# Measures of dependence for Ornstein-Uhlenbeck processes with tempered stable distribution

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# Measures of dependence for Ornstein-Uhlenbeck processes with tempered stable distribution

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**Abstract**

In this paper we investigate the dependence structure for Ornstein-Uhlenbeck processes with totally skewed tempered stable structure. They are natural extension of Ornstein-Uhlenbeck processes with $\alpha-$stable (and Gaussian) distribution. However for the $\alpha$-stable models the covariance is not defined therefore in order to compare the structure of dependence of Ornstein-Uhlenbeck with tempered stable and $\alpha-$stable structure we analyze another measures of dependence defined for infinitely divisible processes such as L$\acute{e}$vy correlation cascade and codifference. We show that for analyzed processes the L$\acute{e}$vy correlation cascade goes faster to zero as in the stable case, while the codifference of the $\alpha$-stable Ornstein-Uhlenbeck process has the same form as in the tempered case.

Key words: truncated L$\acute{e}$vy flight, tempered stable, Ornstein-Uhlenbeck process, structure of dependence
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## 1 Introduction

In modern mathematical finance continuous time models play a crucial role because they allow handling unequally spaced data and even high frequency data, which are realistic for liquid data. The probably most famous example is the

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Ornstein-Uhlenbeck process that was introduced in 1930 by Uhlenbeck and Ornstein [1] as a suitable model in the physical phenomenas. Ornstein and Uhlenbeck proposed this new model as an alternative to the classical Brownian motion in case when some kind of mean reverting tendency is observed in the real data. The Ornstein-Uhlenbeck process have found many applications especially to the real financial data such as interest rates, currency exchange rates, and commodity prices. In finance it is best known in connection with the Vasicek interest rate model, [2].

Many asset pricing models (such as classical Vasicek model) assume that the analyzed data have normal distribution. Unfortunately the assumption of normality is unsatisfactory for many observed data. One approach is to replace the Brownian motion in the Ornstein-Uhlenbeck process by a heavier tailed Lévy process. Many studies have shown that heavy-tailed distributions allow for modelling different kinds of phenomena when the assumption of normality for the observations does not seem not to be reasonable. Especially $\alpha$-stable (stable) distributions have found many practical applications, for instance in finance [3], physics [4] and electrical engineering [5]. The Ornstein-Uhlenbeck process with $\alpha$-stable structure was analyzed in [6] as a suitable model to description of real financial data.

However the stable processes have infinite moments of the second or higher orders therefore there appear many problems especially in applications. In order to overcome this drawback, the processes with tempered stable structure (and their modifications) have been introduced. There are many types of such processes, for example classical tempered stable and modified tempered stable models, see [7, 8, 9]. The classical tempered stable models are known as Truncated Lévy Flight (see for instance [11, 12, 13, 14]), KoBol [15] and CGMY processes [16, 17]. They found many applications especially in finance, see [18, 19], biology [20], physics to description of diffusion and relaxation [21] and turbulence [22] as well as in plasma physics [23], see also [24, 25].

In this paper we consider the Ornstein-Uhlenbeck processe with tempered stable structure that is a natural extensions of Ornstein-Uhlenbeck with Brownian or stable Lévy motion. One of the important steps towards constructing an appropriate mathematical model for the real-life data is covariance. However for the stable models the covariance is not defined therefore in order to compare the structure of dependence of the tempered and $\alpha-$stable Ornstein-Uhlenbeck process we analyze another measure of dependence defined for infinitely divisible processes such as Lévy correlation cascade [28] that is a useful tool for studying the ergodic properties, [29]. We examine the asymptotic behaviour of the aftermentioned measure for considered processes and compare it to the $\alpha-$stable case. As a main result we show that the L$\acute{e}$vy correlation cascade in the considered case goes faster to zero as in stable models. Moreover for the tempered stable and $\alpha$-stable Ornstein-Uhlenbeck process we compare also the another measure, namely codifference [30, 31, 32, 33], measure based on the characteristic function. We prove that for this two analyzed processes this measure indicates the same asymptotic behaviour.

The rest of the paper is organized as follows: In Section 2 we give the definition of considered Ornstein-Uhlenbeck process with tempered stable structure. In order to present the motivation of the paper in Section 3 we describe the real data of turbulence in earth's plasma by using the tempered stable Ornstein-Uhlenbeck process. Then, in Section 4, we review the measures of dependence for infinitely divisible processes: the codifference and the alternative measure called the L$\acute{e}$vy correlation cascade. The measures of dependence for considered processes are studied in Section 5 and their asymptotic behaviour is examined.

# 2 Ornstein-Uhlenbeck process with tempered stable structure

The classical Ornstein-Uhlenbeck process is one of several approaches used to model (with modifications) the real financial data such as interest rates, currency exchange rates, and commodity prices. It is also known as the mean-reverting process, and it is given by the following stochastic differential equation:

$$dY(t) = a(\mu - Y(t))dt + \sigma dB(t), \tag{2.1}$$

where $\{B(t)\}_{t\geq 0}$ denotes the Brownian motion, the parameter $\mu \in R$ represents the equilibrium or mean value supported by fundamentals; $\sigma > 0$- the degree of volatility around it caused by shocks, and $\theta > 0$ - the rate by which these shocks dissipate and the variable reverts towards the mean. If we extend the Brownian motion for the set $(-\infty, 0)$ according to the procedure presented in [34, 35], then we can write the unique solution of equation (2.1):

$$Y(t) = \mu + \sigma \int_{-\infty}^{t} e^{-a(t-u)} dB^*(u), \tag{2.2}$$

where $\{B^*(t)\}_{t\in R}$ is the Brownian motion extended to the set $(-\infty, 0)$.
An extension of the process (2.2) is an $\alpha-$stable Ornstein-Uhlenbeck system defined as follows (see [6, 34]):

$$Y(t) = \mu + \sigma \int_{-\infty}^{t} e^{-a(t-u)} dL_\alpha(u), \tag{2.3}$$

where $\{L_\alpha(t)\}$ is a Lévy process with $\alpha$-stable increments extended to the set $(-\infty, 0)$. Stable Ornstein-Uhlenbeck processes were analyzed for instance in [6, 36] as a models describing real financial data.

In this paper we propose an extension of the aftermentioned Ornstein-Uhlenbeck processes and substitute the Lévy process with $\alpha$-stable (or Gaussian) increments extended to the set $(-\infty, 0)$ by the Lévy process with totally skewed tempered stable structure. In this cases the Ornstein-Uhlenbeck process can be represented by the following stochastic integral:

$$Y(t) = \mu + \sigma \int_{-\infty}^{t} e^{-a(t-u)} dT(u), \tag{2.4}$$

where $\{T(u)\}$ is a Lévy process with totally skewed tempered stable increments extended to the set $(-\infty, 0)$.
An infinitely divisible distribution is called a totally skewed tempered stable (TS) with parameters $\alpha$, $\lambda$ and $C$ if it has no Gaussian component and its Lévy measure is given by, [10]

$$v(dx) = \frac{Ce^{-\lambda x}}{x^{1+\alpha}} 1_{x>0} dx, \tag{2.5}$$

where $\lambda > 0$, $0 < \alpha < 2$ and $C > 0$ for $\alpha > 1$ and $C < 0$ for $\alpha < 1$. The Fourier transform $\phi_{TS}$ of the totally sewed tempered stable distribution is given by the following formula, [10]

$$\phi_{TS}(u) = E\exp(iuT) = \exp\left(C((\lambda - iu)^\alpha - \lambda^\alpha + iu\alpha\lambda^{\alpha-1})\right). \tag{2.6}$$

When $\lambda = 0$, then the random variable $T$ with the Fourier transform given in (2.6) has a totally skewed $\alpha-$stable distribution with the following values of the parameters

$$\alpha,\ \beta = 1,\ \sigma = (|C| cos(\pi * \alpha/2))^{1/\alpha},\ \mu = 0.$$

By using the connection between the TS and corresponding $\alpha-$stable distribution it is easy to find the relation between the probability distribution functions (pdf). Let $p_{TS}(x)$ and $p_S(x)$ be pdf in point $x$ of $TS$ random variable with parameters $\alpha, \lambda, C$ and $\alpha-$stable with appriopriate values of the parameters, respectively, then for $\alpha \neq 1$ we have

$$p_{TS}(x) = e^{-\lambda x + (\alpha-1)c\lambda^\alpha} p_S(x - c\alpha\lambda^{\alpha-1}).$$

For $\alpha = 1$ this relationship takes the form:

$$p_{TS}(x) = e^{-\lambda x + c\lambda} p_S(x - c(1 + ln\lambda)).$$

The main properties as well as the procedures of simulation of the considered tempered stable distribution one can find for instance in [8] and [10].

# 3 Motivation

In order to present the motivation of using the Ornstein-Uhlenbeck process with tempered stable structure we analyse the real data for plasma physics. The data set describes the floating potential fluctuations of earth's plasma expressed in volts. The signal was registered on 15.06.2006 (the time unit - 16 miliseconds) with movable probe in Scrape-Off Layer (SOL) plasma. The small torus radial position was r=11.25 cm. On Fig. 1 we present the analysed real data. The statistical

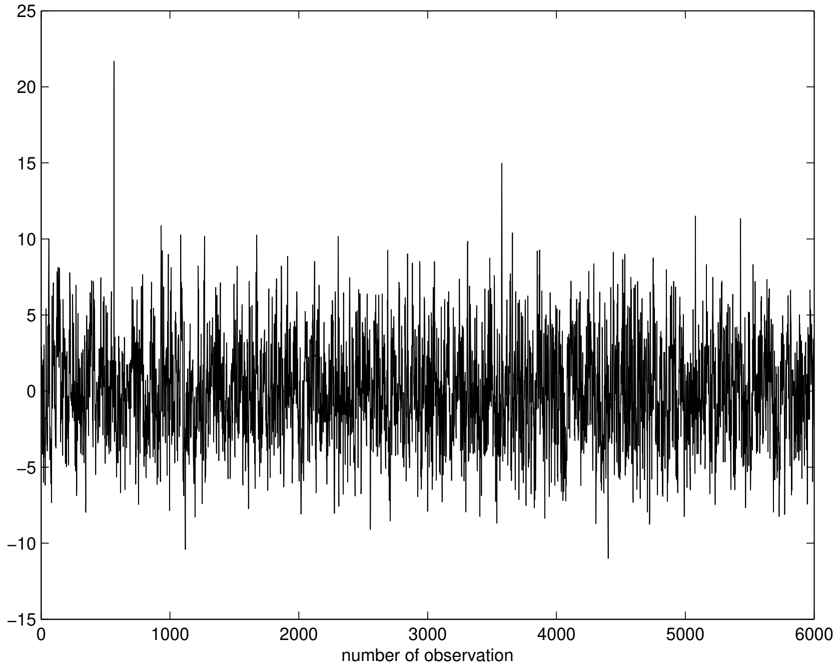


Figure 1: The real data from plasma physics.

tests based on the empirical distribution function, [27], reject the hypothesis that the data can be described by using Gaussian as well as the $\alpha-$stable distribution. Therefore as an alternative we propose to use the tempered stable distribution described in previous Section. The autocorrelation function (ACF) and partial autocorrelation function (PACF) indicate the data can be described by autoregressive model of order 1 (AR(1)), that is a discrete version of the Ornstein-Uhlenbeck process. By using the maximum likelihood method we estimate the parameter $\theta$ of the process given in (2.4). For the simplicity we assume $\mu = 0$ and $\sigma = 1$. The estimation results give $\hat{\theta} = 0.2351$. Moreover by using the method of moments we estimate also the parameters of TS distribution:

$$\hat{\alpha} = 1.8399,\ \hat{\lambda} = 0.1928,\ \hat{C} = 2.1970.$$

On Fig. 2 we present the empirical probability distribution function based on the kernel estimation method of the model's residua that are described by the totally skewned tempered stable distribution as well as the theoretical density function calculated by using the estimated parameters.

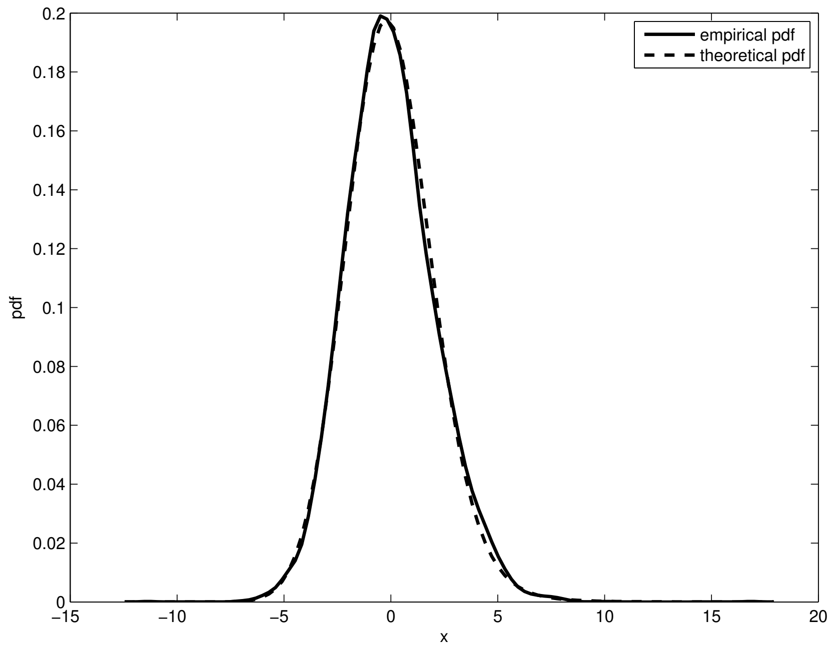


Figure 2: The comparison of the empirical pdf of the rests of the Ornstein-Uhlenbeck model and the theoretical pdf calculated on the basis of the estimated parameters from the TS distribution.

# 4 Measures of dependence for infinitely divisible processes

One of the important tool providing to construction of an appropriate mathematical model for the real-life data is covariance. However for the large class of infinitely divisible processes, namely the strictly $\alpha-$stable, the covariance is not defined. Therefore there appears problem: how in this case describe the dependence structure? One of the answer for this question gave Eliazar and Klafter in [28]. They introduced a new measure of dependence that is defined for infinitely divisible stochastic processes $\{Y(t),\ t \in R\}$ with the following integral representation:

$$Y(t) = \int_X K(t,x)M(dx),$$

where $M$ is an independly scattered infinitely divisible random measure on some measurable space $S$ with control measure $m$, see also [29].
The new measure was introduced as a concept of correlation cascades, which is a promising tool for exploiting the properties of the Poissonian part of $Y(t)$ and the dependence structure of this stochastic process. The L$\acute{e}$vy correlation cascade is defined as follows [28, 29]:

$$C_l(t_1, t_2, \ldots, t_n) = \int_X \Lambda\left(\frac{l}{\min\{K(t_1,x),\ldots,K(t_n,x)\}}\right) m(dx), \qquad (4.7)$$

where the tail function $\Lambda$ is given by

$$\Lambda(l) = \int_{|x|>l} v(dx) \qquad (4.8)$$

and $v$ is a L$\acute{e}$vy measure of the the process $\{Y(t)\}$.
Many significant properties and results connected with the L$\acute{e}$vy correlation cascade for infinitely divisible processes are presented in [28] and [29]. We only mention here the function $C_l(t_1, t_2, \ldots, t_n)$ tells us, how dependent the coordinates of the vector $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are. Therefore, $C_l(t_1, t_2, \ldots, t_n)$ can be considered as an appropriate measure of dependence for the Poissonian part of the infinitely divisible process, [29]. The ergodic property, such as ergodicity, weak mixing and mixing, of a stationary infinitely divisible processes can be described in the language of the L$\acute{e}$vy correlation cascade therefore this measure is a promising tool for studying the dependence structure for this large class of processes.
When the considered process is a moving-average with respect to the L$\acute{e}$vy process $\{Z(t)\}$, i.e. it takes the following form

$$Y(t) = \int_{-\infty}^{t} f(t-u)Z(du),$$

then the L$\acute{e}$vy correlation cascade is defined as follows:

$$C_l(0,t) = \int_t^{\infty} \Lambda\left(\frac{l}{f(y)}\right) dy.$$

The another measure, that is often considered as a tool of the dependence structure description, is the codifference (see for instance [30, 31, 32]). For the stationary infinitely divisible process $\{Y(t)\}$ this measure of dependence is defined as follows:

$$CD(t,0) = CD(Y(t), Y(0)) =$$

$$\log E \exp\{i(Y(t) - Y(0))\} - logE \exp\{iY(t)\} - logE \exp\{-iY(0)\}. \quad (4.9)$$

Codifference carries enough information to detect ergodic properties of the process $\{Y(t)\}$. It is also closely related to the dynamical functional used in [38, 39] to investigate the chaotic behaviour of the considered process. Properties of the measure on can find in [30]. Let us mention here that there is relationship between the asymptotic behaviour of Lévy correlation cascade and codifference, namely for the stationary infinitely divisible process $\{Y(t)\}$ with the Lévy measure $v_0$ of $Y(0)$ without atoms in $2\pi Z$, the following two conditions are equivalent (see Theorem 2 in [29]):

$$\lim_{t\to 0} C_l(t,0) = 0 \text{ for every } l > 0$$

$$\lim_{t\to 0} CD(t,0) = 0.$$

In the next section we consider the Ornstein-Uhlenbeck processes with tempered stable structures. As a main result we show the asymptotic behaviour of this processes in the language of Lévy correlation cascade and compare it with $\alpha-$stable case. Moreover we show that the codifference of the Ornstein-Uhlenbeck process with $\alpha-$stable and tempered stable structure has the same asymptotic properties.

# 5 Structure of dependence of Ornstein-Uhlenbeck process with $\alpha-$ and tempered stable structure

## 5.1 The totally skewned $\alpha$-stable case

Let us consider the Ornstein-Uhlenbeck process with $\alpha$-stable structure given in (2.3). For the simplification we take $\mu = 0$ and $\sigma = 1$. In this case the Lévy measure of the $\alpha$-stable Lévy process $\{L_\alpha(u)\}$ in (2.3) is given by (see [30]):

$$v(dx) = 0.5\left(\frac{1_{x>0}}{x^{1+\alpha}} + \frac{1_{x<0}}{|x|^{1+\alpha}}\right) dx,$$

Therefore the Lévy correlation cascade of the Ornstein-Uhlenbeck process $\{Y(t)\}$ given in (2.3) has the following form [29]:

$$C_l(0,t) = \frac{2}{a\alpha^2 l^\alpha} e^{-a\alpha t}.$$

Moreover the correlation-like measure $r$ defined in (**??**) depends on on parameter (according to the fact that the considered process is stationary) and has the following form

$$r_l(t) = r_l(\tau, \tau + t) = e^{-a\alpha t}.$$

The codifference $CD(t, 0)$ of the Ornstein-Uhlenbeck process with $\alpha$-stable Lévy motion is given by [40]:

$$CD(t, 0) = \frac{1 + e^{-a\alpha t} - |1 - e^{-at}|^{\alpha}}{a\alpha},$$

that for large $t$ and $\alpha > 1$ gives

$$CD(t, 0) \sim const \cdot e^{-at}.$$

## 5.2 Totally skewed tempered stable case

In this case we consider the Ornstein-Uhlenbeck process $\{Y(t), t \in R\}$ with totally skewed tempered stable structure. The $\{T(t)\}$ process in representation (2.4) it is a process with Lévy measure given in (2.5). In the considered case the tail function $\Lambda$ given in (4.8) takes the following form

$$\Lambda(l) = C\lambda^{\alpha}\Gamma(-\alpha, \lambda l),$$

where $\Gamma(s, t)$ is a incomplete gamma function defined as follows

$$\Gamma(s, t) = \int_t^{\infty} x^{s-1} e^{-x} dx. \tag{5.10}$$

Using the form of the $\Lambda$ function we obtain the following form of the Lévy correlation cascade for the tempered stable O-U process defined in (2.4):

$$C_l(t, 0) = \int_t^{\infty} C\lambda^{\alpha}\Gamma(-\alpha, \lambda l e^{au}) du.$$

Let us consider the asymptotic behaviour of such function for $t \to \infty$. Because the incomplete gamma function has the following property

$$\frac{\Gamma(s, x)}{x^{s-1}e^{-x}} \to 1 \text{ for } x \to \infty$$

then we obtain for large $t$

$$C_l(t, 0) \sim C\lambda^{\alpha} \int_t^{\infty} (\lambda l e^{au})^{-\alpha-1} \exp\{-\lambda l e^{au}\} du = \frac{C\lambda^{\alpha}}{a} \int_{\lambda l e^{at}}^{\infty} w^{-\alpha-2} e^{-w} dw$$

$$= \frac{C\lambda^\alpha}{a}\Gamma(-\alpha-1, le^{at}) \sim \frac{C\lambda^\alpha}{a}(\lambda l e^{at})^{-\alpha-2}\exp\{-\lambda l e^{at}\}$$

In this case the codifference defined in (4.9) is given by

$$CD(t,0) =$$

$$C\int_{-\infty}^{0}(\lambda - ie^{as}(e^{-at}-1))^\alpha - (\lambda - ie^{as})^\alpha + 2i\alpha\lambda^{\alpha-1}e^{as}(e^{-at}-1)ds$$

$$-C\int_{-\infty}^{0}(\lambda + ie^{-a(t-s)})^\alpha - \lambda^\alpha ds.$$

By using the following formula

$$(a+b)^\alpha = \sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}a^{\alpha-k}b^k$$

we obtain

$$CD(t,0) =$$

$$C\sum_{k=1}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)ak}\lambda^{\alpha-k}i^k((1-e^{-at})^k - (-1)^k)ds +$$

$$+\frac{2iC\alpha\lambda^{\alpha-1}(e^{-at}-1)}{a} - C\sum_{k=1}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)ak}\lambda^{\alpha-k}i^k e^{-atk}.$$

$$= C\sum_{k=2}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)ak}\lambda^{\alpha-k}i^k((1-e^{-at})^k - (-1)^k - e^{-atk}).$$

When $k$ is even, then the function

$$(1-e^{-at})^k - (-1)^k - e^{-atk}$$

for large $t$ behaves like $ke^{-at}$. When $k$ is odd, then

$$\lim_{t\to\infty}(1-e^{-at})^k - (-1)^k - e^{-atk} = 2.$$

therefore finally we obtain

$$CD(t,0) \sim const \cdot e^{-at} + R,$$

where $R$ is constant.

## 5.3 Classical tempered stable case

Let us consider the process $\{Y(t), t \in R\}$ defined in (2.4) with classical tempered stable structure. For simplification let us take $\mu = 0, \sigma = 1$ in representation (2.4) and consider the CGMY process, i.e. in the definition of L$\acute{e}$vy measure in (**??**) we take $C_1 = C_2 = C > 0$. In this case the tail function $\Lambda$ takes the following form

$$\Lambda(l) = C\lambda_+^\alpha \Gamma(-\alpha, \lambda_+ l) + C\lambda_-^\alpha \Gamma(-\alpha, \lambda_- l),$$

where $\Gamma(s,t)$ is a incomplete gamma function defined as follows

$$\Gamma(s,t) = \int_t^\infty x^{s-1} e^{-x} dx. \tag{5.11}$$

Using the form of the $\Lambda$ function we obtain the following form of the Levy correlation cascade for the tempered stable O-U process defined in (2.4):

$$C_l(t,0) = \int_t^\infty C\lambda_+^\alpha \Gamma(-\alpha, \lambda_+ l e^{au}) + C\lambda_-^\alpha \Gamma(-\alpha, \lambda_- l e^{au}) du.$$

Let us consider the asymptotic behaviour of such function for $t \to \infty$. Because the incomplete gamma function has the following property

$$\frac{\Gamma(s,x)}{x^{s-1}e^{-x}} \to 1 \text{ for } x \to \infty$$

then we obtain for large $t$

$$\begin{aligned}
C_l(t,0) &\sim C\lambda_+^\alpha \int_t^\infty (\lambda_+ l e^{au})^{-\alpha-1} \exp\{-\lambda_+ l e^{au}\} du + C\lambda_-^\alpha \int_t^\infty (\lambda_- l e^{au})^{-\alpha-1} \exp\{-\lambda_- l e^{au}\} du \\
&= \frac{C\lambda_+^\alpha}{a} \int_{\lambda_+ l e^{at}}^\infty w^{-\alpha-2} e^{-w} dw + \frac{C\lambda_-^\alpha}{a} \int_{\lambda_- l e^{at}}^\infty w^{-\alpha-2} e^{-w} dw \\
&= \frac{C\lambda_+^\alpha}{a} \Gamma(-\alpha-1, \lambda_+ l e^{at}) + \frac{C\lambda_-^\alpha}{a} \Gamma(-\alpha-1, \lambda_- l e^{at}) \\
&\sim \frac{C\lambda_+^\alpha}{a} (\lambda_+ l e^{at})^{-\alpha-2} \exp\{-\lambda_+ l e^{at}\} + \frac{C\lambda_-^\alpha}{a} (\lambda_- l e^{at})^{-\alpha-2} \exp\{-\lambda_- l e^{at}\} \\
&= \frac{Ce^{-a(\alpha+2)t}}{al^{\alpha+2}} \left( \frac{\exp\{-\lambda_+ l e^{at}\}}{\lambda_+^2} + \frac{\exp\{-\lambda_- l e^{at}\}}{\lambda_-^2} \right).
\end{aligned}$$

In the simple case when $C = \lambda_+ = \lambda_- = 1$ we have

$$C_l(t,0) \sim \frac{e^{-a(\alpha+2)t}}{2al^{\alpha+2}} \exp\{-l e^{at}\}.$$

In his case the correlation-like measure defined in (**??**) depends only on one parameter. Moreover

$$r_l(t) = r_l(\tau, \tau + t) \sim \exp\{-a(\alpha + 2)t - l(e^{at} - 1)\}.$$

In this case the codifference defined in (4.9) is given by

$$CD(t, 0) =$$

$$\Gamma(-\alpha) \int_{-\infty}^{0} (1 - ie^{-a(t-u)} + ie^{au})^\alpha + (1 + ie^{-a(t-u)} - ie^{au})^\alpha - 2((1 + ie^{au})^\alpha - 1 + (1 - ie^{au})^\alpha) du$$

$$+\Gamma(-\alpha) \int_{0}^{t} ((1 - ie^{-a(t-u)})^\alpha + (1 + ie^{-a(t-u)})^\alpha - 2) du.$$

Therefore we obtain

$$CD(t, 0) \sim \Gamma(-\alpha) e^{-at} i \int_{-\infty}^{0} e^{au} [(1 + ie^{au})^{\alpha-1} - (1 - ie^{au})^{\alpha-1}] du = const \cdot e^{-at}.$$

# 6 Conclusion

In this paper we analyzed structure of dependence of three types of Ornstein-Uhlenbeck processes related to the stable law: classical tempered stable, modified tempered stable and Lamperti stable Ornstein-Uhlenbeck processes. This structure of dependence we described in the language of Lévy correlation cascade as well as the codifference. As a main result we showed that the measure of dependence based on the Lévy measure in the three considered cases goes faster to zero as in stable models. As the conclusion on Figure **??** we present behaviour of the correlation-like measure $r$ for the $\alpha-$stable, classical tempered stable, modified tempered stable and Lamperti stable Ornstein-Uhlenbeck process.

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