On Stability and Efficiency in School Choice Problems

Alcalde, Jose and Romero-Medina, Antonio

University of Alicante

13 February 2011

On Stability and Efficiency in School Choice Problems*

José Alcalde[†]   Antonio Romero-Medina[‡]

February 13, 2011

Abstract

This paper proposes a way to allocate students to schools such that conciliates Pareto efficiency and stability. Taking as a starting point the recent reform proposed by the Boston School Committee, we propose a marginal modification to reach our objective redefine how students are prioritize. Our proposal is to allow schools to prioritize only a small set of students an then use a common priority order for the rest. Under this condition we propose a score based priority ranking that makes the output of the new Boston Mechanism Pareto efficient and stable.

Keywords: School allocation problem, Pareto efficient matching.
Journal of Economic Literature Classification Numbers: C71, C72, D71.

*The authors acknowledge Salvador Barberà, Josep E. Peris and Pablo Revilla for the many discussions we had around this paper. We would also like to acknowledge the financial support by the Institut Valencià d'Investigacions Econòmiques, FEDER and the Spanish Ministerio de Educación y Ciencia under projects SEJ2007-64649 (Alcalde) and ECON2008-027038 (Romero-Medina).

†Corresponding Author. IUDESP, University of Alicante. jose.alcalde@ua.es
‡Universidad Carlos III de Madrid. aromero@eco.uc3m.es

1 Introduction

On the school choice procedures the way municipalities decide the design instrument to allocate student to schools play a key role. In this instruments the municipalities are forced to establish priorities in order to allow break the ties that arise when the system takes into account the parents' interests for the schools. In this present paper we deal with way in which the schools' priority lists are built. We propose a conditions on agents' characteristics that solve the efficiency-equity trade off by proposing a priority structure able to reconcile efficiency and stability.

Municipalities use different criteria to build priority as preference for, otherwise indifferent, centers. When the list of criteria that provides priorities to students is not enough to break all relevant ties the municipalities are forced to use a sort of random process to break the ties.

Even though the criteria used by the schools to prioritize students can be seen as efficiency criteria, the realization of the procedure, to match students and schools' places, could induce inefficient allocations. This is due to the particular procedure employed to allocate the schools' places among the students.

In this sense, we can mainly find two systems: The centralized ones, that focus on stability, and the decentralized ones, focusing on Pareto efficiency. The mechanism used in the Boston area, from 2005, can be seen as a representative example for the first one, whereas the Boston mechanism, used in the same area until that date, is a representative example for the second one. The reader can find a description for both mechanisms in the survey by Sönmez and Ünver (2009). In this paper we concentrate on the former, the Student-Optimal Stable mechanism, (SOSM hence for) which coincides with the realization of the deferred acceptance algorithm in which students send proposals to the schools. This mechanism, introduced by Gale and Shapley (1962), always selects a stable allocation. Moreover, all the students (weakly) prefer this allocation to any other stable allocation. Furthermore, when this mechanism is use, students have no interest on acting strategically.

Along with the Boston area the redesign that has been introduced in several markets have presented the SOSM as the best option for school markets, i.e. for one sided-markets with indifferences. However, the fact this mechanism is, in general, not efficient has trouble scholars (see Kesten, 2010 and Abdulkadiroğlu, Patha, and Roth, 2009). In general it can be shown that the welfare lost due to select the SOSM can be large. There has been several attends to address this problem. In general this attempts accommodate any preexisting priority structure by allowing a final allocation that relax some priorities in order to achieve efficiency (see Kesten, 2010) or allows the agents to trade their initially stable endowments until reaching an allocation that can not be justifiably objected by any student (see Alcalde and Romero-Medina, 2010). In any of those attempts the initial priority structure is, at some point, either modify or ignored. In both case it seem a better design approach to build a priority structure that reconciles efficiency and stability rather than impose priorities that will impose efficiency lost or, ultimately, will be relax or ignore.

The usual way to prioritize students in most school districts to establish priorities comes from the use of a scoring rule. In practical terms the way to deal with the problem of how students would be prioritized is solved by municipalities building a function that associates each student a score for each school. This function often depends on two classes of variables: the relationship between the student and each school includes and the student's characteristics that are not related to any school. Among the first, are the distance from the student's residence to the school; and the number of siblings, already attending to the school. Among the second group we can mention the household income, the student's household size or the presence of any health condition. Further more, this criteria is based on continuous variables that are transform into discrete counterparts in order to be processed. For instance the *distance to school* is a continuous criteria that is transform into a discrete one using "attendance area boundaries" that defines the schools in which each students will be prioritized based on her address. By its definition the *School Attendance Area Policies* assign the same score to students in the same area and forces the procedure to relay on a lottery drawn to build strict priority list.

In our model, we do not assume any particular geographical distribution of students and/or schools, we will assume that schools' priority lists need not to be correlated and each school might exhibit any priority list. This is why, throughout this paper claim that *each school is free to decide how it prioritizes the students*. Accordingly, we refer the policy of accurately redesigning each school attendance area as *schools limited freedom* to prioritize the students. In this paper we show that efficiency and equity are compatible if we proceed as follows: Let us consider two classes of priority lists, namely the *local* and the *school* lists. Then, for a given school, its priority list is used to prioritize (at most) as much students as it can accept. These students will have priority over any other student. The remaining students are prioritized by following the *local committee*'s list. Following this approach, the general recommendation of this paper is to restrict, for each school, the number of students belonging to its *influence area*. Therefore, what the *Local School Committees* would do is to redesign their school attendance areas.

Following, Abdulkadiroğlu et al. (2006), the recent reforms in the allocation mechanisms lies on avoiding the possibility of students' strategic behavior. What it seems to be relevant when designing a mechanism for school allocation problems is to accurately combine stability, efficiency and strategy-proofness. Let us notice that restricting the possible priority lists that schools can propose, our results point out that the use of the new mechanism in the Boston Area always selects stable and efficient allocations and this mechanism is *fully* strategy-proof[1], i.e. not only students have no interest on misrepresenting their preferences, but it should be also expected that schools do not prioritize students in a *wrong* way.[2]

[1]The result follows from Alcalde and Barberà (1994)

[2]In Valencia, Spain, the admission process is similar to the Boston mechanism. The schools prioritize students according to a scoring function that is established by law. Nevertheless, the score that a school assigns to a student is evaluated by the school, and it is not monitored

As Ergin (2002) shows the conditions that reconciles stability and Pareto efficiency are very stringent and involves check the preferences and priorities of agents in both sides of the market. In particular, if each student is free to rank any school as her best school, it is always possible to find non-scarce problems having a stable and Pareto efficient matching. Therefore, if conditions are established on individual characteristics, it is very hard to have in mind natural necessary conditions guaranteeing the existence of a stable and Pareto efficient matching. Our work is closely related to Ergin (2002) and latter extended by Elhers and Erdil (2010) . They characterize priority structures under which the constrained efficient assignments are efficient. Our proposal differs from theirs in the fact that we explore the possibility to propose score based priority rankings.

The rest of the paper is organized as follows. Section 2 introduces the basic framework and provides some definitions which are classical in the literature. Section 3 explores conditions on agents' characteristics under which efficient and stable allocations exist. Our conclusions point out a general difficulty to conciliate both concepts. This difficulty advices us that we should shift the focus on how the allocation problem could be solved. The solution could be found in a reconsideration on how the schools give priorities to the students. We propose a general formulation that can be used as a new way to discuss how the system should be reformed. The way in which this mechanism is described points out how the actual Boston system might be *re-reformed*. Conclusions are gathered to Section 4. Finally, all the proofs are relegated to the Appendix.

2 The School Allocation Problem

The School Allocation Problem family of problems faces two set of non-empty disjoint agents to be called Students and Schools. The set of Students is denoted by $\mathcal{S}$, and has n individuals, i.e. $\mathcal{S} = \{s_1, \ldots, s_i, \ldots, s_n\}$. The set of Schools is denoted by $\mathcal{C}$, and has m elements, i.e. $\mathcal{C} = \{c_1, \ldots, c_j, \ldots, c_m\}$.

Each school has a number of seats (or places), to be distributed among the students, that will be called its capacity. Let $q_{c_j} \geq 1$ denote school c_j's capacity; and let $Q = \{q_{c_1}, \ldots, q_{c_j}, \ldots, q_{c_m}\}$ the vector summarizing schools' capacities. Schools are also endowed a *priorities linear ordering* over the set of students. Let $\pi_{c_j} \in \Re^n$ be the students' ordering for school c_j and Π the $(m \times n)$-matrix summarizing these priorities. Formally, π_{c_j} is described as a n-dimensional vector such that for each $k \in \{1, \ldots, n\}$ there is a unique student s_i for which $\pi_{c_j s_i} = k$; given this description, the j-th row for matrix Π coincides with vector π_{c_j}.

Note that, under our description, no school would consider a student to be inadmissible. Notice that most of the legislative norms impose such a restriction in the way that the schools rank their potential students. Nevertheless, our model might capture the possibility of a student to be inadmissible at a low

by the local administration. This lack of monitoring allows the schools to favor some students. This behavior can be interpreted as a *manipulation* by some schools administrators.

cost: just by introducing a new variable for each school defining the *priority level* of the last admissible student.

On the other side, each student has linear preferences over the set of schools, so that no student will consider two different schools as equivalent (or indifferent), and no school is neither considered as inadmissible by any student. Let ρ_{s_i} denote the schools' ranking induced by student s_i's preferences,[3] and Φ the $(n \times m)$-matrix summarizing these rankings. Note that our model assumes that each student considers all the school as admissible.[4] Nevertheless, we can also reformulate this model by assuming that each student might consider some schools as unacceptable. The essence of this paper is the same in both frameworks.

Therefore, a School Allocation Problem can be described by listing the elements above: $\mathcal{SAP} = \{\mathcal{S}, \mathcal{C}; \Phi, \Pi, Q\}$. We will say that a School Allocation Problem is non-scarce whenever there is enough places to allocate all the students

$$\sum_{c_j \in \mathcal{C}} q_{c_j} \geq n.$$

Given a School Allocation Problem, $\mathcal{SAP}$, a solution for it is an application μ that matches students and schools' places. Such a correspondence is called a matching. Formally,

Definition 1 *A matching for $\mathcal{SAP}$, a School Allocation Problem, is a correspondence μ, applying $\mathcal{S} \cup \mathcal{C}$ into itself, such that:*

(a) For each s_i in $\mathcal{S}$, if $\mu(s_i) \neq s_i$, then $\mu(s_i) \in \mathcal{C}$;

(b) For each c_j in $\mathcal{C}$, $\mu(c_j) \subseteq \mathcal{S}$, and $|\mu(c_j)| \leq q_{c_j}$; [5] *and*

(c) For each s_i in $\mathcal{S}$, and any c_j in $\mathcal{C}$, $\mu(s_i) = c_j$ if, and only if, $s_i \in \mu(c_j)$.

The central solution concept used through the literature is stability, as defined by Balinski and Sönmez (1999). This stability notion coincides with the pair-wise stability introduced by Gale and Shapley (1962). Under our considerations (i.e., each school is acceptable for any student and vice versa), stability is defined as follows.

Definition 2 *A matching for $\mathcal{SAP}$, say μ, is said stable if there is no student-school pair (s_i, c_j) such that*

(a)$\mu(s_i) = s_i$, or $\rho_{s_i c_j} < \rho_{s_i \mu(s_i)}$; and

(b) $|\mu(c_j)| < q_{c_j}$, or $\pi_{c_j s_i} < \pi_{c_j s_h}$ for some $s_h \in \mu(c_j)$.

Throughout this paper, we adopt the convention that $\rho_{s_i \mu(s_i)} = m + 1$ whenever $\mu(s_i) = s_i$.

[3]I.e., $\rho_{s_i c_j} = 3$ indicates that student s_i considers that c_j is her third-best school.

[4]Here, we can also invoke legislative regulations establishing that school attendance is compulsory for the children of certain ages.

[5]Throughout this paper $|T|$ will denote the cardinality of set T.

The idea of instability comes basically from the notion of justified envy. (See Haeringer and Klijn, 2009). Let us consider a matching μ, and let us assume that student s_i prefers to study at school c_j rather that developing her educative formation at her actual school $\mu(s_i)$. If s_i has a priority higher than that of some of the actual students attending school c_j, or this school is still having some vacant, she might claim that the allocation process has been *unfair*.

A second notion that has also been analyzed in this framework is that of efficiency. To introduce appropriately this concept, let us remember that the only role for the schools is to provide educational services needed by the students. Therefore the natural notion of efficiency, as proposed by Balinski and S'onmez (1999) for this framework, is Pareto efficiency (from the students' point of view).

Definition 3 *Given a School Allocation Problem, $\mathcal{SAP}$, we say that matching μ is Pareto efficient if for any other matching μ' there is a student, say s_i, such that*

$$\rho_{s_i\mu(s_i)} < \rho_{s_i\mu'(s_i)}.$$

Note that, for any non-scarce School Allocation Problem, stability and/or efficiency of a matching μ implies that, for each student s_i, $\mu(s_i) \in \mathcal{C}$.

A *matching mechanism* is a regular procedure that associates to each School Allocation Problem a matching for such a problem. A matching mechanism $\mathcal{M}$ is said to be stable if, for any given problem, it always selects a stable matching. Similarly, we say that a matching mechanism is Pareto efficient whenever its outcome is always Pareto efficient, related to its input. It is easy to see that there are stable matching mechanisms. In fact, any of the versions of the deferred-acceptance algorithms proposed by Gale and Shapley (1962) associates a stable matching for the related School Allocation Problem. On the other hand, the now-or-never mechanism introduced by Alcalde (1996) always selects a Pareto efficient matching when the proposals are made by the students.[6]

It is a well known result that it might be an impossible task to conciliate stability and Pareto efficiency, i.e. there is no matching mechanism selecting a stable and Pareto efficient allocation for each School Allocation Problem. In the next section we seek the existence of natural conditions on Φ and Π guaranteeing the existence of matching mechanisms being stable and Pareto efficient.

3 On the existence of stable and Pareto efficient allocations

In this section we explore natural conditions under which stability and Pareto efficiency are compatible for some matching mechanisms. Given that set of stable matchings has a lattice structure and one of its extremes can be reached by applying the student-proposing deferred acceptance algorithm. Therefore, the

[6]The now-or-never mechanism is also known as the Boston mechanism because it was used in the Boston school district.

unique stable matching mechanism that could eventually be Pareto efficient is the so called *Student-Optimal Stable mechanism.* The question is whether on the framework of the school choice problem are conditions able to encompass both the uses of school districts on the priorities matrices under which the students optimal stable matching is always efficient. To provide a positive answer to the above question, let us consider the following condition on schools' priorities:

Definition 4 *We say that the schools' priorities matrix Π satisfies the* ***Common Priorities Condition*** *, CPC in short, if, and only if, there is a (fixed) relation of students' priorities, $\pi = (\pi_{s_1}, \ldots, \pi_{s_i}, \ldots, \pi_{s_n})$, such that for each school c_j, and 'preferred set of students' for such a school, $S^j \subseteq \mathcal{S}$, with $\left|S^j\right| \leq q_{s_j}$, its priority relation satisfies:*

(a) $\pi_{c_j s_i} \leq q_{c_j}$*, for each $s_i \in S^j$; and*

(b) $\pi_{c_j s_i} = \left|S^j\right| + \left|\left\{s_h \in \mathcal{S} \setminus S^j : \pi_{s_h} \leq \pi_{s_i}\right\}\right|$*, for each student $s_i \notin S^j$.*

Let us imagine that students are ordered following a common criterion, which is independent from the schools characteristics and does not depend on the distance from the students' residence to the school, neither on whether the students have or not any sibling at the school, or any similar criteria. Then, once such common priorities have been fixed, each school can modify it by selecting its ideal set of students, whose size must be not higher than the number of its available positions, but the rest of ('non-ideal') students must be prioritized following a *common criterion.*

We can now establish the following result.

Proposition 5 *Let $\mathcal{SAP}$ be a School Allocation Problem. If Π satisfies CPC, then it has only one matching being stable and Pareto efficient.*

We can define a condition, similar to *CPC*, applying to students' characteristics. This is the essence of *CRC.*

Definition 6 *We say that the students' rankings matrix Φ satisfies the* ***Common Ranking Condition****, CRC in short, if, and only if, it is possible to find a linear order over the set of schools $\mathcal{C}$, described by the ranking $\rho = \left(\rho_{c_1}, \ldots, \rho_{c_j}, \ldots, \rho_{c_m}\right)$, such that for each student, s_i, and school c_j,*

$$\rho_{s_i c_j}(s_i) = \begin{cases} 1 + \rho_{c_j} & \text{if } \rho_{c_k} < \rho_{c_j} \\ \rho_{c_j} & \text{if } \rho_{c_k} > \rho_{c_j} \end{cases}$$

where c_k is such that $\rho_{s_i c_k} = 1$.

To introduce the idea underlying *CRC*, let us imagine that there is an unambiguous way to order the schools (that can be derived from some quality index that is commonly accepted), but each student has a preferred school due to some specific criteria (for instance due to its proximity to the student's residence address; or because she has some sibling attending to that school; or because her parents studied at that school; etc.) *CRP* says that all the students will share the same preferences except that they can differ on which her ideal school is.

Unfortunately, as the next example points out, a result similar to Proposition 5 can not be reached just by assuming that students' ranking satisfies *CRC*. We need an extra qualification, which is that all the students share the opinion on which her ideal school is.

Example 7 *Let us consider a School Allocation Problem involving three students and two schools, having one vacant each. Let us assume the ranking and priorities matrices are*

$$\Phi = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}; \text{ and } \qquad \Pi = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Note that this School Allocation Problem satisfies CRC. Nevertheless, the unique stable matching is described as μ, where $\mu(s_1) = c_2$; $\mu(s_2) = c_1$; and $\mu(s_3) = s_3$, which is inefficient. What this example might suggest is that the non existence of a stable and efficient matching is related to the fact that this School Allocation Problem is scarce. Nevertheless, it is not true. Just to show it, let is imagine that there is a new school, c_3 and the new problem, consistent with the previous one, is described by

$$\Phi' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}; \text{ and } \qquad \Pi' = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Note that, for any $q_{c_3} \geq 1$, the unique stable matching is μ', where $\mu'(s_1) = c_2$; $\mu'(s_2) = c_1$; and $\mu'(s_3) = c_3$.

The next result proposes conditions that, when imposed on students' rankings, guarantee the existence of a stable and Pareto efficient matching.

Proposition 8 *Let $\mathcal{SAP}$ be a School Allocation Problem. If Φ satisfies CRC, and all the students exhibit the same ranking, then there is a unique stable and efficient matching.*

Let us mention that the *CRC* was proposed by Alcalde and Barberà (1994) as an example for Top Dominance, a condition guaranteeing the existence of strategy-proof stable matching mechanisms for the marriage problem.[7] It is also easy to check that, given the close (formal) relationship between the School Allocation and the Marriage Problems, *CPC* is also related to the fulfillment of Top Dominance by the schools' declared preferences (or priorities).

[7] A marriage problem, as modeled by Gale and Shapley (1962), can be described as a School Allocation Problem in which all the colleges have just a vacant position.

Definition 9 *Let $\mathcal{R}_i$ be a set of possible rankings for student s_i. We say that $\mathcal{R}_i$ satisfies **Student Top Dominance** if for each c_j and c_k in $\mathcal{C}$, and any two rankings ρ_{s_i} and ρ'_{s_i} in $\mathcal{R}_i$ if $\rho_{s_ic_j} < \rho_{s_ic_k}$ and $\rho'_{s_ic_k} < \rho'_{s_ic_j}$ then*

$$\left\{c_h \in \mathcal{C} \colon \rho_{s_ic_h} \leq \rho_{s_ic_j}\right\} \cap \left\{c_h \in \mathcal{C} \colon \rho'_{s_ic_h} \leq \rho'_{s_ic_k}\right\} = \emptyset.$$

Definition 10 *Let $c_j \in C$ be a school, with admission quota q_{c_j}, and $\mathcal{P}_j$ be a set of possible priorities for such a school. We say that $\mathcal{P}_j$ satisfies **School Top Dominance** if there is a set of students $\mathcal{S}^j \subset \mathcal{S}$, with $\left|\mathcal{S}^j\right| \leq q_{c_j} - 1$, such that for each s_i and s_f in $\mathcal{S}$, and any two priorities π_{c_j} and π'_{c_j} in $\mathcal{P}_j$ if $\pi_{c_js_i} < \pi_{c_js_f}$ and $\pi'_{c_js_f} < \pi'_{c_js_i}$ then*

$$\left\{s_h \in \mathcal{S} \colon \pi_{c_js_h} \leq \pi_{c_js_i}\right\} \cap \left\{s_h \in \mathcal{S} \colon \pi'_{c_js_h} \leq \pi'_{c_js_f}\right\} \subseteq \mathcal{S}^j.$$

The idea that schools have some freedom on how to order their chosen students is behind School Top Dominance. But such a freedom is not full range in the sense that the student which is prioritized to fulfill the college's quota (i.e. $\pi_{c_js_i} = q_{c_j}$) determines which the priority for the rest of students is. In particular, let us observe that, when $q_j = 1$ for each school c_j, the School Top Dominance introduced in Definition 10 coincides with the Top Dominance as defined by Alcalde and Barberà (1994).

The next example points out a general difficulty to generalize the notions of *CPC* and/or *CRC* having positive results similar to the ones proposed in Propositions 5 and 8.

Example 11 *Let us consider a School Allocation Problem involving three students and three schools, having one vacant each. Let us suppose that schools priorities are described by matrix, satisfying School Top Dominance,*

$$\Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Now, let us assume that all the students agree that school c_2 is their worst option. In such a case, and when imposing Student Top Dominance, we have the following two possibilities:

(*a*) *All the students exhibit the same ranking, and*

(*b*) *Students might have opposite opinions on how to rank schools c_1 and c_3.*

Note that if the first case holds Proposition 8 applies and thus, there is a unique (assortative) stable matching, which is also Pareto efficient. Nevertheless, if it is not the case, the rankings matrix might be

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

and the unique stable matching that this problem has is not Pareto efficient. As a conclusion we can say that when the students exhibit rankings satisfying Top Dominance and there is a common opinion on ranking some school as the best one (or the worst one) we can have any of the following two conclusions:

(i) *If we add an anonymity condition, as Alcalde and Barberà (1994) proposed, to the Top Dominance idea, all the students will exhibit the same ranking. Therefore, the conclusion is that there is a unique (assortative) stable matching which is also efficient, as established in Proposition 8.*

(ii) *On the contrary, if such an anonymity is not demanded, we cannot guarantee the existence of a stable and Pareto efficient matching.*

We proposes a way to reformulate how students would be prioritized. As we have mentioned before students are prioritized following a scoring function that can be seen as a function which is separable on two classes of variables. The first type of variables lies on the relationship between any student and each school. Following this conception, it is expected that each student obtains different scores from two schools and also that each school assigns different scores to any two students. Relative to the second class of variables, they are not related to any school. Therefore, we can identify the score that each student obtain, from a school, relative to the first scoring function, with its score by that school;[8] whereas the score assigned to each student, that does not depend on their relationship with any school, can be seen as a score assigned by the *Local School Committee*. The above interpretation allows us to define two scoring functions, to be called the *School Scoring Function*, CS, and the *Local Scoring Function*, LS.

The last question that we deal with, related to this *score-based* way to build schools' priority list, lies on how to combine the *School Scoring Function* and the *Local Scoring Function* to define a *Global Scoring Function*. Following the idea beyond the procedure used by the Seattle School Board, let us define, for each school, its *Priority students* as those for which the *School Scoring Function* is the relevant function for deciding their score. Therefore, we can formally describe the *Global Score* for an student as follows.[9]

Definition 12 *Let $\mathcal{S}$ and $\mathcal{C}$ be the sets of students and colleges respectively. We define the* ***Global Score Function*** *induced by the School Scoring Function $CS : \mathcal{S} \times \mathcal{C} \to \Re_+$; the Local Scoring Function $LS : \mathcal{S} \to \Re_+$; and the Priority*

[8]Let us remember that the scores are determined by the *Local School Committees*, and also that schools do not determine their priority lists. Thus, we are *abusing interpretation* when saying that the school determines the score. We hope the reader not to be induced to confusion due to such an abuse.

[9]The reader can think on different ways to describe a *Global Scoring Function* by, for instance, adding the Local and the School Scoring Functions. Nevertheless, what it is relevant to reach our positive results formalized in Theorem 14 is not related to the particular composition but on the fact that it induces schools priorities satisfying CPC.

Function $P : \mathcal{C} \to 2^{\mathcal{S}}$ *as the function* $GS : \mathcal{S} \times \mathcal{C} \to \Re_+$ *assigning each student-school pair the value*

$$GS(s_i, c_j) = \begin{cases} LS(s_i) + CS(s_i, c_j) & \text{if } s_i \in P(c_j) \\ \\ LS(s_i) & \text{otherwise} \end{cases}$$

To introduce the next definition, for a *Local Scoring Function*, LS, let M denote the maximum score allocated to a student,

$$M = \max_{s_i \in \mathcal{S}} LS(s_i) \tag{1}$$

Definition 13 *Let* $\mathcal{S}$ *and* $\mathcal{C}$ *be the sets of students and colleges respectively, and let* Q *be the vector summarizing school's capacities. Given the Scoring Functions* LS, CS; *and* P, *we say that the Scoring Function* GS, *satisfies the Limited Freedom Condition , LFC henceforth, if it fulfills:*

(a) *For each school* c_j, *and student* s_i,

$$CS(s_i, c_j) > M.$$

(b) *For each school* c_j,

$$|P(c_j)| \leq q_{c_j}, \text{ and}$$

(c) *For any two students* s_i, s_h, $s_i \neq s_h$,

$$LS(s_i) \neq LS(s_h).$$

Note that, under *LFC*, the induced priority ordering for a school, if it chooses the function CS, allows the school to decide which its best students are, to fulfill its available places. Nevertheless, once some student has been declared not to be one of the q_{c_j}-top students, the school's opinion is not taken into account to determine such a student's priority.

A second aspect that we want to remark is that, even though in our model each priority ordering is linear, Definition 13 allows that some students, in the q_{c_j} first positions, share the same priority. Note that this fact does not influence the stability of any matching, even though if it considers that all those students are equaled prioritized. The reader is refer to Ehler and Erdil (2010) for a full characterization of Stable and Efficient allocations in this environment.

Theorem 14 *Let* $\mathcal{S}$ *and* $\mathcal{C}$ *be the sets of students and colleges respectively, and let* Q *be the school's capacities vector. Then for each Scoring Function* GS *that satisfies LFC, and any* $\mathcal{SAP} = \{\mathcal{S}, \mathcal{C}; \Phi, \Pi, Q\}$, *where priorities are induced by* GS, *there exists a unique stable and Pareto efficient matching.*

Note that Theorem 14 establishes that *LFC* is a sufficient condition to guarantee the existence of matchings conciliating the notions of stability and Pareto efficiency. Let us remark that the usual way to prioritize students comes from the employ of some scoring rule. These functions are usually expressed as the addition of two scoring functions. The first one just takes into account the particular characteristics of the students, isolated from the schools, whereas the second one lies on each student's characteristics related to the considered school. This additively separable functional form follows the shape used in Definition 13.

We would like to point out that it is very hard to find a result establishing necessary conditions conciliating stability and Pareto efficiency. In particular, if each student is free to rank any school as her best school, it is always possible to find non-scarce problems having a stable and Pareto efficient matching. Just, consider that students rankings are such that, for each school c_j the number of students for which this school is the first-ranked, i.e. $\rho_{s_i c_j} = 1$, is not greater than q_{c_j}. In such a case, no matter which the matrix Π is, the matching that associates each student her best school is both stable and Pareto efficient. Therefore, if conditions are established on individual characteristics, it is very hard to have in mind natural necessary conditions guaranteeing the existence of a stable and Pareto efficient matching.

4 Concluding Remarks

Let us start this section by quoting the paper by Abdulkadiroğlu et al. (2005). Recent Developments Section.

> "A memorandum from Superintendent Payzant in December 2004 states that BPS plans to change the computerized process used to assign students to schools. Although the task-force report recommended that BPS adopt the TTC assignment algorithm, the School Committee is interested in simulations of both mechanisms and in understanding the extent of preference manipulation under the Boston mechanism. They are also thinking through their philosophical position on the trade-off between stability and efficiency."

This interest for defining a philosophical approach on the trade-off between stability and efficiency is at the origin of a modification in the mechanism used in the Boston Area, decided by July 2005, see Abdulkadiroğlu et al. (2006). As these authors mention, the solution was to adopt a deferred acceptance mechanism because it is strategy-proof. Nevertheless, as we have pointed out in the present paper, this solution is far from solving the trade-off which is at the origin of this reform. In fact the employ of such a solution can be justified because it considers that stability is the central issue. If, moreover, the best that agents can do is to reveal their true characteristics, it is straightforward to conclude, as the Boston School Committee did, that the Student-Optimal Stable mechanism would be adopted.

The main conclusion of this paper is to provide a ways to avoid the efficiency-equity dilemma. It allows to slightly modify the last reform introduced by the Boston School Committee, comes from redefining how the schools influence on designing which students are prioritized. In this sense, the solution comes from the use of scoring rules that, applied to the students' characteristics, fulfill our *Limited Freedom Condition.*

Appendix

I. Proof for Propositions 5 and 8. To prove Propositions 5 and 8, let us start by establishing some claims that will be helpful to understand the structure of stable and efficient matchings.

Claim 15 *Let $\mathcal{SAP}$ be a School Allocation Problem. Then it has, at most, one stable and efficient matching.*

Proof. First of all, and following Martínez et al. (2001), let us note the set of stable matchings has a lattice structure. The operators proposed by Martínez et al. (2001) to prove such a structure are just the students' rankings and the schools' priorities. Therefore, if a stable matching μ is efficient for $\mathcal{SAP}$, it must be the *student optimal stable matching.* ■

Claim 16 *Let $\mathcal{SAP}$ be a School Allocation Problem, and μ be a stable matching for it. Then μ is efficient if, and only if, there is no (non-empty) ordered set of students*

$$\left\{s^1, \ldots, s^i, \ldots, s^k\right\} \subseteq \mathcal{S}, \text{ such that}$$

(a) $\mu\left(s^i\right) \in \mathcal{C}$ for all $s^i \in \mathcal{S}$, and

(b) each student s^i ranks her mate as worse than her next-in-the-order student's mate,

$$\rho_{s^i\mu(s^{i+1})} < \rho_{s^i\mu(s^i)} \textit{ for each } i < k, \textit{ and } \rho_{s^k\mu(s^1)} < \rho_{s^k\mu(s^k)}.$$

Proof. First of all, let us note that if there exists a set of students fulfilling the statement of Claim 16, it is easy to see that the matching is not efficient. To see that, given a matching μ, let us construct μ' such that $\mu'(s_i) = \mu(s_i)$ for each $s_i \notin \left\{s^1, \ldots, s^i, \ldots, s^k\right\}$ and for any student s^i in such a set, $\mu'\left(s^i\right) = \mu\left(s^{i+1}\right)$, modulo k. Note that μ' Pareto dominates μ. On the other hand, let us assume that μ is a stable matching that fails to be efficient. Then, there should be another matching μ' that dominates μ. This is equivalent to say that $\mu' \neq \mu$ and, for each student s_i such that $\mu'(s_i) \neq \mu(s_i)$, $\rho_{s_i\mu'(s_i)} < \rho_{s_i\mu(s_i)}$. Let s^1 be a student such that $\mu'\left(s^1\right) \neq \mu\left(s^1\right)$. Note that, by the Decomposition Lemma,[10]

[10]See, for instance Gale and Sotomayor (1985). Note that Martínez et al. (2000) pointed out that this results is still valid in our framework.

and provided that μ is stable, it must be the case that $\mu\left(s^1\right) \neq s^1$. Let denote $\mu'\left(s^1\right) = c^1 \in \mathcal{C}$. Since μ is stable, and $\rho_{s^1c^1} < \rho_{s^1\mu(s^1)}$, it must be the case that $\left|\mu\left(c^1\right)\right| = q_{c^1}$. Therefore, there should be a student, say $s^2 \in \mu\left(c^1\right) \setminus \mu'\left(c^1\right)$. Since μ' Pareto dominates μ, and $\mu\left(s^2\right) \neq \mu\left(s^2\right)$, there should be an school, say c^2 such that $\mu'\left(s^2\right) = c^2$. Note that if $\mu\left(s^1\right) = c^2$ the result is proved. Otherwise, there should be a student, say $s^3 \in \mu\left(c^2\right) \setminus \mu'\left(c^2\right)$. Let us observe that if $\mu'\left(c^3\right) = \mu\left(c^1\right)$, the result follows. Otherwise, since the set of students is finite, an iterative argument yields the desired result. ■

We can now proceed to prove Propositions 5 and 8.

Proof of Proposition 5.
Let $\mathcal{SAP}$ be a School Allocation Problem. Let us assume that Π satisfies *CPC*. By Claim 15 we now that, if there is a stable and Pareto efficient matching, it must be μ^{SO}, the student optimal stable matching. Since μ^{SO} is stable, let us assume that it is not efficient.

Then, by Claim 16, there must be a matching μ', and an ordered set of students, say $\left\{s^1, \ldots, s^i, \ldots, s^k\right\} = \mathcal{S}'$, such that for each $s^i \in \mathcal{S}'$, $\rho_{s^i\mu'(s^i)} < \rho_{s^i\mu^{SO}(s^i)}$, with $\mu'\left(s^i\right) = \mu^{SO}\left(s^{i+1}\right)$ for each $i \leq k-1$, and $\mu'\left(s^k\right) = \mu^{SO}\left(s^1\right)$.

Since μ^S is stable, we have that, for each student $s^i \in \mathcal{S}'$,

(a) $\pi_{\mu^{SO}(s^{i+1})s^i} > q_{\mu^{SO}(s^{i+1})}$; and

(b) $\pi_{\mu^{SO}(s^{i+1})s^i} > \pi_{\mu^{SO}(s^{i+1})s^{i+1}}$.

where $s^{k+1} = s^1$.

By *CPC*, there must be a priorities order π such that, for each school $c_j \in \mathcal{C}$, and any two students s_i, s_h such that $\min\left\{\pi_{c_js_i},\ \pi_{c_js_h}\right\} > q_{c_j}$, it holds that

$$\left[\pi_{c_js_i} < \pi_{c_js_h}\right] \Leftrightarrow \left[\pi_{s_i} < \pi_{s_h}\right].$$

Therefore, by *CPC* and stability of μ, we have that

$$\pi_{s^1} < \cdots < \pi_{s^i} < \pi_{s^{i+1}} < \cdots < \pi_{s^k} < \pi_{s^1},$$

which contradicts that π might represent a priorities order.

Proof of Proposition 8.
Let $\mathcal{SAP}$ be a School Allocation Problem. Let us assume that Φ satisfies *CRC*, relative to ρ, which is the ranking that all the students exhibit. Note that, in such a case it is straightforward to see that there is a unique stable matching, which is also Pareto efficient.
This matching can be obtained in a simple sequential way. If we denote by $c^t \in \mathcal{C}$ the t-th college according ρ, we proceed as follows:

(1) $\mu^S\left(c^1\right) = \{s_i \in \mathcal{S} \colon \pi_{c^1s_i} \leq q_{c^1}\}$;

(2) $\mu^S\left(c^2\right)$ is the set containing the q_{c^2} prioritized students, according to π_{c^2}, that are not in $\mu^S\left(c^1\right)$; ...

(t) $\mu^S\left(c^t\right)$ is the set containing the q_{c^t} prioritized students, according to π_{c^t}, that are not in $\mu^S\left(c^h\right)$ for any $h < t$.

It is easy to see that this assortative matching is both stable and efficient.

II. Proof for Theorem 14.
To prove Theorem 14, let us consider a School Allocation Problem. Let us assume that there is a 'scoring function' $GS : \mathcal{S} \times \mathcal{C} \rightarrow \Re$ inducing the priorities matrix Π. Note that this is equivalent to say that for each $c_j \in \mathcal{C}$, and any two students s_i and s_h in $\mathcal{S}$, $\pi_{c_j s_i} < \pi_{c_j s_h}$ whenever $GS\left(s_i, c_j\right) > GS\left(s_h, c_j\right)$.
Let us assume that GS satisfies LFC. Then there are three functions $LS : \mathcal{S} \rightarrow \Re_+$, $P : \mathcal{C} \rightarrow 2^{\mathcal{S}}$, and $CS : \mathcal{S} \times \mathcal{C} \rightarrow \Re_+$ decomposing GS as established in Definition 13.

For function LS given, let us define the 'common priority' π as

$$\pi_{s_i} = \left|\left\{s_h \in \mathcal{S} : \ LS\left(s_h\right) \geq LS\left(s_i\right)\right\}\right|.$$

Then, since GS satisfies LFC, and, for each $c_j \in \mathcal{C}$, π_{c_j} is induced by LFC, it follows that Π satisfies CPC related to π. Thus, as Proposition 5 establishes, $\mathcal{SAP}$ has a unique stable and efficient matching.

References

[1] A. Abdulkadiroğlu, P.A. Pathak and A.E. Roth (2009). *Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match.* American Economic Review, Papers and Proceedings, **99**, 1954–1958.

[2] A. Abdulkadiroğlu, P.A. Pathak, A.E. Roth and T. Sönmez (2005). *The Boston Public School Match.* American Economic Review, Papers and Proceedings, **95**, 368–371.

[3] A. Abdulkadiroğlu, P.A. Pathak, A.E. Roth and T. Sönmez (2006). *Changing the Boston School Choice Mechanism: Strategy-proofness as Equal Access* , mimeographed. Harvard University.

[4] J. Alcalde (1996). *Implementation of Stable Solutions to Marriage Markets.* Journal of Economic Theory, **69**, 240–254.

[5] J. Alcalde and S. Barberà (1994). *Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Markets.* Economic Theory, **4**, 417–435.

[6] J. Alcalde and A. Romero-Medina (2010). *Re-Reforming the Bostonian System: A Novel Approach to the Schooling Problem. Mimeo. Available at SSRN: http://ssrn.com/*

[7] M. Balinski and T. Sönmez (1999). A Tale of Two Mechanisms: Student Placement. Journal of Economic Theory, **84**, 73–94.

[8] L. Ehlers and A. Erdil (2010). Efficient assignment respecting priorities. Journal of Economic Theory, **145,** 1269–1282.

[9] A. Erdil and H. Ergin (2008). What's the Matter with Tie-Breaking? Improving Efficiency in School Choice. American Economic Review, **98:3** , 669–689.

[10] Ergin H., Efficient resource allocation on the basis of priorities. Econometrica 70 (2002), 2489–2497.

[11] D. Gale and M. Sotomayor (1985). *Some Remarks on the Stable Matching Problem*. Discrete Applied Mathematics, **11**, 223–232.

[12] D. Gale and L. Shapley (1962). *College Admissions and the Stability of Marriage*. American Mathematical Monthly, **69**, 9–15.

[13] G. Haeringer and F. Klijn (2009). *Constrained School Choice*. Journal of Economic Theory, **144**, 1921–1947.

[14] Kesten (2010). School choice with consent, Quarterly Journal of Economics, **125** (3): 1297–1348.

[15] R. Martínez, J. Massó, A. Neme and J. Oviedo (2000). *Single Agents and the Set of Many-to-One Stable Matchings*. Journal of Economic Theory, **91**, 91–105.

[16] R. Martínez, J. Massó, A. Neme and J. Oviedo (2001). *On the Lattice Structure of the Set of Stable Matchings for a Many-to-One Model*. Optimization, **50**, 439–457.

[17] T. Sönmez and M.U. Ünver (2009). *Matching, Allocation, and Exchange of Discrete Resources*, Jess Benhabib, Alberto Bisin, and Matthew Jackson (eds.) Handbook of Social Economics, Elsevier, forthcoming.