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Entropy in the Creation of Knowledge: a Candidate Source of Endogenous Business Cycles

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Abstract

Two sector growth models, with physical goods and human capital produced under distinct technologies, generally consider a process of knowledge obsolescence / depreciation that is similar to the depreciation process of physical goods. As a consequence, the long term rate of *per capita* growth of the main economic aggregates is constant over time. This rate can be endogenously determined (in endogenous growth models, where production is subject to constant returns) or it can be the result of exogenous forces, like technological progress or population dynamics (in neoclassical growth theory, where decreasing marginal returns prevail).

In this paper, we introduce a new assumption about the generation of knowledge, which involves entropy, i.e., introducing additional knowledge to generate more knowledge becomes counterproductive after a given point. The new assumption is explored in scenarios of neoclassical and endogenous growth and it is able to justify endogenous fluctuations. Entropy in the creation of knowledge will imply that human capital does not grow steadily over time. Instead, cycles of various periodicities are observable for different degrees of entropy. Complete a-periodicity (chaos) is also found for particular values of an entropy parameter. This behaviour of the human capital variable spreads to the whole economy given that this input is used in the production of final goods and, thus, main economic aggregates time paths (i.e., the time paths of physical capital, consumption and output) will also evolve following a cyclical pattern. With this argument, we intend to give support to the view of endogenous business cycles in the growth process, which is alternative to the two mainstream views on business cycles: the RBC theory and the Keynesian interpretation.

Keywords: Growth theory, Endogenous business cycles, Nonlinear dynamics, Entropy, Knowledge.

JEL classification: O41, E32, C61.

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1. Introduction

The original definition of entropy, proposed by Shannon (1948), refers to a measure of uncertainty within some sort of system. Given various possible outcomes, a higher level of entropy implies that the probabilities assigned to each outcome become closer to each other, and in the limit maximum entropy case a complete impossibility of making rational choices arises (all outcomes are equally probable, and thus the choice is simply random). The original work of Shannon (1948) and Shannon and Weaver (1949) was developed in the context of the theory of information, and thus it became a fundamental tool in communication studies, where entropy is thought as the loss of information that occurs with the transmission of some message [see, e.g., Heath and Bryant (2000)]. In a more current use of the term, entropy may be understood as the opposite of synergy, that is, the loss of productivity that occurs when people work together rather than working by themselves.

Within an economic interpretation of this idea, one can associate entropy with the presence of negative marginal returns in the accumulation of a given input. In the present analysis, we associate this comprising notion of entropy with the production of knowledge; the argument is that for a specific type of knowledge there are positive returns until a given level of this input is accumulated, but after a given point additional knowledge is synonymous of a net loss, as the dissemination of such knowledge suffers from decreasing quality. In the same way as the excess of information implies a loss of quality in the transmission of a message, the excess of accumulated knowledge implies that the dissemination of this factor loses quality and part of the input is simply lost.

While our definition of entropy is somehow too inclusive and departs from the Shannon's initial notion of uncertainty, the mathematical concept to be used relies on the original formulation. As we shall see in the next section, the accumulation of the knowledge input is the result of a production process (a trivial production function is assumed), but the following entropy term is associated to the rule of knowledge evolution through time: $E_t = -\tilde{h}_t \cdot \ln \tilde{h}_t$, with $\tilde{h}_t$ the knowledge variable.[1] This term will imply that an accumulation rule characterized by decreasing or constant marginal returns becomes a hump-shaped function which replaces the conventional concave or linear shape.[2] Therefore, we introduce a new way of understanding the creation of knowledge; in our view, this is not only subject to depreciation and obsolescence; it is subject to entropy, since 'common knowledge' can destroy partially the true meaning of the originally generated ideas.

The main implication of a hump-shaped accumulation rule is that with it arises the possibility of 'strange' dynamic behaviour. It is known [see, e.g., May (1976)] that this type of function is able to produce more than the simple long term results of fixed point stability and instability; periodic and a-periodic motion, that is, cycles of different orders and chaos, will be observable for some combinations of parameters values. Under this reasoning, our work is close to the path breaking idea of Day (1982), who explained endogenous cycles within an economic growth framework, by considering a pollution effect that implied that after a given level of accumulated physical capital, the stock of this input began to be destroyed. This is indeed similar to

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[1] In the proposed expression, we consider the natural logarithm instead of the base 2 logarithm of the original Shannon's analysis. As referred in Sato, Akiyama and Crutchfield (2004), this does not change the interpretation of the notion of entropy; it just changes the measure unit [entropy is evaluated in nats (natural digits) instead of bits (binary digits)].

[2] Later, in section 2, we present a phase diagram (figure 8) describing this dynamic rule, which effectively reveals the hump-shaped form.

our argument, which states that too much knowledge introduces entropy in the dissemination of knowledge provoking a destruction of part of the existing knowledge.

The knowledge input is introduced in neoclassical and endogenous growth models, and it intends to give a possible explanation to endogenous business cycles. Optimal growth models are able to explain the long run trends of growth, but they fail to address the issue of economic fluctuations, unless we consider some departure from the Walrasian competitive market structure. In fact, considering that the generation of knowledge is subject to entropy, we are introducing a kind of inefficiency that is able to support the existence of cycles.

The endogenous cycles' literature was introduced by Medio (1979), Stutzer (1980), Benhabib and Day (1981), Day (1982) and Grandmont (1985), in the context not only of intertemporal optimal control models but also under overlapping generations frameworks. This work has gained some strength with the discovery that under increasing returns or a strong externality effect in production, the standard one sector growth model with the consideration of the labour-leisure trade-off is able to generate endogenous fluctuations [see, e.g., Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002) and Coury and Wen (2005)]. Despite the relevance and intuitive appealing of the endogenous business cycles literature, this continues to be somehow marginal relatively to the two main strands of thought about business cycles: the Real Business Cycles theory and the Keynesian view of incomplete markets and nominal rigidities.

In this paper, we intend to contribute to the literature on endogenous cycles, but introducing a new source of fluctuations, which is, as stated, the presence of entropy in the creation of knowledge. The new assumption is worked out under neoclassical and endogenous growth scenarios; these maintain their essential features in terms of the characteristics of the growth process, but for selected parameters values the trends of growth are replaced by more realistic fluctuations around those trends of growth.

The remainder of the paper is organized as follows. Section 2 studies the dynamics of the accumulation of knowledge under entropy. Section 3 introduces the knowledge rule in a neoclassical growth framework, and section 4 considers the same rule in a setup of endogenous growth. Finally, section 5 is destined to a short conclusion.

2. Entropy and Knowledge

Consider $\tilde{h}_t$ a non-rival knowledge variable necessary to produce human capital. This variable will be integrated into growth setups along the next two sections. In this section, we define and study a rule that characterizes its movement over time. A production function for knowledge is assumed, $f(\tilde{h}_t) = \tilde{B}\tilde{h}_t^{\eta}$, where $\tilde{B} > 0$ and $\eta>0$; we leave open the possibility of decreasing, constant or increasing marginal returns in the generation of knowledge. If one considers that this type of knowledge is subject to a conventional process of depreciation / obsolescence, then $\eta<1$ implies that $\tilde{h}_t$ converges to a constant value (zero growth), $\eta=1$ means a positive constant growth long run outcome, and $\eta>1$ is associated with an unstable outcome.

Instead of a simple depreciation / obsolescence process we assume that the knowledge variable is subject to entropy, as defined in the introduction. Therefore, we take the following rule for the accumulation of knowledge over time,

$$\tilde{h}_{t+1} - \tilde{h}_t = \tilde{B}\tilde{h}_t^{\eta} - \tilde{\delta}\tilde{h}_t \ln \tilde{h}_t, \quad \tilde{h}_0 \text{ given} \qquad (1)$$

where $\tilde{\delta}$ is the entropy parameter. Through entropy, we have introduced a nonlinear component in the process of accumulation of knowledge, which will have a dramatic impact over the long term behaviour of the endogenous variable. The dynamics of equation (1) can be studied locally or globally. We begin with a note about local dynamics.

Consider first the particular case η=1; the constant returns case is highlighted separately because it is the only one that allows for a straightforward computation of the steady state and for an explicit stability result. In the presence of constant returns, the absence of entropy ($\tilde{\delta} = 0$) would mean that $\tilde{h}_t$ would not assume a constant steady state value; instead, the variable would grow at a constant positive rate $\tilde{B}$; when one introduces entropy in the process of knowledge creation, one will observe that the entropy term implies an inefficiency that may transform, for some parameter values, the constant growth process into a process of zero growth (and, thus, a constant equilibrium value for $\tilde{h}_t$) or even into a process of periodic or a-periodic motion. Nevertheless, this last possible outcome is not captured by a local analysis. In what concerns local stability, proposition 1 synthesizes the dynamic nature of (1).

Proposition 1. *The knowledge accumulation difference equation with entropy and constant marginal returns has a unique steady state point. This point is stable for* $\tilde{\delta} < 2$*; instability prevails for* $\tilde{\delta} > 2$*; and* $\tilde{\delta} = 2$ *corresponds to a bifurcation point.*

Proof: Let $G(\tilde{h}_t) = (1+\tilde{B}) \cdot \tilde{h}_t - \tilde{\delta}\tilde{h}_t \ln \tilde{h}_t$. The steady state value of $\tilde{h}_t$ is found by solving $G(\tilde{h}_t) = \tilde{h}_t$ in order to the endogenous variable. In a straightforward way, one finds $\tilde{h}^* = e^{\tilde{B}/\tilde{\delta}}$ to be the unique steady state point. To inquire what kind of stability underlies $\tilde{h}^*$, one computes the derivative $\partial G(\tilde{h}_t) / \partial \tilde{h}_t \big|_{\tilde{h}_t = \tilde{h}^*} = 1 - \tilde{\delta}$. Considering $\tilde{\delta} > 0$, that is, that positive entropy exists, we just have to impose $1 - \tilde{\delta} > -1$ to guarantee that the derivative lies inside the unit circle; as a consequence, stability will require $\tilde{\delta} < 2$. Conversely, $\tilde{\delta} > 2$ implies instability. The value $\tilde{\delta} = 2$ indicates a point of transition between stable and unstable areas, and thus respects to a point of bifurcation■

Allowing for a parameter η different from 1, we cannot find an explicit steady state value for $\tilde{h}_t$; this is the solution of $\tilde{h}^* : \tilde{h}_t^{1-\eta} \ln \tilde{h}_t = \tilde{B} / \tilde{\delta}$. The stability result is given by proposition 2.

Proposition 2. *The knowledge accumulation difference equation with entropy is stable for* $\tilde{h}^* < e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$, *unstable for* $\tilde{h}^* > e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$, *and* $\tilde{h}^* = e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$ *respects to a bifurcation point.*

Proof: Consider a generic *G* function, including the possibilities of decreasing, constant and increasing returns: $G(\tilde{h}_t) = \tilde{B}\tilde{h}_t^{\eta} + \tilde{h}_t - \tilde{\delta}\tilde{h}_t \ln \tilde{h}_t$. Computing the derivative, $\partial G(\tilde{h}_t) / \partial \tilde{h}_t \Big|_{\tilde{h}_t = \tilde{h}^*} = 1 - \tilde{\delta} - (1-\eta) \cdot \tilde{\delta} \ln \tilde{h}^*$. Note that in the present case, we have to consider $\tilde{h}^* > 1$, in order to guarantee that $\ln \tilde{h}^* > 0$, a condition that is required for a reasonable steady state result (which corresponds to a positive value of the knowledge variable). Note also that the case studied in proposition 1 is just a particular case of the equation here in appreciation and, thus, assuming η=1 the above derivative reduces to the one in the precedent analysis.

Stability requires $1 - \tilde{\delta} - (1-\eta) \cdot \tilde{\delta} \ln \tilde{h}^* > -1$, what implies $\tilde{h}^* < e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$. Instability will be given by the symmetric condition, $\tilde{h}^* > e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$, and the bifurcation result corresponds to the border case $\tilde{h}^* = e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}}$ ■

One can further explore the local properties of the model, by studying in more detail the nature of the bifurcation point. This can be defined as the point in which the following combination of parameters holds, $\tilde{B} = e^{\frac{2-\tilde{\delta}}{(1-\eta)\cdot\tilde{\delta}}} \cdot \frac{2-\tilde{\delta}}{1-\eta}$. Assuming a constant value for one of the parameters, one can draw in the space of the other parameters a bifurcation line that divides the areas of stability and instability. Consider, as an example, that η=0.5; figure 1 displays the bifurcation line and the regions of stability / instability in the space ($\tilde{B}, \tilde{\delta}$).

In figure 1, the region of stability (*S*) is located below the bifurcation line (*bif*), while the region of instability (*U*) is located above this line. To generalize the result, one presents the same graphic as in figure 1, now highlighting different bifurcation lines for different possibilities regarding the value of η (figure 2).

Figure 2 is illustrative about the nature of the bifurcation. For constant returns in the accumulation of knowledge, the bifurcation is not dependent on $\tilde{B}$ (only on $\tilde{\delta}$). Decreasing and increasing returns mean different slopes of the bifurcation line; in the first case, this is a negatively sloped curve in the space ($\tilde{B}, \tilde{\delta}$), and it becomes positively sloped for η>1. In either case, the region of stability will locate below the line.

The local stability analysis of equation (1) has allowed to understand how the consideration of entropy changes the dynamics of a simple accumulation process. Without entropy, a stability result (η<1) or an instability outcome (η<1) would characterize the dynamics of (1) independently of the value of other parameters (in particular, $\tilde{B}$). With entropy, independently of assuming decreasing, constant or increasing returns, a bifurcation that separates regions of stability and instability is always identified. The bifurcation will not depend on the productivity parameter $\tilde{B}$ only in the special case of constant marginal returns. Therefore, except for η=1, the value of $\tilde{B}$ is decisive for the stability result that is obtained, alongside with the value of the entropy parameter. When η=1, this is indeed a special case because stability is determined only by the fact that $\tilde{\delta}$ is above or below 2.

Local analysis gives important guidance about the stability properties of the model. Nevertheless, this analysis cannot capture some fundamental features of the dynamics of (1). To be precise, one has to engage in a global analysis, which will evidence that the unstable region does not imply necessarily a divergence towards infinity; periodic and a-periodic motion is found. Global dynamics can only be understood through numerical examples and are better revealed if analyzed graphically. The following set of figures allows for a thorough understanding of the dynamics of (1).[3]

We begin by presenting the areas of stability and instability identified in the local analysis, from a global analysis point of view. Three figures are drawn, for different values of η (figure 3 for η=0.5; figure 4 for η=1; and figure 5 for η=1.5). In what concerns the stability area, the local analysis results are confirmed, but the unstable region does not translate immediately a divergence towards infinity; after the bifurcation, the system undergoes a phase of periodic cycles of doubling order and chaos before arriving to the divergence result.

With the global analysis, we expand the possibility of long term outcomes for the time evolution of the knowledge variable. The knowledge variable can be subject to endogenous fluctuations as a result of introducing entropy (recall once again that without entropy we would have stability under decreasing returns and instability otherwise).

To better understand the properties of (1), mainly in the regions where cycles of various orders arise, a set of other graphical representations are shown. Figures 6 and 7 present bifurcation diagrams; in both figures, relating to different parameters, it is clear the period doubling route to chaos, which occurs in the areas identified in previous figures. Figures 8 and 9 use the same set of parameters values to characterize a situation of chaos. In figure 8 we draw a phase diagram, where chaotic motion is clearly identified as a result of the hump-shaped form of the relation between the knowledge variable in two consecutive time moments. Figure 9 displays the time series of the knowledge variable in the long run (after 1,000 observations). Finally, figure 10 resorts to the most usual measure of chaos to emphasize its presence. Lyapunov characteristic exponents (LCEs) measure the exponential divergence of nearby orbits; a positive LCE is synonymous of divergence of nearby orbits or sensitive dependence on initial conditions, a phenomenon that is generally identified with chaotic motion. In the figure in appreciation, we find a positive Lyapunov exponent for the same values of parameter $\widetilde{B}$ for which we have identified before the presence of no fixed point or any kind of periodic cycles (this figure can be compared with figure 7).

3. Neoclassical Growth

Assume that knowledge variable $\widetilde{h}_t$ is an input into the production of human capital. The human capital per capita variable, h_t, evolves over time according to accumulation rule (2),

$$h_{t+1} - h_t = B \cdot [(1-u) \cdot h_t]^{\theta} \cdot \widetilde{h}_t^{\,\zeta} - \delta h_t, \quad h_0 \text{ given} \tag{2}$$

In (2), B>0 is a productivity index, u is the share of human capital used in the production of physical goods (and thus 1-u represents the share of human capital associated with the production of this form of capital), and δ>0 is a depreciation rate.

[3] The various figures relating global analysis are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

Two particular cases of equation (2) are studied. In this section, we concentrate in the absence of long term positive endogenous growth (the growth process has neoclassical features), while in section 4 an endogenous growth setup is assumed. The difference in analysis is determined by the elasticity parameters θ and ζ.

In this section, we take constant returns to scale in the production of human capital ($0<\theta<1$ and $\zeta=1-\theta$); for the circumstances described in section 2 in which $\tilde{h}_t$ had a long run stability solution (fixed point or periodic or a-periodic motion around the steady state point), the model displays neoclassical features, in the sense that there is not a process of sustained positive growth that is endogenously determined. Endogenous variables will tend to long run constant values or they will converge to a long term position where endogenous fluctuations around a constant mean persist over time.

The endogenous growth model of the next section considers the knowledge variable as an externality over a constant returns equation of human capital accumulation ($\theta=1$ and $\zeta>0$). In this case, the growth problem will exhibit a long run constant growth rate (for $\tilde{h}_t$ converging to a fixed point), or a long run scenario with growth cycles, i.e., economic aggregates will grow at an average constant rate, but endogenous fluctuations will characterize the motion of the growth rate (and not only the motion of the capital and consumption aggregates themselves).

Let us concentrate for now in the case $\theta\in(0,1)$ and $\zeta=1-\theta$. Consider a standard utility maximization intertemporal problem under an infinite horizon and a discount factor $\beta<1$,

$$\underset{c}{Max}\sum_{t=0}^{+\infty}U(c_t)\cdot\beta^t \tag{3}$$

The utility function is assumed under a simple concave form, $U(c_t)=\ln c_t$, where c_t stands for per capita consumption, and problem (3) is subject to three constraints; these are the knowledge equation in (1), the human capital equation in (2), and the third is a physical capital accumulation constraint, with a Cobb-Douglas production function that exhibits constant returns to scale,

$$k_{t+1}-k_t=Ak_t^{\alpha}\cdot(uh_t)^{1-\alpha}-c_t-\delta k_t,\ \ k_0 \text{ given} \tag{4}$$

The physical capital variable, k_t, is a per capita variable, parameter $A>0$ is the productivity index in the final goods sector, $\alpha\in(0,1)$ represents the output – physical capital elasticity, and $\delta>0$ is the depreciation rate (that, for simplicity, is considered the same as in the human capital constraint).

Solving the optimal control problem (3) subject to (1), (2) and (4), we find, after the computation of optimality conditions, the following equation translating the time evolution of the consumption variable,

$$c_t=\beta c_{t-1}\cdot\left[1-\delta+\alpha A\cdot\left(\frac{uh_t}{k_t}\right)^{1-\alpha}\right] \tag{5}$$

In order to simplify the dynamic analysis, we make the following assumption: the initial level of consumption chosen by the representative agent is already the steady state level, $c_0 = c^*$. Under this assumption, one can establish, through (5), a linear relation between the capital variables, which is,

$$k_t = u \cdot \left[\frac{\alpha A}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \cdot h_t \tag{6}$$

In section 2, one has observed that an explicit equilibrium value of $\tilde{h}_t$ is attainable only for η=1. With this parameter value it is straightforward the computation of steady state values for our various variables. The following results are obtained,

$$h^* = \left[\frac{B}{\delta} \cdot (1-u)^{\theta} \cdot e^{\left[(1-\theta)\cdot \tilde{B}/\tilde{\delta}\right]} \right]^{1/(1-\theta)} \tag{7}$$

$$k^* = u \cdot \left[\frac{\alpha A}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \cdot \left[\frac{B}{\delta} \cdot (1-u)^{\theta} \cdot e^{\left[(1-\theta)\cdot \tilde{B}/\tilde{\delta}\right]} \right]^{1/(1-\theta)} \tag{8}$$

$$\begin{aligned} c^* = A^{1/(1-\alpha)} \cdot &\left\{ \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{\alpha/(1-\alpha)} - \delta \cdot \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \right\} \\ &\cdot \left[\frac{B}{\delta} \cdot (1-u)^{\theta} \cdot e^{\left[(1-\theta)\cdot \tilde{B}/\tilde{\delta}\right]} \right]^{1/(1-\theta)} \end{aligned} \tag{9}$$

In expressions (7) to (9) some meaningful results are easy to identify: for instance, technology (*A* and *B*) contribute to higher steady state accumulated quantities of both forms of capital, while higher depreciation implies a fall in accumulated capital and in consumption.

The long run outcomes of capital (physical and human) are determined by the behaviour of the knowledge variable, $\tilde{h}_t$. We have seen, in section 2, that such behaviour is directly influenced by the values of $\tilde{\delta}, \tilde{B}$ and η. Therefore, steady state results (7) and (8) are not accomplished in every circumstance. To illustrate a few possible equilibrium results, we draw the evolution of output in the long term, under different combinations of parameters values (figures 11 and 12). We just mention the values of the parameters in the knowledge equation, given that the others (A, B, θ, u, δ, α and β) are not relevant from a qualitative point of view. We take $\tilde{B} = 1, \tilde{\delta} = 2.25$ and $\eta = 0.5$ (figure 11) and $\tilde{B} = 0.5, \tilde{\delta} = 2.7$ and $\eta = 1.25$ (figure 12). The output variable that is represented corresponds to the income generated by the final goods production function, that is, $y_t = Ak_t^{\alpha} \cdot (uh_t)^{1-\alpha} = A^{1/(1-\alpha)} \cdot u \cdot \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{\alpha/(1-\alpha)} \cdot h_t$.

We conclude that in a neoclassical growth model with entropy in the creation of knowledge endogenous business cycles characterizing the time evolution of output emerge, under some circumstances that define the process of knowledge accumulation.

4. Endogenous Growth

The model in section 3 presented neoclassical features in the sense that economic aggregates displayed zero average growth as a long run solution. Now, considering θ=1 and ζ>0, the knowledge variable is introduced in the growth model as a positive externality over the accumulation of human capital, which is subject to a constant marginal returns technology. Therefore, the model has endogenous growth features, meaning that capital and output will grow at a positive (constant in average) growth rate. With the new assumption, the inclusion of the knowledge variable implies endogenous fluctuations in the growth rates.

Consider the same problem as in section 3, so that equation (5) is once again found through the computation of first order conditions. Here, we define variables that do not grow in the long run; these are, following conventional endogenous growth analysis [see, e.g., Barro and Sala-i-Martin (1995)], a consumption – physical capital ratio, $\psi_t \equiv c_t / k_t$, and a physical capital – human capital ratio, $\omega_t \equiv k_t / h_t$. From (2), (4) and (5) we obtain,

$$\psi_{t+1} = \frac{\beta \cdot \left[1 - \delta + \alpha A \cdot (u/\omega_{t+1})^{1-\alpha}\right]}{A \cdot (u/\omega_t)^{1-\alpha} - \psi_t + (1-\delta)} \cdot \psi_t \tag{10}$$

$$\omega_{t+1} = \frac{A \cdot (u/\omega_t)^{1-\alpha} - \psi_t + (1-\delta)}{B \cdot (1-u) \cdot \tilde{h}_t^{\zeta} + (1-\delta)} \cdot \omega_t \tag{11}$$

Steady state values ψ^* and ω^*, which can be determined from (10) and (11), will be constant values for a constant equilibrium of knowledge, $\tilde{h}^*$. Once again, to obtain long run time trajectories that are, in average, constant, we take a simplifying assumption regarding consumption, which in this case is $\psi_0 = \psi^*$. To understand the dynamics of the capital ratio, we present figure 13, for $\tilde{B} = 0.5, \tilde{\delta} = 2.7$ and $\eta = 1.25$. For these values, we know that chaotic motion is present.

Figure 13 is drawn for a capital ratio; each one of the capital variables, and also the per capita output, will grow at a positive rate (that in average is constant); figure 14 illustrates precisely the endogenous growth character of the model by representing, for the same set of parameters values, the growth rate of the income variable,

$$\frac{y_{t+1} - y_t}{y_t} = \left(\frac{\omega_{t+1}}{\omega_t}\right)^{\alpha} \cdot \frac{h_{t+1}}{h_t} - 1 =$$
$$= \left[A \cdot (u/\omega_t)^{1-\alpha} - \psi^* + (1-\delta)\right]^{\alpha} \cdot \left[B \cdot (1-u) \cdot \tilde{h}_t^{\zeta} + (1-\delta)\right]^{1-\alpha} - 1$$

Looking at figure 14, we understand the relevance of the eventual presence of entropy in the generation of knowledge. This might lead to endogenous growth cycles that nevertheless do not disturb the positive growth trend (thus, turning the endogenous growth paradigm more realistic).

5. Final Remarks

We have assumed a knowledge variable with special features. Knowledge is generated through an accumulation process, but it is also subject to entropy: larger quantities of this input imply, after some point, that the accumulated amount of knowledge begins to decline. The impact of this process of knowledge accumulation over the growth of the main economic aggregates will depend on the way this variable is linked with the generation of human capital.

If the knowledge variable is included in a human capital production function with constant returns to scale, the growth model can be interpreted as a neoclassical growth setup: capital and output will grow at a long term zero rate (on average), that is, one can present long term time series for the economic variables that will have a constant mean. These time series are not necessarily constant over time; for some parameters values, periodic and chaotic cycles are obtained. Under this setup, endogenous cycles can coexist with neoclassical growth.

If the knowledge variable emerges as an externality over the production of human capital, the endogenous growth attributes of the original growth model are maintained, in the sense that capital and output grow at positive rates in the long term. These rates are constant over time for some parameters values but for others they will fluctuate around a constant value. Thus, in the case of endogenous growth, entropy in knowledge creation implies endogenous cycles characterizing the growth rates of capital and output.

In synthesis, entropy in knowledge can be understood as a source of endogenous business cycles, and it was introduced in growth models without changing the fundamental properties of the growth process, which remains, respectively, neoclassical or endogenous.

References

Barro, R. J. and X. Sala-i-Martin (1995). *Economic Growth*. New York: McGraw-Hill.

Benhabib, J. and R. H. Day (1981). "Rational Choice and Erratic Behaviour." *Review of Economic Studies*, vol. 48, pp. 459-471.

Christiano, L. and S. Harrison (1999). "Chaos, Sunspots and Automatic Stabilizers." *Journal of Monetary Economics*, vol. 44, pp. 3-31.

Coury, T. and Y. Wen (2005). "Global Indeterminacy and Chaos in Standard RBC Models." *University of Oxford and Cornell University working paper*.

Day, R. H. (1982). "Irregular Growth Cycles." *American Economic Review*, vol. 72, pp.406-414.

Grandmont, J. M. (1985). "On Endogenous Competitive Business Cycles." *Econometrica*, vol. 53, pp. 995-1045.

Guo, J. T. and K. J. Lansing (2002). "Fiscal Policy, Increasing Returns and Endogenous Fluctuations." *Macroeconomic Dynamics*, vol. 6, pp. 633-664.

Heath, R. L. and J. Bryant (2000). *Human Communication Theory and Research: Concepts, Contexts and Challenges*. New Jersey: Lawrence Erlbaum Associates.

May, R. M. (1976). "Simple Mathematical Models with Very Complicated Dynamics." *Nature*, vol. 261, pp. 459-467.

Medio, A. (1979). *Teoria Nonlineare del Ciclo Economico*. Bologna: Il Mulino.

Sato, Y.; E. Akiyama and J. P. Crutchfield (2004). "Stability and Diversity in Collective Adaptation." *Santa Fe Institute Working Paper* 04-08-025.

Schmitt-Grohé, S. (2000). “Endogenous Business Cycles and the Dynamics of Output, Hours, and Consumption.” *American Economic Review*, vol. 90, pp. 1136-1159.
Shannon, C. (1948). “A Mathematical Theory of Communication.” *The Bell System Technical Journal*, vol. 27, pp. 379-423, pp. 623-656.
Shannon, C. and W. Weaver (1949). *The Mathematical Theory of Communication*. Urbana, IL: University of Illinois Press.
Stutzer, M. J. (1980). “Chaotic Dynamics and Bifurcations in a Macro-Model.” *Journal of Economic Dynamics and Control*, vol. 2, pp. 353-376.

Figures

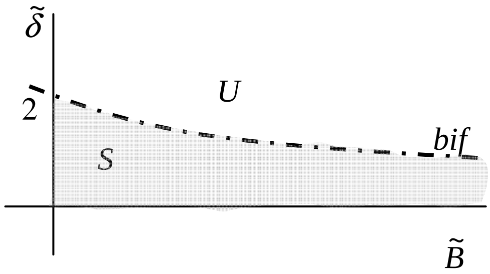

Figure 1 *Areas of stability / instability (η=0.5).*

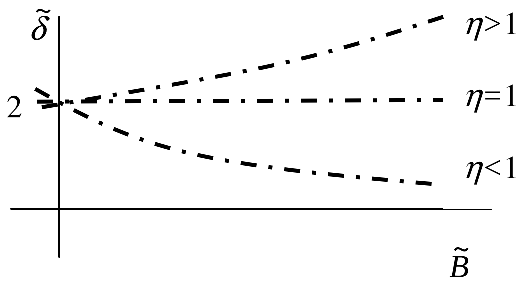

Figure 2 *Bifurcation lines for different values of η.*

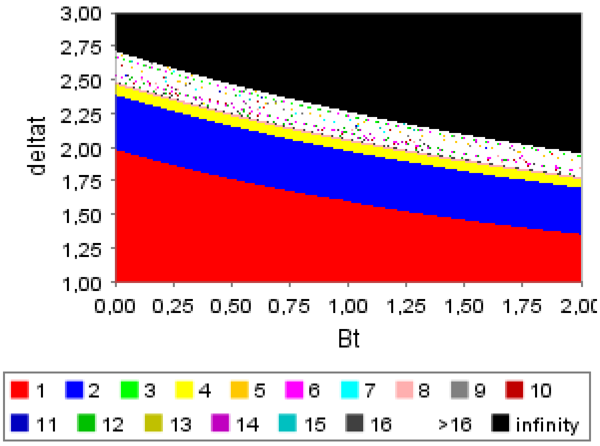

Figure 3 *Global Dynamics in the parameters space (η=0.5).*

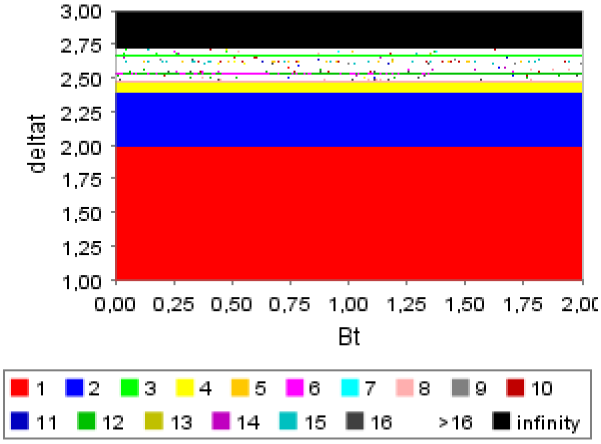

Figure 4 *Global Dynamics in the parameters space (η=1).*

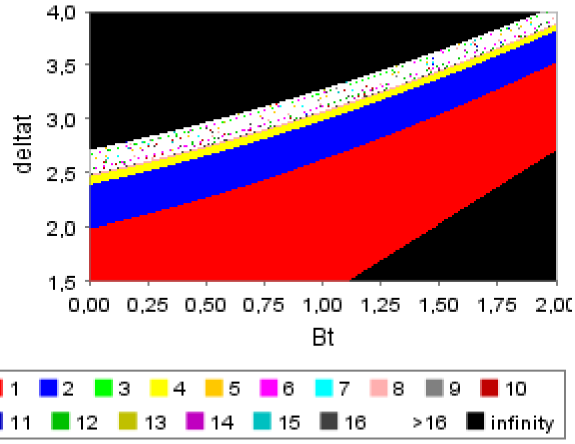

Figure 5 *Global Dynamics in the parameters space (η=1.5).*

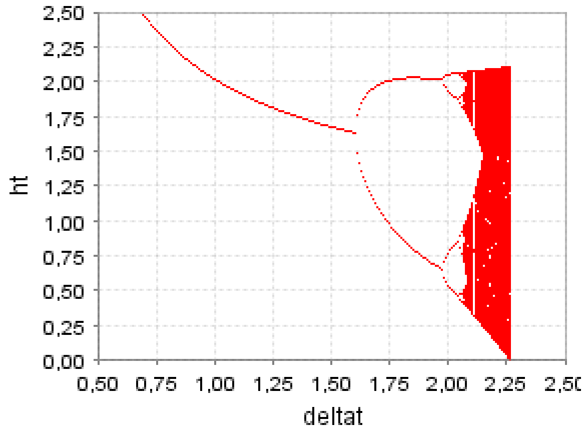

Figure 6 *Bifurcation diagram* $(\tilde{h}_t, \tilde{\delta})$ $(\tilde{B} = 1; \eta = 0.5)$.

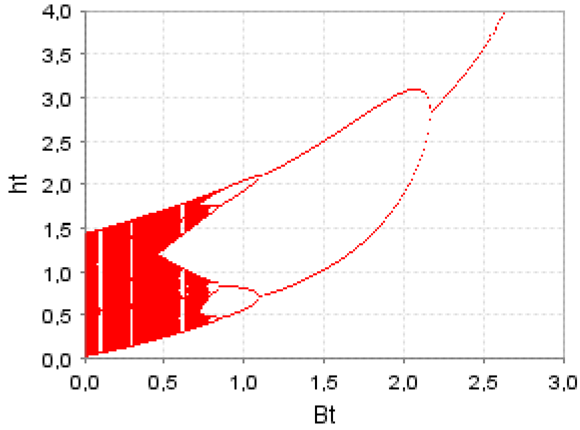

Figure 7 *Bifurcation diagram* $(\tilde{h}_t, \tilde{B})$ $(\tilde{\delta} = 2.7; \eta = 1.25)$.

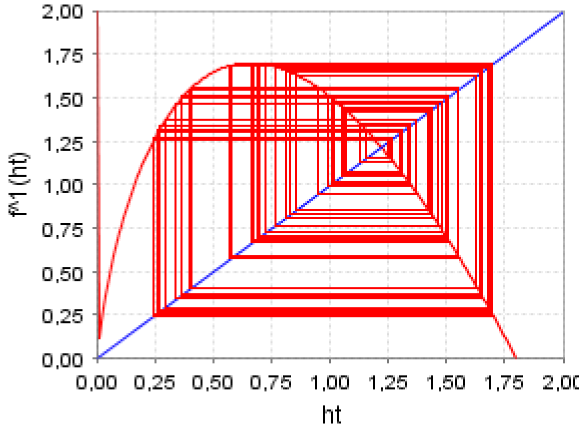

Figure 8 *Phase diagram* $(\tilde{B} = 0.5; \tilde{\delta} = 2.7; \eta = 1.25)$.

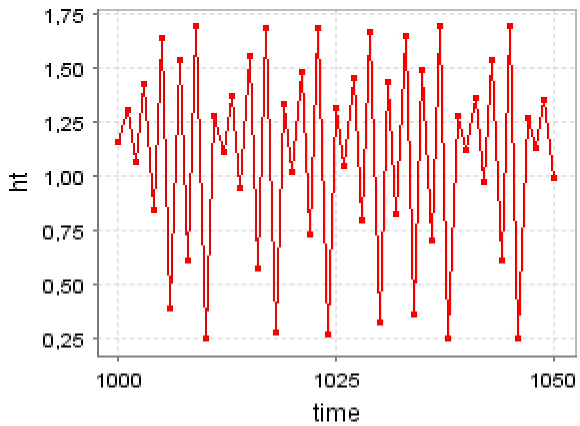

Figure 9 *Long run time path* $(\tilde{B} = 0.5; \tilde{\delta} = 2.7; \eta = 1.25)$.

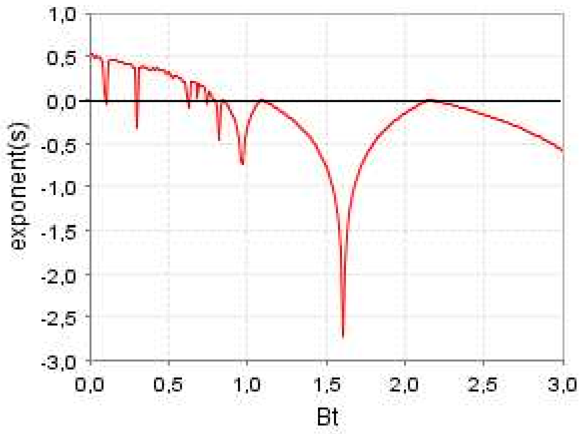

Figure 10 *Lyapunov characteristic exponents* $(\tilde{h}_t, \tilde{B})$ $(\tilde{\delta} = 2.7; \eta = 1.25)$.

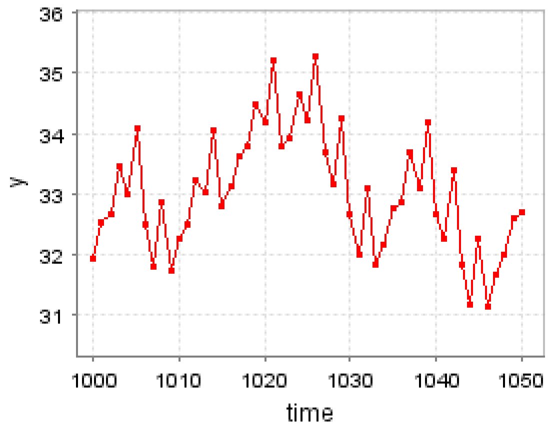

Figure 11 *Long run time path of the output variable in the neoclassical model* $(\tilde{B} = 1; \tilde{\delta} = 2.25; \eta = 0.5)$.

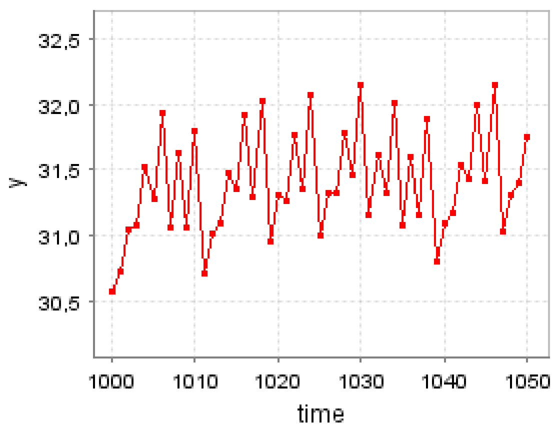

Figure 12 *Long run time path of the output variable in the neoclassical model* $(\tilde{B} = 0.5; \tilde{\delta} = 2.7; \eta = 1.25)$.

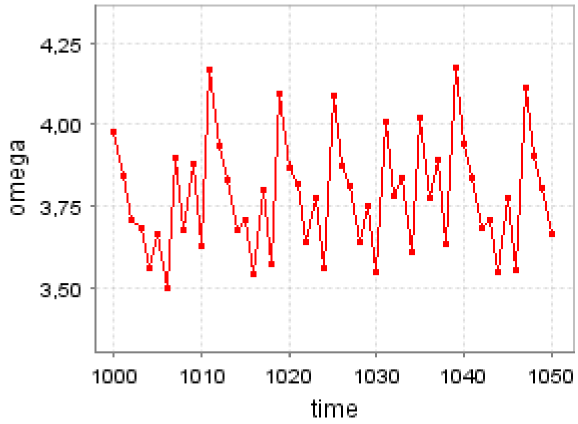

Figure 13 *Long run time path of the capital ratio in the endogenous growth model* $(\tilde{B} = 0.5; \tilde{\delta} = 2.7; \eta = 1.25)$.

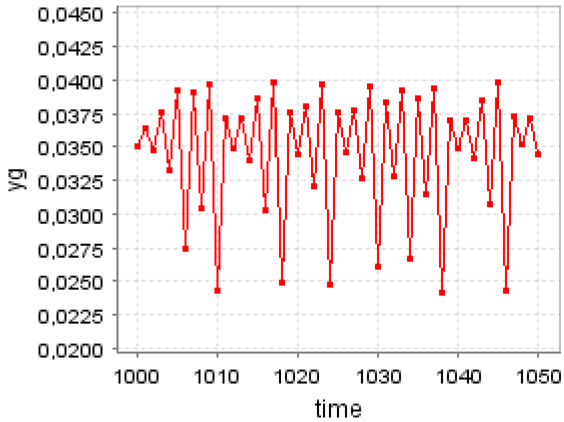

Figure 14 *Long run time path of the output growth rate in the endogenous growth model* $(\tilde{B} = 0.5; \tilde{\delta} = 2.7; \eta = 1.25)$.