Counterparty Risk Subject To ATE

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Abstract

Credit ATE (Additional Termination Event) clause is a counterparty risk mitigant that allows banks to terminate and close out bilateral derivative contracts if the credit rating of the counterparty falls below the trigger level. Since credit default is often preceded by rating downgrades, credit ATE can significantly reduces the counterparty credit risk by early terminating exposure. However, there is still the risk that counterparty may default without going through severe downgrade. This article presents a practical model for valuating CVA subject to ATE.

Keywords: Counterparty Risk, Credit Value Adjustment, Rating Transition, Rating Trigger, Additional Termination Event.

1. Introduction

Counterparty credit risk refers to the risk that a counterparty to a bilateral financial derivative contract may fail to fulfill its contractual obligation causing financial loss to the non-defaulting party. Only over-the-counter (OTC) derivative contracts are subject to counterparty risk. Exchange traded derivatives have very little counterparty risk because the exchange or a clearing house is the central counterparty to both parties to the transaction. With an exchange/clearing house as the central counterparty, the two counterparties to a transaction are not directly exposed to each other's default risk, thereby eliminating the counterparty risk so long as the exchange/clearing house does not default. Exchanges/clearing houses are well protected by the financial industry.[2]

From the perspective of a bank, when the counterparty defaults, the portfolio of all OTC derivative contracts between the bank and the counterparty is marked-to-market (MTM) at the time of default. If the value of the portfolio is negative to the bank, the bank is obligated to pay the full MTM value to the defaulting counterparty. If, however, the value is positive to the bank, the bank will recover only a percentage of that MTM value, usually after a lengthy bankruptcy proceeding.[3] If the recovered amount is less than 100%, ignoring the time value, the bank suffers a credit loss. This potential credit loss due to

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[2] Some clearing house uses haircut to set margin requirement. Clearing funds from participants provide additional protection in extreme market condition when margins are insufficient.

[3] A lengthy bankruptcy proceeding can be costly to the bank as the money cannot earn interests or be invested.

the possibility the counterparty may default must be factored into the deal price. In a similar way, the bank may also benefit from its own risk of bankruptcy.

Essentially, counterparty risk management is managing counterparty exposure. Recently, an increasingly popular exposure control mechanism is to use some form of break clause that allows the bank to terminate the portfolio with the counterparty in the event that some pre-agreed condition is breached by the counterparty. Such a break clause is called ATE. ATE can take different forms (Gregory 2010). A frequently used ATE trigger specification is credit rating. In a credit ATE clause, a credit rating trigger is specified for the counterparty. When the counterparty is downgraded to or below the trigger rating, the bank is entitled to terminate and close out all trading positions. Because a default is often preceded by significant credit downgrade, the credit ATE can effectively reduce counterparty risk by terminating exposure prior to default. One might view that credit ATE creates a sort of right-way exposure where the counterparty exposure is eliminated if the credit quality worsens significantly. From the modeling standpoint, ATE may also be considered as "lossless default" where the contracts are terminated with full recovery, as contrasted with default where the contracts are terminated with loss. However, as pointed out by Gregory (2010), ATE might actually drive the counterparty into default if the positions are closed out and the counterparty is significantly net out-of-the-money (OTM) on those positions.

While credit ATE can significantly reduce counterparty risk, it does not completely eliminate it as it is still possible that the counterparty can default without triggering the ATE event. This is evident that firms may default before being significantly downgraded by the rating agencies. In other words, the residual counterparty risk in the presence of credit ATE comes from the possibility of counterparty default without ever crossing the credit ATE trigger.

Counterparty risk modeling has attracted much attention because of the recent credit crisis. Alavian *et al* (2009), Gregory (2010), Pykhtin and Zhu (2007) provided excellent overview of counterparty risk management practice and CVA pricing. Zhu and Lomibao (2005) proposed a conditional valuation method for exposures of path-dependent derivatives. An overview of various types of ATE can be found in (Gregory 2010). Recently, Yi (2010) proposed a model for bilateral CVA subject to credit rating trigger. In their model, the time of hitting the credit trigger and jump-to-default are modeled as Poisson process.

In this paper, we present a practical model specifically for evaluating bilateral CVA (BCVA) of a portfolio subject to a credit ATE trigger. The discretized formulation naturally leads to a rating-based Markov chain model. We assume that portfolio termination and close-out is mandatory once the credit ATE trigger for either party is breached. For exposition convenience, all trades in the portfolio are assumed to be associated with the same ATE trigger rating, so that once a party is downgraded to or below its ATE trigger, the entire portfolio is terminated. The focus of the paper is modeling the effect of the credit ATE trigger on bilateral CVA. We suggest extension of the model to more complicated situations.

We assume that obligors with the same credit rating have the same rating transition probabilities. As such, the spread difference within a rating class is ignored.[4] We use the normal copula model for joint

[4] Spread difference within a rating class can vary substantially. It is often observed that the credit spread of a lower rated firm can be lower than that of a higher rated firm.

rating transition and default. The risk neutral generator matrices are obtained by calibrating the historical generator matrix to rating-based generic CDS spread curves.[5] To model the credit ATE trigger, we make the ratings that are equal to or below the ATE trigger rating absorbing states. This guarantees that default paths that result in credit loss do not cross the ATE trigger at any time, and enables to calculate the first passage time of the ATE trigger in terms of the ATE rating transition probabilities.

The rest of the paper is organized as follows. Section 2 presents the mathematical formulations of the bilateral CVA in continuous time and their discretization. We point out how the base model can be extended to deal with margins and multiple ATE triggers. Section 3 outlines formulae for calculating the joint transition and conditional joint default probabilities subject to credit ATE trigger. We introduce the ATE transition matrix and describe how to calculate the probability of hitting ATE trigger in terms of transition probabilities. We define the ATE factor profile. Section 4 shows numerical results. Section 5 concludes the paper. Detailed derivation is shown in the appendices.

2. The Model

Throughout this article, we refer the two parties to the underlying derivative trades in the portfolio as the bank, denoted by B, and the counterparty, denoted by C. We use the term party to refer both B and C if it applies to both. We value the portfolio from the bank's perspective. As such, a positive portfolio value or in-the-money (ITM) means the counterparty owes the bank money, and a negative portfolio value or out-of-the-money (OTM) means the reverse.

2.1 Nomenclatures

Before describing the model, we define the notations that will be used throughout this article without further explanation.

- *UCVA*: Unilateral CVA.
- *BCVA*: Bilateral CVA.
- $\tilde{Q}(t,T)$: Risk-neutral transition matrix from *t* to *T*.
- $\tilde{Q}_\omega(t,T)$: Risk-neutral ATE transition matrix from *t* to *T*.
- $\Xi = \{1,2,\cdots,K-1,K\}$: Rating state space where 1 refers to the highest rating class (e.g. AAA/Aaa) and K-1 the lowest rating class. K is the default state. D is also used to refer to the default state.[6]
- $\beta_k(t), k = B,C$: The rating of party *k* at time *t*.
- $\omega_k,\ k = B,C$: The ATE trigger rating of party *k*. If $\beta_k(t)$ crosses ω_k, the portfolio is terminated.
- $\Omega_k = \{\omega_k, \omega_k + 1, \cdots, K-1\} = \{j | j \geq \omega_k, j \neq D\}, k = B,C$: Set of ratings that are equal to or worse than the ATE trigger rating of party k.
- $\Pi_\mathrm{k} = \{1,2,\cdots,\omega_\mathrm{k} - 1\} = \Xi \backslash \{\Omega_\mathrm{k} \cup \{\mathrm{D}\}\}, \mathrm{k} = \mathrm{B},\mathrm{C}$: Set of ratings that are better than ATE trigger of *k*.
- $\eta_k = Min\{t | \beta_k(t) \in \Omega_k\}, k = B,C$: The first time that the party *k* crosses its ATE trigger rating.

[5] For pricing credit risk of a firm without market CDS spread, banks create rating-based generic CDS curves that map credit ratings to market CDS spreads. Generic CDS curve is essentially some average market quotes of CDS spread grouped by credit rating.

[6] The default state *D* is not a valid credit rating. However, transition matrix typically includes the default state for mathematical convenience.

- $\eta = Min(\eta_B, \eta_C)$: The first-to-ATE time of both the bank B and the counterparty C.
- $\tau_k, k = B, C$: The default time of party *k* without ever crossing the trigger (ATE default time).
- $\tau = Min(\tau_B, \tau_C)$: The first-to-default time of both the bank and the counterparty.
- $W(s, q)$: Present (time *t*) value of the cashflow on the portfolio between *s* and *q* where *t* is the valuation time and the portfolio final maturity is *T*.
- $R_k, k = B, C$: Recovery rate of party *k*.
- $\delta_k = 1 - R_k, k = B, C$: Loss-Given-Default (LGD) of party *k*.
- $f^+ = Max(f, 0)$, $f^- = Max(-f, 0)$, and $f = f^+ - f^-$.

2.2 Formulation of CVA with Credit ATE

As stated earlier, this paper is about modeling credit ATE. The key to modeling credit ATE is recognizing the only default scenario that can result in credit loss is if the defaulting party was never downgraded to or below its ATE trigger rating prior to default. In other words, the default paths leading to credit loss must jump from a rating higher than the ATE trigger directly to the default state without ever crossing the ATE trigger. Therefore, any sensible model must make sure that, for party *k,* those paths of rating transition that ever cross into Ω_k are excluded from the default probability. This is the guiding principle of our model.

We consider a portfolio of derivative contracts which are uncorrelated with the credit quality of either the bank or the counterparty.[7] As a result, the portfolio mark-to-market value $W(t, T)$ is independent of either party.[8]

When a party (either the bank or the counterparty) defaults, one of the following scenarios applies:

1) If $\tau > T$, no credit loss will incur to either party as the first-to-default event occurs after the final maturity of the portfolio.
2) If $\eta < \tau$, the first-to-ATE event occurs before the first-to-default event. Since the portfolio is terminated at the first-to-ATE time (mandatory termination), the exposure to either party at first-to-default is zero and hence no credit loss to either party.
3) If $\tau \leq Min\,\{\eta, T\}$, the first-to-default event happens before the portfolio expiry date and the first-to-ATE event. From the bank's perspective, the rule of default settlement is as follows:
    a. If the counterparty defaults first, $\tau_C < \tau_B$, then
        i. If the portfolio value at default $W(\tau_C, T) > 0$, i.e. the counterparty owes the bank money. The bank will receive from the counterparty the amount $R_C W(\tau_C, T)^+$.[9]
        ii. Else, if $W(\tau_C, T) < 0$, the bank owes the counterparty money. The bank pays the counterparty the full portfolio value $W(\tau_C, T)^-$.

[7] Examples are interest rate swaps, caps, swaptions, FX options, equity options. However, CDS or CDO tranches where the reference entities are correlated with either party do not belong to this category.
[8] See Brigo and Chourdakis (2008) for a model of unilateral CVA of CDS when the counterparty and the CDS reference entity are correlated. In their case, the portfolio value strongly correlated with the counterparty.
[9] Brigo and Morini (2010) argue that if the residual deals are taken over by another party, the default settlement amount should be different than the risk-free amount as the credit risk of the replacing counterparty should be factored in. We assume the risk-free portfolio value at default to be the settlement amount.

b. Conversely, if the bank defaults first, $\tau_B < \tau_C$, then
   i. The bank pays the counterparty $R_B W(\tau_B, T)^-$ if the bank owes the counterparty.
   ii. Else, the bank receives $W(\tau_B, T)^+$ if the counterparty owes the bank.
c. In the relatively rare situation where both the bank and the counterparty default at the same time, $\tau = \tau_C = \tau_B$, the bank would pay $R_B W(\tau, T)^-$ to the counterparty if $W(\tau, T) < 0$, or receive $R_C W(\tau, T)^+$ from the counterparty if $W(\tau, T) > 0$. Simultaneous defaults are less frequent than single default, and loss to the bank due to counterparty default and the benefit to the bank from its own default often cancel out to a significant extent.

The present risky value of the portfolio can be expressed as

$$\bar{V}(t,T) = E_t \left\{ \begin{array}{c} 1(\eta \geq \tau)1(\tau > T)W(t,T) + 1(\eta < \tau)W(t,T) + \\ 1(\eta \geq \tau)1(\tau \leq T) \left\{ \begin{array}{c} 1(\tau = \tau_C < \tau_B)[R_C W(\tau_C, T)^+ - W(\tau_C, T)^-] \\ +1(\tau = \tau_B < \tau_C)[W(\tau_B, T)^+ - R_B W(\tau_B, T)^-] \\ +W(t,\tau) - 1(\tau = \tau_C = \tau_B)W(\tau, T) \end{array} \right\} \end{array} \right\} \tag{1}$$

where $W(t, \tau)$ is the present value of cashflow on the portfolio from time *t* to the first-to-default time τ, and $W(\tau, T)$ is the present portfolio value at default time τ.

Using the relation $W(t,T) = W(t,\tau) + W(\tau,T)$ and $W(\tau,T) = W(\tau,T)^+ - W(\tau,T)^-$, we rewrite Eqn. (1) in a more intuitive form

$$\bar{V}(t,T) = E_t\{W(t,T)\} - E_t \left\{ 1(\eta \geq \tau)1(\tau \leq T) \left\{ \begin{array}{c} 1(\tau = \tau_C)\delta_C W(\tau_C, T)^+ \\ -1(\tau = \tau_B)\delta_B W(\tau_B, T)^- \end{array} \right\} \right\} \tag{2}$$

Eqn. (2) shows that the fair expected present value of the portfolio is the risk-free value of the portfolio minus an adjustment due to default by either party or both. This adjustment is commonly referred to as the credit value adjustment or CVA.

The risk-free portfolio value $E_t\{W(t,T)\}$ is independent of the ATE. This is expected because when the portfolio is terminated due to breach of ATE trigger, there is no credit loss. One might think of ATE event as a default with immediate full recovery, or lossless default.

Eqn. (2) extends the CVA formulation without ATE (for example, Gregory (2009), Alavian *et al* (2009)) to ATE. For example, if the portfolio $V(\tau, T)$ in Eqn. (5) of Gregory (2009) is replaced with $1(\eta \geq \tau)W(\tau, T)$, we obtain Eqn. (2) above. Put it another way, the exposure in the presence of ATE is contingent upon the ATE event not occurring before first-to-default. Eqn. (2) is general in the sense that it is valid for any ATE specification.[10] However, the presence of the indicator function $1(\eta \geq \tau)$ makes pricing CVA subject to ATE much more difficult because the first-to-ATE time η and the first-to-default time τ are generally correlated. Since $1(\eta \geq \tau)$ often decreases as the credit quality of either party worsens, ATE creates a sort of "right-way" risk exposure, in the sense that the exposure is non-increasing as the party's credit quality deteriorates.

[10] ATE can be specified as credit rating trigger, portfolio market value trigger, etc. (Gregory 2010). Different ATE types give rise to different η.

The bilateral CVA, denoted by BCVA, is the net expected loss or gain due to default by the counterparty and/or by the bank itself,

$$BCVA = E_t\left\{1(\eta \geq \tau)1(\tau \leq T)\begin{Bmatrix} 1(\tau = \tau_C)\delta_C W(\tau_C, T)^+ \\ -1(\tau = \tau_B)\delta_B W(\tau_B, T)^- \end{Bmatrix}\right\} \tag{3}$$

BCVA contains two terms. The first term is the credit loss which the bank will suffer if the counterparty defaults first prior to the first-to-ATE time and the final portfolio maturity. This is referred to as the unilateral CVA, denoted by *UCVA*. The second term represents the gain by the bank when it defaults first prior to the first-to-ATE time and the final portfolio maturity. This second term is referred to as the debt value adjustment, or DVA, as it is the benefit to the bank on its own debt. So the BCVA is the difference between UCVA and DVA. The BCVA can be negative if DVA exceeds UCVA. An example of negative BCVA is a portfolio of short position in options. In this case, the bank is always OTM if it has collected all option premiums from the counterparty.

The UCVA can be obtained by setting $\eta_B = \infty$ and $\tau_B = \infty$ in Eqn. (3), (see Remark 2.3)

$$UCVA = E_t\{1(\eta_C \geq \tau_C)1(\tau_C \leq T)\delta_C W(\tau_C, T)^+\} \tag{4}$$

Remark 2.1**:** The BCVA without ATE can be recovered by setting the ATE trigger equal to the default state ($\omega_B = \omega_C = D$), meaning that default is the only event that can terminate the portfolio prior to the final maturity. In this case, we have $\eta = \tau$, and hence $1(\eta \geq \tau) \equiv 1$.

Remark 2.2**:** It is conceivable that deal contracts may require that both S&P and Moody's ratings breach the ATE rating trigger. Ratings of these two rating agencies can occasionally differ (split ratings) although the difference is usually no more than one rating notch. We do not consider split ratings and refer to the paper of Lando and Mortensen (2005).

Remark 2.3**:** Although the credit risk of the bank does not appear explicitly in the UCVA formula (4), it does not necessarily imply that the credit quality of the bank has no influence on the UCVA. If we take the first term on the right-hand-side of Eqn. (3), the default time τ_C is given by a correlated default model. Through the correlation, the bank's credit risk implicitly influences (adjusts) the unilateral CVA.

Remark 2.4**:** In Eqn. (3), simultaneous default by B and C is implied by the condition $\tau = \tau_C = \tau_B$ and is modeled by the same joint transition/default model. This approach is appropriate under normal market condition when simultaneous default is infrequent. The likelihood of a simultaneous default tends to increase significantly when the credit market is under severe stress or in crisis mode. When the markets are in crisis, the systemic default risk is substantial. Simultaneous default can be handled by specifically modeling a common default time τ (Gregory 2009).

Remark 2.5**:** Another approach of modeling credit ATE is using Poisson process where one needs to model at least default process with ATE. Yi (2010) pointed out the difficulties in calibrating their model as the available market data do not contain credit trigger information. The ATE transition matrix proposed later in this paper may be helpful in this regard as it is adjusted for credit ATE trigger.

2.3 Discretization

For numerical discretization, we make the following assumptions:

(1) The risk-free portfolio value $W(t, T)$ is independent of the credit rating of either party. This assumption is mainly to simplify exposition. The model can easily be extended to rating dependent exposure, such as rating dependent margining.
(2) All obligors of a given rating are considered to have the same transition probabilities. Heterogeneity in credit spread between obligors with the same rating is ignored. Consequently, we use rating specific credit spreads rather than firm specific spreads. This is a restriction of rating based credit models.

We divide the time domain (t, T) into N sub-intervals, $t = t_0 < t_1 < \cdots < t_N = T$. We consider the loss due to default in time period $t_{k-1} < t \le t_k$ based on the exposure at t_k. In the discrete setting, we do not distinguish the time of default within the same time period, nor do we the ATE time. Consequently, if party *C* defaults in the interval $(t_{k-1}, t_k]$ and if $\eta_C > t_{k-1}$, then we say $\eta_C \ge \tau_C$. Therefore, we have the following set relation

$$\{\eta_C \ge \tau_C, \tau_C \in (t_{k-1}, t_k]\} \cong \{\eta_C > t_{k-1}, \tau_C \in (t_{k-1}, t_k]\} \quad (5)$$

By definition, $\eta_C = Min\{t | \beta_C(t) \in \Omega_C\}$ is the first time the rating of *C* crosses the ATE trigger rating and migrates into set Ω_C. If we are sure that once in Ω_C the counterparty C has no chance to either default or come out of Ω_C, or equivalently, if we specify the ratings in Ω_C to be absorbing states, we have

$$\{\eta_C \le t\} = \{\beta_C(t) \in \Omega_C\} = \cup_{j \in \Omega_C} \{\beta_C(t) = j\} \quad (6)$$

Base on similar reasoning, we also have

$$\{\eta_C > t\} = \{\beta_C(t) \in \Pi_C\} = \cup_{j \in \Pi_C} \{\beta_C(t) = j\} \quad (7)$$

The set relations (5-7) are important because they allow us to express the probability of ATE hitting time $P(\eta_C \le t)$ or the survival probability $P(\eta_C > t)$ in terms of the transition probability $P\{\beta_C(t) = j \in \Pi_C\}$ which is the probability of counterparty C migrating from its current rating to rating j at a future time t under the restriction that the path of migration cannot cross the ATE trigger. It is easy to see that the transition probability $P\{\beta_C(t) = j\}$ is significantly easier to calculate than $P(\eta_C \le t)$.[11] In the following, we will use these set relations to calculate the probabilities of the first-to-ATE time and the first-to-default time when subject to credit ATE trigger.

2.3.1 Unilateral CVA Discretization

Using relation (5), the Euler discretization of the *UCVA* of Eqn. (4) is

$$UCVA = \delta_C \sum_{k=1}^{N} P\{\tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}\} E_t\left\{W\left((t_k, T)\right)^+\right\} \quad (8)$$

[11] This is analogous to calculating the distribution of stock price hitting barrier vs the terminal stock price. The probability of hitting time or the first-passage-time is much harder to calculate.

From the law of total probability and using Eqn. (7), we can calculate the probability in Eqn. (8) in terms of the ATE transition probability and one-period conditional default probability

$$P\{\tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}\} = \sum_{j\in\Pi_C} P(\beta_C(t_{k-1}) = j)P(\tau_C \le t_k | \beta_C(t_{k-1}) = j) \quad (9)$$

where $P(\tau_C \le t_k | \beta_C(t_{k-1}) = j)$ is the one-step conditional default probability with the starting rating *j*. These probabilities can be calculated from the martingale ATE rating transition matrix which will be described in the following. We note that all default and transition when subject to credit ATE trigger is calculated using the martingale ATE transition matrix.

Substituting Eqn. (9) into Eqn. (8), we obtain

$$UCVA = \delta_C \sum_{k=1}^{N} E_t\left\{W\big((t_k, T)\big)^+\right\} \sum_{j\in\Pi_C} P(\beta_C(t_{k-1}) = j)P(\tau_C \le t_k | \beta_C(t_{k-1}) = j) \quad (10)$$

It is clear that the unilateral CVA of a portfolio subject to ATE trigger rating ω_C is entirely due to the possibility of counterparty C jumping to default without going through any rating that is not better than the credit ATE trigger ω_C.

2.3.2 Bilateral CVA Discretization

Analogously, the Euler discretization of BCVA of Eqn. (3) is[12]

$$BCVA = \left\{\begin{array}{c} \delta_C \sum_{k=1}^{N} P\{\tau_B > t_k, \tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} E_t\{W(t_k, T)^+\} \\ -\delta_B \sum_{k=1}^{N} P\{\tau_C > t_k, \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} E_t\{W(t_k, T)^-\} \\ + \sum_{k=1}^{N} P\{\tau_C \in (t_{k-1}, t_k], \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ \times \left\{\delta_C E_t\{W(t_k, T)^+\} - \delta_B E_t\{W(t_k, T)^-\}\right\} \end{array}\right\} \quad (11)$$

The first term represents the expected loss to the bank when the counterparty defaults first. The second term is the benefit the bank would gain if the bank itself defaults first. The third term is the expected PL when both B and C default simultaneously. In a normal market, the effect of simultaneous defaults is usually, but not always, small compared with the first two terms due to the cancelation effect and the fact that the probability of simultaneous default is small relative to single default.[13]

The forward default probabilities in Eqn. (11) are conditional on no prior first-to-ATE event happening, for once an ATE event occurred, the portfolio is terminated and subsequent default will not cause credit loss and CVA.

We now describe how to calculate the forward conditional default probability of counterparty conditional on no prior ATE event, $P\{\tau_B > t_k, \tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\}$, and the forward conditional simultaneous default probability $P\{\tau_C \in (t_{k-1}, t_k], \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\}$. By symmetry, other forward probabilities can be obtained from these formulas.

[12] In literatures, $E_t\{W(t_k, T)^+\}$ and $E_t\{W(t_k, T)^-\}$ are commonly referred to as expected positive exposure (EPE) and expected negative exposure (ENE), respectively.

[13] However, when the credit market is under stress and the systemic risk is high, simultaneous default risk can be high. In such a case, simultaneous default should be specifically modeled (Gregory 2009).

First, we write the forward probability of counterparty defaulting first as

$$P\{\tau_B > t_k, \tau_C \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} = P\{\tau_C > t_{k-1}, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\}$$

$$-P\{\tau_C > t_k, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \quad (12)$$

Using the relation $\{\eta > t\} \cap \{\eta \geq \tau\} \supseteq \{\tau > t\} \cap \{\eta \geq \tau\}$ and the law of total probability, we have

$$P(\tau_C > t_k, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_C > t_k, \tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (13)$$

where the transition probability $P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i)$ is calculated using Eqn. (B.5), and the conditional survival probability $P(\tau_C > t_k, \tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i)$ is from Eqn. (37).

$$P(\tau_C > t_{k-1}, \tau_B > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (14)$$

where $P(\tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i)$ is computed using Eqn. (34).

From the law of total probability, we can express the one-period conditional joint default probability as

$$P\{\tau_C \in (t_{k-1}, t_k], \tau_B \in (t_{k-1}, t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_C \leq t_k, \tau_B \leq t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad (15)$$

Derivation of Eqns. (12-14) is provided in Appendix A. These formulas are based on the assumption of set Ω_k being absorbing, enabling expressing probability of first-passage-time of the trigger by the ATE rating transition probability.

We emphasize that rating transition and conditional default probabilities are based on the condition that no ATE trigger has been breached prior to default. To this end, we make sure that once a rating crosses its ATE trigger, it will not transit further. There is no possibility that a party can be downgraded to or below its ATE trigger and subsequently be upgraded or default.[14] This can be easily achieved by making any rating class in the set $\Omega_k, k = B, C$ an absorbing state. As a result, the default probability is reduced as it is no longer possible to default from ratings in the set Ω_k. As shown numerically in section 3.6, given a transition matrix and an ATE trigger rating, we can quantify the amount of reduction in default probability due to credit ATE.

[14] This approach is similar to importance sampling where the probability distribution is altered to exclude those paths leading to zero payoff. Here, we exclude the paths that cross the credit ATE trigger.

2.4 Model Extensions

We have used the uncollateralized, rating independent portfolio value $W(t,T)$ to derive exposure. This significantly simplifies notation. In doing so, we implied that (a) all trades in the portfolio are nettable under a single netting agreement and (b) the portfolio is not collateralized or no margin requirement.[15] In this section, we suggest extension of the model to more practical and complicated portfolios. First, we note that extending the model to multiple netting agreements is straightforward (Pykhtin and Zhu 2007) since there is still only one ATE rating trigger.

2.4.1 Extension to Rating-Based Margin Threshold

Banks often impose margin threshold on their trading partners. Margin threshold is the maximum positive portfolio value the counterparty does not need to pose collateral. The purpose of margin threshold is to limit the bank's exposure to the counterparty. Margin threshold is often rating dependent where it decreases as the counterparty's credit quality declines.

Let $H(j)$ denote the margin threshold for the counterparty with rating *j*, the UCVA becomes

$$UCVA = E_t\{1(\eta_C \geq \tau_C)1(\tau_C \leq T)\delta_C \times Min[W(\tau_C,T)^+, H(\beta_C(\tau_C -))]\} =$$

$$\delta_C \sum_{k=1}^{N} \sum_{j\in\Pi_C} E_t\left\{Min\left(W\big((t_k,T)\big)^+, H(j)\right)\right\} P(\beta_C(t_{k-1}) = j)P(\tau_C \leq t_k | \beta_C(t_{k-1}) = j) \quad (16)$$

where $\beta_C(\tau_C -)$ is the rating of *C* immediately before default. We assume the margin is determined at t_{k-1} for exposure time t_k. The corresponding BCVA can be derived similarly.

2.4.2 Extension to Multiple ATE Triggers

Banks are increasingly including a credit ATE clause in the trading agreement. This can create a portfolio consisting of deals with heterogeneous ATE trigger ratings. If the netting agreement is based on ATE trigger where deals are nettable if they are subject to the same ATE trigger, the above model can be directly applied to each ATE trigger.

If, however, a single netting agreement covers multiple ATE triggers, extension is a little tricky. In this case, the unilateral CVA is

$$UCVA = E_t\left\{1(\tau_C \leq T)\delta_C\left[\sum_j 1(\eta^j \geq \tau_C)W_j(\tau_C,T)\right]^+\right\} \quad (17)$$

where η^j is the first hitting time of the *j*th ATE trigger, denoted by ω^j, and W_j is the total value of all deals subject to the *j*th ATE trigger. Note that Eqn. (17) includes deals without an ATE trigger, *j=0*. As stated earlier, the non-ATE case is treated by setting $\omega^0 = K$. Because the default state is the only termination trigger, we have $\eta^0 = \tau_C$ and $1(\eta^0 \geq \tau_C) \equiv 1$.

Recognizing all ATE rating triggers are associated with the same counterparty, breaching of a higher ATE trigger must occur no later than a lower ATE trigger. If we index the ATE triggers by $\omega^1 > \omega^2 > \cdots$, then

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[15] Collateralization and margin agreement are usually specified in CSA (Credit Support Annex). See Alavian *et al* (2009) and Gregory (2010) for overviews of CSA.

the relation $\eta^1 \geq \eta^2 \geq \cdots$ must hold, taking into consideration that multiple ATE's may be breached simultaneously.

A general discrete formulation for Eqn. (17) may be difficult. Here, we briefly consider a portfolio subject to two ATE triggers and the non-ATE trigger. The possible ATE hitting time scenarios for this case are

$$\{\eta^1 \geq \eta^2 \geq \tau_C\},\ \{\eta^1 \geq \tau_C > \eta^2\}, \text{ and } \{\eta^1 < \tau_C, \eta^2 < \tau_C\} \tag{18}$$

By the definition of credit ATE, $W_j(\tau_C, T) = 0$ when $\eta^j < \tau_C$. Consequently, we have

$$UCVA = \delta_C \sum_{k=1}^{N} \left\{ \begin{array}{c} E_t(W_0 + W_1 + W_2)^+ P(\eta^1 \geq \eta^2 > t_{k-1}, t_{k-1} < \tau_C \leq t_k) \\ +E_t(W_0 + W_1)^+ P(\eta^1 > t_{k-1}, \eta^2 \leq t_{k-1}, t_{k-1} < \tau_C \leq t_k) \\ +E_t\left({W_0}^+\right) P(\eta^1 \leq t_{k-1}, \eta^2 \leq t_{k-1}, t_{k-1} < \tau_C \leq t_k) \end{array} \right\} \tag{19}$$

where $W_j = W_j(t_k, T)$.

A complication in calculating $P(\eta^1 \geq \eta^2 > t_{k-1}, t_{k-1} < \tau_C \leq t_k)$ is the need to account for possibility of $\omega^2 \leq \beta_C(t_{k-1}) < \omega^1$, i.e. there are two distinct ATEs, each ATE trigger breach terminates only part of the portfolio. We do not elaborate further and leave the details to a future work.

3. Transition and Conditional Default Probability

Eqns. (8-15) show that CVA calculation involves two components. One is the calculation of EPE and ENE at the time nodes t_k. The other involves the calculation of the martingale ATE transition and the default probability conditional on that the ATE trigger has not been breached. Since this paper focuses on modeling ATE trigger and assumes that the portfolio has a unique ATE trigger for each party, we will not elaborate on exposure calculation, and refer to the paper of Zhu and Lomibao (2005) for background and further references. In this section, we describe a model for calculating rating transition probability and default probability conditional on no prior violation of ATE trigger.

The presence of credit ATE trigger necessitates modeling rating transition in addition to default. As a result, we adopt rating based models. As stated previously that, under the credit rating ATE clause, the portfolio between the bank and its counterparty is terminated and closed out once either party has crossed its respective ATE trigger. This implies that we only need to consider those paths of transition that never cross its ATE trigger. This is achieved by making the rating class set Ω_k absorbing.

The model is comprised of the following steps:

1) Choose a historical transition matrix, usually the one-year average transition matrix published by Moody's or Standard & Poor's. The transition matrix may be preprocessed to smooth out irregular behavior (Lando and Mortensen 2005). Calculate the corresponding historical generator matrix using either the JLT (Jarrow *et al* 1997) method or the IRW (Israel *et al* 2001) method.
2) Calibrate the historical generator matrix to the generic CDS spread term structure for each rating using the JLT method. This generates the martingale generator matrix.

3) For the ATE trigger rating ω_k of party *k*, modify the martingale generator matrix by setting rows from ω_k to $K-1$ to zero. The resulting ATE martingale generator matrix guarantees that the paths resulting in credit loss, or loss paths, cannot cross the trigger ω_k.
4) Calculate the required transition probabilities based on the martingale ATE generator matrix.
5) For BCVA, the joint ATE transition probabilities are evaluated using the normal copula model.

3.1 The Historical Transition Matrix

Although its shortcomings for credit derivative pricing are well documented (Schonbucher 2003), virtually all rating-based credit pricing models use a statistical transition matrix estimated from historical corporate default experience. A major reason is that it is simply impractical to imply all entries of the rating transition matrix from market price data. A historical transition matrix provides a structure upon which (model based) adjustments are made such that the (rating based) model prices match the market prices. This adjustment transforms the historical transition probability into the martingale transition probability. This transformation is necessary as CVA is the risk-neutral price of counterparty default risk.

Given a historical transition matrix *P*, we calculate its generator matrix denoted by Λ.[16] Unfortunately, majority of the historical transition matrices do not admit a valid generator as they fail the test of Theorem 3.1(c) of IRW, thereby guaranteeing non-existence of a valid generator matrix.[17]

One solution is to smooth the empirical transition matrix to avoid obvious violation of Theorem 3.1 of IRW, and then calculate a generator matrix. Lando and Mortensen (2005) (LM here after) proposed a method where the smoothing algorithm imposes some constraints based on economic considerations. Since Theorem 3.1 of IRW are sufficient but not necessary conditions for non-existence of a valid generator matrix, the smoothed transition matrix still does not guarantee a valid generator matrix.

Another approach is to assume a generator exists and to search for one. Methods of generator matrix calculation can be found, for example, in IRW, JLT, and Kreinin *et al* (KS here after) (2001). The JLT method is the simplest and guarantees to give a valid generator. The IRW method is shown to be more accurate as it results in a smaller fitting error to the historical transition matrix, but is more computationally involved.

In this paper, we also assume a generator matrix exists. We apply the JLT method (p. 505) to estimate a generator matrix from the Moody's one-year average transition matrix (Moody's 2009) proportionally adjusted for WR. We follow the suggestion of JLT (p. 504) that the martingale generator matrix be expressed as a product of the historical generator matrix and a time-dependent diagonal matrix which can be interpreted as risk premium. Next section describes the calibration of martingale generator matrix.

[16] A generator matrix $\Lambda=(\lambda_{ij})$ of transition matrix *P* is a $K\times K$ matrix with the property (1) $P=Exp(\Lambda)$, and (2) $\lambda_{i,j\neq i}\geq 0, \lambda_{ii}=-\sum_{j\neq i}\lambda_{ij}$, where λ_{ij} is the intensity of jump to rating *j* from *i* in an infinitesimal time interval.
[17] KS examined 32 empirical matrices and found that all passed the test of Theorem 3.1 (a) and (b) of IRW, but most failed test of (c). If the smoothed matrix is without zero entries, it passes test of (c).

3.2 Martingale Generator Matrix Calibration

Since CVA is an adjustment to the fair value of the portfolio, CVA valuation requires risk-neutral transition probabilities. Change from the actuarial rating transition probabilities to the risk-neutral ones is by fitting to the market prices of traded instruments, CDS premiums and bond, broken down by rating.

With the JLT method, the martingale transition matrix is assumed to be a product of a diagonal matrix, interpreted as risk premium, and the historical matrix. The martingale transition matrix is then used to fit the prices of bond of each rating. JLT method is simple and has clear interpretation. However, as shown by LM, the scheme is unstable as the risk premia can be very large especially for high ratings, resulting in unrealistic or even negative rating transition probabilities. Kijima and Komoribayashi (KK) (1998) proposed an alternative fitting procedure that avoids the issue of stability. However, their method results in counterintuitive behavior that the adjustments to non-default transition probabilities, whether downgrade or upgrade, are always opposite to the adjustment to the PD. For example, if the PD for AAA is adjusted upwards, the probabilities of transition to all ratings are adjusted downwards. Based on economic intuition, LM suggested adjusting downgrade probability in the same direction of PD adjustment and upgrade probability in the opposite direction. This is a very appealing idea as one would expect a higher downgrade probability when PD is higher. LM showed that their method yielded more reasonable transition probabilities than JLK and KK methods. A price to pay is that the fitting scheme becomes more complex.

We assume for each rating class there is a generic term structure of CDS premium. Assuming a constant recovery rate, rating-based PD term structure, denoted by $PD(t,\beta)$, can be obtained by bootstrapping the generic CDS curves.[18] We obtain the one-step martingale generator matrix by equating, at time t_k and for rating β, the cumulative martingale default probability with $PD(t_k,\beta)$.

Let $\tilde{Q}(t,T)$ be the martingale transition matrix from t to T, and recall that the martingale generator $\widetilde{\Lambda}(t)$ is the product of the historical generator Λ and a diagonal matrix[19]

$$\widetilde{\Lambda}(t) = U(t)\Lambda \text{ where } U(t) = Diag(\mu_1(t), \mu_2(t), \cdots, \mu_{K-1}(t), 1) \tag{20}$$

Assuming matrix $U(t)$ to be piecewise constant, the incremental martingale transition matrix over the period (t_{k-1}, t_k) is

$$\tilde{Q}(t_{k-1}, t_k) = \left(\tilde{q}_{ij}(t_{n-1}, t_n)\right)_{K\times K} = Exp\{U(t_{k-1})\Lambda\Delta t_k\} \tag{21}$$

Since the rating transition is a Markov chain, the cumulative martingale transition matrix over the period $(0, t_n)$ can be obtained recursively

$$\tilde{Q}(0, t_n) = \prod_{k=1}^{n} \tilde{Q}(t_{k-1}, t_k) = \tilde{Q}(0, t_{n-1})\tilde{Q}(t_{n-1}, t_n) \tag{22}$$

[18] An alternative is to calibrate to the CDS spreads or bond prices directly.

[19] The last element of $U(t)$ does not matter because the last row of the historical generator Λ is zero.

Since the rating-based default probability at t_n is given by the *K*th (last) column of $\tilde{Q}(0,t_n)$, provided $\tilde{Q}(0,t_{n-1})$ is known, the risk premium matrix $U(t_{k-1})$ satisfies

$$\tilde{Q}(0,t_{n-1})\begin{pmatrix}\tilde{q}_{1K}(t_{n-1},t_n)\\ \tilde{q}_{2K}(t_{n-1},t_n)\\ \vdots\\ \tilde{q}_{K-1,K}(t_{n-1},t_n)\\ 1\end{pmatrix}=\begin{pmatrix}PD(t_n,1)\\ PD(t_n,2)\\ \vdots\\ PD(t_n,K-1)\\ 1\end{pmatrix} \tag{23}$$

Eqn. (21) is nonlinear and requires matrix exponential calculation which can be expensive.[20] For a small time period Δt_k, Eqn. (21) can be approximated by the first-order expansion

$$\tilde{Q}(t_{n-1},t_n)=Exp\{U(t_{k-1})\Lambda\Delta t_k\}\approx I+U(t_{k-1})\Lambda\Delta t_k \tag{24}$$

Substituting the last column of Eqn. (24)

$$\begin{pmatrix}\tilde{q}_{1K}(t_{n-1},t_n)\\ \tilde{q}_{2K}(t_{n-1},t_n)\\ \vdots\\ \tilde{q}_{K-1,K}(t_{n-1},t_n)\\ 1\end{pmatrix}=\Delta t_n\begin{pmatrix}\mu_1(t_{n-1})\lambda_{1K}\\ \mu_2(t_{n-1})\lambda_{2K}\\ \vdots\\ \mu_{K-1}(t_{n-1})\lambda_{K-1,K}\\ 0\end{pmatrix}+\begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix} \tag{25}$$

into Eqn. (23), and solve the resulting equation for matrix $U(t_{k-1})$, we obtain

$$\begin{pmatrix}\mu_1(t_{n-1})\\ \mu_2(t_{n-1})\\ \vdots\\ \mu_{K-1}(t_{n-1})\\ 0\end{pmatrix}=\mathrm{A}\tilde{Q}(0,t_{n-1})^{-1}\begin{pmatrix}PD(t_n,1)\\ PD(t_n,2)\\ \vdots\\ PD(t_n,K-1)\\ 0\end{pmatrix}/\Delta t_n \tag{26}$$

where (notice that λ_{jK} is the *jK*-th entry of the generator matrix Λ)

$$\mathrm{A}=Diag\left(1/\lambda_{1K},1/\lambda_{2K},\cdots,1/\lambda_{K-1,K},0\right)_{K\times K} \tag{27}$$

Since λ_{jK} is the default intensity of rating *j* in an infinitesimal period, a non-zero default intensity, $\lambda_{jK}>0$, cannot correspond to a zero cumulative probability, $p_{jK}=0$. It must be $\lambda_{jK}=0$ if $p_{jK}=0$. The implication is that the above calibration algorithm breaks down if $p_{jK}=0$ for some *j*. Note that the cumulative default probability is not necessarily zero even if the corresponding default intensity is. Eqns. (25) and (26) also show why JLT method can produce very large risk premia.

The JLT method guarantees positive (zero) λ_{jK} if the rating *j*'s historical default probability p_{jK} is positive (zero). Unfortunately, the one-year rating transition matrices published by Moody's and S&P have consistently shown zero one-year PD for Aaa/AAA rating, although the five-year PD is not zero. The zero one-year PD for Aaa/AAA rating is an issue of data reliability which we do not address here.

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[20] Matrix exponential functionality is not available in many software packages. Developing efficient matrix exponential computer codes is not trivial.

Lando and Mortensen (2005) proposed a sophisticated method to transform the original historical transition matrix into a smoothed one that satisfies the imposed constraints justified by economic arguments. Their results showed (Appendix C of LM) that the smoothed matrix has no zero PD. JLT method simply assigned a reasonable non-zero value to p_{jK} whenever it is equal to zero.

We believe that a sophisticated approach, such as the LM method, is warranted as it produces a smoother historical transition matrix. Since the transition matrix is used as an input, there is virtually no computational cost for enterprise application except the initial cost of developing methodology and computer program. Nevertheless, for the sake of simplicity, we adopt the simpler JLT approach in our numerical computation. First, we "arbitrarily" assign a value to the one-year PD for Aaa in the historical transition matrix. We then calculate the historical generator matrix Λ from P using the JLT method. The historical generator Λ guarantees no zero entry in the last column so the diagonal matrix A of Eqn. (27) is well defined.

Having calibrated the risk premium matrix $U(t_{n-1})$, we proceed to calculate the one-period incremental risk-neutral transition matrix $\tilde{Q}(t_{n-1}, t_n)$ using Eqn. (24). The time t_n martingale transition matrix $\tilde{Q}(0, t_n)$ is obtained from Eqn. (22). However, $\tilde{Q}(0, t_n)$ cannot be directly used for CVA calculation when credit ATE triggers are present as it did not exclude the possibility that the obligor can breach the ATE trigger and recover. This brings us to the subject of adjusting the generator matrix to account for ATE trigger.

Remark 3.1: Since the risk premium matrix $U(t)$is time-dependent but deterministic, the transition process is Markov but time inhomogeneous.

3.3 The ATE Generator and Transition Matrices

The main idea of using a credit ATE trigger as counterparty risk mitigant is that it may eliminate the bank's exposure to counterparty by terminating the portfolio when the counterparty is downgraded below the ATE trigger.

Effective modeling of ATE rating trigger requires that the set Ω_k be absorbing, so that the probability of jump to default without crossing ATE trigger – credit loss probability – is not overstated.[21] This can be achieved by modifying either the martingale transition matrix or the martingale generator matrix. The difference is that the generator matrix is for continuous time Markov chain, and the transition matrix is for discrete Markov chain. Under the continuous Markov chain setting there is absolutely no possibility to migrate out of Ω_k, whereas in the discrete Markov chain migration into and then out of Ω_k in the same period is still possible. As will be shown in section 4.1, both the probability of default and the probability of not breaching ATE are smaller with the generator matrix approach than with the transition matrix approach.

Given the ATE trigger rating ω, we define the risk-neutral ATE generator matrix as

$$\widetilde{\Lambda}_\omega(t) = \Psi(\omega)\widetilde{\Lambda}(t) = \Psi(\omega)U(t)\Lambda \qquad (28)$$

[21] If the loss probability is overstated, CVA is overstated, defeating the purpose of ATE.

where

$$\Psi(\omega) = \begin{pmatrix} I_{(\omega-1)\times(\omega-1)} & 0 \\ 0 & 0 \end{pmatrix}_{K\times K} \tag{29}$$

is a matrix operator that sets the rows of the operand equal or below ω to zero.

The one-period risk-neutral ATE transition matrix is defined as

$$\tilde{Q}_\omega(t_{k-1}, t_k) = Exp\{\tilde{\Lambda}_\omega(\mathrm{t_{k-1}})\Delta t_k\} \cong I + \tilde{\Lambda}_\omega(\mathrm{t_{k-1}})\Delta t_k \tag{30}$$

Clearly, for rows of $\tilde{Q}(t_{k-1}, t_k)$ below $\omega - 1$, the off-diagonal entries are equal to zero and the diagonal entries are equal to one. This guarantees that the set $\Omega = \{\omega, \omega + 1, \cdots, K - 1\}$ be absorbing.

The time t_n martingale ATE transition matrix is

$$\tilde{Q}_\omega(0, t_n) = \prod_{k=1}^{n} \tilde{Q}_\omega(t_{k-1}, t_k) \tag{31}$$

Given the current (t = 0) counterparty rating $\beta_C(0)$, we can calculate the probability in Eqn. (10) as

$$P(\beta_C(t_{n-1}) = j) = \{\tilde{Q}_{\omega_C}(0, t_{n-1})\}_{\beta_C(0),j} \tag{32}$$

$$P(\tau_C \le t_n | \beta_C(t_{n-1}) = j) = \{\tilde{Q}_{\omega_C}(t_{n-1}, t_n)\}_{jK} \tag{33}$$

where the rating $j \in \Pi_C$. In our model, rating belong to the rating set Ω_C do not contribute to the CVA calculation.

3.4 The Joint Rating Transition

If the rating transition of the bank is independent of that of the counterparty, the bilateral CVA is simply the difference of the two standalone CVAs, one for counterparty risk and the other (DVA) for bank's own default risk, each evaluated using the unilateral CVA model without regarding the other.

However, as shown by Gregory (2009), default correlation between the parties can significantly impact the CVA, even the unilateral CVA. For calculating bilateral CVA under the rating-based Markov chain model described above, it is convenient to adopt a normal copula model for correlated rating transition.[22] The details are given in Appendices B and C. Here we give the formulae for joint rating transition and conditional joint default probabilities.

The formulation for time t_n transition probability $P(\beta_C(t_n) = j, \beta_B(t_n) = i)$ is given by Eqn. (B.5). For rating pair $(j, i) \in \Pi_C \times \Pi_B$ and from Eqn. (B.6), we obtain

$$P(\tau_B > t_n | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) = 1 - P(\tau_B \le t_n | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i)$$

$$= 1 - \sum_{m \in \Xi} P(\beta_B(t_n) = D, \beta_C(t_n) = m | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) \tag{34}$$

[22] Other copula models, such as t-copula copula, can also be used. The t-copula has tail dependency resulting in a larger joint default probability than the normal copula which has no tail dependency.

Let $\Gamma = \Xi \backslash D = \{1,2,\cdots,K-1\}$ be the set of all non-default ratings. Using Eqn. (B.5) and the set relation

$$\{\tau_B \leq t_n\} = \{\beta_B(t_n) = K\} \text{ and } \{\tau_B > t_n\} = \{\beta_B(t_n) \in \Gamma\} \tag{35}$$

we obtain

$$\begin{aligned} &P(\beta_B(t_n) = D, \beta_C(t_n) = m | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) \\ &\quad = N_2(\sigma_m^C, \sigma_K^B; \rho) - N_2(\sigma_{m+1}^C, \sigma_K^B; \rho) \end{aligned} \tag{36}$$

where $N_2(x, y; \rho)$ is the standard bivariate normal distribution function, and

$$\begin{aligned} &P(\tau_C > t_n, \tau_B > t_n | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) = \\ &\textstyle\sum_{(a,b)\in\Gamma\times\Gamma} P(\beta_C(t_n) = a, \beta_B(t_n) = b | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i) = \\ &\textstyle\sum_{(a,b)\in\Gamma\times\Gamma} \{N_2(\sigma_a^C, \sigma_b^B; \rho) - N_2(\sigma_{a+1}^C, \sigma_b^B; \rho) - N_2(\sigma_a^C, \sigma_{b+1}^B; \rho) + N_2(\sigma_{a+1}^C, \sigma_{b+1}^B; \rho)\} \end{aligned} \tag{37}$$

where

$$\sigma_m^k = N^{-1}\left(\textstyle\sum_{q=m}^K \tilde{Q}_{\omega_k}(t_{n-1}, t_n)_{\beta_k(t_{n-1}),q}\right), \;\; k = B, C \tag{38}$$

3.5 The ATE Factors

The purpose of credit ATE is to eliminate counterparty exposure when counterparty credit risk increases. As a result, credit ATE reduces CVA. Obviously, the extent of CVA reduction depends on the exposure profiles $EPE = E_t\{W(t_k,T)^+\}$ and $ENE = E_t\{W(t_k,T)^-\}$. Generally, ATE is more effective for portfolio with long final maturity. As portfolio exposure calculation is often expensive, it is desirable to be able to estimate the percentage of CVA reduction given the credit ratings of both parties and the placement of the credit ATE triggers. This can be accomplished by the ATE factors.

The ATE factors are multipliers that, when applied to each period CVA without ATE, produces CVA subject to ATE. ATE factors depend only on the ratings of the two parties and their respective credit ATE triggers.

Notice that the BCVA without ATE trigger is given by

$$BCVA_{No-ATE} = \begin{Bmatrix} \delta_C \sum_{k=1}^N P\{\tau_B^R > t_k, \tau_C^R \in (t_{k-1}, t_k]\} E_t\{W(t_k,T)^+\} \\ -\delta_B \sum_{k=1}^N P\{\tau_C^R > t_k, \tau_B^R \in (t_{k-1}, t_k]\} E_t\{W(t_k,T)^-\} \\ + \sum_{k=1}^N P\{\tau_B^R \in (t_{k-1}, t_k], \tau_C^R \in (t_{k-1}, t_k]\} \\ \times \{\delta_C E_t\{W(t_k,T)^+\} - \delta_B E_t\{W(t_k,T)^-\}\} \end{Bmatrix} \tag{39}$$

where τ_k^R is the default time without ATE for party $k = B, C$. Remember, as we emphasized repeatedly in this paper, that CVA with ATE is calculated using the risk-neutral ATE transition matrix, and τ_k^R and τ_k follow different distributions.

We define bilateral ATE factor profiles for the bank *B* and the counterparty *C*,

$$R^{C,ATE}(t_{k-1},t_k) = \left\{ \begin{matrix} P\{\tau_B > t_k, \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ +P\{\tau_B \in (t_{k-1},t_k], \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \end{matrix} \right\}$$

$$\div \left\{ P\{\tau_B^R > t_k, \tau_C^R \in (t_{k-1},t_k]\} + P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R \in (t_{k-1},t_k]\} \right\} \quad (40)$$

$$R^{B,ATE}(t_{k-1},t_k) = \left\{ \begin{matrix} P\{\tau_B \in (t_{k-1},t_k], \tau_C > t_k, \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \\ +P\{\tau_B \in (t_{k-1},t_k], \tau_C \in (t_{k-1},t_k], \eta_C > t_{k-1}, \eta_B > t_{k-1}\} \end{matrix} \right\}$$

$$\div \left\{ P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R > t_k\} + P\{\tau_B^R \in (t_{k-1},t_k], \tau_C^R \in (t_{k-1},t_k]\} \right\} \quad (41)$$

As mentioned earlier, while the paths of τ_k cannot go through Ω_k, there is no such restriction on τ_k^R. The possible paths of τ_k are a subset of those of τ_k^R. Hence, τ_k^R and τ_k follow different distributions where $P(\tau_k^R \in (t_{k-1},t_k]) \geq P(\tau_k \in (t_{k-1},t_k], \eta_k > t_{k-1})$. Therefore, we have the bounds

$$0 \leq R^{C,ATE}(t_{k-1},t_k) \leq 1 \text{ and } 0 \leq R^{B,ATE}(t_{k-1},t_k) \leq 1 \quad (42)$$

In terms of the ATE factor profiles, we can rewrite the BCVA formulation (11) as

$$BCVA = \left\{ \begin{matrix} \delta_C \sum_{k=1}^N R^{C,ATE}(t_{k-1},t_k) P\{\tau_B^R > t_k, \tau_C^R \in (t_{k-1},t_k]\} E_t\{W(t_k,T)^+\} \\ -\delta_B \sum_{k=1}^N R^{B,ATE}(t_{k-1},t_k) P\{\tau_C^R > t_k, \tau_B^R \in (t_{k-1},t_k]\} E_t\{W(t_k,T)^-\} \end{matrix} \right\} \quad (43)$$

The bilateral ATE factor profiles may be useful as guideline in setting ATE trigger, as well as adapting existing CVA calculation computer programs to CVA with ATE.

4. Numerical Results

We now present numerical results to show the difference between the transition matrix, generator matrix based transition matrix, and the ATE transition matrix. For this purpose and for clarity, we use an artificial matrix as the input transition matrix.

4.1 ATE Transition Matrix: An Example

This example shows that an ATE trigger reduces the probabilities of portfolio remaining alive or default with loss. This effect is more pronounced if the ATE transition matrix is based on the ATE generator since the generator does not allow any possibility that the rating can transit from the ATE trigger rating or below. This latter observation is the reason we use the risk-neutral ATE generator to calculate the risk-neutral transition matrix. The fact that ATE reduces the probability of default is the fundamental reason for CVA reduction.

Suppose we are given a transition matrix P of four rating classes A, B, C and D where D is the default state. The generator matrix of P is Λ.

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} -0.551 & 0.3535 & 0.1294 & 0.0678 \\ 0.1531 & -0.822 & 0.4719 & 0.1972 \\ 0.1767 & 0.4482 & -1.046 & 0.4213 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Without ATE trigger, the two-period transition probability is the probability of transition from one rating to another or default over the two period time where any non-default rating can transit to any rating or default. For example, rating A may transit to rating C at the end of period one and from rating C back to rating A in the second period. Thus, over the two periods, rating A stays where it starts consists of three possible transition paths, $A \to A \to A$, $A \to B \to A$ and $A \to C \to A$. The two-period probability of $A \to A$ is the sum of the probabilities of these three paths. The non-ATE two-period transition matrix is

$$P2 = P \times P = \begin{pmatrix} 0.39 & 0.24 & 0.14 & 0.23 \\ 0.13 & 0.31 & 0.19 & 0.37 \\ 0.12 & 0.20 & 0.21 & 0.47 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4.1.1 Case 1: High ATE Trigger Rating

When the ATE trigger rating is *B*, the one- and two-period ATE transition matrices directly from *P* are

$$P^B = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}; \ and \qquad P2^B = P^B \times P^B = \begin{pmatrix} 0.36 & 0.32 & 0.16 & 0.16 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix}$$

Comparing the two-period ATE transition matrix $P2^B$ with the non-ATE matrix *P2* shows that ATE trigger at rating B

1) Reduces the probability of remaining at rating A and default; and
2) Increases the probabilities of crossing the trigger rating.

The ATE generator matrix Λ^B and the associated two-period ATE transition matrix $\tilde{Q}^B(0,2)$ are

$$\Lambda^B = \begin{pmatrix} -0.551 & 0.3535 & 0.1294 & 0.0678 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}; \ \tilde{Q}^B(0,2) = Exp(2\Lambda^B) = \begin{pmatrix} 0.3324 & 0.4285 & 0.1569 & 0.0822 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

Comparing with $P2^B$, we see that in $\tilde{Q}^B(0,2)$,

a) The probability of default and remaining at A is further reduced, and reduction in PD is significant;
b) The probability of crossing the ATE trigger rating B is further increased.

Again, we emphasize that the underlying reason for this pattern is that $P2$ prohibits only inter-period transition from ratings *B* and *C* back to rating *A*. But it implicitly permits intra-period rating transition from *B* and *C*. On the contrary, $\tilde{Q}^B(0,2)$ strictly prohibits intra-period migration from *B* and *C*.

4.1.2 Case 2: Low ATE Trigger Rating

We now move the trigger rating a notch lower from rating *B* to rating *C*. We want to illustrate that rating *C* as the trigger is less effective than rating *B* in the sense that PD with loss is greater in Case 2 than in Case 1, and the probability of crossing the trigger is lower in Case 2 than in Case 1.

When rating *C* is ATE trigger, the one- and two-period ATE transition matrices directly from *P* are

$$P^C = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \; and \quad P2^C = P^C \times P^C = \begin{pmatrix} 0.38 & 0.22 & 0.20 & 0.20 \\ 0.11 & 0.27 & 0.31 & 0.31 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix}$$

The ATE generator matrix Λ^C and its associated two-period ATE transition matrix $\tilde{Q}^C(0,2)$ are

$$\Lambda^C = \begin{pmatrix} -0.551 & 0.3535 & 0.1294 & 0.0687 \\ 0.1531 & -0.822 & 0.4719 & 0.1972 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}; \quad \tilde{Q}^C(0,2) = Exp(2\Lambda^C) = \begin{pmatrix} 0.3632 & 0.1879 & 0.3045 & 0.1443 \\ 0.0814 & 0.2189 & 0.4922 & 0.2075 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

It can be easily verified that all conclusions from Case 1 still hold.

We see from comparing $\tilde{Q}^C(0,2)$ with $\tilde{Q}^B(0,2)$ that

1) The two-period default probability of rating A is 0.1443 when the trigger is *C* (Case 2) which is significantly higher than the PD of 0.0822 when the trigger is B (Case 1).
2) The total ATE termination probability is 0.5854 in Case 1, and is 0.3045 in Case 2. So there is a much greater likelihood that ATE occurs in Case 1 than Case 2.

Since CVA is a measure of loss upon default and the likelihood of default, and that ATE termination is equivalent to lossless default, CVA is smaller in Case 1. This example explains why ATE is more effective when the trigger rating is higher.

5. Conclusions

We have presented a rating-based Markov chain model for valuation of bilateral CVA of a derivative portfolio subject to a credit ATE trigger. The model is comprised of several key components. First, as most rating-based credit risk model, we take a (one-year) historical rating transition matrix which is readily available from the major rating agencies. From this historical transition matrix, we compute the historical generator matrix.

Second, we use the JLT method to calculate the risk-neutral generator matrix by calibrating to the market implied CDS spread curves broken down by rating.

Third, the credit ATE trigger is modeled by the risk-neutral ATE transition matrix. Assuming mandatory termination and close-out of the portfolio upon first breaching of ATE trigger, the risk-neutral ATE generator matrix is obtained by assigning zero value to the appropriate rows of the risk-neutral generator matrix of the 2nd step. The ATE transition probabilities permit to transform calculating the probability of first crossing the ATE trigger into calculating transition probability.

The ATE transition matrix is comprised of the transition probability from one rating to another without ever crossing the ATE trigger. ATE transition matrix is appropriate since the default under ATE is the jump-to-default from above the ATE trigger rating where jump means jumping over all ratings equal to and below the ATE trigger. The paths of jump-to-default are only a subset of all possible paths that lead to default. Hence, the PD under ATE is lower than the actual PD, as shown by a numerical example. It is this reduction in PD that mitigates the counterparty default risk.

Fourth, we use the normal copula model for joint rating transition where the marginal rating transition thresholds are mapped to the rating transition probability of each party viewed standalone.

We introduce the ATE factor profile defined as the ratio of PD with ATE and PD without ATE. The ATE factor profile depends only on the firm's credit ratings and the ATE triggers, and does not involve the actual portfolio composition.

Appendix A: Rating Transition and Conditional Default Probability

In this section, we prove Eqns. (12-14). Again, we emphasize that, with credit ATE, our method assumes the rating classes not higher than the ATE trigger are absorbing states. For party *k*, any rating transition path leading to a live rating - either the default or another rating above the ATE trigger - cannot pass through the set Ω_k at any time.

From the set relation

$$\{\tau_B > t_k, \tau_C > t_k, \eta_B > t_{k-1}, \eta_C > t_{k-1}\} = \{\tau_B > t_k, \tau_C > t_k\} \cap \{\eta_B > t_{k-1}, \eta_C > t_{k-1}\}$$

$$= \cup_{\substack{j \in \Pi_C \\ i \in \Pi_B}} \{\{\tau_B > t_k, \tau_C > t_k\} \cap \{\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i\}\} \quad \text{(A.1)}$$

and the law of total probability, we obtain

$$P(\tau_B > t_k, \tau_C > t_k, \eta_B > t_{k-1}, \eta_C > t_{k-1}) =$$

$$\sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k, \tau_C > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad \text{(A.2)}$$

This proves Eqn. (13).

To prove Eqn. (14), noticing the set relation

$$\{\tau_B > t_k, \tau_C > t_{k-1}, \eta_B > t_{k-1}, \eta_C > t_{k-1}\} = \{\tau_B > t_k, \tau_C > t_{k-1}\} \cap \{\eta_B > t_{k-1}, \eta_C > t_{k-1}\}$$

$$= \cup_{\substack{j \in \Pi_C \\ i \in \Pi_B}} \{\{\tau_B > t_k\} \cap \{\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i\}\} \quad \text{(A.3)}$$

where we have used the set relation $\{\beta_C(t_{k-1}) \in \Pi_C\} \supseteq \{\tau_C > t_{k-1}\}$.

Again, by virtue of the law of total probability, Eqn. (A.3) implies that

$$P(\tau_B > t_k, \tau_C > t_{k-1}, \eta_B > t_{k-1}, \eta_C > t_{k-1})$$

$$= \sum_{\substack{j \in \Pi_C \\ i \in \Pi_B}} P(\beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) P(\tau_B > t_k | \beta_C(t_{k-1}) = j, \beta_B(t_{k-1}) = i) \quad \text{(A.4)}$$

which proves Eqn. (14).

Appendix B: A Copula Model for Joint Rating Transition

We describe a simple one-factor normal copula model for joint rating transition of the counterparty and the bank. This model is an extension of the model used in the credit VaR model (RiskMetrics 1997).

Let X_k denote the credit quality index of party *k*. in the one-factor model, X_k is decomposed into a systemic component *Y* (market factor) that is common to both parties and a firm specific component ϵ_k (idiosyncratic factor)

$$X_k = \sqrt{\rho}Y + \sqrt{1-\rho}\epsilon_k, k = B, C \tag{B.1}$$

where *Y* and ϵ_k are assumed to follow the standard normal distribution. ρ is the asset correlation that measures the co-movement of the credit indices. Eqn. (B.1) is used for modeling the conditional joint default over the one-step (t_{n-1}, t_n) as well as the long step $(0, t_n)$ rating transition matrix. As such, the correlation ρ perhaps should be associated with a maturity tag and calibrated accordingly.

Suppose we know the risk-neutral ATE transition matrix $\tilde{Q}_{\omega_k}(0, t_n)$ and the current rating $\beta_k(0)$ for party *k*, we calculate $P(\beta_C(t_n) = j, \beta_B(t_n) = i)$ where $j \in \Pi_C, i \in \Pi_B$. The use of ATE transition matrices guarantees there would be no prior breach of ATE trigger by party *k*.

We map X_k onto a grid of rating change from the current rating $\beta_k(0)$. The grid size is determined by the risk-neutral ATE transition matrix $\tilde{Q}_{\omega_k}(0, t_n)$ or $\tilde{Q}_{\omega_k}(t_{n-1}, t_n)$. To this end, we introduce the time t_n rating transition thresholds for party *k* by[23]

$$-\infty = \alpha_{K+1}^k < \alpha_K^k < \alpha_{K-1}^k < \cdots < \alpha_2^k < \alpha_1^k = \infty \tag{B.2}$$

Given the initial rating $\beta_k(0)$, the grid thresholds are determined such that $\beta_k(t_n) = j$ when $X_k \in \left(\alpha_{j+1}^k, \alpha_j^k\right)$. In the normal copula method, the thresholds are calculated recursively by

$$P\left(\alpha_{j+1}^k < X_k \le \alpha_j^k\right) = N\left(\alpha_j^k\right) - N\left(\alpha_{j+1}^k\right) = \tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),j} \tag{B.3}$$

where $\tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),j}$ is the $(\beta_k(0), j)$-*th* entry of $\tilde{Q}_{\omega_k}(0, t_n)$, and $N(x)$ and $N^{-1}(x)$ are, respectively, the standard normal distribution function and the standard inverse normal distribution function.

Recall that X_k obeys the standard normal distribution, solving for α_j^k recursively we obtain

$$\alpha_j^k = N^{-1}\left(\textstyle\sum_{i=j}^K \tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),i}\right), \; j = K, \cdots, 2 \tag{B.4}$$

Having obtained the rating transition thresholds for both the bank and the counterparty, the joint transition probability from the initial joint rating state $\left(\beta_C(0), \beta_B(0)\right) \in \Pi_C \times \Pi_B$ to the joint rating state at t_n $\left(\beta_C(t_n), \beta_B(t_n)\right) = (j, i)$ is

$$P(\beta_C(t_n) = j, \beta_B(t_n) = i) = P\left\{\alpha_{j+1}^C < X_C \le \alpha_j^C, \alpha_{i+1}^B < X_B \le \alpha_i^B\right\}$$

---

[23] Actually, $\alpha_1^k = \infty$ is a result of $\sum_{i=1}^K \tilde{Q}_{\omega_k}(0, t_n)_{\beta_k(0),i} = 1$.

$$= N_2(\alpha_j^C, \alpha_i^B; \rho) - N_2(\alpha_{j+1}^C, \alpha_i^B; \rho) - N_2(\alpha_j^C, \alpha_{i+1}^B; \rho) + N_2(\alpha_{j+1}^C, \alpha_{i+1}^B; \rho) \quad \text{(B.5)}$$

Setting the current time to t_{n-1} and following the same procedure, we obtain the one-step conditional joint default probability

$$P(\tau_C \le t_n, \tau_B \le t_n | \beta_C(t_{n-1}) = j, \beta_B(t_{n-1}) = i)$$

$$= N_2\left(N^{-1}\left(\tilde{Q}_{\omega_C}(t_{n-1}, t_n)_{jK}\right), N^{-1}\left(\tilde{Q}_{\omega_B}(t_{n-1}, t_n)_{iK}\right); \rho\right) \quad \text{(B.6)}$$

Eqn. (B.6) is the base formula from which other conditional joint probabilities in section 3.4 can be obtained.

Appendix C: Correlation Estimation

If the credit index X_k in Eqn. (B.1) is interpreted as the log asset return of the party k, the parameter ρ is the pair-wise asset return correlation between the two parties. The correlation is an unobservable parameter and needs to be estimated. Asset correlation estimation is very difficult.

Here, we mention several methods for estimating asset return correlation. These are by no means the only asset correlation estimation models.

Other than picking a fixed number, the simplest estimating/forecasting method is to use the equity correlation as proxy for asset correlation for the equity correlation is readily available from the equity price. In the CreditMetrcs method, a firm's equity return is assumed to be a weighted average of the returns of country/industry indices with the weights specified based on the firm's industry participation (RiskMetrics 1997). A major drawback of using equity correlation as a proxy is it ignores the significant difference between equity and asset, especially for financial firms (Zeng and Zhang, 2001a).

Another approach is to infer the pair-wise asset correlation from the joint and the single name default probabilities of the two parties.

Zeng and Zhang (2001b) assessed the performance of three widely used asset correlation estimation models - historical models, average models and factor models. They concluded that the KMV's Global Correlation Model performed best.

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