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Abstract

We prove the existence of a Pareto optimal state of a society with non-binary personal preferences. To our knowledge, this is the weakest set of conditions under which the existence of a Pareto optimal state has been proven. In our theory everybody in society engages in maximization as a *personal* act of volitional choice based on non-binary preferences, as in Sen (1997). The resultant equilibrium belongs to a unanimity-based *nonempty social maximal* set. Our generalization exposes the fact that such equilibria support discrimination, which is a surprising, though serious, indictment of relying exclusively on the Pareto principle in social evaluation. (100 words)

Keywords: Non-binary choice, Non-binary preferences, Maximization, Pareto optimality

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1. Introduction

We prove the existence of a Pareto optimal state of a finite society with non-binary personal preferences. To our knowledge, our statement constitutes the weakest set of conditions under which the existence of a Pareto optimal state has been proven to date. This paper can be seen as extending to the case of a *finite society* the work of Amartya Sen (1997) on maximization as a *personal* act of volitional choice based on non-binary preferences, which is an enormous aid in our construction of a theory of a *social interactional outcome*, construed here as a generalization of the very concept of equilibrium in economic theory. Such an "equilibrium" social outcome is defined as the existence of a nonempty *social interactional maximal* set.[1]

How does our theory differ from mainstream economic theory? Arrow (1998), as always, exhibits his perspicacity as follows,

> "The answer depends in part as to what we mean by economic theory. Certainly, "rational choice theory" is broader than "economic theory." Rational choice theory means that the individual actors act rationally (that is, by maximizing according to a complete ordering) within the constraints imposed by preferences, technology, and beliefs, and by the institutions which determine how individual actions interact to determine outcomes. Further, the beliefs are themselves formed by some kind of rational process. By economic theory, we mean that in some sense, markets are the central institution in which individual actions interact and that other institutions are of negligible importance."

Since a market supports *impersonal* exchange, in the competitive case it is irrelevant whether a seller sells to one demander or another, nor does it matter to a buyer as to from which seller the purchase is made. All inter-personal differences in *identities of distinct individuals* are obliterated in both rational choice theory and present-day economic theory. Binariness of personal preferences is decidedly too restrictive a requirement for socio-economic analysis of several types of collective outcomes in society. This is because "inter-personal differences" are defined as equivalent to "differences in the order in which the persons rank order their respective feasible alternatives." This banishes from the conceptual framework *all other* considerations of personal characteristics including forms of identification with distinct socio-economic groupings that exist in society. Operationally, therefore, a person cannot be black or white, or male or female, and so on, both in rational choice theory and in mainstream economic theory, simply because these are considerations that are *tertiary* to the binariness of personal preferences.

The principal reason that a new formal theory needs construction is that in mainstream economic theory, a person in a society (1) is identified exclusively by a *binary* relation of (weak) preference, (2) some restrictions are placed on this relation that generates a complete (or sometimes incomplete) ordering of the set of feasible personal alternatives of choice, (3) the sole motivation of the person is to maximize personal preference so as to determine personal choice, and (4) a set of additional sufficient restrictions are imposed to precipitate the existence of a cohesive *social (un)interactional outcome*, which is typically referred to as a *general equilibrium*.

To be able to investigate social phenomena where gender or race or any other such differences in personal identities are operationally significant, a release from the present confining format of economic theory is needed. We are aided, as noted above, in our effort by

[1] To be distinguished from a nonempty personal maximal set.

the groundbreaking work of Amartya Sen (1997) on the theory of rational *personal* behavior that is based on *non-binary personal preferences,* thereby setting the stage for us to formulate a theory that goes considerably beyond the rather confining mainstream economic theory that is alluded to by Arrow. To that we now proceed.

2. Preliminaries

For a given set V_i^j, let $R_i(V_i^j)$ be person *i*'s binary relation of weak preference that stands for "at least as good as", which is defined on a finite set S_i of alternatives social states, and V_i^j is a background set on which the binary relation R_i is dependent, with $i = 1, \dots, n$, and $j = 1, \dots, k_i$ specifying the possible parametric variations, V_i^j, of person *i*'s background set. Here, $n \geq 2$ is finite, $k_i \geq 2$ is finite, and S_i has at least three elements.

For $R_i(V_i^j)$, we can define the asymmetric part $P_i(V_i^j)$ that stands for "strict preference", and the symmetric part $I_i(V_i^j)$ that stands for "indifference" as follows.

DEFINITION 1: $(\forall i, \forall j \ \& \forall x, y \in S_i)$: $[[xR_i(V_i^j)y] \& \sim[yR_i(V_i^j)x]] \leftrightarrow [xP_i(V_i^j)y]$.

DEFINITION 2: $(\forall i, \forall j \ \& \forall x, y \in S_i)$: $[[xR_i(V_i^j)y] \& [yR_i(V_i^j)x]] \leftrightarrow [xI_i(V_i^j)y]$.

In this context, it is important to note that a variation in a *tertiary* consideration, viz., a parametric variation in the background set, can, in general, alter the order of personal preference insofar as $(\forall i, \forall x, y \in S_i \ \& \ \exists j = l, m, l \neq m)$: $[xP_i(V_i^l)y]$ & $[yP_i(V_i^m)x]$, are both admissible, thereby rendering $R_i(V_i^j)$ a non-binary relation. In fact, Sen (1997) develops the theory of maximization as a *personal* act of non-binary choice based on non-binary personal preference. Our purpose is not to examine the issues that arise in the context of binariness or non-binariness *per se*; Sen has already done that in his groundbreaking work in the context of personal choice.

Our purpose here, instead, is to examine if a *social interaction outcome* based on *non-binary* personal preferences exists, and whether it is Pareto optimal. We prove that, in fact, there exists such an outcome, and that it is Pareto optimal, under some extremely mild conditions, at least judging by the restrictions imposed in the literature to demonstrate the existence of alternative types of equilibria that are based on mainstream utility theory and choice theory. Clearly, such a social outcome is a generalization that goes considerably beyond all of the alternative concepts of equilibrium in general equilibrium theory or in game theory that are predicated on binary personal preferences. There are three significant implications.

First, notice that since $R_i(V_i^j)$ a non-binary relation insofar as $(\forall i, \forall x, y \in S_i \ \& \ \exists j = l, m, l \neq m)$: $[xP_i(V_i^l)y]$ & $[yP_i(V_i^m)x]$ are both admissible, not a single person is *indecisive* over the pair (*x, y*). In fact, everybody is perfectly decisive in strictly preferring *x* over *y* when the conditions that obtain are V_i^l, and unanimously equally decisive in preferring *y* over *x* under distinct conditions V_i^m. However, the pair of alternatives *x* and *y* are still not rank comparable if background set variations are also accommodated in the conceptual framework in which to think and reason about such tertiary considerations.

Second, identities, prejudices and biases, and shared identities and biases, can all be operationally captured by the personal background sets, which can then, in a one-way direction, have influence over the form that personal preferences may take. This is a new route that is opened up , for every person is society, to have personal beliefs and values influenced by, though not entirely determined by, social norms and customs.

Third, non-binariness of personal preferences violates some standard behavioral assumptions of rational choice theory such as the *Weak Axiom of Revealed Preference*, according to which, if x is chosen from $S \subseteq X$, and $x \in T \subseteq S$, then x must not be rejected in choice from T in favor of a distinct $y \in T$. The theory of non-binary choice based on preference can, in fact, accommodate this violation of *WARP* as well, and thus its scope and reach extend well beyond the present-day theories of existence of equilibria –general or in a game - that rely entirely on standard binary choice theory for determining behavior of an agent or a player. This has substantive implications for enriching the class of social phenomena that can be explained axiomatically. Moreover, such discourse would be occurring in a conceptual framework that is sufficiently rich that it permits us to bring reason to bear on such an examination.

3. Existence of a Pareto Optimal State

To achieve our objective, we utilize three lemmas in Sen (1970) with relatively minor generalizations to prove an existence theorem.[2] First, however, some definitions are in order.

DEFINITION 3: $R_i(V_i^j)$ is reflexive over S_i if and only if $(\forall i, \forall j \;\&\forall x \in S_i)$: $\left[xR_i(V_i^j)x\right]$.

DEFINITION 4: *Acyclicity*: holds if and only if $(\forall i, \forall j \;\&\; \forall x_1,\ x_2, \dots, x_l\ \in S_i,)$: $[(x_1 P_i(V_i^j)x_2$ & $x_2 P_i(V_i^j)x_3, \dots \& \; x_{l-1} P_1(V_i^j)x_l)]\&(l \geq 3) \rightarrow\ x_1 R(V_i^j)x_l$.

DEFINITION 5: A ranking relation that is reflexive and acyclical is called an *acyclic-ordering*.

Let $J = \bigcup_{i=1}^{n}(\bigcup_{j=1}^{k_i} V_i^j)$, and $S = \bigcap_{i=1}^{n} S_i \neq \emptyset$, and assume that S has at least three elements.

DEFINITION 6: A social interaction outcome *rule* is a functional relation f that assigns exactly one social ranking $R(S,J)$ of S to an inter-personal non-binary preference profile, such that

$R(S,J) = f\left(R_1(V_1^j), \dots, R_n(V_n^j)\right)$, where $\forall i, j$: $R_i(V_i^j)$ is an acyclic-ordering of S_i.[3]

By $P(S,J)$we denote the asymmetric part of $R(S,J)$.We next turn to unanimity under all possible variations of the background set to define Pareto preference.

DEFINITION 7A: $\forall j \;\&\forall x, y \in S$: $\left[\forall i \colon x\ R_i(V_i^j)\ y\right] \leftrightarrow x\ \hat{R}(S,J)\ y$.

DEFINITION 7B: $\forall j \;\&\forall x, y \in S$: $\left[\forall i \colon x\ P_i(V_i^j)\ y\right] \leftrightarrow x\ \hat{P}(S,J) y$.

Remark: Definition 7A is a generalization of the Pareto 'preference or indifference' rule to non-binary personal preferences over the set S of alternative social states, denoted by $\hat{R}(S,J)$, and similarly, Definition 7B is a generalization of the Pareto 'strict preference' rule, denoted by $\hat{P}(S,J)$.

DEFINITION 8: A social state x in S is *Pareto optimal* if and only if it is not Pareto dominated by any state y in S in accordance with Definition 7B.

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[2] *Lemma 3.2* below is Sen's (1997) own generalization to non-binary preferences (under transitivity). *Lemma 3.1* below is our generalization to non-binary preferences, and *Theorem 1* is our generalization of a lemma in Sen (1970), also to the case of non-binary preferences.

[3] A social interaction outcome rule is a generalization of what Sen (1970) calls a collective choice rule.

Finally, using Definitions 6, *7A*, and *7B*, and by requiring that $[x\ R(S,J)\ y] \leftrightarrow [x\ \hat{R}(S,J)y]$ and $[x\ P(S,J)\ y] \leftrightarrow [x\ \hat{P}(S,J)y]$, we can obtain a maximal social interaction outcome by using the following two lemmas.

LEMMA 3.1. *$\hat{R}(S,J)$ is an acyclic-ordering of S.*

Proof: (See Sen (1970, Lemma 2*a, p.29)).

$\forall j\ \&\ \forall x \in S$, since by Definition 6, $\forall i: x\ R_i(V_i^j)\ x$, it follows that $\hat{R}(S,J)$ is reflexive. Also,

$$(\forall x_1,\ x_2, \dots, x_l\ \in S): [(x_1\hat{P}(S,J)x_2\ \&\ x_2\hat{P}(S,J)x_3, \dots \&\ x_{l-1}\hat{P}(S,J)x_l)] \rightarrow\ x_1 R(V_i^j)x_l$$

$$\rightarrow [(x_1\hat{R}(S,J)x_2\ \&\ x_2\hat{R}(S,J)x_3, \dots \&\ x_{l-1}\hat{R}(S,J)x_l)]$$

$$\rightarrow \left[\forall j\ \&\ \left\{\forall i: x_1\ R_i(V_i^j)\ x_2\ \&\ x_2\ R_i(V_i^j)\ x_3 \dots\ x_{l-1}\ R_i(V_i^j)\ x_l\right\}\right]$$

$$\rightarrow \forall j\ [\forall i: x_1\ R_i(V_i^j)\ x_l]$$

$$\rightarrow x_1\ \hat{R}(S,J)x_l\ .\ \blacklozenge$$

Next, consider

DEFINITION 9: $M(\hat{R},S,J) = \{x|x \in S\ \&\ \sim[\exists y \in S: y\ \hat{P}(S,J)\ x]\}$.

Remark: The social interaction maximal set of socially undominated elements of *S* is fully captured by Definition 9 with respect to the Pareto rule $\hat{P}(S,J)$, which is the asymmetric part given in Definition 7B .

DEFINITION 10: A social interaction outcome *x* exists if and only if $x \in M(\hat{R},S,J) \neq \emptyset$.

Remark: Definition 10 defines the concept of general equilibrium in this society, and is referred to here as a social interaction outcome.

LEMMA 3.2. *The maximal set is non-empty for every finite set acyclically-ordered by a non-binary preference relation.*

Proof: (See Sen (1970, Lemma 1*b, p.11, and Sen (1997)). Let $S = \{x_1, \dots, x_m\}$. Let $a_1 = x_1$, and follow the recursive rule $x_{q+1}\ \hat{P}(S,J)x_q \rightarrow a_{q+1} = x_{q+1}$, and $a_{q+1} = a_q$, otherwise, so that by construction, x_m is a maximal element. ♦

Remark: Note that non-binariness of *personal* preferences, in the sense that $(\forall i, \forall x, y \in S_i\ \&\ \exists j = l, m, l \neq m)$: $\left[xP_i(V_i^l)y\right]\ \&\ [yP_i(V_i^m)x]$ are both admissible, poses no problem for obtaining a nonempty *social* maximal set since the personal non-comparability of a pair of alternatives in *S* is rendered irrelevant for defining the maximal set. This, of course, is not true of the social *optimal* set of best elements that is defined as $C(\hat{R},S,J) = \{x|x \in S\ \&\ \forall y \in S: x\ \hat{R}(S,J)y\}$, which would necessarily be empty if $(\forall i, \forall x, y \in S_i\ \&\ \exists j = l, m, l \neq m)$: $\left[xP_i(V_i^l)y\right]\ \&\ [yP_i(V_i^m)x]$ are both admissible.

Thus, requiring maximizing behavior as an act of volitional personal choice, instead of the more demanding optimization, does have an advantage in the case of non-comparability arising from non-binariness of personal preferences. In fact, it should not come as a surprise that once there is a social acyclic-ordering which ranks at least one pair of alternatives, though not necessarily all such pairs, if and only if these two alternatives are comparable over all individuals and over all background sets, there must be an element which is Pareto undominated and thus Pareto optimal. This would also follow from Zorn's lemma.

In the case of personal choice theory, Sen (1997) exploits precisely this combination of (A) non-binariness of preferences (and the entailed partially non-comparable ranking), *and* demanding (B) maximizing personal behavior, that precipitates the existence of a maximal element despite non-comparability. The existence of an optimal element is more demanding, in that it requires (a) optimizing personal behavior, which yields the existence of such an optimal element only if (b) all personal preferences are binary, and they are reflexive transitive and also complete – this is the formulation of standard rational choice theory – both in general equilibrium theory and in game theory. We merely take Sen's (1997) work one step further to obtain a social interactional, rather than a personal, nonempty maximal set, and hence have the following result.

THEOREM 3.1. *For every set of non-binary personal preferences* $\left(R_1(V_1^j), \ldots, R_n(V_n^j)\right)$ *over a finite set S of alternative social states, where* $\forall i, j: R_i(V_i^j)$ *is an acyclic-ordering, there exists a nonempty maximal social interaction set* $M(\hat{R}, S, J)$ *that contains at least one Pareto-optimal state.*

Proof: (See Sen (1970, Lemma 2*e, p.30)). By Lemma 1, the Pareto preference or indifference relation $\hat{R}(S, J)$ is an acyclic-ordering of the set S of alternative social states. And considering Definitions 8, 9 and 10, it follows that the Pareto-optimal subset of S is equivalent to the social maximal set $M(\hat{R}, S, J)$. Further, since S is finite, and $\hat{R}(S, J)$ acyclically-orders it, by Lemma 2, $M(\hat{R}, S, J)$ is nonempty. Hence a non-binary personal preferences based social interaction outcome exists, and it is Pareto optimal. ♦

4. Concluding Remarks

Notice that, unlike the case of the existence of equilibrium in an Arrow-Debreu exchange economy, and unlike the case of existence of Nash equilibrium, for our existence result we do not impose any requirements other than reflexivity and acyclicity on personal preferences. Moreover, moving from *form* to alternative *interpretations* of the background set, and by considering parametric variations of this set, many of the inadequacies in explanations of social and economic phenomena entailed by binariness are entirely jettisoned, and indeed replaced by a much more comprehensive conceptual structure that still exhibits the unanimity property embodied in the Pareto rule.

By specifying in each personal background set both (a) community values and beliefs (such as giving one's set to a senior citizen, or not grabbing the largest slice of cake), and (b) several observable marks of identification (such as life expectancy, gender, race, and so on), our theory of a society can provide formal explanations for the existence of social outcomes that can exhibit racial or gender discrimination.

Since these discriminatory social states also are Pareto optimal, advocacy of such optimality implies tolerating discrimination, which constitutes a somewhat surprising, though rather serious, indictment of Pareto optimality as an objective worthy of pursuit.[4]

[4] This technical note can also be seen as serving to provide the formal basis of the claims made in Naqvi (2010).

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