \sigma^2\alpha).$$

We can state the following

Theorem 1. *(i) if $x_0 \ge \theta$, then, for $t > 0$, $\rho_t(\alpha, x_0; t) > 0$, and the effective discount rate increases on $[0, +\infty)$;*

(ii) if $x_0 \le \theta - \sigma^2\alpha/\kappa$, then, for $t > 0$, $\rho_t(\alpha, x_0; t) < 0$, and the effective discount rate decreases on $[0, +\infty)$: the hyperbolic discounting is observed;

(iii) if $\theta - \sigma^2\alpha/\kappa < x_0 < \theta$, then, for $0 < t < t^* = -(1/\kappa)\ln[(\theta - x_0)\kappa/(\sigma^2\alpha)]$, $\rho_t(\alpha, x_0; t) > 0$, hence, the effective discount rate increases on $[0, t^*]$; the hyperbolic discounting is observed on $[t^*, +\infty)$.

Corollary 2. *The effective discount rate $\rho(\alpha, x_0; t) > 0$ iff*

$$r + (\theta - x_0)\alpha\kappa e^{-\kappa t} - \frac{\sigma^2\alpha^2}{2} e^{-2\kappa t} > 0. \tag{4.13}$$

If $x_0 < \theta - \sigma^2\alpha e^{-kt}/(2\kappa)$, then the idiosyncratic discount rate is positive, and (4.13) holds even if the standard discount rate $r = 0$. In particular, the idiosyncratic discount rate is positive if the discounting is hyperbolic for all t (case (ii) in Theorem 3). If $x_0 \ge \theta - \sigma^2\alpha e^{-kt}/(2\kappa)$ then the standard discount rate must be positive for (4.13) to hold.

Recall that if (4.1) is satisfied, the gain-loss asymmetry is observed. Straightforward calculations show that for the OU process, (4.1) is equivalent to

$$\frac{\sigma^2(\alpha + 1)}{2\kappa} e^{-\kappa t}\left(1 - e^{-2\kappa T}\right) > (\theta - x_0)\left(1 - e^{-\kappa T}\right). \tag{4.14}$$

So, if $x_0 > \theta$, the gain-loss asymmetry is always observed. For small T, (4.14) becomes

$$\frac{\sigma^2(\alpha + 1)}{\kappa} e^{-\kappa t} > \theta - x_0.$$

In particular, poor agents who always exhibit hyperbolic discounting must not be too poor to discount gains more than losses, i.e., the following inequalities must hold:

$$\theta - \sigma^2(\alpha + 1)/\kappa < x_0 \leq \theta - \sigma^2\alpha/\kappa.$$

5. Random date environmental catastrophe and willingness to pay

Individual discounting in the uncertain world is by all means interesting and important. Social discounting has become vitally important due to the necessity of cost-benefits analysis of environmental policies (see, for example, discussion in [4], [22], [33], [36]–[38] and references therein). The Stern Review [33] uses an artificially low discount rate to justify rationality to suffer the cost of efforts to mitigate the consequences of climate change now in order not to suffer much greater losses later. According to other studies, immediate adoption of a stringent abatement policy is unjustified (see, for example, [22], [35], and [21], and references therein). Gollier [12] claims that the rate at which environmental impacts should be discounted is in general different from the one at which monetary benefits should be discounted. He estimates that changes in the environment should be discounted at 1.5%, whereas changes in consumption should be discounted at 3.2%.

Optimal design of a GHG abatement policy is extremely complicated due to high degree of uncertainty concerning environmental damage, efficiency of GHG reduction technologies, availability of new technologies in long time horizons. In particular, Weitzman [36], [37] argues that we will never have enough information about the right-hand tail of the probability distribution for temperature changes because the distributions are fat-tailed. Weitzman's analysis has unpleasant welfare implications: if the social welfare function is CRRA, the society should be willing to sacrifice all current consumption to avoid the global warming in the future. Pindyck [27] calibrates a thin-tailed displaced gamma distribution for temperature change using the data from the IPCC studies and estimates the fraction of consumption that the society would be willing to give up now and throughout the future in order to ensure that the temperature increase at a specific horizon is limited to a certain level. His results advocate slow beginning of abatement policies. Dismal results in [36] and [37] in are due to the fact that in case of the CRRA utility function, the marginal utility grows unboundedly if the consumption becomes vanishingly small. Pindyck [28] imposes a bound on the marginal utility so that even if the temperature distribution is fat tailed, the expected marginal utility remains finite. Furthermore, [28] shows that depending on the bound and other parameters of the model, thin-tailed distributions may generate higher willingness to pay than fat-tailed distributions.

Evaluation of the willingness to pay (WTP) is useful for a policy analysis. The goal of our exercise below (built, to some extent, on [27]) is to study how the willingness to pay depends on the society's wealth, growth opportunities, beliefs about effectiveness of abatement policies, and time horizon over which catastrophic environmental events may happen.

Assume that there are two sources of uncertainty: economic uncertainty driven by productivity shocks and (therefore) stochastic consumption and environmental uncertainty. Studies [36], [37],[27], and [28] concentrate on uncertainty over temperature change caused by GHG emissions. We believe that there is the lack of knowledge not only about the right tail of the distribution for temperature change, but also about the critical level of GHG emissions that may trigger catastrophic losses of the welfare. Neither do we know when this critical level may be reached. Therefore, in our stylized model, we associate environmental uncertainty with an unknown date environmental catastrophe.

Suppose that the country's aggregate consumption follows a geometric process: $b_t = e^{X_t}$, where a random variable X_t is specified by the moment-generating function

$$\mathrm{MGF}(x,t,\gamma) = E\left[e^{\gamma X_t}\,\middle|\, X_0 = x\right].$$

Let

$$\Xi(x,t,\gamma) = \ln \mathrm{MGF}(x,t,\gamma).$$

Assume that the social utility function is a CRRA function

$$u(b_t) = \frac{b_t^{1-\alpha}-1}{1-\alpha},\ \alpha \in (0,+\infty).$$

Parameter α represents the relative risk aversion.

Suppose that T is a random variable independent of the process X. $T \sim \exp(\lambda)$ is the random time of a catastrophe. If the catastrophe happens, the country will experience a loss of a fraction, $\epsilon_m \in (0,1)$, of the aggregate consumption so that only a fraction $1-\epsilon_m$ of it will remain from date T onwards. This assumption accords with [27]. Pindyck [27] argues that some of the effects of global warming may be permanent; in addition certain resources will have to be allocated to reduce the impact of dramatic environmental changes.

Assume that the size of the loss can be reduced if the stream of investment in new technologies, etc. is made now. Due to the stream of investment costs, the country will lose a fraction $\epsilon_M \in (0,\epsilon_m)$ of the aggregate consumption, so that the country will have the fraction $1-\epsilon_M$ of the aggregate consumption from now on until T. Let the loss reduction be $(\epsilon_m - \epsilon_M)(1-e^{-aT})$, where $a > 0$, which means that the loss reduction depends on how far the date of the catastrophe is. After the catastrophe, the remaining fraction of aggregate consumption will be $1-\epsilon_m + (\epsilon_m - \epsilon_M)(1-e^{-aT})) = 1-\epsilon_M + (\epsilon_M - \epsilon_m)e^{-aT}$. Parameter a characterizes effectiveness of abatement policies. If a is large, then it is possible almost to eliminate the loss incurred during the catastrophic event. If a is small, the loss reduction is negligible. One can also view $1/a$ as the average time it takes abatement process to produce desirable effects.

Let x be the current realization of the underlying process X. For any given T, the present value of the country without any policy change (business as usual (BAS) scenario), the present value of the country is

$$v_0(x,T) = \int_0^T e^{-rt} E^x[u(b_t)]dt + \int_T^\infty e^{-rt} E^x[u((1-\epsilon_m)b_t)]dt.$$

If the loss reducing investment takes place, then the present value of the country for each T is:

$$v(x,T) = \int_0^T e^{-rt} E^x[u((1-\epsilon_M)b_t)]dt + \int_T^\infty e^{-rt} E^x \left[u\left((1-\epsilon_M + (\epsilon_M - \epsilon_m)e^{-aT})b_t\right)\right] dt.$$

Taking expectation w.r.t. T, we write the present value of the country as

$$V_0(x) = \int_0^\infty v_0(x,T) e^{-\lambda T} dT,$$

if no action is taken to limit environmental damages, and

$$V(x) = \int_0^\infty v(x,T) e^{-\lambda T} dT,$$

if investment in loss reducing technologies takes place.

We define the country's willingness to pay as the maximum fractional loss of the aggregate consumption the society would agree to pay now for loss reduction in the future. By definition, this is the value ϵ_M which equates $V_0(x)$ and $V(x)$. In order to find this ϵ_M, we will linearize each of the functions $v_0(x,T)$ and $v(x,T)$.

$$v_0(x,T) = \int_0^\infty e^{-rt} E^x[u(b_t)]dt - \epsilon_m \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt \tag{5.1}$$

$$\begin{aligned} v(x,T) &= \int_0^\infty e^{-rt} E^x[u(b_t)]dt - \epsilon_M \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\ &\quad + (\epsilon_M - \epsilon_m) e^{-aT} \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt \end{aligned} \tag{5.2}$$

In the Appendix, we show that $V_0(x) = V(x)$ iff

$$\frac{\epsilon_M}{\epsilon_m} = 1 - \frac{(\lambda + a)\int_0^\infty e^{-(r+\lambda)t} E^x[u'(b_t)b_t]dt}{\int_0^\infty \left(a e^{-rt} + \lambda e^{-(r+\lambda+a)t}\right) E^x[u'(b_t)b_t]dt}. \tag{5.3}$$

For the CRRA utility function, the last equation becomes

$$\frac{\epsilon_M}{\epsilon_m} = 1 - \frac{(\lambda + a)\int_0^\infty e^{-(r+\lambda)t} E^x\left[e^{(1-\alpha)X_t}\right] dt}{\int_0^\infty \left(a e^{-rt} + \lambda e^{-(r+\lambda+a)t}\right) E^x\left[e^{(1-\alpha)X_t}\right] dt}.$$

We proceed using the cumulant-generating function of X_t:

$$\frac{\epsilon_M}{\epsilon_m} = 1 - \frac{(\lambda + a)\int_0^\infty e^{-(r+\lambda)t+\Xi(x,t,1-\alpha)}dt}{a\int_0^\infty e^{-rt+\Xi(x,t,1-\alpha)}dt + \lambda\int_0^\infty e^{-(r+\lambda+a)t+\Xi(x,t,1-\alpha)}dt}. \tag{5.4}$$

Notice that $r-\frac{1}{t}\Xi(x,t,1-\alpha)$ is the social discount rate under scenario when no catastrophic event ever happens; $r+\lambda-\frac{1}{t}\Xi(x,t,1-\alpha)$ is the social discount rate under scenario when a catastrophic event is possible, but no abatement policy is in place; $r+\lambda-\frac{1}{t}\Xi(x,t,1-\alpha)$ is the social discount rate when a catastrophic event is possible and abatement policy is in effect. Observe that a sufficient condition for all integrals in (5.4) to be finite is that $r-\frac{1}{t}\Xi(x,t,1-\alpha)$ is positive and bounded away from zero.

If the underlying uncertainty is Gaussian, then

$$\Xi(x,t,1-\alpha) = (1-\alpha)x + t\Psi(1-\alpha),$$

and the willingness to pay is independent of the current wealth of the country. Indeed, in this case,

$$\frac{\epsilon_M}{\epsilon_m} = \frac{\lambda}{r+\lambda-\Psi(1-\alpha)} \cdot \frac{a}{r+a-\Psi(1-\alpha)} \tag{5.5}$$

(see the Appendix for derivation). Now the social discount rate under scenario when no catastrophic event ever happens is $r-\Psi(1-\alpha)$; we assume that it is positive to ensure finiteness of all integrals in (5.4). Then WTP is an increasing function in the growth rate of the expected utility of consumption, $\Psi(1-\alpha)$. Notice that if the growth rate of the expected utility of consumption is negative (which seems to be a widespread perception of the population in the developed countries) then the social discount rate is positive even if the rate of pure time preference $r = 0$. On the contrary, if $\Psi(1-\alpha) > 0$ (say, BRIC), then the rate of pure time preference must be positive and sufficiently large for the social discount rate, $r-\Psi(1-\alpha)$, to be positive.

We will study the relationship between the willingness to pay and the country's current wealth using the OU model. Then

$$\Xi(x,t,1-\alpha) = (1-\alpha)e^{-\kappa t}x + \frac{\sigma^2(1-\alpha)^2}{4\kappa}(1-e^{-2\kappa t}) + \theta(1-\alpha)(1-e^{-\kappa t}).$$

We need to calculate the integrals of the form

$$I(\gamma) := I(r,a,\lambda,\alpha,\sigma,\kappa,\theta;x\,\gamma) := \int_0^\infty e^{-\gamma t+A+By(t)-Cy(t)^2}dt,$$

where $\gamma(\in\{r, r+\lambda, r+\lambda+a\}) > 0$, $y(t) = e^{-\kappa t}$,

$$C = \sigma^2(1-\alpha)^2/(4\kappa), \quad B = (1-\alpha)(x-\theta), \quad A = C + \theta(1-\alpha).$$

We change the variable $y = e^{-\kappa t}$ and integrate by parts:

$$\begin{aligned} I(\gamma) &= -\frac{1}{\kappa}\int_0^\infty e^{(\kappa-\gamma)t+A+By(t)-Cy(t)^2} de^{-\kappa t} \\ &= \frac{e^A}{\kappa}\int_0^1 y^{\gamma/\kappa-1}e^{By-Cy^2}dy \\ &= \frac{e^A}{\gamma}\left[e^{B-C} - \int_0^1 y^{\gamma/\kappa}(B-2Cy)e^{By-Cy^2}dy\right] \end{aligned}$$

The integral above can be easily calculated using, say, trapezoid rule. If $\gamma/\kappa < 2$, then, to improve convergence, the exponential can be expanded into the Taylor series, the first one or two terms subtracted, and the corresponding integrals evaluated explicitly. The remaining integrand will be of class $C^s[0,1]$, where $s > 2$, and the error bound of the trapezoid rule justified. However, since we do not aim at high accuracy, we will apply the trapezoid rule to the integral above directly.

5.1. **Numerical example.** We match the parameters of the OU process to similar parameters used in [13]. Namely, we set the central tendency $\theta = 0.018$, the coefficient of mean reversion $\kappa = 0.7$, and the volatility $\sigma = 0.036$. We use the same coefficient of relative risk aversion, $\alpha = 2$, as in [13] as well. To study dependence of WTP on the current (log) wealth, we set $\lambda = 0.01$, and $a = 0.02$, which means that the expected time of the catastrophe in 100 years from now[3], and abatement policies may be expected to produce desirable effect in 50 years from now. In Fig.1, we plot ϵ_M/ϵ_m as functions of $x = \log b_0$ for $r = 0.015, 0.03, 0.045$ (left panel) and the sensitivity of WTP with respect to the current (log) wealth. We see that WTP as measured by this ratio increases in the current (log) wealth and decreases in r. The sensitivity decreases in x and r, hence the countries whose current wealth level is below the central tendency are more likely to increase their WTP if extra wealth is added. Therefore if the developed countries keep their promise to deliver "fast start" funding to the poor countries the world can be expected to make progress on practical measures for tackling global warming. However the sums so far committed are much smaller than the initial pledge which is easy to understand because the financial crisis made many people in the developed word to feel much poorer and decrease their WTP.

In Fig. 2, we plot the dependence of WTP (left panel) and sensitivity (right panel) on $1/a$, the expected time until the abatement policy will produce an effect for the country of average current wealth. We see that WTP decreases with $1/a$ and r. Thus, if the society anticipates that the abatement policy will produce an

[3]Pindyck [27] fits his model to the IPCC data so that the global temperature increases by 5°C in 100 years.

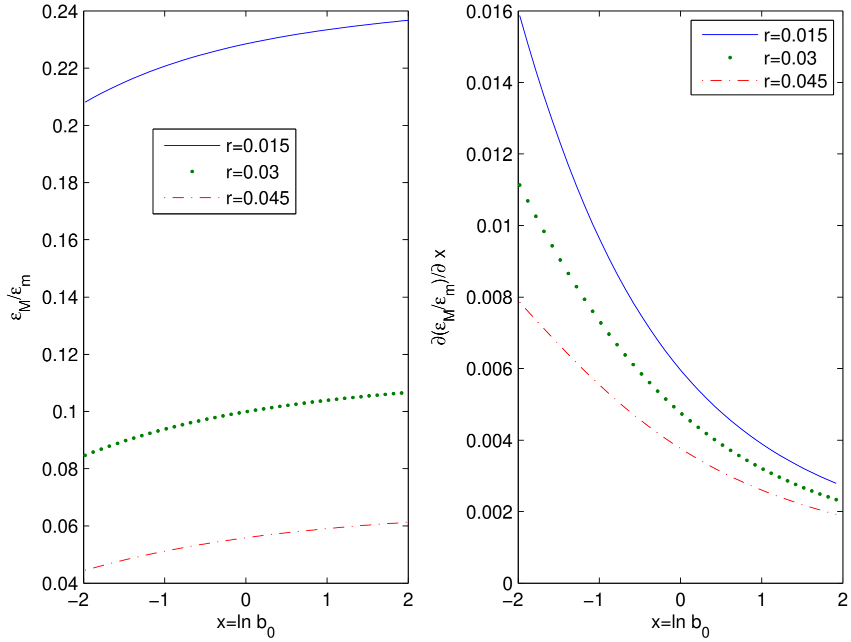

Figure 1. WTP (left panel) and sensitivity (right panel) as functions of x and r. Parameters: $\alpha = 2, \theta = 0.018, \sigma = 0.036, \kappa = 0.7, \lambda = 0.01, a = 0.02$.

effect after a longer time interval, they will be less willing to suffer costs. We also see that the absolute value of sensitivity to the technical improvements (measured by the expected time until abatement policy produces effect) decreases in r and $1/a$, the latter meaning that further improvements in efficiency of abatement technologies influence significantly the propensity to suffer costs. Finally, in Fig 3., we plot the joint dependence on λ and $1/a$. We see that the willingness to pay increases in λ and decreases in $1/a$, and the absolute value of sensitivity w.r.t. $1/a$ increases in λ and decreases in $1/a$. The main message of this numerical example is that one of the most important factors determining WTP is the society's beliefs about efficiency of the abatement policy. Small WTP is consistent with beliefs that it will take quite a long time for the abatement policy to produce the desirable effect on the environment. Notice also that even the highest WTP in the Fig.1-3, is still a small fraction of the aggregate consumption, because according to [6], damage estimates from several IAMs, yield a range of 0.5% to 2% of lost GDP for the change in global temperature, $\Delta T = 3^{\mathrm{o}}$C, and 1% to 8% of lost GDP for $\Delta T = 5^{\mathrm{o}}$C. Even if the loss of GDP amounts to 8%, the highest ratio ϵ_M/ϵ_m in our example is about 0.25, which corresponds to WTP of 2% of the GDP, which agrees with estimates in [27].

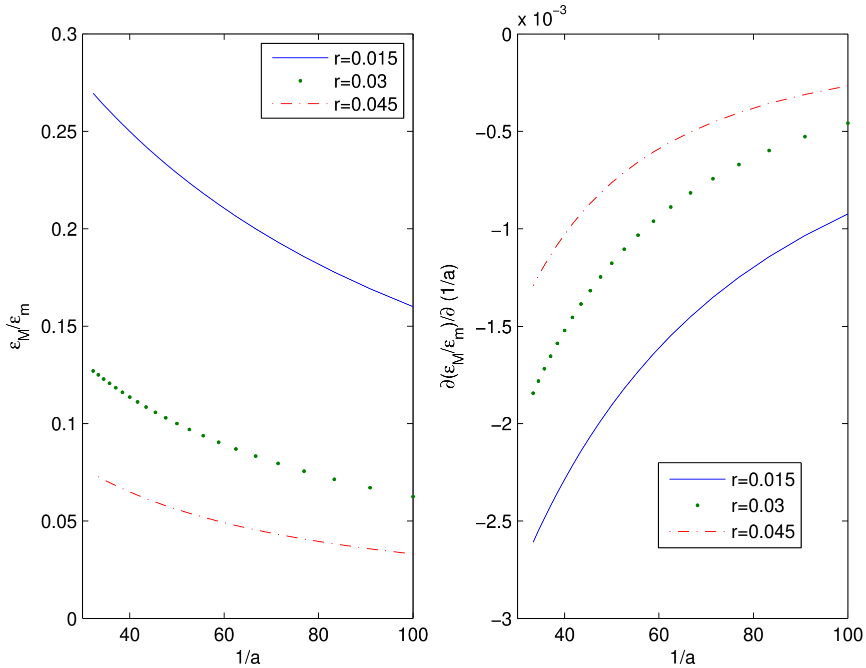

FIGURE 2. WTP (left panel) and sensitivity (right panel) as functions of $1/a$ and r. Parameters: $\alpha = 2, \theta = 0.018, \sigma = 0.036, \kappa = 0.7, \lambda = 0.01, x = 0.018$.

6. Conclusion

This paper shows that discounted utility anomalies can be explained within the standard discounted utility framework if there is uncertainty about, for example, the agent's income. We introduced the notion of the term structure of absolute risk aversion and demonstrated that, for a general utility function satisfying usual conditions, the gain-loss asymmetry and delay-speedup asymmetry for small gains and losses follow from a natural assumption that the term structure is normal, that is, non-decreasing. The gain-loss asymmetry can be observed even if the discounting is exponential. In order to observe non-exponential discounting, i.e., to have a time-dependent effective discount rate, the cumulant-generating function of the underlying stochastic variable must be a non-linear function of time. In particular, the hyperbolic discounting takes place if the cumulant-generating function is a convex function in time.

We used as model examples of stochastic income with non-linear cumulant-generating function the CIR and OU mean-reverting models. For these models, the shape of the effective discount rates depends on the current income. If the agent is rich (the current income b_0 is higher than the central tendency θ), then

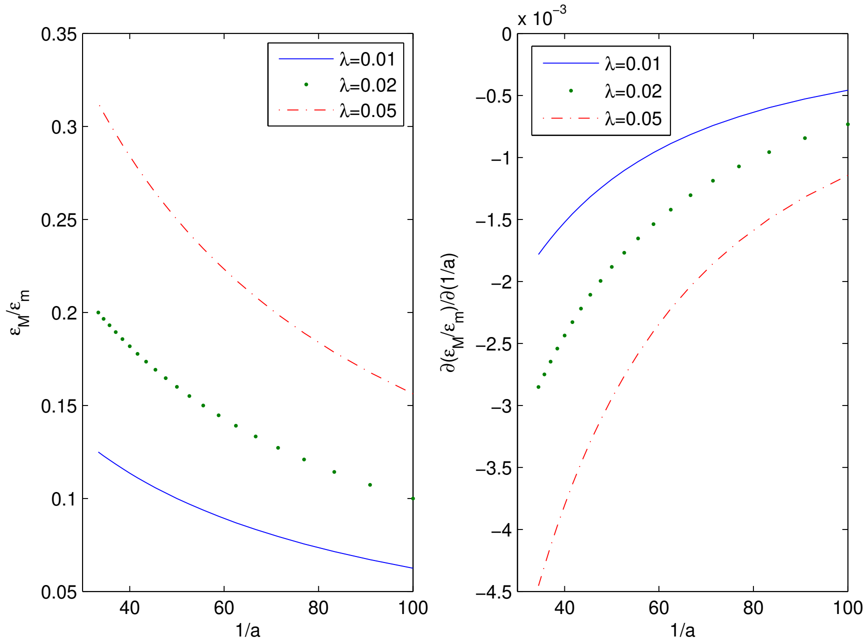

FIGURE 3. WTP (left panel) and sensitivity (right panel) as functions of $1/a$ and λ. Parameters: $\alpha = 2, \theta = 0.018, \sigma = 0.036, \kappa = 0.7, r = 0.01, x = 0.018$.

the effective discount rate increases in time (as the borrowing rate for a sound corporation), and no hyperbolic discounting is observed. If the agent is poor so that his current income is less than the long-run average by a certain non-zero margin (which depends on the risk attitude, type of uncertainty and the parameters of the income process): $b_0 \le \theta_1 < \theta$, then the effective discount rate decreases with time, and the hyperbolic discounting is observed. Finally, if the agent is neither rich nor too poor: $\theta_1 < b_0 < \theta$, then there exists $t^* > 0$ such that the hyperbolic discounting is observed over the interval $[t^*, +\infty)$ but is not observed on $[0, t^*]$; as $b_0 \downarrow \theta_1$, $t^* \to +0$ (the poorer the agent becomes, the higher is the probability that the hyperbolic discounting will be observed in an experiment).

As an application, we calculated WTP for reduction of a future loss caused by an environmental catastrophe in a model driven by economic and environmental uncertainty. The model shows that wealthier countries are more willing to pay for abatement policies, but WTP in poor countries is more sensitive to changes in wealth. Another important factor which determines the WTP is the society's beliefs concerning efficiency of abatement policies.

Our approach may potentially have other interesting applications such as, for example, contingent valuation of environmental goods. The contingent valuation method involves the use of sample surveys to elicit the willingness of respondents to pay for environmental programs or projects. For the history of the contingent valuation method and contingent valuation debate see [29], and [17]. According to [29], one of the most influential papers in natural resource and environmental economics was "Conservation Reconsidered" by [19]. That paper suggested that the difference between willingness-to-pay and willingness-to-accept compensation for "grand scenic wonders" may be large indeed. [16] presented a deterministic model that demonstrates that the differences in the willingness-to-pay and willingness-to-accept are due to the lack of substitutes for a public good. According to our results, compensation for losses requested by individuals is higher than the price the same individuals agree to pay for gains due to the presence of uncertainty. Thus, when facing a question of the sort "How much should the government pay for the damage to an endangered species", the same individual will name a greater price than when asked a question of the sort "How much should the government pay to preserve an endangered species." Long-lived environmental problems such as global warming, and native-exotic species protection are other potential applications of our results. In the context of the global warming, our model explains why it is natural that the poor countries do not want to commit themselves to costly emission reductions.

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Appendix A. Appendix

A.1. **Discount factors for losses.** As before, we want to derive the relation between T, m, m' and b_t, which makes the agent indifferent between the dated losses (m', t) and $(m, t+T)$. Using the Taylor expansion of order 2, we obtain the following approximation

$$L(T;m;m';b_t) \;=\; \delta^T\left\{-mE[u'(b_{t+T})]+\frac{m^2}{2}E[u''(b_{t+T})]\right\} +m'E[u'(b_t)]-\frac{(m')^2}{2}E[u''(b_t)], \tag{A.1}$$

whence $L(T;m;m';b_t)\ge 0$ iff

$$m'+(m')^2\frac{E[-u''(b_t)]}{2E[u'(b_t)]}\ge P(t,t+T)m+m^2\delta^T\frac{E[-u''(b_{t+T})]}{2E[u'(b_t)]}.$$

Letting $t=0$, and using the linear terms only, we derive that $L(T,m;m';b_0)\ge 0$ iff

$$m'\ge P(0,T)m=\delta^T\frac{E[u'(b_T)]}{u'(b_0)}m;$$

equivalently,

$$u'(b_0)\ge\frac{\delta^T E[u'(b_T)]m}{m'}.$$

By concavity of u, the last inequality holds iff $b_0\le K_l$, where

$$K_l=\phi\left(\frac{\delta^T E[u'(b_T)]m}{m'}\right),$$

and $\phi=(u')^{-1}$. We conclude that $(m',0)\succeq(m,T)$ if and only if $b_0\ge K_l$, i.e, relatively rich agents prefer to expedite the loss.

We want to find m'_l such that $L(T;m;m'_l;b_t)=0$. It follows from (A.1) that m'_l is a solution of the quadratic equation

$$-\delta^T\left\{mE[u'(b_{t+T})]+\frac{m^2}{2}E[-u''(b_{t+T})]\right\}+ +m'E[u'(b_t)]+\frac{(m')^2}{2}E[-u''(b_t)]=0. \tag{A.2}$$

The change of sign $m,m'\mapsto -m,-m'$ turns (3.3) into (A.2), therefore, (3.7) becomes

$$m'_l=mP(t,t+T)\left[1+\frac{m}{2}\left(-P(t,t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]}+\frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right]. \tag{A.3}$$

Assuming that (3.8) holds and $P(t,t+T)<1$, we obtain that the coefficient at m inside the square brackets is positive.

It follows from (A.3) that, modulo $o(m^2)$ term, the money discount factor $\mathcal{D}_l(t,T;m)=m'_l/m$ for losses is

$$\mathcal{D}_l(t,T;m)=P(t,t+T)\left[1-\frac{m}{2}\left(P(t,t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]}-\frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right],$$

which is eqrefdisclcrra.

A.2. **Proof of** (5.3)**.** Using the approximations (5.1) and (5.2) above, we derive

$$
\begin{aligned}
V_0(x) &= \int_0^\infty e^{-\lambda T} dT \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\
&\quad -\epsilon_m \int_0^\infty e^{-\lambda T} dT \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt; \\
V(x) &= \int_0^\infty e^{-\lambda T} dT \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\
&\quad -\epsilon_M \int_0^\infty e^{-\lambda T} dT \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\
&\quad +(\epsilon_M - \epsilon_m) \int_0^\infty e^{-(\lambda+a)T} dT \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt.
\end{aligned}
$$

Evidently, $V(x) = V_0(x)$ iff

$$
\begin{aligned}
&\epsilon_m \int_0^\infty e^{-\lambda T} dT \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\
&= \epsilon_M \int_0^\infty e^{-\lambda T} dT \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \\
&-(\epsilon_M - \epsilon_m) \int_0^\infty e^{-(\lambda+a)T} dT \int_T^\infty e^{-rt} E^x[u'(b_t)b_t]dt.
\end{aligned}
$$

Changing the order of integration, we arrive at

$$
\begin{aligned}
&\epsilon_m \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \int_0^t e^{-\lambda T} dT \\
&= \epsilon_M \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \int_0^\infty e^{-\lambda T} dT \\
&-(\epsilon_M - \epsilon_m) \int_0^\infty e^{-rt} E^x[u'(b_t)b_t]dt \int_0^t e^{-(\lambda+a)T} dT.
\end{aligned}
$$

Integration over T gives

$$
\begin{aligned}
&\epsilon_M \frac{1}{\lambda} \int_0^\infty e^{-(r+\lambda)t} E^x[u'(b_t)b_t]dt \\
&+\epsilon_M \int_0^\infty e^{-rt} \left(\frac{1-e^{-\lambda t}}{\lambda} - \frac{1-e^{-(\lambda+a)t}}{\lambda+a} \right) E^x[u'(b_t)b_t]dt \\
&= \epsilon_m \int_0^\infty e^{-rt} \left(\frac{1-e^{-\lambda t}}{\lambda} - \frac{1-e^{-(\lambda+a)t}}{\lambda+a} \right) E^x[u'(b_t)b_t]dt,
\end{aligned}
$$

equivalently

$$\begin{aligned}
&\epsilon_M(\lambda+a)\int_0^\infty e^{-(r+\lambda)t}E^x[u'(b_t)b_t]dt+\\
&+\epsilon_M\int_0^\infty \left(ae^{-rt}-(\lambda+a)e^{-(r+\lambda)t}+\lambda e^{-(r+\lambda+a)t}\right)E^x[u'(b_t)b_t]dt=\\
&=\epsilon_m\int_0^\infty \left(ae^{-rt}-(\lambda+a)e^{-(r+\lambda)t}+\lambda e^{-(r+\lambda+a)t}\right)E^x[u'(b_t)b_t]dt.
\end{aligned}$$

Simplifying further, we obtain

$$\begin{aligned}
&\epsilon_M\int_0^\infty \left(ae^{-rt}+\lambda e^{-(r+\lambda+a)t}\right)E^x[u'(b_t)b_t]dt=\\
&=\epsilon_m\int_0^\infty \left(ae^{-rt}-(\lambda+a)e^{-(r+\lambda)t}+\lambda e^{-(r+\lambda+a)t}\right)E^x[u'(b_t)b_t]dt,
\end{aligned}$$

whence (5.3) follows.

A.3. **Proof of** (5.5).

$$\begin{aligned}
\frac{\epsilon_M}{\epsilon_m} &= 1-\frac{(\lambda+a)\int_0^\infty e^{-(r+\lambda-\Psi(1-\alpha))t}dt}{a\int_0^\infty e^{-(r-\Psi(1-\alpha))t}dt+\lambda\int_0^\infty e^{-(r+\lambda+a-\Psi(1-\alpha))t}dt}\\
&= 1-\frac{\frac{\lambda+a}{r+\lambda-\Psi(1-\alpha)}}{\frac{a}{r-\Psi(1-\alpha)}-\frac{\lambda}{r+\lambda+a-\Psi(1-\alpha)}}\\
&= 1-\frac{(\lambda+a)(r-\Psi(1-\alpha))(r+\lambda+a-\Psi(1-\alpha))}{(r+\lambda-\Psi(1-\alpha))\left(a(r+\lambda+a-\Psi(1-\alpha))-\lambda(r-\Psi(1-\alpha))\right)}\\
&= 1-\frac{(\lambda+a)(r-\Psi(1-\alpha))(r+\lambda+a-\Psi(1-\alpha))}{(\lambda+a)(r+\lambda-\Psi(1-\alpha))(r+a-\Psi(1-\alpha))}\\
&= 1-\frac{(r-\Psi(1-\alpha))(r+\lambda+a-\Psi(1-\alpha))}{(r+\lambda-\Psi(1-\alpha))(r+a-\Psi(1-\alpha))}\\
&= \frac{(r+\lambda-\Psi(1-\alpha))(r+a-\Psi(1-\alpha))-(r-\Psi(1-\alpha))(r+\lambda+a-\Psi(1-\alpha))}{(r+\lambda-\Psi(1-\alpha))(r+a-\Psi(1-\alpha))}\\
&= \frac{\lambda\left((r+a-\Psi(1-\alpha))-(r-\Psi(1-\alpha))\right)}{(r+\lambda-\Psi(1-\alpha))(r+a-\Psi(1-\alpha))}\\
&= \frac{\lambda}{r+\lambda-\Psi(1-\alpha)}\cdot\frac{a}{r+a-\Psi(1-\alpha)}.
\end{aligned}$$