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OPTIMAL STOPPING IN LÉVY MODELS, FOR NON-MONOTONE DISCONTINUOUS PAYOFFS *

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Abstract. We give short proofs of general theorems about optimal entry and exit problems in Lévy models, when payoff streams may have discontinuities and be non-monotone. As applications, we consider exit and entry problems in the theory of real options, and an entry problem with an embedded option to exit.

Key words. optimal stopping, Lévy processes, non-monotone discontinuous payoffs

AMS subject classifications. 60G40, 60G51, 91B25, 91G20

1. Introduction. The paper presents a general approach for valuation of and optimal exercise strategies for contingent claims of American type. Such problems arise frequently in economics (real options) and finance (American options). See, e.g., [26, 12] for analysis of various situations and list of references.

In the majority of publications on optimal stopping problems, instantaneous payoff functions such as $(e^{X_t} - K)_+$, $(X_t - K)_+$, $(K - e^{X_t})_+$ and $(K - X_t)_+$ were considered; see, for example, [16] (random walks) and [7, 23, 2, 3, 1] (Lévy models). In [8, 9, 10, 11, 13, 12], we developed a general approach to optimal stopping problems based on the representation of an instantaneous payoff as the expected present value (EPV) of a payoff stream. Under fairly weak conditions on the payoff stream, we derived simple formulas for the optimal stopping time in the class of stopping times of the threshold type; in [14, 15], the method was generalized to Markov-modulated Lévy models. Later, the representation of the payoff as the EPV of a continuous stream was used in [32, 17], and the monotonicity condition was relaxed. In [24, 25, 21, 17], options with instantaneous payoffs $(X_t)_+^\nu$ are studied; this type of payoffs leads to non-trivial mathematical problems but it is rather artificial for applications in economics and finance. The last remark concerns the examples in [32, 17] as well. For general results on irreversible investment, see [28] and the bibliography therein. In diffusion context, fairly general non-trivial results related to partially reversible investment problems and problems with non-smooth streams are available (see, e.g., [20, 22, 34] and the bibliography therein) but the methodology of these papers is difficult to adjust to Lévy models.

In the present paper, we give short proofs of general stopping theorems for measurable payoff streams and discontinuous instantaneous payoffs and relax the monotonicity condition further. This allows us to consider options with more complicated payoff structure. The results have natural counterparts in random walk models and admit generalizations to regime-switching models and optimal stopping problems with finite time horizon. This can be done as in [12, 14, 15] for the case of monotone payoff functions.

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The general methodology of the paper can be applied to analyze more involved problems with strategic interactions and ambiguity, which were studied previously in pure diffusion models (see, e.g., [27, 18, 19, 4] and the bibliography therein.)

The rest of the paper is organized as follows. In Section 2, we introduce the necessary notation and recall general formulas for the EPV of streams which accrue (or will start to accrue), when a certain boundary is reached or crossed, in terms of the EPV operators under supremum and infimum processes. In Section 3, we prove a series of general theorems for irreversible exit problems, which have solutions of the threshold type, and, in Sections 4 and 5, we consider entry problems with payoff streams and instantaneous payoffs, respectively. In Section 6, we apply the general theorems derived in the main body of the paper to solve an investment problem with the embedded option to exit. The main difficulty stems from the fact that once the embedded problem of optimal exit is solved, the entry problem can be formulated as an entry problem with the non-monotone payoff stream. Section 7 concludes.

2. Notation and auxiliary results.

2.1. Lévy processes: main objects. A Lévy process $X = \{X_t\}$ on $\mathbb{R}$ is defined in terms of the generating triplet $(\sigma^2, b, F(dx))$, where σ^2 is the (instantaneous) variance of the Brownian Motion (BM) component, $F(dy)$ is the Lévy density (density of jumps), and $b \in \mathbb{R}$. The characteristic exponent $\psi(\xi)$ is definable from $\mathbb{E}\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)}$. An important general relation between ψ and L, the infinitesimal generator of X, is $Le^{ix\xi} = -\psi(\xi)e^{ix\xi}$.

If X is a BM with embedded compound Poisson jumps or, more generally, if the jump part is a finite variation process, then the Lévy-Khintchine formula for ψ can be written in the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_{\mathbb{R}\setminus 0} (1 - e^{i\xi y})F(dy) \tag{2.1}$$

(in the general case, the Lévy-Khintchine formula has an additional term, see, e.g. [31]), the infinitesimal generator of X acts on sufficiently regular functions as follows

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x) + \int_{\mathbb{R}\setminus 0} (u(x+y) - u(x))F(dy), \tag{2.2}$$

and b can be interpreted as the drift. As one of the simplest examples, the reader may have in mind the double-exponential jump-diffusion (DEJD) model with

$$F(dy) = c_+\lambda_+ e^{-\lambda_+ y}\mathbb{1}_{(0,+\infty)}(y)dy + c_-(-\lambda_-)e^{-\lambda_- y}\mathbb{1}_{(-\infty,0)}(y)dy, \tag{2.3}$$

where $\lambda_- < 0 < \lambda_+$ and $c_\pm \geq 0$; if $c_+ = 0$ (respectively, $c_- = 0$), then there are no positive (respectively, negative) jumps. Substituting (2.3) into (2.1), we find

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi - \frac{ic_+\xi}{\lambda_+ - i\xi} - \frac{ic_-\xi}{\lambda_- - i\xi}. \tag{2.4}$$

For Lévy processes with the jump part of infinite variation, the action of L and Lévy-Khintchine formula are more involved, and the interpretation of b is not so straightforward. See, e.g., [31].

Given any $q > 0$, we let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable, independent of X, with mean q^{-1}. The normalized resolvent of X is given by

$$(2.5) \qquad (\mathcal{E}_q f)(x) = \mathbb{E}[x + X_{T_q}] := \mathbb{E}\left[\int_0^{+\infty} q e^{-qt} f(x + X_t) dt\right].$$

We call $\mathcal{E}_q$ the expected present value operator (operator, which calculates the expected present value of a stream of payoffs) or EPV-operator. Note that

$$(2.6) \qquad \mathcal{E}_q = q(q - L)^{-1}$$

as operators in appropriate function spaces (see [10, 12, 11] for details).

2.2. Wiener-Hopf factorization and EPV operators. The *supremum process* $\overline{X}$ and the *infimum process* $\underline{X}$ of X are given by by $\overline{X}_t = \sup_{0\le s\le t} X_s$, $\underline{X}_t = \inf_{0\le s\le t} X_s$. Replacing in (2.5) X with $\overline{X}$ and $\underline{X}$, we define the EPV-operators $\mathcal{E}^\pm_q$ under supremum and infimum processes, respectively. It can be shown that $\mathcal{E}_q f$ and $\mathcal{E}^\pm_q f$ are well-defined for a nonnegative measurable (or an arbitrary bounded measurable) function f on $\mathbb{R}$. Being expectation operators, $\mathcal{E}_q$ and $\mathcal{E}^\pm_q$ are positive operators. For illustration, the reader may have in mind the following simple examples:

EXAMPLE 2.1. In the BM model, let $\beta^\pm = \beta^\pm_q$ be the positive and negative roots of the characteristic equation

$$(2.7) \qquad q - \Psi(\beta) = 0.$$

Then $\mathcal{E}^\pm_q = I^\pm_{\beta^\pm}$, where $I^\pm_{\beta^\pm}$ denotes the following convolution operators with the exponentially decaying kernels:

$$(2.8) \qquad I^+_{\beta^+} u(x) = \int_0^{+\infty} \beta^+ e^{-\beta^+ y} u(x+y) dy,$$

$$(2.9) \qquad I^-_{\beta^-} u(x) = \int_{-\infty}^{0} (-\beta^-) e^{-\beta^- y} u(x+y) dy.$$

EXAMPLE 2.2. If X is spectrally negative, that is, there are no positive jumps, then (2.7) has a unique positive root $\beta^+ = \beta^+_q$, and $\mathcal{E}^+_q = I^+_{\beta^+}$; if X is spectrally positive, that is, there are no negative jumps, then (2.7) has a unique negative root $\beta^- = \beta^-_q$, and $\mathcal{E}^-_q = I^-_{\beta^-}$.

EXAMPLE 2.3. In the case of DEJD model with the Lévy density (2.3), equation (2.7) has two negative and two positive roots $\beta^\pm_j = \beta^\pm_j(q), j = 1, 2$, and

$$(2.10) \qquad \mathcal{E}^\pm_q = \sum_{j=1,2} a^\pm_j(q) I^\pm_{\beta^\pm_j(q)},$$

where $a^\pm_j(q) > 0$ and $a^\pm_1(q) + a^\pm_2(q) = 1$ (see [11, 12]) For efficient realizations of the EPV operators for a general Lévy process, see [6].

In the proofs of optimal stopping results, we will systematically use the following properties of the EPV-operators.

LEMMA 2.4. *a) EPV-operators $\mathcal{E}_q$ and $\mathcal{E}^\pm_q$ are positive.*

b) If $u(x) = 0 \ \ \forall\, x \in (-\infty, h)$, then $\mathcal{E}^-_q u(x) = 0 \ \ \forall\, x \in (-\infty, h)$, and the same statement holds with $(-\infty, h]$ instead of $(-\infty, h)$.

c) If $u(x) = 0 \ \ \forall\, x \in (h, +\infty)$, then $\mathcal{E}^+_q u(x) = 0 \ \ \forall\, x \in (h, +\infty)$, and the same statement holds with $[h, +\infty)$ instead of $(h, +\infty)$.

d) Statements b) and c) hold for the inverses $(\mathcal{E}_q^-)^{-1}$ and $(\mathcal{E}_q^+)^{-1}$, respectively, if u is sufficiently regular[1] Note that in all cases, which we will consider, functions are sufficiently regular so that d) is applicable. For details, see [9, 11, 12].

Denote

$$\phi_q^+(\xi) = \mathbb{E}\left[e^{i\xi\overline{X}_{T_q}}\right], \ \phi_q^-(\xi) = \mathbb{E}\left[e^{i\xi\underline{X}_{T_q}}\right]. \tag{2.11}$$

The form of the Wiener-Hopf factorization (WHF) formula that is commonly used in probability theory is as follows:

$$\mathbb{E}\big[e^{i\xi X_{T_q}}\big] = \mathbb{E}\big[e^{i\xi\overline{X}_{T_q}}\big] \cdot \mathbb{E}\big[e^{i\xi\underline{X}_{T_q}}\big], \qquad \forall\, \xi \in \mathbb{R}. \tag{2.12}$$

Calculating the LHS in (2.12) and using (2.11), we obtain an equivalent form

$$q/(q + \psi(\xi)) = \phi_q^+(\xi)\phi_q^-(\xi), \qquad \forall\, \xi \in \mathbb{R}. \tag{2.13}$$

If the Lévy density exponentially decays at infinity, then (2.13) admits the analytic continuation into a strip around the real axis.

The operator version of the Wiener-Hopf factorization states that

$$\mathcal{E}_q = \mathcal{E}_q^+\mathcal{E}_q^- = \mathcal{E}_q^-\mathcal{E}_q^+, \tag{2.14}$$

as operators in spaces of measurable semi-bounded functions. In the proofs of optimal stopping results, we will systematically use Lemma 2.4, equalities (2.6), (2.14) and their corollaries such as

$$\mathcal{E}_q^+(q - L) = q(\mathcal{E}_q^-)^{-1}, \quad \mathcal{E}_q^-(q - L) = q(\mathcal{E}_q^+)^{-1}. \tag{2.15}$$

2.3. Stochastic expressions. For a stopping time τ and a measurable f, define

$$V_{\rm ex}(\tau; f; x) = q^{-1}\mathbb{E}[\mathbb{1}_{T_q<\tau} f(x + X_{T_q})] =: \mathbb{E}\left[\int_0^{\tau-0} e^{-qt} f(x + X_t)dt\right], \tag{2.16}$$

$$V_{\rm en}(\tau; f; x) = q^{-1}\mathbb{E}[\mathbb{1}_{\tau\le T_q} f(x + X_{T_q})] =: \mathbb{E}\left[\int_\tau^{+\infty} e^{-qt} f(x + X_t)dt\right]. \tag{2.17}$$

To ensure finiteness, we assume that

$$\mathbb{E}[|f(x + X_{T_q})|] < \infty \ \forall\ x; \tag{2.18}$$

in some cases, this condition can be relaxed. Expression in (2.16) is the EPV of stream $f(X_t)$, which will be abandoned at time τ; expression in (2.17) is the EPV of stream $f(X_t)$, which will start to accrue at time τ. If one chooses τ to maximize $V_{\rm ex}(\tau; f; x)$ (resp., $V_{\rm en}(\tau; f; x)$), we obtain the exit and entry problem, respectively. Since

$$V_{\rm ex}(\tau; f; x) = q^{-1}\mathcal{E}_q f(x) + V_{\rm en}(\tau; -f; x), \tag{2.19}$$

[1] If process X has non-trivial BM component, then it is necessary to require that u is continuous at h and piece-wise differentiable. Indeed, as the examples of BM and DEJD show, $(\mathcal{E}_q^-)^{-1}$ and $(\mathcal{E}_q^+)^{-1}$ may have terms, which are differential operators of order 1. Therefore, for $u = \mathbb{1}_{(-\infty,0)}$, we have $u(x) = 0, x \ge 0$, but $(\mathcal{E}_q^\pm)^{-1}u(0) = d^\pm\delta$, where δ is the Dirac delta function (unit mass at 0), and $d^\pm$ are constants. If the process has no BM component, then the continuity condition at h can be relaxed.

the exit problem with stream f is equivalent to the entry problem with stream $-f$, and optimality conditions for one problem can be easily reformulated in terms of optimality conditions for the other problem.

A natural generalization of both problems is optimal timing to swap one stream for another one, that is, maximization of

$$V(\tau; f_0, f_n; x) = V_{\rm ex}(\tau; f_0; x) + V_{\rm en}(\tau; f_n; x). \tag{2.20}$$

This problem is equivalent to maximization of $V_{\rm ex}(\tau; f_0 - f_n; x)$, or, alternatively, to maximization of $V_{\rm en}(\tau; f_n - f_0; x)$. We leave to the reader evident reformulations of the optimal stopping results for the exit and entry problems below for the case of the option to swap one stream for another one.

For $h \in \mathbb{R}$, denote by $\tau_h^{\pm}$ the first entrance time into $[h, +\infty)$ and $(-\infty, h]$, respectively. The aim of the paper is to formulate general conditions, which ensure that the optimal stopping time is of the threshold type, namely, of the form $\tau_h^{\pm}$. The main ingredient of the proof are the following general formulas established in [8, 9, 10, 11, 13, 12] for wide classes of Lévy processes, and in [6] for an arbitrary Lévy process.

THEOREM 2.5. *Let $h \in \mathbb{R}$ and let f be a measurable function, which is either semi-bounded or satisfies (2.18). Then*

$$V_{\rm ex}(\tau_h^-; f; x) = q^{-1}\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ f(x), \tag{2.21}$$

$$V_{\rm en}(\tau_h^-; f; x) = q^{-1}\mathcal{E}_q^- \mathbb{1}_{(-\infty,h]}\mathcal{E}_q^+ f(x), \tag{2.22}$$

$$V_{\rm ex}(\tau_h^+; f; x) = q^{-1}\mathcal{E}_q^+ \mathbb{1}_{(-\infty,h)}\mathcal{E}_q^- f(x), \tag{2.23}$$

$$V_{\rm en}(\tau_h^+; f; x) = q^{-1}\mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}\mathcal{E}_q^- f(x). \tag{2.24}$$

For completeness, we recall the proof, which is similar to the standard proof of (2.12). Assuming that $X_0 = 0$, we use the obvious identity $X_{T_q} = (X_{T_q} - \underline{X}_{T_q}) + \underline{X}_{T_q}$, two important facts [29, p. 81]:

(i) the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ are identical in law,

(ii) the random variables $\underline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ are independent,

and definitions of the EPV-operators:

$$\begin{aligned} qV_{\rm ex}(\tau_h^-; f; x) &= \mathbb{E}[\mathbb{1}_{\tau_h^- < T_q} f(x + X_{T_q})] \\ &= \mathbb{E}[\mathbb{1}_{x+\underline{X}_{T_q} > h} f(x + \underline{X}_{T_q} + (X_{T_q} - \underline{X}_{T_q}))] \\ &= \mathbb{E}[\mathbb{1}_{x+\underline{X}_{T_q} > h} \mathcal{E}_q^+ f(x + \underline{X}_{T_q})] \\ &= \mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ f(x). \end{aligned}$$

The result is (2.21). Using (2.14), we derive

$$\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ f(x) = \mathcal{E}_q f(x) + \mathcal{E}_q^- \mathbb{1}_{(-\infty,h]}\mathcal{E}_q^+ (-f)(x),$$

therefore, applying (2.19) and (2.21), we obtain (2.22). Finally, changing the direction on the real axis and replacing the infimum process with the supremum process and vice versa, we derive (2.22) and (2.24).

2.4. Optimal stopping lemmas. From now on, the standing assumptions are

ASSUMPTION 1. *Function f is measurable and satisfies (2.18).*

Assumption 2. *X is a Lévy process satisfying (ACP)-property, with non-trivial supremum and infimum processes.*

In [31, pp.288-289], the reader can find several equivalent definitions of (ACP)-property. One of these is: for any $f \in L_\infty(\mathbb{R})$, $\mathcal{E}_q f$ is continuous. A sufficient condition is: for some $t > 0$, the transition measure $\mathbb{P}_{X_t}$ is absolutely continuous.

For a Borel set B, let τ_B be the first entrance time into B. The proofs of the following general lemmas are based on the Dynkin formula, which is applicable to function V if LV is universally measurable. The definition of a universally measurable function can be found in [31]. For our purposes, it suffices to know that a Borel function is universally measurable. In the optimal stopping lemmas and theorems below, we will formulate conditions of optimality in class $\mathcal{M}$ of stopping times satisfying $\tau < \infty$, a.s., and, under weaker conditions, in the class of stopping times of the threshold type.

Lemma 2.6. *Let B be a Borel set such that*

$$W_{\rm ex}(\tau_B; f; \cdot) := (q-L)V_{\rm ex}(\tau_B; f; \cdot) \quad \text{is universally measurable;} \tag{2.25}$$

$$W_{\rm ex}(\tau_B; f; x) \;=\; f(x), \quad x \in \mathbb{R} \setminus B, \quad \text{a.e.;} \tag{2.26}$$

$$W_{\rm ex}(\tau_B; f; x) \;\geq\; f(x), \quad x \in B, \quad \text{a.e.;} \tag{2.27}$$

$$V_{\rm ex}(\tau_B; f; x) \;\geq\; 0, \quad \forall\, x. \tag{2.28}$$

Then τ_B maximizes $V_{\rm ex}(\tau; f; \cdot)$ in $\mathcal{M}$.

Proof. Let τ be a stopping time. Then, using Dynkin's formula and (2.26)–(2.28), we obtain

$$\begin{aligned} V_{\rm ex}(\tau_B; f; x) &= \mathbb{E}\left[\int_0^\tau e^{-qt}(q-L)V_{\rm ex}(\tau_B; f; x+X_t)dt\right] + \mathbb{E}\left[e^{-q\tau}V_{\rm ex}(\tau_B; f; X_\tau)\right] \\ &\geq \mathbb{E}\left[\int_0^\tau e^{-qt} f(x+X_t)dt\right]. \end{aligned}$$

With $\tau = \tau_B$, we obtain the equality, which means that τ_B is optimal. $\square$ For the entry problem, the evident analog is

Lemma 2.7. *Let B be a Borel set such that*

$$W_{\rm en}(\tau_B; f; \cdot) := (q-L)V_{\rm en}(\tau_B; f; \cdot) \quad \text{is universally measurable;} \tag{2.29}$$

$$W_{\rm en}(\tau_B; f; x) \;=\; 0, \quad x \in \mathbb{R} \setminus B, \quad \text{a.e.;} \tag{2.30}$$

$$W_{\rm en}(\tau_B; f; x) \;\geq\; 0, \quad x \in B, \quad \text{a.e.;} \tag{2.31}$$

$$V_{\rm en}(\tau_B; f; x) \;\geq\; q^{-1}\mathcal{E}_q f(x), \quad \forall\, x. \tag{2.32}$$

Then τ_B maximizes $V_{\rm en}(\tau; f; \cdot)$ in $\mathcal{M}$.

Remark 2.8. a) Lemma 2.6 and (2.25)– (2.28) with f are equivalent to Lemma 2.7 and (2.29)–(2.32) with $-f$ on the strength of (2.19).

b) If X satisfies (ACP)-property and B is closed, then (2.26) and (2.30) follow from the generalization of the Black-Scholes equation [9, Thm. 2.12]).

3. Irreversible exit.

3.1. Exit problem: optimality conditions for τ_h^-.

Let there exist h such that

$$\mathcal{E}_q^+ f(x) \leq 0, \quad x \leq h, \quad \text{and} \quad \mathcal{E}_q^+ f(x) \geq 0, \quad x \geq h. \tag{3.1}$$

In the remaining part of the subsection, we use Lemma 2.6 to derive several groups of sufficient conditions of optimality of τ_h^-.

THEOREM 3.1. *Let there exist $h \in \mathbb{R}$ such that (3.1) holds. Then*

(a) τ_h^- maximizes $V_{\mathrm{ex}}(\tau; f; x)$ in the class of stopping times of the threshold type.

(b) If, in addition,

$$f(x) + \int_{h-x}^{\infty} V_{\mathrm{ex}}(\tau_h^-; f; x+y)F(dy) \le 0, \quad x < h, \quad \text{a.e.}, \tag{3.2}$$

then τ_h^- maximizes $V_{\mathrm{ex}}(\tau; f; x)$ in $\mathcal{M}$.

Denote the LHS in (3.2) by $U_{\mathrm{ex}}(\tau_h^-; f; x)$, and call this function the remorse index: if the remorse index is non-positive in the action region, the exit is optimal, and there is no reason to regret the decision to exit.

Proof. (a) If h is the exit boundary, then $V_{\mathrm{ex}}(\tau; f; x)$ is given by (2.21). Since $\mathcal{E}_q^-$ is a positive operator, the RHS in (2.21) is maximized if and only if $w := \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ f$ is maximized. Condition (3.1) ensures that w is maximized.

(b) It follows from a generalization of the Black-Scholes equation [9, Thm. 2.12] that (2.26) holds. Condition (2.28) is immediate from (2.21) and (3.1). Next, for $x \le h$, we have $V_{\mathrm{ex}}(\tau_h^-; f; x) = 0$, therefore, taking (2.2) into account, we conclude that (2.27) is equivalent to (3.2). Since f and $V_{\mathrm{ex}}(\tau_h^-; f; \cdot)$ are measurable, $U_{\mathrm{ex}}(\tau_h^-; f; \cdot)$ is a measurable function on $(-\infty, h)$. Hence,

$$W_{\mathrm{ex}}(\tau_h^-; f; \cdot) = f - U_{\mathrm{ex}}(\tau_h^-; f; \cdot)$$

is measurable on $(-\infty, h)$. Since $W_{\mathrm{ex}}(\tau_h^-; f; x) = f(x), x > h$, and f is measurable, (2.25) holds. ∎ Below, we discuss simple conditions, which imply the most involved condition (3.2), hence, optimality of τ_h^- in $\mathcal{M}$.

THEOREM 3.2. *Let there exist $h \in \mathbb{R}$ such that (3.1) holds, and let $U_{\mathrm{ex}}(\tau_h^-; f; \cdot)$ be non-decreasing on $(-\infty, h)$. Then (3.2) holds, and τ_h^- maximizes $V_{\mathrm{ex}}(\tau; f; x)$ in $\mathcal{M}$.*

REMARK 3.3. The main idea is that under a weak condition (3.1), the remorse index is non-positive in some neighborhood of the boundary of the inaction region. The monotonicity condition ensures that the remorse index is non-positive on the whole action region. The advantage of the monotonicity condition is that, in many cases, this condition can be easily verified.

Proof. Since $U_{\mathrm{ex}}(\tau_h^-; f; \cdot)$ is non-decreasing on $(-\infty, h)$, it suffices to prove that $U_{\mathrm{ex}}(\tau_h^-; f; h-0) \le 0$. Assume that $U_{\mathrm{ex}}(\tau_h^-; f; h-0) > 0$. Then there exists $b < h$ such that $U_{\mathrm{ex}}(\tau_h^-; f; x) > 0, b < x < h$. To see that this is impossible, extend $U_{\mathrm{ex}}(\tau_h^-; f; x)$ by zero to a function on $\mathbb{R}$. Then, a.e.,

$$\begin{aligned}\mathcal{E}_q^+ U_{\mathrm{ex}}(\tau_h^-; f; x) &= \mathcal{E}_q^+ f(x) - \mathcal{E}_q^+ W_{\mathrm{ex}}(\tau_h^-; f; x)\\ &= \mathcal{E}_q^+ f(x) - \mathcal{E}_q^+ (q-L)q^{-1}\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ f(x).\end{aligned}$$

Using $\mathcal{E}_q = q(q-L)^{-1}$ and the Wiener-Hopf factorization formula (2.14), we obtain

$$\mathcal{E}_q^+ U_{\mathrm{ex}}(\tau_h^-; f; x) = \mathbb{1}_{(-\infty,h]}\mathcal{E}_q^+ f(x). \tag{3.3}$$

Due to (3.1), the RHS in (3.3) is non-positive, but, if $U_{\mathrm{ex}}(\tau_h^-; f; x) > 0$, for $b < x < h$ (and $U_{\mathrm{ex}}(\tau_h^-; f; x) = 0$, for $x \ge h$), then the LHS is positive at some x, contradiction. ∎ In the next three theorems, we give several sets of conditions on f, $V_{\mathrm{ex}}(\tau_h^-; f; x)$ and $F(dy)$, which imply that $U_{\mathrm{ex}}(\tau_h^-; f; x)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 3.4. *Let f be a non-decreasing function which changes sign. Then*

(i) there exists h such that (3.1) holds, and
(ii) τ_h^- maximizes $V_{\rm ex}(\tau; f; x)$ in $\mathcal{M}$.

Proof. Since f is non-decreasing and changes sign, $\mathcal{E}_q^+ f$ enjoys these properties, hence, h satisfying (3.1) exists. From (2.21) and positivity of $\mathcal{E}_q^+$, it follows that $V_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing, hence, $U_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing. □

THEOREM 3.5. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (3.1) holds;
(ii) f is non-decreasing on $(-\infty, h)$;
(iii) $V_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing on $(h, +\infty)$.
Then τ_h^- maximizes $V_{\rm ex}(\tau; f; x)$ in $\mathcal{M}$.

Proof. Under conditions (ii)-(iii), $U_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing on $(-\infty, h)$. □

THEOREM 3.6. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (3.1) holds;
(ii) f is non-decreasing on $(-\infty, h)$;
(iii) measure $F(dy)$ is non-increasing on $(0, +\infty)$ in the following sense: for any Borel set $A \subset (0, +\infty)$ and $x > 0$, $F(A + x) \le F(A)$.
Then τ_h^- maximizes $V_{\rm ex}(\tau; f; x)$ in $\mathcal{M}$.

Proof. Changing the variables $x + y \mapsto y$, we obtain

$$\int_{h-x}^{\infty} V_{\rm ex}(\tau_h^-; f; x+y) F(dy) = \int_h^{\infty} V_{\rm ex}(\tau_h^-; f; y) F(dy - x).$$

On the strength of (3.1) and (2.21), $V_{\rm ex}(\tau_h^-; f; y) \ge 0, y \ge h$, and, on the strength of (iii), $F(dy - x)$ does not decrease as x increases. Hence, the second term in (3.2) is non-decreasing on $(-\infty, h)$. The first term is non-decreasing by (ii). □

REMARK 3.7. If the restriction of $F(dy)$ on $(0, +\infty)$ has the density: $F(dy) = p_+(y)dy$, then (iii) is equivalent to the condition that p_+ in non-increasing on $(0, +\infty)$. Monotonicity conditions can be relaxed further.

THEOREM 3.8. *Let (3.1) hold, and let there exist $\gamma \in \mathbb{R}$ such that $U_{{\rm ex},\gamma}(\tau_h^-; f; x) := e^{\gamma x} U_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing on $(-\infty, h)$. Then τ_h^- is an optimal exit time in $\mathcal{M}$.*

Proof. Multiplying (3.2) by $e^{\gamma x}$, we obtain an equivalent condition $U_{{\rm ex},\gamma}(\tau_h^-; f; x) \le 0$, $\forall\ x < h$, a.e. If $U_{\rm ex}(\tau_h^-; f; h - 0) > 0$, then $U_{\rm ex}(\tau_h^-; f; x) > 0$ in some left neighborhood of h, which contradicts (3.1) (the proof is the same as in Theorem 3.2). Thus, $U_{\rm ex}(\tau_h^-; f; h - 0) \le 0$, $U_{{\rm ex},\gamma}(\tau_h^-; f; h - 0) \le 0$, and $U_{{\rm ex},\gamma}(\tau_h^-; f; x) \le 0, x < h$. Equivalently, (3.2) holds, and τ_h^- is an optimal exit time by Theorem 3.1. □

THEOREM 3.9. *Let (3.1) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-decreasing on $(-\infty, h)$;
(ii) $V_{{\rm ex},\gamma}(\tau_h^-; f; x) := e^{\gamma x} V_{\rm ex}(\tau_h^-; f; x)$ is non-decreasing on $(h, +\infty)$.
Then τ_h^- is an optimal exit time in $\mathcal{M}$.

Proof. We multiply (3.2) by $e^{\gamma x}$ and obtain an equivalent condition

$$f_\gamma(x) + \int_{h-x}^{\infty} V_{{\rm ex};\gamma}(\tau_h^-; f; x+y) e^{-\gamma y} F(dy) \le 0, \quad x < h \text{ a.e.} \tag{3.4}$$

Under conditions (i)-(ii), both terms on the LHS are non-decreasing; hence, remorse index $U_{{\rm ex},\gamma}(\tau_h^-; f; x)$ is non-decreasing as well.□

EXAMPLE 3.10. Consider a multiplant firm in a declining industry. For simplicity, in this example, the scrap value is 0. As the stochastic factor decreases, the profit flow falls, but, at sufficiently low levels of profits, the firm may start a capacity-reduction process, and the profit flow will increase because the multiplant owner internalizes the positive externality (price rise) that closing one of her plants has on her return from others (see [33] for exit with multiplant firms without uncertainty). However, under fairly general conditions, this anticipated improvement of fortune makes no impact on the exit decision of the firm.[2]

To be more specific, we assume that $f(x) = -A - e^{\alpha x}, x \le 0$, and $f(x) = e^x - B, x > 0$, where $A \ge 0, \alpha, B > 0$, $-A - 1 \le 1 - B$, $\alpha \le 1 + A$, and $\kappa^+_q(1) := \mathbb{E}[e^{\overline{X}_{T_q}}] < B$.

Then (2.18) holds, and (3.1) is satisfied with $h = \ln\left(B/\kappa^+_q(1)\right) > 0$. Set $\gamma = -\alpha/(1+A)$. Then function f_γ is non-decreasing. Indeed, for $x < 0$,

$$e^{-\gamma x} f'_\gamma(x) = e^{-\gamma x}(-\gamma A e^{\gamma x} - (\gamma + \alpha) e^{(\gamma+\alpha)x}) \ge -\gamma A - \gamma - \alpha = 0,$$

since $\gamma < 0$ and $\gamma + \alpha \ge 0$. For $x > 0$, $f'_\gamma(x) \ge 0$ because $\gamma \in [-1, 0]$.

Define operators $\mathcal{E}^{\pm,\gamma}_q = e^{\gamma x}\mathcal{E}^\pm_q e^{-\gamma x}$. Both are convolution operators with non-negative (generalized) kernels. Since f_γ is non-decreasing,

$$V_{\mathrm{ex},\gamma}(\tau^-_h; f; \cdot) = q^{-1}\mathcal{E}^{-,\gamma}_q \mathbb{1}_{(h,+\infty)} \mathcal{E}^{+,\gamma}_q f_\gamma$$

is non-decreasing as well. Thus, conditions (ii)-(iii) of Theorem 3.9 are satisfied, and τ^-_h is an optimal exit time.

Remark 3.11. a) Condition "$\mathcal{E}^+_q f$ is non-decreasing" fails in a neighborhood of $-\infty$, and this condition is imposed in [17] for an equivalent entry problem with the reward function $-f$.

b) We allow f to be discontinuous at 0, which can be interpreted as a drop of profitability.

Theorem 3.12. *Let (3.1) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-decreasing on $(-\infty, h)$;
(ii) measure $e^{-\gamma y} F(dy)$ is non-increasing on $(0, +\infty)$
Then τ^-_h is an optimal exit time in $\mathcal{M}$.

Proof. Since $V_{\mathrm{ex},\gamma}(\tau^-_h; f; x) \ge 0$ for all x, the second term on the LHS of (3.4) is non-decreasing; the first term is non-decreasing by (i).□

Example 3.13. Theorem 3.12 is applicable to a firm, whose profit flow $f(X_t)$ is subject to adverse shocks at high levels of the underlying stochastic factor (e.g., demand). For instance, at a certain level, competitors using alternative technologies will start entering the market, so that the profit flow drops and may even decrease further because of the flow of new competitors; the other possibility is a profit stream that may become negative over a certain interval at high levels of the stochastic factor but is positive in a neighborhood of $+\infty$. Then (3.1) can be satisfied.

At low levels, we may allow for discontinuities and non-monotonicity as in Example 3.10, and allow for non-monotonicity of $V_{\mathrm{ex}}(\tau^-_h; f; x)$ if condition (ii) of Theorem 3.12 is satisfied.

Theorems above allow for non-monotone payoffs but exclude jumps of the payoff down in the action region. However, this kind of situation is important for exit problems in declining industry with firms of finite size: as some firms exit, the market

[2] For example, in case of a duoply, if the capacity of each of the two plants of the two plant firm is higher than the capacity of a single plant firm, then the multiplant firm closes both of its plants before the other firm closes its plant, see [33]. We leave for the future study construction of an industry equilibrium with infinitesimally small heterogeneous firms, which supports the price dynamics with these properties.

share of the remaining firms, hence, their profits, jump up. Below, we formulate and prove two simple analogs of Theorems 3.6 and 3.12, which allow one to consider payoffs with jumps down. For simplicity, we consider the case of only one jump down in the action region; the reader can easily generalize the theorem below to the case of several jumps.

THEOREM 3.14. *Let the following four conditions hold*
(i) there exists $h \in \mathbb{R}$ *such that (3.1) holds;*
(ii) there exists $h' < h$ *such that* f *is non-decreasing on* $(-\infty, h')$ *and on* (h', h)*;*
(iii) $U_{\rm ex}(\tau_h^-; f; h'-0) \le 0$*;*
(iv) measure $F(dy)$ *is non-increasing on* $(0, +\infty)$.
Then τ_h^- *maximizes* $V_{\rm ex}(\tau; f; x)$ *in* $\mathcal{M}$.

Proof. The proof of Theorem 3.6 can be repeated word by word to prove that the no-remorse index $U_{\rm ex}(\tau_h^-; f; x)$ does not decrease on $(-\infty, h')$ and on (h', h). Since $U_{\rm ex}(\tau_h^-; f; h'-0) \le 0$, we have $U_{\rm ex}(\tau_h^-; f; x) \le 0, x < h'$. Finally, the main argument used in the proof of Theorem 3.2 can be repeated word by word to prove that $U_{\rm ex}(\tau_h^-; f; x) \le 0$, for $x \in (h', h)$. □ Modifying the proof of Theorem 3.12, we derive from Theorem 3.14

THEOREM 3.15. *Let there exist* $\gamma \in \mathbb{R}$ *and* $h' < h$ *such that*
(i) (3.1) holds;
(ii) f_γ *is non-decreasing on* $(-\infty, h')$ *and on* (h', h)*;*
(iii) $U_{\rm ex}(\tau_h^-; f; h'-0) \le 0$*;*
(iv) measure $e^{-\gamma y} F(dy)$ *is non-increasing on* $(0, +\infty)$.
Then τ_h^- *is an optimal exit time in* $\mathcal{M}$.

3.2. Exit problem: optimality conditions for τ_h^+. Changing the direction on the real axis, and, therefore, replacing the supremum process and non-decreasing functions with the infimum process and non-increasing functions, we obtain the counterparts of the results of Subsection 3.1. The analog of (3.1) is

$$\mathcal{E}_q^- f(x) \ge 0, \quad x \le h, \quad \text{and} \quad \mathcal{E}_q^- f(x) \le 0, \quad x \ge h. \tag{3.5}$$

Condition (3.2) becomes: on $(h, +\infty)$, a.e.,

$$f(x) + \int_{-\infty}^{h-x} V_{\rm ex}(\tau_h^+; f; x+y) F(dy) \le 0. \tag{3.6}$$

Denote the LHS in (3.6) by $U_{\rm ex}(\tau_h^+; f; x)$. The interpretation of this function as the remorse index is the same as in the previous subsection – only the form of the action region is different.

THEOREM 3.16. *Let there exist* $h \in \mathbb{R}$ *such that (3.5) holds. Then*
(a) τ_h^+ *maximizes* $V_{\rm ex}(\tau; f; x)$ *in the class of stopping times of the threshold type.*
(b) If, in addition, (3.6) holds, then τ_h^+ *maximizes* $V_{\rm ex}(\tau; f; x)$ *in* $\mathcal{M}$.

THEOREM 3.17. *Let there exist* $h \in \mathbb{R}$ *such that (3.5) holds, and let* $U_{\rm ex}(\tau_h^+; f; \cdot)$ *be non-increasing on* $(h, +\infty)$. *Then (3.6) holds, and* τ_h^+ *maximizes* $V_{\rm ex}(\tau; f; x)$ *in* $\mathcal{M}$. In the next three theorems, we give several sets of conditions on f, $V_{\rm ex}(\tau_h^+; f; x)$ and $F(dy)$, which imply that $U_{\rm ex}(\tau_h^+; f; x)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 3.18. *Let* f *be a non-increasing function, which changes sign. Then*
(i) there exists h *such that (3.5) holds, and*
(ii) τ_h^+ *maximizes* $V_{\rm ex}(\tau; f; x)$ *in* $\mathcal{M}$.

THEOREM 3.19. *Let the following three conditions hold*

(i) there exists $h \in \mathbb{R}$ such that (3.5) holds;
(ii) f is non-increasing on $(h, +\infty)$;
(iii) $V_{\rm ex}(\tau_h^+; f; x)$ is non-increasing on $(-\infty, h)$.
Then τ_h^+ maximizes $V_{\rm ex}(\tau; f; x)$ in $\mathcal{M}$.

THEOREM 3.20. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (3.5) holds;
(ii) f is non-increasing on $(h, +\infty)$;
(iii) measure $F(dy)$ is non-decreasing on $(-\infty, 0)$ in the following sense: for any Borel set $A \subset (-\infty, 0)$ and $x < 0$, $F(A + x) \le F(A)$.
Then τ_h^+ maximizes $V_{\rm ex}(\tau; f; x)$ in $\mathcal{M}$.

REMARK 3.21. If the restriction of $F(dy)$ on $(-\infty, 0)$ has the density: $F(dy) = p_-(y)dy$, then (iii) is equivalent to the condition that p_- in non-decreasing on $(-\infty, 0)$. Monotonicity conditions can be relaxed further.

THEOREM 3.22. *Let (3.5) hold, and let there exist $\gamma \in \mathbb{R}$ such that $U_{{\rm ex},\gamma}(\tau_h^+; f; x) := e^{\gamma x} U_{\rm ex}(\tau_h^+; f; x)$ is non-increasing on $(h, +\infty)$. Then τ_h^+ is an optimal exit time in $\mathcal{M}$.*

THEOREM 3.23. *Let (3.5) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-increasing on $(h, +\infty)$;
(ii) $V_{{\rm ex},\gamma}(\tau_h^+; f; x) := e^{\gamma x} V_{\rm ex}(\tau_h^+; f; x)$ is non-increasing on $(-\infty, h)$.
Then τ_h^+ is an optimal exit time in $\mathcal{M}$.

THEOREM 3.24. *Let (3.5) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-increasing on $(h, +\infty)$;
(ii) measure $e^{-\gamma y} F(dy)$ is non-decreasing on $(-\infty, 0)$.
Then τ_h^+ is an optimal exit time in $\mathcal{M}$. We leave to the reader generalizations of Theorems 3.20 and 3.24 for the case of the payoff $f(x)$, which jumps up at some point in the action region (cf. Theorems 3.14 and 3.15).

4. Irreversible entry.

Assumptions 1-2 continue to hold.

4.1. Entry problem: optimality conditions for τ_h^-.

The optimal entry theorems are obtained by trivial reformulations of the optimal exit theorems because

$$V_{\rm en}(\tau; f; x) = q^{-1}\mathcal{E}_q f(x) + V_{\rm wait.en}(\tau; f; x),$$

where $V_{\rm wait.en}(\tau; f; x) = V_{\rm ex}(\tau; -f; x)$ is the value of waiting to enter (until time τ). Thus, we can use the theorems for the exit problem with $-f$ and $V_{\rm wait.en}(\tau; f; x) = V_{\rm ex}(\tau; -f; x)$ instead of f and $V_{\rm ex}(\tau; f; x)$. Each statement of the form "f is non-decreasing" becomes "f is non-increasing", and (3.1) and (3.2) become

$$\mathcal{E}_q^+ f(x) \ge 0, \quad x \le h, \quad \text{and} \quad \mathcal{E}_q^+ f(x) \le 0, \quad x \ge h, \tag{4.1}$$

and

$$-f(x) + \int_{h-x}^{\infty} V_{\rm wait.en}(\tau_h^-; f; x+y) F(dy) \le 0, \quad x < h, \ \text{a.e.}, \tag{4.2}$$

respectively. An equivalent form of (4.2) in terms of f only is

$$f(x) + q^{-1} \int_{h-x}^{\infty} (\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)} \mathcal{E}_q^+ f)(x+y) F(dy) \ge 0, \quad x < h, \ \text{a.e..} \tag{4.3}$$

Denote the LHS in (4.2) by $U_{\rm en}(\tau_h^-; f; x)$, and call this function the remorse index: if the remorse index is non-positive in the action region, the exit is optimal, and there is no reason to regret the decision to enter.

THEOREM 4.1. *Let there exist $h \in \mathbb{R}$ such that (4.1) holds. Then*

(a) τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in the class of stopping times of the threshold type.
(b) If, in addition, (4.2) holds, then τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in $\mathcal{M}$.

THEOREM 4.2. *Let there exist $h\in\mathbb{R}$ such that (4.1) holds, and let $U_{\rm en}(\tau_h^-;f;\cdot)$ be non-decreasing on $(-\infty,h)$. Then (4.2) holds, and τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in $\mathcal{M}$.* In the next three theorems, we give several sets of conditions on f, $V_{\rm wait.en}(\tau_h^-;f;x)$ and $F(dy)$, which imply that $U_{\rm en}(\tau_h^-;f;\cdot)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 4.3. *Let f be a non-increasing function, which changes sign. Then*
(i) there exists h such that (4.1) holds, and
(ii) τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in $\mathcal{M}$.

THEOREM 4.4. *Let the following three conditions hold*
(i) there exists $h\in\mathbb{R}$ such that (4.1) holds;
(ii) f is non-increasing on $(-\infty,h)$;
(iii) $V_{\rm wait.en}(\tau_h^-;f;x)$ is non-decreasing on $(h,+\infty)$.
Then τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in $\mathcal{M}$.

THEOREM 4.5. *Let the following three conditions hold*
(i) there exists $h\in\mathbb{R}$ such that (4.1) holds;
(ii) f is non-increasing on $(-\infty,h)$;
(iii) measure $F(dy)$ is non-increasing on $(0,+\infty)$.
Then τ_h^- maximizes $V_{\rm en}(\tau;f;x)$ in $\mathcal{M}$. Monotonicity conditions can be relaxed further.

THEOREM 4.6. *Let (4.1) hold, and let there exist $\gamma\in\mathbb{R}$ such that $U_{{\rm en},\gamma}(\tau_h^-;f;\cdot):=e^{\gamma x}U_{\rm en}(\tau_h^-;f;\cdot)$ is non-decreasing on $(-\infty,h)$. Then τ_h^- is an optimal entry time in $\mathcal{M}$.*

THEOREM 4.7. *Let (4.1) hold, and let there exist $\gamma\in\mathbb{R}$ such that*
(i) $f_\gamma(x):=e^{\gamma x}f(x)$ is non-increasing on $(-\infty,h)$;
(ii) $V_{{\rm wait.en},\gamma}(\tau_h^-;f;x):=e^{\gamma x}V_{\rm wait.en}(\tau_h^-;f;x)$ is non-decreasing on $(h,+\infty)$.
Then τ_h^- is an optimal entry time in $\mathcal{M}$.

THEOREM 4.8. *Let (4.1) hold, and let there exist $\gamma\in\mathbb{R}$ such that*
(i) $f_\gamma(x):=e^{\gamma x}f(x)$ is non-increasing on $(-\infty,h)$;
(ii) measure $e^{-\gamma y}F(dy)$ is non-increasing on $(0,+\infty)$
Then τ_h^- is an optimal entry time in $\mathcal{M}$. We leave to the reader generalizations of Theorems 4.5 and 4.8 for the case of the payoff $f(x)$, which jumps up at some point in the action region (cf. Theorems 3.14 and 3.15).

4.2. Entry problem: optimality conditions for τ_h^+. We use the theorems for the exit problem in Subsection 3.2 with $-f$ and $V_{\rm wait.en}(\tau;f;x)=V_{\rm ex}(\tau;-f;x)$ instead of f and $V_{\rm ex}(\tau;f;x)$. Each statement of the form "f is non-increasing" becomes "f is non-decreasing", and (3.5) and (3.6) become

$$\mathcal{E}_q^- f(x)\le 0,\quad x\le h,\quad \text{and}\quad \mathcal{E}_q^- f(x)\ge 0,\quad x\ge h, \tag{4.4}$$

and

$$-f(x)+\int_{-\infty}^{h-x}V_{\rm wait.en}(\tau_h^+;f;x+y)F(dy)\le 0,\quad x>h,\ \text{a.e.}, \tag{4.5}$$

respectively. An equivalent form of (4.5) in terms of f only is

$$f(x)+q^{-1}\int_{-\infty}^{h-x}(\mathcal{E}_q^+\mathbb{1}_{(-\infty,h)}\mathcal{E}_q^- f)(x+y)F(dy)\ge 0,\quad x>h,\ \text{a.e.}. \tag{4.6}$$

The LHS in (4.5) is denoted $U_{\rm en}(\tau_h^+; f; x)$ and called the remorse index. The interpretation is the same as in the preceding subsection only the action region is different.

THEOREM 4.9. *Let there exist $h \in \mathbb{R}$ such that (4.4) holds. Then*
(a) τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in the class of stopping times of the threshold type.
(b) If, in addition, (4.5) holds, then τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in $\mathcal{M}$.

THEOREM 4.10. *Let there exist $h \in \mathbb{R}$ such that (4.4) holds, and let $U_{\rm en}(\tau_h^+; f; \cdot)$ be non-increasing on $(h, +\infty)$. Then (4.5) holds, and τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in $\mathcal{M}$.*
In the next three theorems, we give several sets of conditions on f, $V_{\rm wait.en}(\tau_h^+; f; x)$ and $F(dy)$, which imply that $U_{\rm en}(\tau_h^+; f; \cdot)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 4.11. *Let f be a non-decreasing function, which changes sign. Then*
(i) there exists h such that (4.4) holds, and
(ii) τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in $\mathcal{M}$.

THEOREM 4.12. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (4.4) holds;
(ii) f is non-decreasing on $(h, +\infty)$;
(iii) $V_{\rm wait.en}(\tau_h^+; f; x)$ is non-increasing on $(-\infty, h)$.
Then τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in $\mathcal{M}$.

THEOREM 4.13. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (4.4) holds;
(ii) f is non-decreasing on $(h, +\infty)$;
(iii) measure $F(dy)$ is non-decreasing on $(-\infty, 0)$.
Then τ_h^+ maximizes $V_{\rm en}(\tau; f; x)$ in $\mathcal{M}$. Monotonicity conditions can be relaxed further.

THEOREM 4.14. *Let (4.4) hold, and let there exist $\gamma \in \mathbb{R}$ such that $U_{{\rm en},\gamma}(\tau_h^+; f; x) := e^{\gamma x} U_{\rm en}(\tau_h^+; f; x)$ is non-increasing on $(h, +\infty)$. Then τ_h^+ is an optimal entry time in $\mathcal{M}$.*

THEOREM 4.15. *Let (4.4) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-decreasing on $(h, +\infty)$;
(ii) $V_{{\rm wait.en},\gamma}(\tau_h^+; f; x) := e^{\gamma x} V_{\rm wait.en}(\tau_h^+; f; x)$ is non-increasing on $(-\infty, h)$.
Then τ_h^+ is an optimal entry time in class $\mathcal{M}$.

THEOREM 4.16. *Let (4.4) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $f_\gamma(x) := e^{\gamma x} f(x)$ is non-decreasing on $(h, +\infty)$;
(ii) measure $e^{-\gamma y} F(dy)$ is non-decreasing on $(-\infty, 0)$.
Then τ_h^+ is an optimal entry time in class $\mathcal{M}$.

EXAMPLE 4.17. Consider the investment into a plant, which will produce a widget of a new kind. The investment cost I is fixed, and the profit flow is an increasing function of the underlying stochastic factor, say, the demand for new widgets of this kind. However, as the demand increases, competitors enter the market, with newish versions of the widget. Hence, eventually, the profit flow will start to decline. We assume that the profit flow will stabilize at level $A \geq qI$. To be more specific, we model the profit flow as $\Pi(x) = Ae^x, x \leq 0$, $\Pi(x) = xe^{-x} + A, x > 0$. The entry problem is equivalent to the option with the non-monotone payoff stream $f(x) = \Pi(x) - qI$. Assume that $\kappa_q^-(1) := \mathbb{E}[e^{\underline{X}_{T_q}}] > qI/A$; then, on the negative half-axis, there exists a unique $h = \ln(qI/(A\kappa_q^-(1))$ such that $\mathcal{E}_q^- f(x) \leq 0, x \leq h$, $\mathcal{E}_q^- f(x) \geq 0, x \in (h, 0)$. Since $f(x) \geq f_1(x) := A\min\{e^x, 1\}$, and f_1 is non-decreasing on $\mathbb{R}$, we have $\mathcal{E}_q^- f_1(0) \leq \mathcal{E}_q^- f_1(x) \leq \mathcal{E}_q^- f(x), x \geq 0$; but $\mathcal{E}_q^- f_1(0) = \mathcal{E}_q^- f(0) > 0$. Hence, (4.4) is satisfied. Clearly, there exist $\gamma > 0$ such that $xe^{(\gamma-1)x} + (A - qI)e^{\gamma x}$ is increasing; the smallest one is $\gamma = 1/(1+A-qI)$. If $e^{-\gamma y}F(dy)$ is non-decreasing on $(-\infty, 0)$, all conditions of Theorem 4.13 are satisfied, and $h = \ln(qI/(A\mathbb{E}[e^{\underline{X}_{T_q}}])$ is the optimal entry threshold. We leave to the leader the generalizations of Theorems 4.13 and 4.16 for the case of the payoff $f(x)$, which jumps down at some point in the action region (cf. Theorems 3.14 and 3.15).

5. Problems with instantaneous payoffs.

5.1. Relation to entry problems. In this section, we consider an option to get an instantaneous payoff $G(X_t)$. If τ is an optimal exercise time, then the option value is

$$V_{\mathrm{inst}}(\tau; G; x) = \mathbb{E}^x[e^{-q\tau}G(X_\tau)].$$

If $G(X_t)$ can be represented as the EPV of a stream $f(X_t)$: $G(x) = q^{-1}\mathcal{E}_q f(x)$, then the reformulation of the results for the entry problems is straightforward. In particular, using the Wiener-Hopf factorization formula $\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+$, we can write formally $\mathcal{E}_q^\pm f = q(\mathcal{E}_q^\mp)^{-1}G$, and derive from (2.23) and (2.24)

$$V_{\mathrm{inst}}(\tau_h^-; G; x) = \mathcal{E}_q^- \mathbb{1}_{(-\infty,h]}(\mathcal{E}_q^-)^{-1}G(x), \tag{5.1}$$

$$V_{\mathrm{inst}}(\tau_h^+; G; x) = \mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}(\mathcal{E}_q^+)^{-1}G(x). \tag{5.2}$$

Under weak regularity conditions on G and process X, one can define $w_\pm(x) = (\mathcal{E}_q^\pm)^{-1}G(x)$ without resorting to f and prove (5.1) and (5.2) (see [9, 12, 11] for a detailed analysis). For instance, if, on a semi-bounded interval $(A, +\infty)$, $G(x) = e^{\beta x}$, and $\kappa_q^+(\beta) := \mathbb{E}\left[e^{\beta \overline{X}_{T_q}}\right] < \infty$, then

$$w_+(x) = (\mathcal{E}_q^+)^{-1}G(x) = \kappa_q^+(\beta)^{-1}e^{\beta x}, \quad x > A. \tag{5.3}$$

Similarly, if, on a semi-bounded interval $(-\infty, -A)$, $G(x) = e^{\beta x}$, and $\kappa_q^-(\beta) := \mathbb{E}\left[e^{\beta \underline{X}_{T_q}}\right] < \infty$, then

$$w_-(x) = (\mathcal{E}_q^-)^{-1}G(x) = \kappa_q^-(\beta)^{-1}e^{\beta x}, \quad x < -A. \tag{5.4}$$

For more general G, $w_\pm$ can be calculated using the Fourier transform technique [9]. Another possibility, which can be realized for wide classes of Lévy processes, is to represent $(\mathcal{E}_q^\pm)^{-1}G(x)$ in the form

$$(\mathcal{E}_q^+)^{-1}G(x) = c_{q0}^+ G(x) - c_{q1}^+ G'(x) - \int_0^{+\infty} G(x+y)k_q^{+-}(y)dy, \tag{5.5}$$

$$(\mathcal{E}_q^-)^{-1}G(x) = c_{q0}^- G(x) - c_{q1}^- G'(x) - \int_{-\infty}^0 G(x+y)k_q^{--}(y)dy, \tag{5.6}$$

where $c_{q0}^\pm$ and $c_{q1}^\pm$ are constants, and $k_q^{\pm -}$ are sufficiently regular functions on $(0, +\infty)$ and $(0, -\infty)$, respectively. In BM model, $c_{q0}^\pm = 1, c_{q1}^\pm = 1/\beta^\pm$, and $k_q^{\pm -}(y) = 0$; in DEJD model,

$$c_{q0}^\pm = \frac{\lambda_\pm(\beta_1^\pm + \beta_2^\pm)}{\beta_1^\pm \beta_2^\pm}, \quad c_{q1}^\pm = \frac{\lambda_\pm}{\beta_1^\pm \beta_2^\pm}, \quad k_q^{\pm -}(y) = \frac{\mp\lambda_\pm(\lambda_\pm - \beta_1^\pm)(\lambda_\pm - \beta_2^\pm)}{\beta_1^\pm \beta_2^\pm}e^{-\lambda_\pm y}.$$

For Lévy processes with the Lévy density given by exponential polynomials on positive and negative half-axes, representations (5.5) and (5.6) are derived in [9, 12, 11].

5.2. Optimality of τ_h^-. The results below are obtained from the results of Subsection 4.1 replacing f, $\mathcal{E}_q^+ f$, $V_{\mathrm{en}}(\tau; f; x)$ and $V_{\mathrm{wait.en}}(\tau; f; x)$ with $(q-L)G$, $(\mathcal{E}_q^-)^{-1}G$, $V_{\mathrm{inst}}(\tau; G; x)$ and $V_{\mathrm{wait.en}}(\tau; G; x) = V_{\mathrm{inst}}(\tau; G; x) - G(x)$, respectively. To avoid any changes in the proofs, we assume that there exists a measurable f satisfying the no-bubble condition (2.18) such that $G = q^{-1}\mathcal{E}_q f$ although this condition can be relaxed. Conditions (4.1) and (4.2) become

$$(\mathcal{E}_q^-)^{-1}G(x) \ge 0, \quad x \le h, \quad \text{and} \quad (\mathcal{E}_q^-)^{-1}G(x) \le 0, \quad x \ge h, \tag{5.7}$$

and

$$-(q-L)G(x) + \int_{h-x}^{\infty} V_{\mathrm{wait.inst}}(\tau_h^-; G; x+y)F(dy) \le 0, \quad x < h, \text{ a.e.}, \tag{5.8}$$

respectively. An equivalent form of (5.8) in terms of G only is

$$(q-L)G(x) + \int_{h-x}^{\infty} (\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)} (\mathcal{E}_q^-)^{-1} G)(x+y)F(dy) \ge 0, \quad x < h, \text{ a.e.}. \tag{5.9}$$

The LHS in (5.8) is denoted $U_{\mathrm{inst}}(\tau_h^-; f; x)$ and called the remorse index.

THEOREM 5.1. *Let there exist $h \in \mathbb{R}$ such that (5.7) holds. Then*
(a) τ_h^- maximizes $V_{\mathrm{inst}}(\tau; f; x)$ in the class of stopping times of the threshold type.
(b) If, in addition, (5.8) holds, then τ_h^- maximizes $V_{\mathrm{inst}}(\tau; f; x)$ in $\mathcal{M}$.

THEOREM 5.2. *Let there exist $h \in \mathbb{R}$ such that (5.7) holds, and let $U_{\mathrm{inst}}(\tau_h^-; f; \cdot)$ be non-decreasing on $(-\infty, h)$. Then (5.8) holds, and τ_h^- maximizes $V_{\mathrm{inst}}(\tau; f; x)$ in $\mathcal{M}$.* In the next three theorems, we give several sets of conditions on G, $V_{\mathrm{wait.inst}}(\tau_h^-; G; x)$ and $F(dy)$, which imply that $U_{\mathrm{inst}}(\tau_h^-; f; \cdot)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 5.3. *Let $(q-L)G$ be a non-increasing function, which changes sign. Then*
(i) there exists h such that (5.7) holds, and
(ii) τ_h^- maximizes $V_{\mathrm{inst}}(\tau; G; x)$ in $\mathcal{M}$.

THEOREM 5.4. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (5.7) holds;
(ii) $(q-L)G$ is non-increasing on $(-\infty, h)$;
(iii) $V_{\mathrm{wait.inst}}(\tau_h^-; G; x)$ is non-decreasing on $(h, +\infty)$.
Then τ_h^- maximizes $V_{\mathrm{inst}}(\tau; G; x)$ in $\mathcal{M}$.

THEOREM 5.5. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ such that (5.7) holds;
(ii) $(q-L)G$ is non-increasing on $(-\infty, h)$;
(iii) measure $F(dy)$ is non-increasing on $(0, +\infty)$.
Then τ_h^- maximizes $V_{\mathrm{inst}}(\tau; G; x)$ in $\mathcal{M}$. Monotonicity conditions can be relaxed further.

THEOREM 5.6. *Let (5.7) hold, and let there exist $\gamma \in \mathbb{R}$ such that $U_{\mathrm{inst},\gamma}(\tau_h^-; f; x) := e^{\gamma x} U_{\mathrm{inst}}(\tau_h^-; f; x)$ is non-decreasing on $(-\infty, h)$. Then τ_h^- maximizes $V_{\mathrm{inst}}(\tau; G; x)$ in $\mathcal{M}$.*

THEOREM 5.7. *Let (5.7) hold, and let there exist $\gamma \in \mathbb{R}$ such that*
(i) $((q-L)G)_\gamma(x) := e^{\gamma x}(q-L)G(x)$ is non-increasing on $(-\infty, h)$;
(ii) $V_{\mathrm{wait.inst},\gamma}(\tau_h^-; G; x) := e^{\gamma x} V_{\mathrm{wait.inst}}(\tau_h^-; G; x)$ is non-decreasing on $(h, +\infty)$.
Then τ_h^- maximizes $V_{\mathrm{inst}}(\tau; G; x)$ in $\mathcal{M}$.

THEOREM 5.8. *Let (5.7) hold, and let there exist $\gamma \in \mathbb{R}$ such that*

(i) $((q-L)G)_\gamma(x) := e^{\gamma x}(q-L)G(x)$ *is non-increasing on* $(-\infty, h)$*;*
(ii) *measure* $e^{-\gamma y}F(dy)$ *is non-increasing on* $(0, +\infty)$
Then τ_h^- *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$. We leave to the reader generalizations of Theorems 5.5 and 5.8 for the case of $(q-L)G(x)$, which jumps up at some point in the action region (cf. Theorems 3.14 and 3.15).

5.3. Optimality conditions for τ_h^+. The results below are obtained from the results of Subsection 4.2 replacing f, $\mathcal{E}_q^- f$, $V_{\mathrm{en}}(\tau; f; x)$ and $V_{\mathrm{wait.en}}(\tau; f; x)$ with $(q-L)G$, $(\mathcal{E}_q^+)^{-1}G$, $V_{\mathrm{inst}}(\tau; G; x)$ and $V_{\mathrm{wait.en}}(\tau; G; x) = V_{\mathrm{inst}}(\tau; G; x) - G(x)$, respectively. To avoid any changes in the proofs, we assume that there exists a measurable f satisfying the no-bubble condition (2.18) such that $G = q^{-1}\mathcal{E}_q f$ although this condition can be relaxed. Conditions (4.4) and (4.5) become

$$(\mathcal{E}_q^+)^{-1}G(x) \le 0, \quad x \le h, \quad \text{and} \quad (\mathcal{E}_q^+)^{-1}G(x) \ge 0, \quad x \ge h, \tag{5.10}$$

and

$$-(q-L)G(x) + \int_{-\infty}^{h-x} V_{\mathrm{wait.inst}}(\tau_h^+; G; x+y)F(dy) \le 0, \quad x > h, \text{ a.e.,} \tag{5.11}$$

respectively. An equivalent form of (5.11) in terms of G only is

$$(q-L)G(x) + \int_{-\infty}^{h-x} (\mathcal{E}_q^+ \mathbb{1}_{(-\infty,h)}(\mathcal{E}_q^+)^{-1}G)(x+y)F(dy) \ge 0, \quad x > h, \text{ a.e..} \tag{5.12}$$

The LHS in (5.11) is denoted $U_{\mathrm{inst}}(\tau_h^+; f; x)$ and called the remorse index.

THEOREM 5.9. *Let there exist* $h \in \mathbb{R}$ *such that (5.10) holds. Then*
(a) τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; f; x)$ *in the class of stopping times of the threshold type.*
(b) If, in addition, (5.11) holds, then τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; f; x)$ *in* $\mathcal{M}$*.*

THEOREM 5.10. *Let there exist* $h \in \mathbb{R}$ *such that (5.10) holds, and let* U *be non-increasing on* $(h, +\infty)$*. Then (5.11) holds, and* τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$*.* In the next three theorems, we give several sets of conditions on f, $V_{\mathrm{wait.inst}}(\tau_h^+; G; x)$ and $F(dy)$, which imply that $U_{\mathrm{inst}}(\tau_h^+; f; \cdot)$ is non-decreasing. The simplest sufficient condition is given in

THEOREM 5.11. *Let* $(q-L)G$ *be a non-decreasing function, which changes sign. Then*
(i) there exists h *such that (5.10) holds, and*
(ii) τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$*.*

THEOREM 5.12. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ *such that (5.10) holds;*
(ii) $(q-L)G$ *is non-decreasing on* $(h, +\infty)$*;*
(iii) $V_{\mathrm{wait.inst}}(\tau_h^+; G; x)$ *is non-increasing on* $(-\infty, h)$*.*
Then τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$*.*

THEOREM 5.13. *Let the following three conditions hold*
(i) there exists $h \in \mathbb{R}$ *such that (5.10) holds;*
(ii) $(q-L)G$ *is non-decreasing on* $(h, +\infty)$*;*
(iii) measure $F(dy)$ *is non-decreasing on* $(-\infty, 0)$*.*
Then τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$*.* Monotonicity conditions can be relaxed further.

THEOREM 5.14. *Let (5.10) hold, and let there exist* $\gamma \in \mathbb{R}$ *such that* $U_{\mathrm{inst},\gamma}(\tau_h^+; f; x) := e^{\gamma x}U_{\mathrm{inst}}(\tau_h^+; f; x)$ *is non-increasing on* $(h, +\infty)$*. Then* τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau; G; x)$ *in* $\mathcal{M}$*.*

THEOREM 5.15. *Let (5.10) hold, and let there exist* $\gamma \in \mathbb{R}$ *such that*

(i) $((q-L)G)_\gamma(x) := e^{\gamma x}(q-L)G(x)$ *is non-decreasing on* $(h,+\infty)$*;*
(ii) $V_{\mathrm{wait.inst},\gamma}(\tau_h^+;G;x) := e^{\gamma x}V_{\mathrm{wait.inst}}(\tau_h^+;G;x)$ *is non-increasing on* $(-\infty,h)$*.*
Then τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau;G;x)$ *in* $\mathcal{M}$*.*

THEOREM 5.16. *Let (5.7) hold, and let there exist* $\gamma\in\mathbb{R}$ *such that*
(i) $((q-L)G)_\gamma(x) := e^{\gamma x}(q-L)G(x)$ *is non-decreasing on* $(h,+\infty)$*;*
(ii) measure $e^{-\gamma y}F(dy)$ *is non-decreasing on* $(-\infty,0)$*.*
Then τ_h^+ *maximizes* $V_{\mathrm{inst}}(\tau;G;x)$ *in* $\mathcal{M}$*.* We leave to the reader generalizations of Theorems 5.13 and 5.16 for the case of $(q-L)G(x)$, which jumps down at some point in the action region (cf. Theorems 3.14 and 3.15).

6. Entry with an embedded option to exit.

6.1. Post-investment value of the investment project. Consider the firm's manager, who contemplates the investment into a plant, which will yield a revenue stream $R(X_t) = Ge^{X_t}$, the operational cost stream being constant $c>0$. Should the firm decide to abandon the project in the future, it can do it at any moment. The scrap value C may be either positive or negative; for instance, if the clean-up of the contaminated site is required by law, then C is negative. Once the project is in operation, the exit problem is equivalent to the option to abandon a stream $f(X_t) = Ge^{X_t} - c - qC$. Assume that $c+qC>0$. Function $f(x)$ is non-decreasing, and it changes sign, hence, Theorem 3.4 is applicable. This theorem states that there exists h satisfying (3.1), and $\tau^-_{h_*}$ is the optimal exit time. Since f is increasing, the h, denote it h_*, is unique. It is given by

$$e^{h_*} = (c+qC)/(G\kappa_q^+(1)). \tag{6.1}$$

The post-investment value of the investment project can be written in several forms

$$\begin{aligned}
V(x) &= C + V_{\mathrm{ex}}(\tau^-_{h_*}; R-c-qC; x) && (6.2)\\
&= C + q^{-1}\mathcal{E}_q(R-c-qC)(x) + V_{\mathrm{en}}(\tau^-_{h_*}, c+qC-R, x) && (6.3)\\
&= q^{-1}\mathcal{E}_q(R-c)(x) + V_{\mathrm{en}}(\tau^-_{h_*}, c+qC-R, x) && (6.4)\\
&= C + q^{-1}(\mathcal{E}_q f_1)(x), && (6.5)
\end{aligned}$$

where, a.e.,

$$\begin{aligned}
f_1(x) &= f(x) + W_{\mathrm{en}}(\tau^-_{h_*}; -f; x) && (6.6)\\
&= f(x) - q^{-1}(q-L)\mathcal{E}_q^- \mathbb{1}_{(-\infty,h_*]}\mathcal{E}_q^+ f(x). && (6.7)
\end{aligned}$$

6.2. Timing investment. Assume that the fixed investment cost $I>C$; then entry is non-optimal at $x\le h_*$. Timing investment, the manager maximizes $V_{\mathrm{inst}}(\tau;G;x)$, where $G(x) = V(x) - I$. In order to apply one of the theorems of Subsection 5.3, we impose two additional conditions:

(i) $k^-(x)$, the pdf of $\underline{X}_{T_q}$, satisfies

$$k^-(x) = \int \mu^-(d\beta)(-\beta)e^{-\beta x}, \quad x<0, \tag{6.8}$$

where $\mu^-(d\beta)$ is a non-negative measure supported at a subset of $(-\infty,0)$ (by Bernstein's theorem, (6.8) is equivalent to *absolute monotonicity* of $k^-(x)$);

(ii) the inverse $(\mathcal{E}_q^+)^{-1}$ admits the representation

$$(\mathcal{E}_q^+)^{-1} = c_{q0}^+ - c_{q1}^+\partial_x - K^{+-}, \tag{6.9}$$

where $c_{q0}^+, c_{q1}^+ > 0$, and K_q^{+-} is the convolution operator with the non-negative kernel k^{+-}, which is monotone on $(0, +\infty)$.

Conditions (i)–(ii) hold in BM and DEJD model, and for many other classes of Lévy processes. See (5.5)-(5.6).

THEOREM 6.1. *Let (6.8) and (6.9) hold. Then*

a) there exists the unique $h^{**} > h_*$ such that (5.10) holds with $h = h^{**}$, hence, $\tau_{h^{**}}^+$ is the optimal entry time in the class of stopping times of the threshold type;

b) if, in addition, measure $F(dy)$ is non-decreasing on $(-\infty, 0)$, then $\tau_{h^{**}}^+$ is an optimal entry time in $\mathcal{M}$.

Proof. a) Using (6.8) and (6.1), we derive, for $x > h_*$,

$$\begin{aligned}
&V_{\mathrm{en}}(\tau_{h_*}^-, c + qC - R, x)\\
&= q^{-1}\mathcal{E}_q^- \mathbb{1}_{(-\infty,h_*]}\mathcal{E}_q^+(c + qC - R)(x)\\
&= q^{-1}(\mathcal{E}_q^- \mathbb{1}_{(-\infty,h_*]}(c + qC - G\kappa_q^+(1)e^{\cdot}))(x)\\
&= q^{-1}\int \mu^-(d\beta)\int_{-\infty}^0 dy\,(-\beta)e^{-\beta y}\mathbb{1}_{(-\infty,h_*]}(x+y)(c + qC - G\kappa_q^+(1)e^{x+y}))\\
&= (c/q + C)\int \mu^-(d\beta)\int_{-\infty}^0 dy\,(-\beta)e^{-\beta y}\mathbb{1}_{(-\infty,h_*]}(x+y)(1 - e^{x+y-h_*}))\\
&= (c/q + C)\int \mu^-(d\beta)\int_{-\infty}^{h_*-x} dy\,(-\beta)e^{-\beta y}(1 - e^{x+y-h_*}))\\
&= (c/q + C)\int \mu^-(d\beta)\frac{e^{\beta(x-h_*)}}{1-\beta}.
\end{aligned}$$

Substituting into (6.3) and using $(\mathcal{E}_q^+)^{-1}\mathcal{E}_q = \mathcal{E}_q^-$, we calculate

$$\begin{aligned}
w(x) &:= (\mathcal{E}_q^+)^{-1}(V(x) - I)\\
&= -c/q - I + q^{-1}\kappa_q^-(1)Ge^x + (c/q + C)\int \mu^-(d\beta)\frac{e^{\beta(x-h_*)}}{1-\beta}.
\end{aligned}$$

Since the support of $\mu(d\beta)$ is a subset of $(-\infty, 0)$, and, for $\beta < 0$, $\kappa_q^+(\beta)^{-1} > 0$, we see that the second derivative of $w(\ln S)$ w.r.t. S is positive, hence, $w(\ln S)$ is convex, and to prove that equation $w(h) = 0$ has a unique root on $(h_*, +\infty)$, it suffices to show that $w(h_* + 0) < 0$ and $w(+\infty) > 0$.

Using the Wiener-Hopf factorization formula, (6.5) and (6.6), we obtain

$$\begin{aligned}
w(x) &= -(I - C) + q^{-1}\mathcal{E}_q^- f(x) + q^{-1}\mathcal{E}_q^- W_{\mathrm{en}}(\tau_h^-; -f; \cdot)(x)\\
&= q^{-1}\mathcal{E}_q^-(R - c - qI)(x) + q^{-1}\mathcal{E}_q^- W_{\mathrm{en}}(\tau_h^-; -f; \cdot)(x)
\end{aligned}$$

Since $W_{\mathrm{en}}(\tau_h^-; -f; x) = 0, x \ge h_*$, the second term on the RHS above vanishes as $x \to +\infty$, and the first term tends to $+\infty$ because $R(x)$ does. Hence, $w(+\infty) = +\infty$.

It remains to show that $w(h_* + 0) < 0$. Suppose that $w(h_* + 0) \ge 0$. Then, for $x > h_*$, $w(x) \ge 0$, and, therefore,

$$V_{\mathrm{inst}}(\tau_{h_*}^+; V - I; x) = \mathcal{E}_q^+ \mathbb{1}_{(h_*,+\infty)}(\mathcal{E}_q^+)^{-1}(V - I)(x) \ge 0, \quad x > h_*.$$

Hence, the value of entry at the threshold h_*, at cost I, is non-negative. But $V(f;C;h_*) = C < I$, contradiction.

b) Conditions (i) and (iii) of Theorem 5.13 hold, therefore, it remains to prove that $(q-L)G$ is non-decreasing on $(h^{**}, +\infty)$. On the strength of (6.5) and (6.7),

$$(q-L)G(x) = (q-L)(V-I)(x) = f(x) - q^{-1}(q-L)\mathcal{E}_q^- \mathbb{1}_{(-\infty,h_*]}\mathcal{E}_q^+ f(x) - q(I-C),$$

hence, for $x > h^{**}$,

$$(q-L)G(x) = f(x) - q(I-C) = e^x - c - qI.$$

The RHS defines an increasing function. □

7. Conclusion. In the paper, we derived a series of optimal stopping theorems for options with non-monotone discontinuous payoff streams and options with instantaneous payoffs, in Lévy driven models, and, as an application, solved the investment problem with an embedded option to exit. The results have natural analogs for random walks, and can be generalized for regime-switching models.

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