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Abstract

The paper introduces a notion of complementarity (substitutability) of two signals which requires that in all decision problems each signal becomes more (less) valuable when the other signal becomes available. We provide a general characterization which relates complementarity and substitutability to a Blackwell comparison of two auxiliary signals. In a setting with a binary state space and binary signals, we find an explicit characterization that permits an intuitive interpretation of complementarity and substitutability. We demonstrate how these conditions extend to more general settings.

Keywords: Complementarity, substitutability, value of information, Blackwell ordering.

JEL Classification No.: C00, C44, D81, D83

1 Introduction

Suppose that two signals are available to a decision maker, and that each signal contains some information about an aspect of the world that is relevant to the decision maker's choice. In this paper we ask under which conditions these two signals are *substitutes*, and under which conditions they are *complements*. Roughly speaking, we mean by this that the incentive to acquire one signal decreases as the other signal becomes available (in the case of substitutes), or that it increases as the other signal becomes available (in the case of complements).

The incentives to acquire signals depend, of course, on the decision for which the information will be used. When we call signals complements or substitutes in this paper, then we mean that the conditions described above are satisfied *in all decision problems*. Hence we say that signals are substitutes if in all decision problems the value of each signal decreases as the other signal becomes available. The signals are complements if in all decision problems the value of each signal increases as the other signal becomes available. The conditions that we shall provide will thus not refer to any particular decision problem, but only to the joint distribution of signals, conditional on the various possible states of the world. We identify features of the joint distribution of signals that are necessary or sufficient for these signals to be substitutes or complements. Our approach is in the spirit of Blackwell's (1951) comparison of the informativeness of signals.

Two examples indicate how signals can be complements in our sense.

Example 1. *Signal 1 is the encrypted messages that your enemy's military commanders send to each other. Signal 2 is the encryption code. The encryption code is of no value by itself, unless you also have the messages that were sent. Equally, the messages sent are of no value by themselves without the encryption code. Together, however, the two signals are of positive value.*[1]

Example 2. *Signal 1 is the weather forecast for tomorrow. Signal 2 is information about the qualification of the forecaster. Knowing the qualification of the forecaster is of no value by itself, unless you also have the forecaster's forecast. But if you have the weather forecast, then it is potentially valuable to know the qualification of the forecaster. Symmetrically, the value of the forecast increases if you know the qualification of the forecaster.*

[1] The following is a special case of Example 1 in which the signal and the state are real numbers: $s = \omega + \varepsilon$, where s is a signal, ω is the state, and ε is random noise that is independent of the state. Here, ε is the "language" in which s expresses ω, and hence, if we regard ε as a signal, s and ε are complementary signals.

A special feature of these two examples is that signal 2 is independent of the information that the decision maker is interested in, that is, the decision-relevant state of the world. Signal 2 only provides information about signal 1. In Example 1, signal 2 provides the "language" in which signal 1 is expressed. In Example 2, signal 2 provides the "strength" of signal 1. Therefore, signal 2 has positive value only if the decision maker also has access to signal 1. Otherwise, it has zero value. This makes the two signals complements.

Our focus in this paper will be on the case when neither of the two signals is uninformative by itself, but instead both signals provide by themselves information about the state of the world.[2] Signals can be complements in this case, too. One result of this paper shows for a setting in which each signal is informative by itself, with two possible states of the world, and two possible realizations per signal, that complementary signals are characterized by a property that is related to a feature of Example 1. The result says that in the specified setting, signals are complements if and only if there are a state and a realization of each signal so that if received by themselves, each realization increases the probability of the state in comparison to the prior, yet if received together, the two signal realizations decrease the probability of the state. We refer to this as "meaning reversal:" the meaning of each realization is reversed when received together.

If a signal realization by itself raises the decision maker's subjective probability of a state, but if there is a realization of the other signal such that the two signal realizations, if observed together, lower the probability of that state, then there must be a different realization of the other signal that has the opposite effect: If observed with the same realization of the first signal, the decision maker's probability of the state must be raised in comparison to the prior.[3] Thus, the "meaning" of the realization of the first signal depends on the realization of the second signal. This is a weaker property than the property that the first signal provides the "language" in which the second signal is expressed, because in that case one would expect that the meaning of *all* realizations of the first signal depends on the realization of the second signal. By contrast, we obtain this feature only for *at least one* realization of the first signal. "Meaning reversal" is restrictive in another way, however. It requires a particular form of dependence of the meaning of the realization of one signal on the realization of the other signal: If the second signal by itself has the

[2]In economic applications it often seems artificial to assume that signals that are entirely uninformative. For example, it would be contrived to formalize the situation that motivates Dow and Gorton (1993), and that we describe later in this Introduction, assuming uninformative signals.

[3]This is because the prior probability is the expected value of the posterior probabilities.

same meaning as the first signal by itself, then the meaning of the joint realization is the opposite of the meaning that the two signal realizations have by themselves. Meaning reversal is thus related to, but different from the main feature of Example 1.[4]

An example of meaning reversal in the economics literature is provided by Dow and Gorton (1993). A technology company is observed by two analysts. One analyst learns whether the company's lead engineer is leaving the company to create an independent competitor, and the other learns whether the technology that the engineer is working on is likely to succeed. If the technology is likely to succeed and the engineer stays, then this is good news for the company's value. If the technology is likely to fail, and the engineer leaves, that is also good news because the company is likely to stay dominant in its market. However, the remaining cases are bad news about the company's value, because either a competitor with a promising technology is created, or because a dubious project will be continued further. The interpretation of each analyst's signal may be reversed by the other analyst's signal.

The reversal result that we have just illustrated will be shown in this paper for the setting with two states and two realizations per signal only. However, we also explore the extent to which it generalizes. We show that in many cases it is necessary for complementarity of signals that the meaning of the realization of one signal can be reverted by a realization of the other signal. We also show that this condition is in general not sufficient.

The next example illustrates how signals can be substitutes.

Example 3. *Signals 1 and 2 are the advice offered by two advisors. They both work with the same sources, and will tell you exactly the same thing. Then each of them will have positive value, but once you have heard what one of them says, you do not derive any additional benefit from hearing what the other one says.*

In the setting with two states and two realizations per signal that we referred to earlier a property that is related to perfect correlation is necessary and sufficient for signals to be substitutes. Interestingly, however, this condition is weaker than perfect correlation. Roughly speaking, it requires that conditional on observing certain realizations of one signal, the other signal does not provide further information to the decision maker. In the

[4] In the general model, in which both signals are informative, there is no case of complements that is similar to Example 2. We explain in Remark 4 below that Example 2 is not robust to perturbations which introduce a small informational content of signal 2 by itself.

general setting, with arbitrarily many states and signal realizations, a similar condition is necessary, but not sufficient for signals to be substitutes.

The results described so far provide interesting, but partial insights into the nature of the complementarity and substitutability relations among signals. We also offer in this paper completely general characterizations of complements and substitutes. These results show that two signals are complements (resp. substitutes) if and only if, among two other signals that are derived from the two original signals, one dominates the other in the sense of Blackwell (1951), that is, is more valuable in all decision problems. We thus reduce the problem of determining whether two signals are complements (resp. substitutes) to the problem of determining whether among two other signals one Blackwell dominates the other. This is useful because it allows us to use well-known characterizations of Blackwell dominance to find out whether two signals are complements (resp. substitutes).

After establishing this general result, the paper's objective is to obtain characterizations of complements and substitutes that offer more immediate insights than the general result does. This leads us to the results outlined earlier in the Introduction. As is well known, Blackwell comparisons are qualitatively different in the case of two states, and in the case of three or more states, with the case of two states being much easier to study. In the same way we find that we have particularly strong results for the case of only two states, and somewhat weaker result for the general case, as explained above.

Many pairs of signals are neither complements nor substitutes if our definitions are used. This is because our definitions of these terms require certain conditions to be true in *all* decision problems. This is in the spirit of Blackwell's comparison whose ordering of signals is incomplete. More signals will satisfy the conditions for being substitutes or complements if we restrict attention to smaller classes of decision problems. In the context of Blackwell's original work this line of investigation has been pursued by Lehmann (1988), Persico (2000), Athey and Levin (2001) and Jewitt (2007). A similar research agenda is feasible in our context, and we present in this paper a first step in this direction.

Radner and Stiglitz (1984) consider a setting in which a one-dimensional real parameter indexes the quality of a signal. They show that the value of the signal in any decision problem is weakly increasing but not everywhere concave as the quality of information increases. In particular, a non-concavity occurs for any decision problem in the neighborhood of the parameter value for which the signal is uninformative. Non-concavity of the value of a signal as the quality of the signal improves indicates increasing returns to scale in information. It may be possible to interpret an improvement in the quality of a signal as "making a further signal available," and one might be able to interpret a non-concavity of the value of information as a complementarity between an existing signal, and a further signal that might be made available. We have not yet explored whether we can make these analogies precise.

The idea that signals may be complements or substitutes has previously appeared in some applied work. An example is the paper by Sarvary and Parker (1997), who take consumers' valuations of signals as exogenously given, and focus on competition among information providers. Complementarity and substitutability of signals has previously also been referred to in an auction context by Milgrom and Weber (1982) and in a voting context complementarity of voters' information has been emphasized by Persico (2004). In auction or voting contexts, different signals are held by different individuals, whereas our paper focuses on a single person decision problem. All papers listed, moreover, consider complementarity or substitutability in very specific settings, whereas we seek characterizations of signals that are in all decision problems complements or substitutes.

The paper is organized as follows: Section 2 provides definitions. Section 3 contains our result on the relation between substitutability, complementarity, and Blackwell comparisons. Section 4 studies in detail the special case that there are only two states of the world. Section 5 investigates the extent to which the findings of Section 4 generalize when the number of states of the world is arbitrary. In Section 6 we specialize to the case in which the state of the world is a real number, and the utility function is linear in the state of the world. The results of Section 4 can be transferred to this setting. Section 7 concludes. Some of the proofs are relegated to the appendix.

2 Definitions

The state of the world is a random variable $\tilde{\omega}$ with realizations ω in a finite set Ω which has at least two elements. The probability distribution of $\tilde{\omega}$ is denoted by π. Without loss of generality we assume that each state in Ω occurs with the same probability: $\pi(\omega) = 1/|\Omega|$ for all $\omega \in \Omega$.[5] Two signals are available: $\tilde{s}_1$ with realizations s_1 in the finite set S_1 where S_1 has at least two elements, and $\tilde{s}_2$ with realizations s_2 in the finite set S_2 where S_2 also

[5] Our results in Sections 3-5 would not be different if the prior was any other distribution with support Ω. This follows from the relation between our analysis and the Blackwell comparison of signals that is pointed out in Proposition 1 below, and from the fact that the Blackwell comparison of signals is independent of the prior as long as the prior has full support. In Section 6, by contrast, the prior distribution of the state will matter, and we shall then relax the assumption of a uniform distribution.

has at least two elements. We assume without loss of generality that $S_1 \cap S_2$ is empty. The joint distribution of signals $\tilde{s}_1$ and $\tilde{s}_2$ conditional on the state being $\omega \in \Omega$ is denoted by $p_{12,\omega}$. The probability assigned by this distribution to some realization $(s_1, s_2) \in S_1 \times S_2$ is denoted by $p_{12,\omega}(s_1, s_2)$. The unconditional distribution of $(\tilde{s}_1, \tilde{s}_2)$ is denoted by $\bar{p}_{12}$ and is given by: $\bar{p}_{12}(s_1, s_2) = \sum_{\omega \in \Omega} p_{12,\omega}(s_1, s_2)\pi(\omega)$ for all $(s_1, s_2) \in S_1 \times S_2$. The probability distribution on Ω conditional on observing signal realization $(s_1, s_2) \in S_1 \times S_2$ (where $\bar{p}_{12}(s_1, s_2) > 0$) is denoted by q_{s_1,s_2} and is given by:

$$q_{s_1,s_2}(\omega) = \pi(\omega)\frac{p_{12,\omega}(s_1, s_2)}{\bar{p}_{12}(s_1, s_2)} \text{ for all } \omega \in \Omega. \tag{1}$$

For $i = 1, 2$ the marginal distribution of signal $\tilde{s}_i$ conditional on the state being $\omega \in \Omega$ is denoted by $p_{i,\omega}$. The probability assigned by this distribution to some realization $s_i \in S_i$ is denoted by $p_{i,\omega}(s_i)$. For $i = 1, 2$ the unconditional distribution of $\tilde{s}_i$ is denoted by $\bar{p}_i$ and it is given by: $\bar{p}_i(s_i) = \sum_{\omega \in \Omega} p_{i,\omega}(s_i)\pi(\omega)$ for all $s_i \in S_i$. Without loss of generality we assume that $\bar{p}_i(s_i) > 0$ for all $s_i \in S_i$. For $i = 1, 2$ the probability distribution on Ω conditional on observing signal realization $s_i \in S_i$ is denoted by q_{s_i} and is given by:

$$q_{s_i}(\omega) = \pi(\omega)\frac{p_{i,\omega}(s_i)}{\bar{p}_i(s_i)} \text{ for all } \omega \in \Omega. \tag{2}$$

We shall say that signal $\tilde{s}_i$ is "informative" if there is at least one $s_i \in S_i$ such that $q_{s_i} \neq \pi$. We shall say that signal $\tilde{s}_i$ is "informative conditional on signal realization $s_j \in S_j$" (where $j \neq i$) if there is at least one $s_i \in S_i$ such that $q_{s_1,s_2} \neq q_{s_j}$. We shall make the following:

Assumption 1. *For every $i \in \{1, 2\}$, if signal $\tilde{s}_i$ is not informative, then there is a signal realization $s_j \in S_j$ (where $j \neq i$) such that signal $\tilde{s}_i$ is informative conditional on signal realization s_j.*

This assumption rules out signals that are of no potential use to the decision maker.

To define when the two signals are substitutes or complements we need some auxiliary definitions.

Definition 1. *A* decision problem *is a pair (A, u) where A is some finite set of actions and u is a utility function: $u : A \times \Omega \to \mathbb{R}$.*

Definition 2. *For given decision problem (A, u):*

- *The* value of not having any signal *is:*

$$V_\emptyset(A,u) \equiv \max_{a\in A} \sum_{\omega\in\Omega} \left(u(a,\omega)\pi(\omega)\right). \tag{3}$$

- *For* $i \in \{1,2\}$ *the* value of having signal $\tilde{s}_i$ alone *is:*

$$V_i(A,u) \equiv \sum_{s_i\in S_i} \bar{p}_i(s_i) \max_{a\in A} \sum_{\omega\in\Omega} (u(a,\omega)q_{s_i}(\omega)). \tag{4}$$

- *The* value of having both signals *is:*

$$V_{12}(A,u) \equiv \sum_{s_1\in S_1} \sum_{s_2\in S_2} \bar{p}_{12}(s_1,s_2) \max_{a\in A} \sum_{\omega\in\Omega} (u(a,\omega)q_{s_1,s_2}(\omega)). \tag{5}$$

The two key definitions of this paper are:

Definition 3. *Signals* $\tilde{s}_1$ *and* $\tilde{s}_2$ *are* substitutes *if for all decision problems* (A,u) *we have:*

$$V_1(A,u) - V_\emptyset(A,u) \geq V_{12}(A,u) - V_2(A,u) \tag{6}$$

and

$$V_2(A,u) - V_\emptyset(A,u) \geq V_{12}(A,u) - V_1(A,u). \tag{7}$$

Definition 4. *Signals* $\tilde{s}_1$ *and* $\tilde{s}_2$ *are* complements *if for all decision problems* (A,u) *we have:*

$$V_{12}(A,u) - V_2(A,u) \geq V_1(A,u) - V_\emptyset(A,u) \tag{8}$$

and

$$V_{12}(A,u) - V_1(A,u) \geq V_2(A,u) - V_\emptyset(A,u). \tag{9}$$

Note that the two inequalities in Definition 3, and also the two inequalities in Definition 4, are equivalent.

For a simple interpretation of the inequalities in Definitions 3 and 4 suppose that the decision maker's not explicitly modeled *overall* utility is additive in the expected utility from decision problem (A,u) and money. Then the inequalities in Definitions 3 and 4 compare the decision maker's willingness to pay for signals in different scenarios. For example, the inequality in Definition 3 says that the willingness to pay for signal $\tilde{s}_1$ is larger if signal $\tilde{s}_2$ is not available than if it is available. It seems natural to call signals substitutes in this case. Without postulating the existence of money, and additive utility, one could interpret the inequalities in Definitions 3 and 4 using an idea in von Neumann and Morgenstern (1953, p. 18). They argue that inequalities that involve differences of von Neumann Morgenstern utilities reflect differences in the *intensity* of a preference. For example, in the case of Definition 3, this interpretation says that the preference for having signal $\tilde{s}_1$ over not having signal $\tilde{s}_1$ is *more intense* when signal $\tilde{s}_2$ is not present than when it is present. This interpretation of the difference of von Neumann Morgenstern utilities is not universally accepted, however.[6]

We next offer formalizations of the three examples informally described in the Introduction using the framework of this section. It will be obvious that the signals in Examples 1 and 2 are complements and in Example 3 substitutes.

Example 1. $\Omega = \{a, b\}$, $S_1 = \{\alpha, \beta\}$, $S_2 = \{\hat{\alpha}, \hat{\beta}\}$. *Signal 2 determines the language that signal 1 uses. If signal 2 has realization $\hat{\alpha}$, then signal 1 uses signal realization α to indicate state a, and signal realization β to indicate state b. If signal 2 has realization $\hat{\beta}$ then the signal 1 uses the reverse language. The code corresponding to $\hat{\alpha}$ is used with probability φ. Independent of the language, signal 1 indicates the state correctly with probability $\rho \in [0.5, 1]$ and provides an incorrect signal with the remaining probability. We display the two conditional distributions $p_{12,a}$ and $p_{12,b}$ in Figure 1. Rows correspond to realizations of signal $\tilde{s}_1$, and columns correspond to realizations of signal $\tilde{s}_2$.*

	$\hat{\alpha}$	$\hat{\beta}$
α	$\varphi\rho$	$(1-\varphi)(1-\rho)$
β	$\varphi(1-\rho)$	$(1-\varphi)\rho$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	$\varphi(1-\rho)$	$(1-\varphi)\rho$
β	$\varphi\rho$	$(1-\varphi)(1-\rho)$

$\omega = b$

Figure 1: Example 1 (signals are complements)

Example 2. $\Omega = \{a, b\}$, $S_1 = \{\alpha, \beta\}$, $S_2 = \{\hat{\alpha}, \hat{\beta}\}$. *Signal 1 indicates the state correctly with probability $\rho \in [0.5, 1]$ if signal 2 has realization $\hat{\alpha}$ and it indicates the state correctly with probability $\hat{\rho} \in [0.5, 1]$ if signal 2 has realization $\hat{\beta}$. Signal 1 always uses the code by which α indicates that the state is a, and β indicates that the state is b. Signal 2 has*

[6] Luce and Raiffa (1957, p. 32) regard this interpretation as a fallacy, whereas Binmore (2009, p. 67) is sympathetic to this interpretation. We return to von Neumann and Morgenstern's argument in more detail in Remark 1 below.

	$\hat{\alpha}$	$\hat{\beta}$
α	$\varphi\rho$	$(1-\varphi)\hat{\rho}$
β	$\varphi(1-\rho)$	$(1-\varphi)(1-\hat{\rho})$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	$\varphi(1-\rho)$	$(1-\varphi)(1-\hat{\rho})$
β	$\varphi\rho$	$(1-\varphi)\hat{\rho}$

$\omega = b$

Figure 2: Example 2 (signals are complements)

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ	0
β	0	$1-\rho$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	$1-\rho$	0
β	0	ρ

$\omega = b$

Figure 3: Example 3 (signals are substitutes)

realization $\hat{\alpha}$ with probability $\varphi \in (0,1)$. The conditional distributions of the signals are shown in Figure 2.

Example 3. $\Omega = \{a, b\}$, $S_1 = \{\alpha, \beta\}$, $S_2 = \{\hat{\alpha}, \hat{\beta}\}$. *Each signal indicates the state correctly with probability $\rho \in [0.5, 1)$ and incorrect with the remaining probability. Signals are perfectly correlated. The code that the signals use is that α and $\hat{\alpha}$ indicate that the state is a, and β and $\hat{\beta}$ indicate that the state is b. The conditional distributions of the signals are shown in Figure 3.*

3 Connection With Blackwell Dominance

To obtain a general characterization of signals that are complements or substitutes, we define two auxiliary signals, $\tilde{s}_S$ and $\tilde{s}_C$. Informally, the signal $\tilde{s}_S$ can be described as follows. An unbiased coin is tossed. If "heads" comes up, the decision maker is informed about the realization of $\tilde{s}_1$. If "tails" comes up, the decision maker is informed about the realization of $\tilde{s}_2$. Formally, the signal $\tilde{s}_S$ has realizations in the set $S_S \equiv S_1 \cup S_2$.[7] For given state $\omega \in \Omega$, the probability that $\tilde{s}_S$ has realization $s_1 \in S_1$ is $p_{S,\omega}(s_1) \equiv \frac{1}{2} p_{1,\omega}(s_1)$, and the probability that $\tilde{s}_S$ has realization $s_2 \in S_2$ is $p_{S,\omega}(s_2) \equiv \frac{1}{2} p_{2,\omega}(s_2)$. The unconditional distribution of $\tilde{s}_S$ is denoted by $\bar{p}_S$ and is given by $\bar{p}_S(s_S) = \pi(\omega) \sum_{\omega\in\Omega} p_{S,\omega}(s_S)$ for all $s_S \in S_S$. The conditional distribution on Ω conditional on observing signal realization

[7]Recall that we assume that $S_1 \cap S_2$ is empty.

$s_S \in S_S$ is the distribution q_{s_S} that was defined in equation (2).

The second auxiliary signal, $\tilde{s}_C$, is intuitively constructed as follows. An unbiased coin is tossed. If "heads" comes up, the decision maker is informed about the realizations of $\tilde{s}_1$ and $\tilde{s}_2$. If "tails" comes up, the decision maker receives no information. Formally, the signal $\tilde{s}_C$ has realizations in the set $S_C \equiv (S_1 \times S_2) \cup \{N\}$. Here, the symbol N denotes the case that the decision maker receives no information. For given state $\omega \in \Omega$, the probability that $\tilde{s}_C$ has realization $(s_1, s_2) \in S_1 \times S_2$ is $p_{C,\omega}(s_1, s_2) \equiv \frac{1}{2} p_{12,\omega}(s_1, s_2)$, and the probability that $\tilde{s}_C$ has realization N is $p_{C,\omega}(N) \equiv \frac{1}{2}$. The unconditional distribution of $\tilde{s}_C$ is denoted by $\bar{p}_C$ and is given by $\bar{p}_S(s_S) = \pi(\omega) \sum_{\omega \in \Omega} p_{S,\omega}(s_S)$ for all $s_S \in S_S$. The conditional distribution on Ω conditional on observing signal realization $s_C \in S_C$ is the distribution q_{s_C} that was defined in equation (1) if $s_C \in S_1 \times S_2$, and it is the prior distribution π if $s_C = N$. We shall write for this distribution also q_N.

Definition 5. *For given decision problem* (A, u) *and for* $k \in \{S, C\}$*, the* value of having signal $\tilde{s}_k$ *is:*

$$V_k(A, u) \equiv \sum_{s_k \in S_k} \bar{p}_k(s_k) \max_{a \in A} \sum_{\omega \in \Omega} (u(a, \omega) q_{s_k}(\omega)). \tag{10}$$

Proposition 1. *(i) Signals* $\tilde{s}_1$ *and* $\tilde{s}_2$ *are substitutes if and only if signal* $\tilde{s}_S$ Blackwell dominates *signal* $\tilde{s}_C$*, i.e. in all decision problems* (A, u)*:*

$$V_S(A, u) \geq V_C(A, u). \tag{11}$$

(ii) Signals $\tilde{s}_1$ *and* $\tilde{s}_2$ *are complements if and only if signal* $\tilde{s}_C$ Blackwell dominates *signal* $\tilde{s}_S$*, i.e. in all decision problems* (A, u)*:*

$$V_C(A, u) \geq V_S(A, u). \tag{12}$$

Proof. For part (i) note that in all decision problems (A, u) the inequality that defines substitutes, $V_1(A, u) - V_\emptyset(A, u) \geq V_{12}(A, u) - V_2(A, u)$ is equivalent to: $\frac{1}{2}(V_1(A, u) + V_2(A, u)) \geq \frac{1}{2}(V_{12}(A, u) + V_\emptyset(A, u))$. But by definition the expression on the left hand side is the same as V_S, and the expression on the right hand side is the same as V_C. Thus (i) follows. The proof of part (ii) is analogous. □

Remark 1. *This result is related to von Neumann and Morgenstern's (1953, p. 18) discussion of the meaning of comparisons of utility differences to which we alluded before.*

Roughly speaking,[8] *their argument is as follows. If $a, b, c,$ and d are outcomes, then the comparison of utility differences $u(a) - u(b) > u(c) - u(d)$ can be inferred from choices among lotteries because it is equivalent to: $0.5u(a) + 0.5u(d) > 0.5u(b) + 0.5u(c)$, and hence to the preference of the lottery that gives a and d each with probability 0.5 over the lottery that gives b and c each with probability 0.5. This preference can be interpreted as an expression of an intensity of preferences because it means that, starting from a lottery that gives b and d each with probability 0.5 the decision maker rather has b be replaced by a than d by c. Hence the step from b to a seems larger to the decision maker than the step from c to d. In our setting a, b, c, d are replaced by signals on which the decision maker can base a choice. Our Proposition 1 is then a formal statement of the way in which choices among lotteries express, according to von Neumann and Morgenstern, comparisons of utility differences.*

Blackwell and Girshick (1954, Theorem 12.2.2.) offer a variety of characterizations of Blackwell dominance. A well-known characterization is that one signal Blackwell dominates another if the dominated signal is a garbling of the dominating signal (Theorem 12.2.2., Condition (2), Blackwell and Girshick (1954)). Another condition is that the posteriors resulting from the dominating signal are a mean-preserving spread of the posteriors resulting from the dominated signal (Theorem 12.2.2., Condition (5), Blackwell and Girshick (1954)). We now show an example in which this latter condition can be used to easily verify that signals are complements *and* substitutes.

Example 4. $\Omega = \{a, b\}, S_1 = \{\alpha, \beta, \gamma\}$ *and* $S_2 = \{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$. *The distributions $p_{12,a}$ and $p_{12,b}$ are displayed in the same way as in Examples 1-3. Intuitively, in this example signal $\tilde{s}_1$ reveals the state with probability ρ. If signal $\tilde{s}_1$ reveals the state, signal $\tilde{s}_2$ is not informative and has realization $\hat{\gamma}$. Similarly, signal $\tilde{s}_2$ reveals the state with probability $1 - \rho$, and if signal $\tilde{s}_2$ reveals the state, signal $\tilde{s}_1$ is not informative and has realization γ. To verify that signals are complements and substitutes in Example 4 one can easily check that $\tilde{s}_S$ and $\tilde{s}_C$ imply identical posterior distributions: $(\frac{1}{2}, \frac{1}{2})$ with probability 0.5, and $(1, 0)$ and $(0, 1)$ with probability 0.25 each. Therefore, by the characterization of Blackwell dominance quoted above, $\tilde{s}_S$ (weakly) Blackwell dominates $\tilde{s}_C$ and vice versa, and by Proposition 1 signals are complements and substitutes.*

[8] Our rendition of von Neumann and Morgenstern's argument follows Binmore's (2009, p. 67). Von Neumann and Morgenstern's original argument is slightly different, involving only 3 outcomes. In substance it is the same.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	ρ
β	0	0	0
γ	$1-\rho$	0	0

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	0
β	0	0	ρ
γ	0	$1-\rho$	0

$\omega = b$

Figure 4: Example 4 (signals are complements and substitutes)

4 The Case of Two States

It is easier to verify Blackwell dominance when there are only two states of the world, and therefore beliefs are one-dimensional, than when there are more than two states of the world, and therefore beliefs are multi-dimensional. The qualitative difference between the one-dimensional case and the case of two or more dimensions is explained in Section 12.4 of Blackwell and Girshick. In the one-dimensional case the convex value function[9] arising from an arbitrary decision problem can be approximated arbitrarily closely by linear combinations of a very simple subclass of piecewise linear, convex functions. No such approximation result is known in the two or more-dimensional case. The relevance of having a dense class of simple value functions is that one can correspondingly restrict attention to a simple class of decision problems when checking Blackwell dominance.

The results cited in the previous paragraph, and the close connection between our concepts and Blackwell dominance shown in the previous section, motivate why we begin our study here with the case in which there are only two states of the world. We label them: $\Omega = \{a, b\}$. The key property of the two states model is that we can restrict attention to two action decision problems where $A = \{T, B\}$ and u is given by Figure 5.

	$\omega = a$	$\omega = b$
T	0	x
B	$1-x$	0

Figure 5: A two action decision problem

Lemma 1. *In the two states model, signals are complements (substitutes) if and only if*

[9] Value functions map posterior beliefs into the expected utility that the decision maker obtains when holding those beliefs and choosing optimally. Every decision problem gives rise to a convex value function.

they are complements (substitutes) in all two action decision problems given by Figure 5 with $x \in (0,1)$.

Proof. The main argument in the proof of Theorem 12.4.1. in Blackwell and Girshick (1954) demonstrates that in the two states case a signal $\tilde{s}$ Blackwell dominates another signal $\tilde{s}'$ if and only if $\tilde{s}$ is more valuable than $\tilde{s}'$ in all two action problems of the form shown in Figure 5.[10] We can then apply Proposition 1 to infer Lemma 1. □

4.1 Substitutes

We focus in this subsection on the case that each signal has only two realizations. The next section will offer a strong characterization of substitutes in the case of arbitrary finite state and signal spaces. We denote the realizations of signal $\tilde{s}_1$ by α and β, and those of signal $\tilde{s}_2$ by $\hat{\alpha}$ and $\hat{\beta}$. We shall focus on the case that each signal realization individually is informative, that is, leads to a posterior belief that is different from the prior belief. It is easy to see that otherwise there can't be substitutes. This will also be shown in general in Proposition 5 below. Without loss of generality we assume that observing α or $\hat{\alpha}$ (resp. β or $\hat{\beta}$) alone raises the decision maker's belief that the state is a (resp. b): $q_\alpha(a) > \pi(a)$ and $q_{\hat{\alpha}}(a) > \pi(a)$. We refer to the model with two states and two realizations per signal if it satisfies the assumptions introduced in this paragraph as the "binary-binary" model.

Proposition 2. *In the binary-binary model, signals are substitutes if and only if the joint realizations $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ each have strictly positive prior probability, and*

$$q_{\alpha,\hat{\alpha}}(a) = \max\{q_\alpha(a), q_{\hat{\alpha}}(a)\}, \text{ and} \tag{13}$$

$$q_{\beta,\hat{\beta}}(b) = \max\{q_\beta(b), q_{\hat{\beta}}(b)\}. \tag{14}$$

Call a realization of a single signal "extreme" if it provides the strongest evidence for state a, or state b, among all four individual signal realizations. The conditions in Proposition 2 say that conditional on an extreme realization of a signal the other signal is not informative. Thus, in the binary-binary model, substitutability amounts to a form of conditional uninformativeness of signals.

[10]Blackwell and Girshick's proof refers to a decision problem that is as in Figure 5 but with the first row of payoffs replaced by $(-(1-x), x)$, where $x \in (0,1)$, and the second row of payoffs replaced by $(0,0)$. The same argument that Blackwell and Girshick use can be used to demonstrate that a signal $\tilde{s}$ Blackwell dominates another signal $\tilde{s}'$ if and only if $\tilde{s}$ is more valuable than $\tilde{s}'$ in all two action problems of the form shown in Figure 5. We omit the details.

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ	φ
β	$\mu\varphi'$	$\mu\rho'$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	$\eta\rho$	$\eta\varphi$
β	φ'	ρ'

$\omega = b$

Example 5 (α and β are extreme signal realizations)

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ	0
β	φ	$1-\rho-\varphi$

$\omega = a$

	$\hat{\alpha}$	$\hat{\beta}$
α	ρ'	0
β	φ'	$1-\rho'-\varphi'$

$\omega = b$

Example 6 (α and $\hat{\beta}$ are extreme signal realizations)

Figure 6: Two different types of substitutes

Signal distributions that satisfy the conditions of Proposition 2 can be classified into two types. For signal distributions of the first type the two extreme realizations are different realizations of the same signal, whereas for signal distributions of the second type, the two extreme realizations are realizations of two different signals. We illustrate these two types in Figure 6.

Example 5 illustrates the first type. We show the case in which both extreme signal realizations come from signal $\tilde{s}_1$. It then has to be the case that, conditional on the realization of signal $\tilde{s}_1$, signal $\tilde{s}_2$ is always not informative. This happens if conditional on any realization of signal $\tilde{s}_1$, the likelihood ratios of joint signal realizations are the same for all realizations of signal $\tilde{s}_2$. The corresponding information structure is displayed in Example 5 where the likelihood ratios are denoted by η and μ which are both less than 1.[11] Note that the perfect correlation Example 3 is a special case of Example 5 where $\varphi = \varphi' = 0$, $\rho = \rho'$ and $\mu\rho' = \eta\rho = 1 - \rho$.

Example 6 illustrates the second type of signal distributions that make signals substitutes. In this type, the two extreme realizations come from different signals. We show the

[11]Of course, the entries in each table in Example 5 have to sum to one. Moreover, since $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ occur with positive probability, we have $\rho, \rho' > 0$ while $\varphi, \varphi' \geq 0$. Finally, to satisfy our assumption that $\hat{\alpha}$ indicates state a, we need that $\rho + \mu\varphi' \geq \eta\rho + \varphi'$.

case in which α and $\hat{\beta}$ are the extreme realizations. In this case, signals are substitutes if and only if signal $\tilde{s}_1$ is not informative conditional on $\hat{\beta}$, and signal $\tilde{s}_2$ is not informative conditional on α. It is not hard to see that this is equivalent to the realization $(\alpha, \hat{\beta})$ having zero probability in both states. Accordingly, the information structure is of the form shown in Example 6.[12] Note that our earlier Example 3 is a special case of Example 6 where $\varphi = \varphi' = 0$ and $\rho' = 1 - \rho$.

We prove the sufficiency of the conditions in Proposition 2 in the Appendix. The proof is by calculation, using the fact that according to Lemma 1 we can restrict attention to a one parameter class of decision problems with two actions only. We show the necessity of the conditions in Proposition 2 in the next section, where we shall derive the necessity from a more general result that is proved using a simple, geometric argument.

Remark 2. *Among all pairs of conditional joint distributions of signals $\tilde{s}_1$ and $\tilde{s}_2$ in the binary-binary model the ones shown in Figure 6 are rare. One way of saying this formally is to identify pairs of conditional joint distributions of the two signals with vectors in 8-dimensional Euclidean space, and to endow the set of all joint distributions with the relative Euclidean topology. The set of distributions that are* not *like the distributions in Figure 6 is then an open and dense subset of the set of all joint distributions, and is thus generic. This is mathematically obvious given Proposition 2. It may also appear to be intuitively plausible given how stringent the requirement that defines substitutes is, however, as we will point out below, in the same topological sense, complements, although their definition seems equally stringent, are* not *rare.*

4.2 Complements

We begin again by considering the binary-binary model introduced in the previous subsection. Note that our earlier Examples 1 and 2 would be special cases of the binary-binary model had we not ruled out in the previous subsection the case that at least one of the signals is not informative. Indeed, it is obvious that, whenever at least one signal by itself is not informative, signals are complements. We state this simple observation in the next section as Proposition 9. We now focus on the case that both signals are by themselves informative.

[12]In accordance with Proposition 2 we need $\rho, \rho' > 0$ and $\varphi, \varphi' \geq 0$. To satisfy our assumption that α and $\hat{\alpha}$ indicate state a, we need that $\rho \geq \rho'$ and $\rho + \varphi \geq \rho' + \varphi'$. To ensure that α is the strongest signal for state a we need: $\rho\varphi' \geq \rho'\varphi$, and finally, to ensure that $\hat{\beta}$ is the strongest signal for state b we need: $(1-\rho)\varphi' \leq (1-\rho')\varphi$.

Proposition 3. *In the binary-binary model, signals are complements if and only if the joint realizations* $(\alpha, \hat{\alpha})$ *and* $(\beta, \hat{\beta})$ *each have strictly positive probability in at least one state, and one of the following conditions holds:*[13]

$$q_{\alpha,\hat{\alpha}}(a) \leq \pi(a), \quad \textit{or} \tag{15}$$
$$q_{\beta,\hat{\beta}}(b) \leq \pi(b). \tag{16}$$

Inequality (15) says that if the decision maker receives signal $(\alpha, \hat{\alpha})$ the decision maker's posterior probability of state a is less than or equal to the prior $\pi(a)$, even though individually both α and $\hat{\alpha}$ move the decision maker's probability of state a above $\pi(a)$. Inequality (16) says that if the decision maker receives signal $(\beta, \hat{\beta})$ the decision maker's posterior probability of state b is not more than the prior $\pi(b)$, even though individually both β and $\hat{\beta}$ move the decision maker's probability of state b above $\pi(b)$. In both cases, two signals which by themselves move the decision maker's beliefs into one direction, if received together move the decision maker's beliefs into the opposite direction. The "meaning" of these signals is reversed if they are received together.

We prove the sufficiency of the conditions in Proposition 3 in the Appendix. We derive the necessity in the next section from a more general result. Example 7 shows a class of complements. If $\nu > \mu$, the signal realizations α and $\hat{\alpha}$ by themselves raise the decision maker's belief that the true state is a. If $\rho \leq \varphi$, then the joint signal realization $(\alpha, \hat{\alpha})$, by contrast, reduces the decision maker's probability that the true state is a or leaves it unchanged.[14]

Remark 3. *Whereas substitutes are rare, as we noted in Remark 2, complements are not. To express this formally, we again endow the set of all pairs of joint conditional probability distributions of the two signals with the relative Euclidean topology, and note that the set of distributions that correspond to complements has an open subset. For example, a small open ball around a pair of full support distributions that satisfy one of the*

[13] As we note in Remark 5 the two conditions are mutually exclusive. In any particular example, at most one of them can be true.

[14] Example 7 captures all conditional joint probability distributions of the two signals in the binary-binary model for which condition (15) holds, and for which in each state the probabilities of the two signal realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ are the same. (There are, of course, other conditional joint distributions of the two signals for which signals are complements.) All suitable values for the four parameters in Example 7 can be found by making choices allowed in the following procedure: First pick ν such that $0 < \nu < 1$. Then pick $\mu > 0$ such that $2\nu - 1 \leq \mu < \nu$. Then pick $\varphi \geq 0$ such that $2\nu - 1 \leq \varphi \leq \mu$. Finally, pick $\rho \geq 0$ such that $2\nu - 1 \leq \rho \leq \varphi$.

	$\widehat{\alpha}$	$\widehat{\beta}$
α	ρ	$\nu-\rho$
β	$\nu-\rho$	$1+\rho-2\nu$

$\omega = a$

	$\widehat{\alpha}$	$\widehat{\beta}$
α	φ	$\mu-\varphi$
β	$\mu-\varphi$	$1+\varphi-2\mu$

$\omega = b$

Figure 7: Example 7 (signals are complements)

conditions in Proposition 3 as a strict inequality[15] *is a subset of the set of all distributions that correspond to complements. This mathematically trivial fact is intuitively surprising given how stringent the requirement that defines complements is.*

Remark 4. *In the Introduction and in Section 2 we provided Examples 1 and 2 as examples of pairs of signals that are complements. These examples are not special cases of the binary-binary model because one signal, signal 2, is uninformative by itself. However, we can perturb the conditional signal distributions in these examples so that signal 2 is informative. Then we obtain special cases of the binary-binary model. One can verify that if we perturb Example 1 in this way we obtain a pair of signals that exhibits meaning reversal in the sense of Proposition 3. By contrast, Example 2 cannot be perturbed so that it satisfies the meaning reversal condition of Proposition 3. Example 2 is thus not robust. The intuitive reason for this can be seen when considering decision problems of the type shown in Figure 5 in the case that the decision maker, when holding the prior belief, is almost indifferent between the two actions. By choosing x close enough to 0.5 we can construct a decision problem in which signal 2, even if its informational content is very small, has positive value by itself. By contrast, signal 1 may be known to be so strong that no realization of signal 2 can change the decision maker's optimal choice, and therefore, if combined with signal 1, signal 2 has zero marginal value. Thus, in this decision problem, signals are not complements.*

Remark 5. *The two conditions in Proposition 3, inequalities (15) and (16), are mutually exclusive. To see this, suppose (15) were true: $q_{\alpha,\hat{\alpha}}(a) \leq \pi(a)$. Because, by assumption, $q_\alpha(a) > \pi(a)$, signal realization $(\alpha, \hat{\beta})$ then must have positive prior probability, and: $q_{\alpha,\hat{\beta}}(a) > \pi(a)$. Hence $q_{\alpha,\hat{\beta}}(b) < \pi(b)$. But because, also by assumption, $q_{\hat{\beta}}(b) > \pi(b)$, signal realization $(\beta, \hat{\beta})$ then must have positive prior probability and: $q_{\beta,\hat{\beta}}(b) > \pi(b)$, i.e. inequality (16) is false.*

[15] With a suitable choice of parameters in Example 7, condition (15) holds as a strict inequality.

We generalize the sufficiency part of Proposition 3 to obtain a sufficient condition for complementarity in the case when signals have arbitrarily many realizations. Let $\underline{s}_i$ (resp. $\bar{s}_i$) be the realization of signal $\tilde{s}_i$, which provides the weakest (resp. strongest) support for state a: $q_{\underline{s}_i}(a) = \min_{s_i} q_{s_i}(a)$ and $q_{\bar{s}_i}(a) = \max_{s_i} q_{s_i}(a)$. Let

$$x \in X \equiv \left(\max\{q_{\underline{s}_1}(a), q_{\underline{s}_2}(a)\}, \min\{q_{\bar{s}_1}(a), q_{\bar{s}_2}(a)\}\right), \tag{17}$$

that is, x is larger than the smallest posterior probability of a that is induced by any realization of a single signal, and smaller than the largest posterior probability of a induced by any realization of a single signal. We partition the set S_i of realizations of signal $\tilde{s}_i$ into two subsets, depending on whether they induce posterior beliefs $q_{s_i}(a)$ that are smaller or larger than x:

$$S_i^\beta(x) = \{s_i \in S_i \mid q_{s_i}(a) \leq x\}, \quad S_i^\alpha(x) = \{s_i \in S_i \mid q_{s_i}(a) > x\}. \tag{18}$$

Now imagine that, instead of observing each realization of signal $\tilde{s}_i$, the decision maker only observes whether a realization is in one of the two partitions. This amounts to observing a signal with two realizations. We call this binary signal $\tilde{t}_i(x)$ and denote the realization of $\tilde{t}_i(x)$ by $t_i^\beta(x)$ if $s_i \in S_i^\beta(x)$ and by $t_i^\alpha(x)$ if $s_i \in S_i^\alpha(x)$.

Proposition 4. *In the two state case, if for all $x \in X$ the signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements, then the signals $\tilde{s}_1$ and $\tilde{s}_2$ are complements.*

Proof. We denote the expected utility that the decision maker receives when maximizing expected utility in some arbitrary decision problem (A, u) after observing the realization of $\tilde{t}_i(x)$ by $V_{i,x}(A, u)$ and we denote the expected utility that the decision maker receives when maximizing expected utility in decision problem (A, u) after observing the joint realization $(\tilde{t}_1(x), \tilde{t}_2(x))$ by $V_{12,x}(A, u)$. Let the auxiliary signals $\tilde{t}_C(x)$ and $\tilde{t}_S(x)$ be defined analogously to $\tilde{s}_C$ and $\tilde{s}_S$, and denote the expected utility that the decision maker receives when maximizing expected utility in decision problem (A, u) after observing these signals by $V_{C,x}(A, u)$ and $V_{S,x}(A, u)$.

By Lemma 1 it is sufficient to verify complementarity for the two action problem of Figure 5 for all $x \in (0, 1)$. For $x \notin X$, there is at least one signal $\tilde{s}_i$ which is not informative. Hence, signals are obviously complements (see Proposition 9 below). Let $x \in X$, and let (A, u) for the purposes of this proof be the corresponding two action decision problem. By Proposition 1, it is sufficient to show that $V_C(A, u) \geq V_S(A, u)$.

To demonstrate this, we begin with two observations. The first observation is that $V_i(A,u) = V_{i,x}(A,u)$. This is so since in the two action problem at hand, all that matters for the decision maker's optimal action after observing realization s_i of signal $\tilde{s}_i$ is whether the posterior belief $q_{s_i}(a)$ is smaller or larger than x. But this is precisely the information provided by signal $\tilde{t}_i(x)$. We omit the formal proof. The second observation is that, evidently, the signal $(\tilde{s}_1, \tilde{s}_2)$ is (weakly) more informative than the signal $(\tilde{t}_1(x), \tilde{t}_2(x))$. Hence, $V_{12}(A,u) \geq V_{12,x}(A,u)$. Using these two observations, we can deduce:

$$\begin{aligned}
V_C(A,u) &= 0.5V_{12}(A,u) + 0.5V_{\emptyset}(A,u) \\
&\geq 0.5V_{12,x}(A,u) + 0.5V_{\emptyset}(A,u) \\
&= V_{C,x}(A,u) \\
&\geq V_{S,x}(A,u) \\
&= 0.5V_{1,x}(A,u) + 0.5V_{2,x}(A,u) \\
&= 0.5V_1(A,u) + 0.5V_2(A,u) \\
&= V_S(A,u),
\end{aligned} \tag{19}$$

where the inequality in the fourth line follows because by assumption $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements. This proves the claim. $\square$

To use Proposition 4 in practice one notices that the distribution of $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are determined by the underlying distributions of $\tilde{s}_1$ and $\tilde{s}_2$, and that Proposition 3 characterizes when $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements. The following example illustrates how Proposition 4 can be applied.

Example 8. *$\Omega = \{a, b\}$ and for $i = 1, 2$: $S_i = \{s_{i1}, s_{i2}, s_{i3}\}$. The distribution of signals conditional on the state is shown in Figure 8. Note that for $n < m$, the realization s_{in} provides stronger support for state a than the realization s_{im}. Note also that the two signals are symmetric. There are only two partitions into which realizations can be grouped: For $x \in (q_{s_{i3}}(a), q_{s_{i2}}(a))$, we have to consider the signals that arise from the partition $\{\{s_{i3}\}, \{s_{i2}, s_{i1}\}\}$. And for $x \in [q_{s_{i2}}(a), q_{s_{i1}}(a))$, we have to consider the signals that arise from the partition $\{\{s_{i3}, s_{i2}\}, \{s_{i1}\}\}$. The induced signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are described by the information structures in Figure 9. Observe that in both cases shown in Figure 9, the signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are complements since condition (15) from Proposition 3 is met.*

In Subsection 5.3 we give an example that illustrates that the condition in Proposition 4 is not necessary for complementarity.

	s_{21}	s_{22}	s_{23}
s_{11}	6/100	0	24/100
s_{12}	0	40/100	0
s_{13}	24/100	0	6/100

$\omega = a$

	s_{21}	s_{22}	s_{23}
s_{11}	10/100	0	10/100
s_{12}	0	40/100	0
s_{13}	10/100	0	30/100

$\omega = b$

Figure 8: Example 8 (signals are complements)

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	46/100	24/100
$t_1^\beta(x)$	24/100	6/100

$\omega = a$

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	50/100	10/100
$t_1^\beta(x)$	10/100	30/100

$\omega = b$

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	6/100	24/100
$t_1^\beta(x)$	24/100	46/100

$\omega = a$

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	10/100	10/100
$t_1^\beta(x)$	10/100	70/100

$\omega = b$

Figure 9: Signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ for Example 8 and partition $\{\{s_{i3}\}, \{s_{i2}, s_{i1}\}\}$ (top) and partition $\{\{s_{i3}, s_{i2}\}, \{s_{i1}\}\}$ (bottom)

5 The General Case

As we explained in the previous section, the case that there is an arbitrary finite number of states is harder to analyze than the case of two states. In our investigation below, the main results show that the conditions that are necessary and sufficient for substitutes or complements in the binary-binary model are necessary, but not sufficient, for substitutes or complements in the general model.

5.1 Substitutes

We begin by considering the case that at least one signal is not informative. The following result was already anticipated in the previous section.

Proposition 5. *If at least one signal is not informative, then signals are not substitutes.*

Proof. Without loss of generality assume that signal $\tilde{s}_1$ is not informative. Then $V_1(A,u) - V_\emptyset(A,u) = 0$ in all decision problems (A,u). For signals not to be substitutes, we therefore need: $V_{12}(A,u) - V_2(A,u) > 0$ for some decision problem (A,u). By Assumption 1 signal $\tilde{s}_1$ is informative conditional on some realization of signal $\tilde{s}_2$. Therefore there are $s_2 \in S_2$ and two $s_1, s_1' \in S_1$ such that $\bar{p}_{12}(s_1,s_2) > 0$, $\bar{p}_{12}(s_1',s_2) > 0$, and $q_{s_1,s_2} \neq q_{s_1',s_2}$. By the separating hyperplane theorem there are then a vector $r \in \mathbb{R}^{|\Omega|}$ and a number $e \in \mathbb{R}$ such that $q_{s_1,s_2} r > e$ and $q_{s_1',s_2} r < e$. Consider the decision problem (A,u) in which the decision maker has two actions: R and E, and in which the payoff to action R in state ω is given by the ω-th component of r, and the payoff to action E is equal to e in all states of the world. After observing at least one of the joint realizations (s_1,s_2) and (s_1',s_2) an action that was optimal under q_{s_2} will not be optimal under the posterior following observation of the joint realization. Therefore, $V_{12}(A,u) - V_2(A,u) > 0$, and signals are not substitutes. □

Assuming that both signals are informative, we showed in the previous section that in the binary-binary model a necessary and sufficient condition for substitutes is that a signal is not informative conditional on the other signal having a realization that induces extreme posteriors. We now show that a similar condition is in general necessary for substitutes. We begin with a useful auxiliary result. For any subset C of a finite-dimensional Euclidean space we denote by "*co* C" the convex hull of C.

Lemma 2. *If signals are substitutes, then for every* $(s_1,s_2) \in S_1 \times S_2$ *such that* $\bar{p}_{12}(s_1,s_2) > 0$*:*

$$q_{s_1,s_2} \in co\left\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\right\}. \tag{20}$$

Proof. By part (i) of Proposition 1, if signals are substitutes, $\tilde{s}_S$ Blackwell dominates $\tilde{s}_C$. By condition (5) of Theorem 12.2.2. in Blackwell and Girshick (1954) this means that the posteriors after observing $\tilde{s}_S$ are a mean-preserving spread of the posteriors after observing $\tilde{s}_C$. Therefore, the posteriors after observing $\tilde{s}_C$ are contained in the convex hull of the posteriors after observing $\tilde{s}_S$. This implies Lemma 2. □

We now state our main result on substitutes. Recall that an element of a convex set C is called an "extreme point" of C if it is not a convex combination of two different elements of C where each of these elements has strictly positive weight.

Proposition 6. *Suppose signals are substitutes. If for some* $i \in \{1,2\}$ *and some* $s_i^* \in S_i$ *the vector* $q_{s_i^*}$ *is an extreme point of* $co\left\{q_{s_k} | k \in \{1,2\}, s_k \in S_k\right\}$, *then signal* $\tilde{s}_j$ *(where* $j \neq i$*) is not informative conditional on signal realization* s_i^**.*

Observe that the condition in Proposition 6 is a generalization of the condition in Proposition 2. Proposition 2 is for the binary-binary model only, and it shows for that model that the condition is necessary and sufficient for substitutes. For the general case, by contrast, Proposition 6 only asserts the necessity of the condition. An example that we present in subsection 5.3 will show that the condition in Proposition 6 is in general not sufficient for substitutes.

Proof. Indirect. Suppose $q_{s_i^*,s_j} \neq q_{s_i^*}$ for some $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$. By standard properties of posteriors $q_{s_i^*}$ can be written as a convex combination of the vectors $q_{s_i^*,s_j}$ $(s_j \in S_j)$. We can infer that $q_{s_i^*,s_j} \neq q_{s_i^*}$ for at least two $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$, and that both of these vectors $q_{s_i^*,s_j}$ receive positive weight in the convex combination that makes up $q_{s_i^*}$. By Lemma 2 for every $s_j \in S_j$ with $\bar{p}_{12}(s_i^*, s_j) > 0$ the vector $q_{s_i^*,s_j}$ is an element of $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$. We have thus inferred that $q_{s_i^*}$ can be written as the convex combination of at least two different elements of $co\,\{q_{s_i} | i \in \{1,2\}\, s_i \in S_i\}$ where each element receives positive weight. Next, one can easily see that this implies that one can also express $q_{s_i^*}$ as the convex combination of *exactly two* different elements of $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$ where each element receives positive weight. This contradicts our assumption that $q_{s_i^*}$ is an extreme point of $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$. ☐

The following result adds to Proposition 6 the observation that if signals are substitutes there are at least two signal realizations to which the condition of Proposition 6 applies.

Corollary 1. *Suppose signals are substitutes. Then there are $i, j \in \{1,2\}$ and $s_i^* \in S_i$, $s_j^* \in S_j$ such that $q_{s_i^*} \neq q_{s_j^*}$, $k \neq i$ implies that signal $\tilde{s}_k$ is not informative conditional on s_i^*, and $k \neq j$ implies that signal $\tilde{s}_k$ is not informative conditional on s_j^*.*

Proof. By Proposition 5, if signals are substitutes, at least one signal is informative, and therefore the set $\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$ has at least two elements. Hence $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$ has at least two extreme points: by the Krein-Milman Theorem (Ok, 2007, p. 659) $co\,\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$ is the closed convex hull of its extreme points, and if it had only one extreme point, it would therefore have to have only one element. By Milman's Converse to the Krein-Milman Theorem (Ok, 2007, p. 660), all extreme points of this set are elements of $\{q_{s_i} | i \in \{1,2\}, s_i \in S_i\}$. The result follows from Proposition 6. ☐

We can use Proposition 6 to prove the necessity of the condition in Proposition 2.

Proof of the Necessity Part of Proposition 2. The necessity of the condition in Proposition 2 is an immediate consequence of Proposition 6 once we show that the signal realizations $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ have strictly positive probability in some state. We prove this indirectly. Suppose the realization $(\alpha, \hat{\alpha})$ had zero probability in both states. Then $p_{1,\omega}(\alpha) = p_{12,\omega}(\alpha, \hat{\beta})$ for both ω, and, because each realization of each signal occurs with strictly positive prior probability: $p_{12,\omega}(\alpha, \hat{\beta}) > 0$ for some ω. Hence, the posterior conditional on observing $(\alpha, \hat{\beta})$ is well-defined and we have $q_{\alpha,\hat{\beta}}(a) = q_\alpha(a)$. Now suppose that $\hat{\beta}$ provides the weakest individual evidence for state a, i.e. $q_{\hat{\beta}}(a)$ is an extreme point of the convex hull of $\{q_{s_i}(a) \mid i \in \{1,2\}, s_i \in S_i\}$. Then because $p_{12,\omega}(\alpha, \hat{\beta}) > 0$ for some ω, we can apply Proposition 6 and infer that $q_{\alpha,\hat{\beta}}(a) = q_{\hat{\beta}}(a)$. Hence, the two previous equalities yield $q_\alpha(a) = q_{\hat{\beta}}(a)$, a contradiction to our assumption that α indicates state a, and $\hat{\beta}$ indicates b. If, on the other hand, β provides the weakest individual evidence for state a, then an analogous argument yields the contradiction $q_{\hat{\alpha}}(a) = q_\beta(a)$. A similar argument shows that $(\beta, \hat{\beta})$ cannot have zero probability in both states. □

Clearly, if signals are perfectly correlated, then they are substitutes. In the remainder of this subsection we ask when the converse is true, i.e. when substitutes need to be perfectly correlated. We begin by defining perfect correlation formally.

Definition 6. *Signals $\tilde{s}_1$ and $\tilde{s}_2$ are* perfectly correlated *if there is a one-to-one function $f : S_1 \to S_2$ such that*

$$\bar{p}_{12}(s_1, s_2) > 0 \Leftrightarrow s_2 = f(s_1). \tag{21}$$

The following is obvious:

Proposition 7. *If signals are perfectly correlated, then they are substitutes.*

A converse to this proposition can be proved under additional assumptions.

Proposition 8. *Assume:*

(i) $q_{s_i} \neq q_{s'_i}$ *for all* $i \in \{1,2\}$ *and* $s_i, s'_i \in S_i$ *where* $s_i \neq s'_i$*;*

(ii) $q_{s_i} \notin co\{q_{s_j} | j \in \{1,2\}, s_j \in S_j, q_{s_j} \neq q_{s_i}\}$ *for all* $i \in \{1,2\}$ *and* $s_i \in S_i$*.*

If signals are substitutes, then they are perfectly correlated.

Assumption (i) in Proposition 8 is mild. It requires that no two different signal realizations give rise to the same posterior. Assumption (ii) is more restrictive. It says that the posterior q_{s_i} resulting from any signal realization s_i is not contained in the convex hull of the set of posteriors arising from all signal realizations if one removes from that set any posterior that is identical to q_{s_i}.

Proof. Condition (ii) in Proposition 8 implies that the set of extreme points of $co\{q_{s_i}| i \in \{1,2\}, s_i \in S_i\}$ includes $\{q_{s_i}|i \in \{1,2\}, s_i \in S_i\}$. On the other hand, by Milman's Converse to the Krein-Milman Theorem (Ok, 2007, p. 660), all extreme points of $co\{q_{s_i}| i \in \{1,2\}, s_i \in S_i\}$ are in $\{q_{s_i}|i \in \{1,2\}, s_i \in S_i\}$. Therefore, the set of extreme points of $co\{q_{s_i}|i \in \{1,2\}, s_i \in S_i\}$ equals $\{q_{s_i}|i \in \{1,2\}, s_i \in S_i\}$.

Consider any $s_1 \in S_1$, and suppose $s_2 \in S_2$ is such that $\bar{p}_{12}(s_1, s_2) > 0$. Then, by Proposition 6 we have $q_{s_1,s_2} = q_{s_1}$ and $q_{s_1,s_2} = q_{s_2}$, and therefore $q_{s_1} = q_{s_2}$. There must be at least one such $s_2 \in S_2$, and by assumption (i) in Proposition 8 there can be only one such $s_2 \in S_2$. We define: $f(s_1) \equiv s_2$. This can be done for every $s_1 \in S_1$. The function f satisfies, by construction, $\bar{p}_{12}(s_1, s_2) > 0 \Leftrightarrow s_2 = f(s_1)$. The function f is one-to-one because $f(s_1) = f(s_1')$ implies by construction $q_{s_1} = q_{s_1'}$ which, by condition (i) of Proposition 8 implies $s_1 = s_1'$. □

5.2 Complements

We begin again with the case that at least one signal is not informative. For this case, the following observation is immediate from the definition of complements.

Proposition 9. *If at least one signal is not informative, then signals are complements.*

In the previous section we showed that a form of "meaning reversal" was necessary and sufficient for signals to be complements in the binary-binary example. The next result shows that in general, with more than two states, under an additional assumption meaning reversal is necessary for complements. Note that, unlike in the case of the binary-binary model, the result does not assert that meaning reversal is sufficient for complements. Indeed, in the next subsection we shall show an example where the meaning reversal condition presented in this subsection is satisfied, and where signals are substitutes.

The following result looks formidable. We unpack the statement of the result for the reader in the text that follows the result.

Proposition 10. *Suppose signals are complements. Consider any $r \in \mathbb{R}^{|\Omega|}$. Define $e \equiv r\pi$. If for each $i \in \{1, 2\}$ there is a partition (S_i^E, S_i^R) of S_i such that the following three conditions are satisfied:*

(i) For each $i \in \{1,2\}$*:*

$$e \geq rq_{s_i} \textit{ for all } s_i \in S_i^E \textit{ and } e > rq_{s_i} \textit{ for at least one } s_i \in S_i^E, \tag{22}$$

and

$$rq_{s_i} \geq e \textit{ for all } s_i \in S_i^R \textit{ and } rq_{s_i} > e \textit{ for at least one } s_i \in S_i^R; \tag{23}$$

(ii) For each $k \in \{E, R\}$ *there is at least one* $(s_1, s_2) \in S_1^k \times S_2^k$ *such that*

$$\bar{p}_{12}(s_1, s_2) > 0; \tag{24}$$

(iii) For each $(k, \ell) \in \{(E, R), (R, E)\}$*:*

$$e \geq rq_{s_1,s_2} \textit{ for all } (s_1, s_2) \in S_1^k \times S_2^\ell \textit{ with } \bar{p}_{12}(s_1, s_2) > 0, \tag{25}$$

or

$$rq_{s_1,s_2} \geq e \textit{ for all } (s_1, s_2) \in S_1^k \times S_2^\ell \textit{ with } \bar{p}_{12}(s_1, s_2) > 0; \tag{26}$$

then

$$rq_{s_1,s_2} \geq e \textit{ for some } (s_1, s_2) \in S_1^E \times S_2^E \textit{ with } \bar{p}_{12}(s_1, s_2) > 0, \tag{27}$$

or

$$e \geq rq_{s_1,s_2} \textit{ for some } (s_1, s_2) \in S_1^R \times S_2^R \textit{ with } \bar{p}_{12}(s_1, s_2) > 0. \tag{28}$$

This result indicates in lines (27) and (28) that a form of meaning reversal is a necessary condition for complementarity. To interpret the result suppose the decision maker wants to learn from the signals whether the expected utility of a risky action R whose payoffs are given by the vector r is larger or smaller than the expected utility from a safe action E that yields payoff e in all states. Assume that r and e are such that with the prior belief π the decision maker is indifferent between the two actions. We denote the set of realizations of signal $\tilde{s}_i$ which imply a posterior belief for which action E has higher expected utility than action R by S_i^E, and we denote the set of realizations of signal $\tilde{s}_i$ which imply a posterior belief for which action R has higher expected utility than action E by S_i^R. Beliefs for which the decision maker is indifferent can be assigned arbitrarily to one of these two sets.

Signal realizations in S_i^E by themselves indicate that the expected value rq_{s_i} is not larger than e. But according to (27) for some joint realization where both realizations are in S_i^E we have (almost[16]) the reverse: $rq_{s_1,s_2} \geq e$. In the same way, (28) is a form of meaning reversal. At least one of these two meaning reversals must occur according to Proposition 10.

Note, however, that the meaning reversal is necessary only if conditions (24), (25) and (26) hold. Among these, (24) is a mild regularity condition. The remaining two conditions are more restrictive. They refer to the case that the decision maker receives "mixed messages" from the two signal. There are two possible types of mixed messages: the first type is when s_1 is in S_1^E but s_2 is in S_2^R; the second type is when s_1 is in S_1^R but s_2 is in S_2^E. The conditions require that for each of the two types of mixed signals one can say unambiguously which signal is "stronger," irrespective of the specific realization of the signals. Thus either for all mixed realizations of the first type the expected value of action E is at least as large as that of action R, and hence signal $\tilde{s}_1$ is stronger, or for all mixed realizations of the first type the expected value of action R is at least as large as that of action E, and hence signal $\tilde{s}_2$ is stronger. An analogous condition needs to hold for all mixed realizations of the second type, but it is not necessary that the same signal is stronger for mixed realizations of both types.

Proof. Indirect. Assume for some $r \in \mathbb{R}^{|\Omega|}$ and $e \in \mathbb{R}$ there were partitions (S_i^E, S_i^R) (for $i \in \{1, 2\}$) that satisfy the conditions (i)-(iii) of the Proposition, but neither (27) nor (28) were true. Consider the decision problem with two actions, R and E, where the payoff of action R in state ω is given by the ω-th component of r, and the payoff of action E is equal to e in all states of the world. For an arbitrary belief q the expected payoff of action R is rq, and the expected payoff of E is e. By assumption, the prior π is such that $r\pi = e$, that is, the agent is indifferent between the two actions based on the prior. We shall show that the signals are not complements in this decision problem. For the remainder of this proof, (A, u) will denote this particular decision problem.

Suppose for $(k, \ell) = (E, R)$ condition (25) were true, and for $(k, \ell) = (R, E)$ condition (26) were true. Together with the assumption that neither (27) nor (28) are true, we can deduce that, conditional on observing any joint signal realization (s_1, s_2), one optimal action for the decision maker is E whenever $s_1 \in S_1^E$, independent of the realization of signal $\tilde{s}_2$, and R whenever $s_1 \in S_1^R$, again independent of the realization of signal $\tilde{s}_2$. Therefore, $V_{12}(A, u) - V_1(A, u) = 0$. On the other hand, the strict inequalities in conditions (22) and (23), applied to $i = 2$, imply that $V_2(A, u) - V_\emptyset(A, u) > 0$. Thus,

[16] Ignoring the possibility of indifference.

signals are not complements. The case that for $(k,\ell)=(E,R)$ condition (26) is true, and for $(k,\ell)=(R,E)$ condition (25) is true, is analogous, with the roles of signals 1 and 2 swapped.

Now consider the case that for both admissible (k,ℓ) condition (25) holds. We shall calculate $V_1(A,u)-V_\emptyset(A,u)$ and $V_{12}(A,u)-V_2(A,u)$. To calculate these value differences we recall that a positive marginal value from a signal arises only when the signal changes the decision maker's optimal choice. As the prior makes the decision maker indifferent, we can pick the decision maker's choice when holding the prior as is convenient for our proof. We pick it to be R. Then we have:

$$\begin{aligned} V_1(A,u)-V_\emptyset(A,u) &= \sum_{s_1\in S_1^E} \bar{p}_1(s_1)(e-rq_{s_1}) && (29)\\ &= \sum_{s_1\in S_1^E} \sum_{\substack{s_2\in S_2:\\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e-rq_{s_1,s_2}), && (30)\end{aligned}$$

where the second line equals the first because the expected value of the posterior belief conditional on the realizations of both signals is the posterior belief conditional on the realization of signal $\tilde{s}_1$. Focusing again on signal realizations that change the set of optimal choices for the decision maker we also calculate:

$$V_{12}(A,u)-V_2(A,u) = \sum_{s_1\in S_1^E} \sum_{\substack{s_2\in S_2^R:\\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e-rq_{s_1,s_2}). \tag{31}$$

This equation follows from the assumption that (25) holds for both admissible (k,ℓ) and that neither (27) nor (28) are true. Subtracting (31) from (30), we find:

$$\begin{aligned} &V_1(A,u)-V_\emptyset(A,u)-(V_{12}(A,u)-V_2(A,u))\\ =& \sum_{s_1\in S_1^E} \sum_{\substack{s_2\in S_2^E:\\ \bar{p}_{12}(s_1,s_2)>0}} \bar{p}_{12}(s_1,s_2)(e-rq_{s_1,s_2}). \qquad (32)\end{aligned}$$

By condition (24), applied to $k=E$, in Proposition 10, the sum on the right hand side of the last equality is over at least one pair (s_1,s_2). Moreover, because (27) does not hold for any $(s_1,s_2)\in S_1^E\times S_2^E$, this sum is negative, and therefore signals are not complements. The remaining case, when for both admissible (k,ℓ) condition (26) holds, is analogous, with the optimal choice under the prior taken to be E. □

Remark 6. *For the case of two states, we have a sufficient condition for complements in Proposition 4, and a necessary condition for complements in Proposition 10. The example in Subsection 5.3 will show that neither of these results is a complete characterization, that is, the sufficient condition is not necessary, and the necessary condition is not sufficient. For the case of more than two states we only have the necessary condition for complements of Proposition 10, and we don't have a general sufficient condition. It is interesting to observe that the proof of Proposition 10 makes reference only to choice problems in which there are only two actions. The condition in Proposition 10 would therefore even be necessary if we required signals to satisfy the complementarity inequality only in all two action choice problems. One way in which one could try to find a more restrictive necessary condition would be to consider arguments involving decision problems with more than two actions.*

Proposition 10 has the following corollary that provides a necessary condition that is easier to check than the necessary condition in Proposition 10 because no reference is made to the vector r and the number e. Instead, a condition is provided under which a suitable vector r and a number e can be constructed. The proposition's necessary condition does not make the connection with meaning reversal obvious. This is why we have first stated Proposition 10.

Corollary 2. *If signals are complements, then for every signal realization (s_1', s_2') with $\bar{p}_{12}(s_1', s_2') > 0$ we have:*

$$\pi \in co\left\{q_{s_1,s_2} | (s_1, s_2) \in S_1 \times S_2 \setminus \{(s_1', s_2')\} \text{ and } \bar{p}_{12}(s_1, s_2) > 0\right\}. \tag{33}$$

Proof. The proof is indirect. Denote the convex hull to which the corollary refers by C and suppose $\pi \notin C$. Then there is a hyperplane through π that does not intersect with C. Let r be the orthogonal vector of this hyperplane, and define $e \equiv r\pi$. We can choose r such that $rq < e$ for all $q \in C$. We now show that with this choice of r and e the necessary condition of Proposition 10 is violated. For $i = 1, 2$ define $S_i^E \equiv S_i \setminus \{s_i'\}$ and $S_i^R \equiv \{s_i'\}$. We first verify conditions (22) and (23) of Proposition 10. Let $i \in \{1, 2\}$ and $j \neq i$. Because for every $s_i \in S_i^E$ and every $s_j \in S_j$ we have: $q_{s_i,s_j} \in C$, we can conclude: $rq_{s_i,s_j} < e$. Because q_{s_i} is a convex combination of q_{s_i,s_j} for $s_j \in S_j$, this implies: $rq_{s_i} < e$, and thus (22) holds. Now consider $q_{s_i'}$. If this belief satisfied: $rq_{s_i'} \leq e$, then we could infer $r\pi < e$, because π is a convex combination of q_{s_i} for $s_i \in S_i$, which contradicts $e = r\pi$. Therefore: $rq_{s'_i} > e$, which verifies (23). Next, we note that (24) holds by construction, and that also by construction (25) is true for both (k, ℓ). On the other hand, (27) and (28) are violated by construction. Thus, Proposition 10 implies that signals are not complements. $\square$

We now use Corollary 2 to derive the necessity part of Proposition 3.

Proof of the Necessity Part of Proposition 3. We begin by proving that $\bar{p}_{12}(\alpha, \hat{\alpha}) > 0$ and $\bar{p}_{12}(\beta, \hat{\beta}) > 0$. The proof is indirect. Suppose first that both probabilities were zero. Then the signals would be perfectly correlated, and therefore not be complements. Next suppose $\bar{p}_{12}(\alpha, \hat{\alpha}) = 0$ but $\bar{p}_{12}(\beta, \hat{\beta}) > 0$. Because α and $\hat{\alpha}$ occur with strictly positive prior probability probability, we have to have: $\bar{p}_{12}(\alpha, \hat{\beta}) > 0$ and $\bar{p}_{12}(\beta, \hat{\alpha}) > 0$. Because α and $\hat{\alpha}$ indicate that the state is more likely to be a, it must be that $q_{\alpha,\hat{\beta}}(a) > \pi(a)$ and $q_{\beta,\hat{\alpha}}(a) > \pi(a)$. But then the condition of Corollary 2 is violated if we take (s'_1, s'_2) to be $(\beta, \hat{\beta})$. A symmetric argument applies if $\bar{p}_{12}(\alpha, \hat{\alpha}) > 0$ and $\bar{p}_{12}(\beta, \hat{\beta}) = 0$. We conclude that $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ have strictly positive prior probability.

We now prove that $q_{\alpha,\hat{\alpha}}(a) \leq \pi(a)$ or $q_{\beta,\hat{\beta}}(b) \leq \pi(b)$. The proof is indirect. Suppose

$$q_{\alpha,\hat{\alpha}}(a) > \pi(a) \quad \text{and} \quad q_{\beta,\hat{\beta}}(b) > \pi(b). \tag{34}$$

We begin with the case that the two mixed realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ both have strictly positive prior probability so that posteriors conditioning on these signal realizations are well-defined. We go through different possible orderings of the posterior beliefs, and show that none of them is compatible with signals being complements. Consider first the following two cases:

$$q_{\alpha,\hat{\beta}}(a) \geq \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) \leq \pi(a), \tag{35}$$

$$q_{\alpha,\hat{\beta}}(a) \leq \pi(a) \quad \text{and} \quad q_{\beta,\hat{\alpha}}(a) \geq \pi(a). \tag{36}$$

Condition (35) together with (34) implies that in the decision problem of Figure 5 with $x = 0.5 = \pi(a)$, which we shall denote by (A, u) in this proof, the marginal value of signal $\tilde{s}_2$ conditional on signal $\tilde{s}_1$ is zero for both signal realizations of signal $\tilde{s}_1$. Thus, $V_{12}(A, u) - V_1(A, u) = 0$, and signals are not complements (note that $V_2(A, u) - V_\emptyset(A, u) > 0$ by the assumption that signal $\tilde{s}_2$ is informative and $x = 0.5$.) For ordering (36) the argument is the same with the roles of signals 1 and 2 swapped.

We are left with the orderings

$$q_{\alpha,\hat{\beta}}(a) > \pi(a) \qquad \text{and} \qquad q_{\beta,\hat{\alpha}}(a) > \pi(a), \tag{37}$$

$$q_{\alpha,\hat{\beta}}(a) < \pi(a) \qquad \text{and} \qquad q_{\beta,\hat{\alpha}}(a) < \pi(a). \tag{38}$$

If (37) holds in combination with (34), the necessary condition in Corollary 2 is violated if we choose $(s'_1, s'_2) = (\beta, \hat{\beta})$, and if (38) holds in combination with (34), the necessary condition in Corollary 2 is violated if we choose $(s'_1, s'_2) = (\alpha, \hat{\alpha})$.

It remains to discuss the cases in which at least one of $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ does not have strictly positive prior probability. Suppose first that both realizations $(\alpha, \hat{\beta})$ and $(\beta, \hat{\alpha})$ have zero prior probability. This means that signals are perfectly correlated and therefore the marginal value of a signal when the other signal is available is zero. Hence, signals are not complements. Suppose next that $(\alpha, \hat{\beta})$, but not $(\beta, \hat{\alpha})$ has zero probability. If $q_{\beta,\hat{\alpha}}(a) \leq \pi(a)$, then the same argument as for ordering (35) can be used, and if $q_{\beta,\hat{\alpha}}(a) \geq \pi(a)$, the same argument as for ordering (36) can be used. For the remaining case that $(\beta, \hat{\alpha})$, but not $(\alpha, \hat{\beta})$ has zero probability, the argument is analogous. □

5.3 A Counterexample

In this subsection we present an example that shows that the conditions in Proposition 6 for substitutes and Proposition 10 for complements are only necessary, but not sufficient. The example also shows that the sufficient conditions for complements in Proposition 4 are not necessary for complements.

Example 9 is shown in Figure 10.[17] The example is a two state example: $\Omega = \{a, b\}$. Each individual signal $\tilde{s}_i$ has two informative realizations: α_i, β_i, and two not informative realizations: σ_i, σ'_i. Among all individual and joint signal realizations, the posterior belief that the state is a can take on only three values: it equals $1/(1+\lambda) > 1/2$ for the realizations $\alpha_i, (\sigma_1, \sigma_2), (\sigma'_1, \sigma'_2)$; it equals $1/2$ for the realizations σ_i, σ'_i; and it equals $\lambda/(1+\lambda) < 1/2$ for the realizations $\beta_i, (\sigma_1, \sigma'_2), (\sigma'_1, \sigma_2)$.

Lemma 3. *In Example 9 signals are substitutes if $2\varphi \leq \rho$ and complements if $2\varphi \geq \rho$.*

Proof. Individually, a signal is informative with probability $(1+\lambda)\rho$. If it is informative, it induces the same posteriors as a signal with likelihood ratios $1/\lambda$ and λ. Therefore, the

[17]To ensure that all probabilities are non-negative and sum to one, we have to choose the parameters $\rho, \varphi, \lambda \in (0, 1)$ such that $(1+\lambda)(\rho + 2\varphi) = 1$.

	α_2	σ_2	σ_2'	β_2
α_1	ρ	0	0	0
σ_1	0	φ	$\lambda\varphi$	0
σ_1'	0	$\lambda\varphi$	φ	0
β_1	0	0	0	$\lambda\rho$

$\omega = a$

	α_2	σ_2	σ_2'	β_2
α_1	$\lambda\rho$	0	0	0
σ_1	0	$\lambda\varphi$	φ	0
σ_1'	0	φ	$\lambda\varphi$	0
β_1	0	0	0	ρ

$\omega = b$

Figure 10: Example 9 (signals are substitutes if $2\varphi \leq \rho$ and complements if $2\varphi \geq \rho$)

marginal value of an individual signal is the same as the marginal value of a signal with likelihood ratios $1/\lambda$ and λ multiplied by the probability $(1+\lambda)\rho$.

Conditional on being informative, signals are perfectly correlated. Therefore, if one signal is available and is informative, then the other signal's marginal value is zero. On the other hand, if one signal is available and not informative, the other signal induces the same posteriors as a signal with likelihood ratios $1/\lambda$ and λ. Therefore, the marginal value of a signal given the other signal is already available is the same as the marginal value of a signal with likelihood ratios $1/\lambda$ and λ multiplied by the probability that the other signal is not informative, which is $1-(1+\lambda)\rho$.

It follows that signals are substitutes if and only if $1-(1+\lambda)\rho \leq (1+\lambda)\rho$, and signals are complements if and only if $1-(1+\lambda)\rho \geq (1+\lambda)\rho$. With $(1+\lambda)(\rho+2\varphi) = 1$, these conditions are equivalent to $2\varphi \leq \rho$ resp. $2\varphi \geq \rho$. □

We shall now show that the example satisfies, for *all* parameter combinations, the necessary conditions in Proposition 6 for substitutes and Proposition 10 for complements. We shall thus show that neither set of conditions is sufficient. Consider first the conditions in Proposition 6. The realizations of signal $\tilde{s}_i$ which individually induce the most extreme posteriors are α_i and β_i. Conditional on such an extreme realization, signals are perfectly correlated. In particular, once an extreme realization is observed, no realization of the other signal changes the decision maker's belief. This means that the necessary condition for substitutes in Proposition 6 is met for both signals $\tilde{s}_i$. However, for $2\varphi > \rho$, signals are not substitutes.

Next, we show that the example satisfies all conditions of Proposition 10. It is easy to see that for any r and e for which some partition of S_1 and S_2 satisfies condition (i) of Proposition 10, the equation $rq \geq e$ is equivalent to $q(a) \geq 0.5$ or $q(a) \leq 0.5$. Without loss of generality we assume it is equivalent to $q(a) \geq 0.5$. For each of the two sets S_i there are four partitions that satisfy condition (i) of Proposition 10. We must have $\alpha_i \in S_i^R$ and $\beta_i \in S_i^E$, but σ_i and σ_i' can each be allocated to either of the two sets. This yields 16 pairs of partitions, all of which satisfy condition (ii) of Proposition 10. One can check that condition (iii) is violated by the two pairs of partitions for which σ_i and σ_i' are both in S_i^E for some $i \in \{1,2\}$ and σ_j and σ_j' are both in S_j^R for $j \neq i$. Ignoring these two cases, one can check that in all other cases there is some meaning reversal. For example, if $S_1^R = \{\alpha_1, \sigma_1\}, S_1^E = \{\sigma_1', \beta_1\}$, $S_2^R = \{\alpha_2, \sigma_2'\}$, and $S_2^E = \{\sigma_2, \beta_2\}$, then meaning reversal occurs for the signal realizations (σ_1, σ_2'). This shows that the example satisfies the necessary condition for complementarity in Proposition 10. However, for $2\varphi < \rho$, signals are not complements.

The example also demonstrates that the sufficient condition in Proposition 4 for complementarity is not necessary. To see this, pick some x such that $\lambda/(1+\lambda) < x < 0.5$, and note that $S_i^\alpha(x) = \{\alpha_i, \sigma_i, \sigma_i'\}$ and $S_i^\beta(x) = \{\beta_i\}$ for $i = 1, 2$. The information structure for the derived signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ is shown in Figure 11. Observe that $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ are perfectly correlated and therefore are not complements.

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	$1-\lambda\rho$	0
$t_1^\beta(x)$	0	$\lambda\rho$

$\omega = a$

	$t_2^\alpha(x)$	$t_2^\beta(x)$
$t_1^\alpha(x)$	$1-\rho$	0
$t_1^\beta(x)$	0	ρ

$\omega = b$

Figure 11: Signals $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$ for Example 9 and partition $\{\{\sigma_i', \sigma_i, \alpha_i\}, \{\beta_i\}\}$

6 Linear Decision Problems

Many pairs of signals are neither complements nor substitutes. This is because the requirements that define complementarity and substitutability are stringent in that they require the complementarity or substitutability inequalities to hold in *all* decision problems. In applications, only subclasses of decision problems might be of interest. Restricting attention to subclasses of decision problems gives rise to notions of complementarity or substitutability that apply to more pairs of signals. In the literature on the Blackwell comparison of the informativeness of signals, the parallel line of research has been pursued by Lehmann (1988), Persico (2000), Athey and Levin (2001), and Jewitt (2007).

These authors restrict attention to decision problems where states and actions are real numbers. Only decision problems for which the utility function u satisfies some form of monotonicity, for example a single crossing condition, are considered. These authors then provide informativeness comparisons for real valued signals that are assumed to satisfy the monotone likelihood ratio condition. In particular, Jewitt (2007, Propositions 1, 3, and 4) shows that informativeness comparisons in this restricted set-up are equivalent to Blackwell comparisons that are carried out for each pair of possible states, pretending in each case that these two states were the only possible states of the world. Our insights into complementarity and substitutability in the case of only two states can therefore be extended to settings with more than two states if one restricts attention to the same class of decision problems as the authors quoted above do, and if one makes use of the close relation between complementarity, substitutability, and Blackwell comparisons. However, to proceed along these lines one needs to impose conditions on the joint conditional distribution of signals that ensure that both auxiliary signals that were constructed in Section 3 satisfy the monotone likelihood ratio condition. The investigation of such conditions goes beyond the scope of this paper.

We focus instead on a different subclass of decision problems, linear decision problems, for which we can extend our results without strong conditions for the signal distributions. We assume that states, though not necessarily signals, are real numbers, and we only consider decision problems where the utility function is linear in the state. This is satisfied, for example, if the state corresponds to the value of some object, actions correspond to bids or purchase decisions, and utility is additive in the value of the object and money. Our results in this section will show that complementarity or substitutability in the linear model is equivalent to complementarity or substitutability in an auxiliary model in which there are only two states, and in which utility functions are unrestricted. Therefore, our earlier analysis of the two state case can be extended in a straightforward way to obtain an analysis of substitutability and complementarity in linear decision problems.

An important conceptual feature of the analysis that follows is that it depends on the prior distribution π over the state space Ω. In the linear model, signals that are complements or substitutes for one prior need not be complements or substitutes for some other prior. This is different from the case in which we allow all possible utility functions, as in the previous sections. In that case the prior can be fixed, but which prior is chosen does not affect the analysis. These considerations motivate the reference to the prior distribution in the following definition.[18]

Definition 7. *Suppose Ω is a finite subset of $\mathbb{R}$, and suppose $\underline{\omega} \equiv \min \Omega = 0$ and $\bar{\omega} \equiv \max \Omega = 1$. Let π be a prior distribution over Ω. Signals $\tilde{s}_1$ and $\tilde{s}_2$ are* linear substitutes given π *if for all decision problems (A, u) such that for any given $a \in A$ the utility function $u(a, \omega)$ is a linear function of ω we have:*

$$V_1(A, u) - V_\emptyset(A, u) \geq V_{12}(A, u) - V_2(A, u). \tag{39}$$

Signals $\tilde{s}_1$ and $\tilde{s}_2$ are linear complements given π *if for all decision problems (A, u) such that for any given $a \in A$ the utility function $u(a, \omega)$ is a linear function of ω we have:*

$$V_{12}(A, u) - V_2(A, u) \geq V_1(A, u) - V_\emptyset(A, u). \tag{40}$$

Now consider some arbitrary signal $\tilde{s}$ with realizations in the finite set S. We shall associate with $\tilde{s}$ another, auxiliary signal $\hat{s}$ that is defined in an auxiliary model with only two states, $\underline{\omega} = 0$ and $\bar{\omega} = 1$, and which has realizations in the same finite set S in which also the realizations of $\tilde{s}$ are contained. To specify this auxiliary model we thus have to specify the prior probabilities $\hat{\pi}(\underline{\omega})$ and $\hat{\pi}(\bar{\omega})$ of the two states, and the conditional probabilities $\hat{p}_{\underline{\omega}}(s)$ and $\hat{p}_{\bar{\omega}}(s)$ of all signal realizations $s \in S$. We shall specify these below. We shall then apply this construction of an auxiliary signal to the particular case that the signal $\tilde{s}$ is $(\tilde{s}_1, \tilde{s}_2)$ to obtain an auxiliary signal $(\hat{s}_1, \hat{s}_2)$. Our main result will be that $\tilde{s}_1$ and $\tilde{s}_2$ are linear substitutes given a prior π if and only if $\hat{s}_1$ and $\hat{s}_2$ are substitutes, and that an analogous result holds for complements.

We now construct the auxiliary signal $\hat{s}$. We denote by $E[\tilde{\omega}]$ the expected value of $\tilde{\omega}$, and we denote by $E[\tilde{\omega}|s]$ the expected value of $\tilde{\omega}$ conditional on some signal realization s. We set $\hat{\Omega} = \{\underline{\omega}, \bar{\omega}\}$, we specify the prior probabilities $\hat{\pi}$ as follows:

$$\hat{\pi}(\bar{\omega}) = E[\tilde{\omega}] \quad \text{and} \quad \hat{\pi}(\underline{\omega}) = 1 - E[\tilde{\omega}], \tag{41}$$

and we define the conditional signal probabilities by setting for any $s \in S$:

$$\hat{p}_{\bar{\omega}}(s) = \frac{\sum\limits_{\omega \in \Omega} [\pi(\omega) p_\omega(s) \omega]}{E[\tilde{\omega}]} \quad \text{and} \quad \hat{p}_{\underline{\omega}}(s) = \frac{\sum\limits_{\omega \in \Omega} [\pi(\omega) p_\omega(s)(1 - \omega)]}{1 - E[\tilde{\omega}]}. \tag{42}$$

[18]The assumption in this definition that $\min \Omega = 0$ and $\max \Omega = 1$ is a normalizing assumption that is without loss of generality.

It is easy to verify that these are non-negative numbers whose sum is 1.

By construction, the prior expected value of the state in the auxiliary model is the same as in the original model. The next lemma shows that the auxiliary signal that we have constructed also induces the same distribution of conditional expected values of $\tilde{\omega}$ as the original signal.

Lemma 4. *For every signal realization $s \in S$: (i) the prior probability of observing signal realization s is in the auxiliary model the same as it was in the original model; and (ii) the conditional expected value of the state, conditional on observing signal realization s, is in the auxiliary model the same as it was in the original model.*

Proof. To prove (i), note that in the auxiliary model the prior probability of observing s is:

$$\begin{aligned}
& \hat{\pi}(\bar{\omega})\hat{p}_{\bar{\omega}}(s) + \hat{\pi}(\underline{\omega})\hat{p}_{\underline{\omega}}(s) \\
= \; & E[\tilde{\omega}]\frac{\sum\limits_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)\omega]}{E[\tilde{\omega}]} + (1-E[\tilde{\omega}])\frac{\sum\limits_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)(1-\omega)]}{1-E[\tilde{\omega}]} \\
= \; & \sum_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)], \qquad (43)
\end{aligned}$$

which is equal to the prior probability of observing s in the original model.

To prove (ii), note that in the auxiliary model the conditional expected value of the state is:

$$\frac{\hat{\pi}(\bar{\omega})\hat{p}_{\bar{\omega}}(s)}{\sum\limits_{\omega\in\hat{\Omega}}\hat{\pi}(\omega)\hat{p}_{\omega}(s)} = \frac{E[\tilde{\omega}]\frac{\sum_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)\omega]}{E[\tilde{\omega}]}}{\sum\limits_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)]} = \frac{\sum\limits_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)\omega]}{\sum\limits_{\omega\in\Omega}[\pi(\omega)p_{\omega}(s)]} = E[\tilde{\omega}|s], \qquad (44)$$

where we use the assumption that $\underline{\omega} = 0$ and $\bar{\omega} = 1$. $\square$

In the particular case in which signal $\tilde{s}$ is equal to $(\tilde{s}_1, \tilde{s}_2)$, the above construction yields an auxiliary signal $(\hat{s}_1, \hat{s}_2)$ with realizations in $S_1 \times S_2$. We denote the signals that result if the decision maker observes only the first, or only the second, of the two components of the auxiliary signal by $\hat{s}_1$ and $\hat{s}_2$. Since equation (42) is additive in the conditional probabilities $p_{\omega}(s)$, $\hat{s}_i$ and the auxiliary signal associated with $\tilde{s}_i$ have the same distribution conditional on each state. Consequently, $(\hat{s}_1, \hat{s}_2)$, $\hat{s}_1$ and $\hat{s}_2$ induce the same distribution of conditional expected values of the state as $(\tilde{s}_1, \tilde{s}_2)$, $\tilde{s}_1$ and $\tilde{s}_2$. In addition, when utility is linear, the decision maker's expected utility of any action only depends on the expected value of the state ω and not on any other feature of the distribution of ω. This explains the main result of this section.

Proposition 11. *Signals $\tilde{s}_1$ and $\tilde{s}_2$ are linear substitutes given π if and only if the auxiliary signals $\hat{s}_1$ and $\hat{s}_2$ are substitutes. Signals $\tilde{s}_1$ and $\tilde{s}_2$ are linear complements given π if and only if the auxiliary signals $\hat{s}_1$ and $\hat{s}_2$ are complements.*

Remark 7. *Proposition 11 refers to signals $\tilde{s}_1$ and $\tilde{s}_2$ as linear complements or substitutes* given the prior π, *but refers to signals $\hat{s}_1$ and $\hat{s}_2$ as complements and substitutes* without reference to a prior. *This is because in the auxiliary two state model all utility functions are linear with respect to the state, and therefore there is no difference between linear complementarity or substitutability, and complementarity or substitutability in the sense of the previous sections. Moreover, as was emphasized before, the complementarity and substitutability notions of the previous section are independent of the prior. Note that the prior π does, of course, enter into the definition of signals $\hat{s}_1$ and $\hat{s}_2$*

Proof. Consider any decision problem (A, u) where u is linear in ω. There is a related decision problem $(\hat{A}, \hat{u})$ in the auxiliary model where the action set is $\hat{A} = A$, that is, the same as in the original decision problem, and where the utility function $\hat{u}$ is obtained from the utility function u in the original model by setting $\hat{u}(a, \overline{\omega}) = u(a, \overline{\omega})$ and $\hat{u}(a, \underline{\omega}) = u(a, \underline{\omega})$ for any $a \in \hat{A}$. Thus, $\hat{u}$ is the restriction of u to $A \times \{\underline{\omega}, \bar{\omega}\}$. For every decision problem (A, u) there is thus a corresponding decision problem $(\hat{A}, \hat{u})$, but observe also that, vice versa, for every given decision problem $(\hat{A}, \hat{u})$, there is a unique corresponding decision problem (A, u) such that u is linear in ω.

Denote by $\hat{V}_\emptyset(\hat{A}, \hat{u})$ the decision maker's expected utility when choosing optimally in the auxiliary model with no information, for $i = 1, 2$, denote by $\hat{V}_i(\hat{A}, \hat{u})$ the decision maker's expected utility when choosing optimally after observing the realization of $\hat{s}_i$ in the auxiliary model, and denote by $\hat{V}_{12}(\hat{A}, \hat{u})$ the decision maker's expected utility when choosing optimally after observing the realization of $(\hat{s}_1, \hat{s}_2)$ in the auxiliary model. Our proof strategy is to show:

$$\hat{V}_\emptyset(\hat{A}, \hat{u}) = V_\emptyset(A, u),\ \hat{V}_i(\hat{A}, \hat{u}) = V_i(A, u) \text{ (for } i = 1, 2), \hat{V}_{12}(\hat{A}, \hat{u}) = V_{12}(A, u). \qquad (45)$$

This equation immediately implies that the same inequalities that determine whether $\tilde{s}_1$ and $\tilde{s}_2$ are complements or substitutes also determine whether $\hat{s}_1$ and $\hat{s}_2$ are complements or substitutes, and thus proves the result.

To prove our claims we observe first that, when utility is linear, the decision maker's expected utility only depends on the expected value of the state $\tilde{\omega}$, not on the distribution of $\tilde{\omega}$. Therefore, Lemma 4 (ii) implies that, when maximizing expected utility in decision problem (A, u) conditional on some signal realization s, the decision maker obtains the same maximal expected utility in the auxiliary model using the auxiliary signal as he obtained in the original model using the original signal. Second, Lemma 4 (i) says that all signal realizations s have the same prior probability in the auxiliary model as in the original model. Because the ex ante expected utilities to which we refer above are calculated as the sum over all signal realizations of the probability of that signal realization times the maximal expected utility obtainable after observing that signal realization, our assertion follows. □

As a consequence of this proposition, we can check whether any two signals are linear complements or linear substitutes by checking complementarity and substitutability, respectively, in an associated two state model. For this latter purpose we can use the results of Section 4 for two state models. Consider, for instance, a linear model in which individual signals each only have two possible realizations: $S_1 = \{\alpha, \beta\}$ and $S_2 = (\hat{\alpha}, \hat{\beta}\}$, and assume that $E[\tilde{\omega}|\alpha] > E[\tilde{\omega}|\beta]$ and $E[\tilde{\omega}|\hat{\alpha}] > E[\tilde{\omega}|\hat{\beta}]$. Then we can infer from Proposition 2 that signals are substitutes if and only if $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ each have strictly positive probability and $E[\tilde{\omega}|(\alpha, \hat{\alpha})] = \max\{E[\tilde{\omega}|\alpha], E[\tilde{\omega}|\hat{\alpha}]\}$ and $E[\tilde{\omega}|(\beta, \hat{\beta})] = \min\{E[\tilde{\omega}|\beta], E[\tilde{\omega}|\hat{\beta}]\}$. Similarly, we can infer from Proposition 3 that signals are complements if and only if $(\alpha, \hat{\alpha})$ and $(\beta, \hat{\beta})$ each have strictly positive prior probability and $E[\tilde{\omega}|(\alpha, \hat{\alpha})] \leq E[\tilde{\omega}]$ or $E[\tilde{\omega}|(\beta, \hat{\beta})] \geq E[\tilde{\omega}]$. Intuitively, one obtains characterizations in terms of conditional expected values of the state because in the auxiliary model, where the state is either 0 or 1, the conditional probabilities equal conditional expected values, and because conditional expected values in the auxiliary model equal conditional expected values in the original model by part (ii) of Lemma 4.

7 Conclusion

This paper has provided some insights into the nature of substitutability and complementarity relations among signals. Our most general conditions for substitutability and complementarity in the case that there are more than two states are only necessary, not sufficient, and therefore give us only a partial description of signals that are substitutes or complements. As the necessary condition for substitutes is obviously very restrictive, whereas the necessary condition for complements is not obviously as restrictive, perhaps the most intriguing open question is how large the class of complements is if there are more than two states. As explained in Section 6, a further study of complementarity and substitutability in the case that attention is restricted to decision problems in which the decision maker's utility function satisfies some monotonicity condition seems also feasible.

A further line of work is to pursue the implications of complementarity and substitutability in economic settings. In this context it is interesting that complementarity and substitutability of signals may not only matter in single person decision problems, but also in games when agents hold private signals, and each agent's preferences depend on all signal realizations, that is, in contexts with interdependent preferences. Such contexts arise naturally in auctions or in voting games. It seems worthwhile to explore the implications of complementarity and substitutability in those contexts. Finally, complementarity of signals may also matter when agents acquire signals sequentially. In this case, the second signal may be acquired when the agent already knows the realization of the first signal. By contrast, in our setting, each signal is acquired without knowing the realization of the other signal. Extending our results to a setting where agents evaluate signals knowing the realization of other signals is another project for future work.

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A Proof of the Sufficiency Part of Proposition 2

We only consider the case in which the realization α provides the strongest individual evidence for state a: $q_\alpha(a) \geq q_{\hat{\alpha}}(a)$. The other case can be dealt with analogously. There are two further cases: we first consider the case in which β provides the strongest individual evidence for b: $q_\beta(b) \geq q_{\hat{\beta}}(b)$. In this case, conditions (13) and (14) become:

$$q_{\alpha,\hat{\alpha}}(a) = q_\alpha(a), \quad q_{\beta,\hat{\beta}}(a) = q_\beta(a). \tag{46}$$

We now argue that signal $\tilde{s}_2$ does not affect the decision maker's belief if he has observed signal $\tilde{s}_1$. Indeed, if the realization $(\alpha, \hat{\beta})$ has strictly positive probability in some state, then since $q_\alpha(a)$ is a convex combination of $q_{\alpha,s_2}(a)$, $s_2 \in \{\hat{\alpha}, \hat{\beta}\}$, the left equality above implies that $q_{\alpha,\hat{\beta}}(b) = q_\alpha(b)$. Moreover, if $(\alpha, \hat{\beta})$ has zero probability in all states, then clearly the decision maker maintains his belief after having observed the realization α with probability 1. In sum, we have shown that the probability that a realization of signal $\tilde{s}_2$ changes the decision maker's belief if realization α of signal $\tilde{s}_1$ has been observed is zero. Symmetrically, the probability that a realization of signal $\tilde{s}_2$ changes the decision maker's belief if realization β of signal $\tilde{s}_1$ has been observed is zero. But this means that the marginal value of signal $\tilde{s}_2$, if signal $\tilde{s}_1$ is available, is zero in all decision problems. Hence, signals are substitutes.

We next consider the case $q_{\hat{\beta}}(b) \geq q_\beta(b)$. In this case, conditions (13) and (14) become:

$$q_{\alpha,\hat{\alpha}}(a) = q_\alpha(a), \quad q_{\beta,\hat{\beta}}(b) = q_{\hat{\beta}}(b). \tag{47}$$

We first argue that this implies

$$p_{12,a}(\alpha, \hat{\beta}) = p_{12,b}(\alpha, \hat{\beta}) = 0. \tag{48}$$

Indeed, suppose the contrary were true. Then because for i, j, $q_{s_i}(a)$ is a convex combination of $q_{s_i,s_j}(a)$, $s_j \in S_j$, (47) would imply that $q_{\alpha,\hat{\beta}}(a) = q_\alpha(a)$, and $q_{\alpha,\hat{\beta}}(a) = q_{\hat{\beta}}(a)$, a contradiction to our assumption that realization α indicates state a and realization $\hat{\beta}$ indicates state b.

We now demonstrate that signals are substitutes. Suppose first that the realization $(\beta, \hat{\alpha})$ has zero probability in all states. Then (48) implies that signals are perfectly correlated. Therefore, the probability that a realization of one signal changes the decision maker's belief if the other signal is available is zero. Hence, signals are substitutes.

Suppose next that $(\beta, \hat{\alpha})$ has strictly positive probability in some state. (47) together with the fact that for i, j, $q_{s_i}(a)$ is a convex combination of $q_{s_i,s_j}(a)$, $s_j \in S_j$ and the assumption that α provides the strongest and $\hat{\beta}$ the weakest individual evidence for state a implies the ordering:

$$q_{\beta,\hat{\beta}}(a) \;=\; q_{\hat{\beta}}(a) \;\leq\; q_\beta(a) \;\leq\; q_{\beta,\hat{\alpha}}(a) \;\leq\; q_{\hat{\alpha}}(a) \;\leq\; q_\alpha(a) \;=\; q_{\alpha,\hat{\alpha}}(a). \tag{49}$$

We now use Lemma 1 to demonstrate that signals are substitutes. By Lemma 1, it is sufficient to verify that signals are substitutes in all two action problems of Figure 5 for all $x \in (0, 1)$. We show that for any x there is a signal $\tilde{s}_i$ so that $V_{12}(A, u) - V_i(A, u) = 0$

holds for the two action decision problem (A, u) with parameter x.

- $x \leq q_{\beta,\hat{\beta}}(a)$ or $x \geq q_{\alpha,\hat{\alpha}}(a)$: Then all realizations of $(\tilde{s}_1, \tilde{s}_2), \tilde{s}_1, \tilde{s}_2$ induce the same optimal action, so that $V_{12}(A, u) - V_1(A, u) = V_{12}(A, u) - V_2(A, u) = 0$.

- $x \in (q_{\beta,\hat{\beta}}(a), q_{\beta,\hat{\alpha}}(a)]$: Then the probability that a realization of signal $\tilde{s}_1$ moves the decision maker's belief when realization $\hat{\beta}$ of signal $\tilde{s}_2$ has been observed is zero, and no realization of signal $\tilde{s}_1$ changes the optimal action if realization $\hat{\alpha}$ of signal $\tilde{s}_2$ has already been observed. Therefore, $V_{12}(A, u) - V_2(A, u) = 0$.

- $x \in (q_{\beta,\hat{\alpha}}(a), q_{\alpha,\hat{\alpha}}(a)]$: Then the probability that a realization of signal $\tilde{s}_2$ moves the decision maker's belief if realization α of signal $\tilde{s}_1$ has been observed is zero, and no realization of signal $\tilde{s}_2$ changes the optimal action if realization β of signal $\tilde{s}_1$ has already been observed. Therefore, $V_{12}(A, u) - V_1(A, u) = 0$.

B Proof of the Sufficiency Part of Proposition 3

We begin with the observation that the conditions in Proposition 3 imply that all signal realizations have strictly positive prior probability. Suppose, for example, (15) were true and $\bar{p}_{12}(\alpha, \hat{\beta}) = 0$. Then $q_\alpha(a) = q_{\alpha,\hat{\alpha}}(a) \leq \pi(a)$ which would contradict our assumption that $q_\alpha(a) > \pi(a)$. The argument can be completed by repeating this step a number of times.

By Lemma 1, it suffices to verify complementarity for all two action problems described in Figure 5. Below, we shall assume that $x \leq 0.5 = \pi(a)$. If $x \leq 0.5$, then it is optimal under the prior belief to choose B. We shall assume that $q_\beta(a) < x$ and $q_{\hat{\beta}}(a) < x$ so that after observing β or $\hat{\beta}$ it is strictly optimal to choose T. If this were not true, at least one of the signals would by itself never provide a strict incentive to switch away from the action that maximizes expected utility under the prior, and thus this signal by itself would have zero value. Signals would then trivially be complements.

A signal has positive value by itself if it sometimes induces the decision maker to switch to T, and the value of the signal is the expected utility increase arising from these switches. If the decision maker attaches probability $q(a) < x$ to state a, and switches from B to T, then the increase in expected utility is:

$$(1 - q(a))x - q(a)(1 - x) = x - q(a). \tag{50}$$

Observing a second signal realization sometimes induces the decision maker to switch back from T to B. If some signal observation induces the decision maker to hold beliefs $q(a) > x$, and to switch from T to B, then the increase in expected utility is:

$$q(a)(1-x) - (1-q(a))x = q(a) - x. \tag{51}$$

Building on these considerations, we can now calculate for the two action decision problem (A, u) that corresponds to the parameter value x:

$$\begin{aligned} V_2(A,u) - V_\emptyset(A,u) &= \bar{p}_2(\hat{\beta})[x - q_{\hat{\beta}}(a)] \\ &= \bar{p}_{12}(\beta,\hat{\beta})[x - q_{\beta,\hat{\beta}}(a)] + \bar{p}_{12}(\alpha,\hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]. \end{aligned} \tag{52}$$

The first line uses the assumption $q_{\hat{\beta}}(a) < x$. The first and the second line are equal because the expected value of the posterior belief after observing both signal realizations, taking expected values over the realizations of signal 1, is the posterior belief after observing the realization of signal 2 only. We next compute the marginal value of signal $\tilde{s}_2$ when signal $\tilde{s}_1$ is available:

$$\begin{aligned} V_{12}(A,u) - V_1(A,u) &= \bar{p}_{12}(\beta,\hat{\alpha})[q_{\beta,\hat{\alpha}}(a) - x]^+ + \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x]^+ \\ &\quad + \bar{p}_{12}(\alpha,\hat{\alpha})[x - q_{\alpha,\hat{\alpha}}(a)]^+ + \bar{p}_{12}(\alpha,\hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]^+. \end{aligned} \tag{53}$$

Here, we use for any real number z the following notation: $z^+ \equiv z$ if $z \geq 0$, and $z^+ \equiv 0$ if $z < 0$. We have also made use of our assumption $q_\beta(a) < x$.

We now prove first that (15) implies that signals are complements. Condition (15) implies that $q_{\beta,\hat{\alpha}}(a) > 0.5 > x$ because otherwise we could not have $q_{\hat{\alpha}}(a) > 0.5 = \pi(a)$. Thus,

$$V_{12}(A,u) - V_1(A,u) \geq \bar{p}_{12}(\beta,\hat{\alpha})[q_{\beta,\hat{\alpha}}(a) - x]. \tag{54}$$

Therefore, we obtain for the difference:

$$\begin{aligned} & V_{12}(A,u) - V_1(A,u) - (V_2(A,u) - V_\emptyset(A,u)) \\ \geq\ & \bar{p}_{12}(\beta,\hat{\alpha})[q_{\beta,\hat{\alpha}}(a) - x] + \bar{p}_{12}(\beta,\hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha,\hat{\beta})[q_{\alpha,\hat{\beta}}(a) - x]. \end{aligned} \tag{55}$$

Now we add and subtract $\bar{p}_{12}(\alpha,\hat{\alpha})[q_{\alpha,\hat{\alpha}}(a) - x]$ on the right hand side. Using the fact that $\sum_{(s_1,s_2)\in S_1\times S_2} \bar{p}_{12}(s_1,s_2) q_{s_1,s_2}(a) = \pi(a) = 0.5$, the right hand side of (55) becomes equal to

$$0.5 - x - \bar{p}_{12}(\alpha, \hat{\alpha})[q_{\alpha,\hat{\alpha}}(a) - x] \geq 0.5 - x - \bar{p}_{12}(\alpha, \hat{\alpha})[0.5 - x] \geq 0. \tag{56}$$

The first inequality follows because $q_{\alpha,\hat{\alpha}}(a) \leq 0.5$ by (15). The second inequality follows because $x \leq 0.5$ and since $\bar{p}_{12}(\alpha, \hat{\alpha}) < 1$. This establishes that (15) implies that signals are complements.

We next prove that (16) implies that signals are complements. Condition (16) implies: $q_{\beta,\hat{\beta}}(a) \geq \pi(a) = 0.5 \geq x$, and hence we have:

$$V_{12}(A, u) - V_1(A, u) \;\; \geq \;\; \bar{p}_{12}(\beta, \hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha, \hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]^+. \tag{57}$$

Thus,

$$\begin{aligned}
& V_{12}(A, u) - V_1(A, u) - (V_2(A, u) - V_\emptyset(A, u)) \\
\geq \;\; & \bar{p}_{12}(\beta, \hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha, \hat{\beta})[x - q_{\alpha,\hat{\beta}}(a)]^+ \\
& + \bar{p}_{12}(\beta, \hat{\beta})[q_{\beta,\hat{\beta}}(a) - x] + \bar{p}_{12}(\alpha, \hat{\beta})[q_{\alpha,\hat{\beta}}(a) - x] \geq 0.
\end{aligned} \tag{58}$$

The sum in (58) is non-negative since $q_{\beta,\hat{\beta}}(a) \geq \pi(a) = 0.5 \geq x$ by (16), and because the sum of the second and the fourth term is always non-negative. Thus we have again shown that signals are complements.