# On Integration Policies and Schooling

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**Abstract**

This paper proposes a reform for school allocation procedures in order to help integration policies reach their objective. For this purpose, we suggest the use of a *natural* two-step mechanism. The (*stable*) first step is introduced as an adaptation of the deferred-acceptance algorithm designed by Gale and Shapley (1962), when students are divided into two groups. The (*efficient*) second step captures the idea of *exchanging places* inherent to Gale's *Top Trading Cycle*. This latter step could be useful for Municipal School Boards when implementing some integration policies.

# I. Introduction

On March 6, 1961 President John F. Kennedy issued Executive Order 10925. This can be seen as the beginning of a 'new era' against intolerance and discriminatory practices. At that time, the way to fight against some of these 'customs' was by promoting the use of affirmative action policies.

In a static setting, these policies can be seen as discriminatory practices; whereas, from a dynamic perspective, the existence of a certain degree of 'historical discrimination' justifies the use of affirmative action policies, as a necessary previous step, before concentrating on avoiding discriminatory practices. In other words, policies against *circumstantial* discrimination are useless if they are not accompanied by measures to reduce *structural* inequalities.

Evidence suggests that the 'long run' arguments have prevailed, which is why, for instance, the OECD countries explicitly implemented affirmative action measures to reach integration objectives. As a recent example concerning this matter, let us mention the MENA-OECD conference on gender equality in government and business, held in Paris on 4 May 2010.[1] Other illustrative examples include firstly, the presence of several policies to prevent gender gaps in terms of salary and/or political representation and upon which legal reforms in some countries are based; and secondly, the existence of intensive migration movements over recent years that has justified the adoption of certain integration policies (from race, ethnic and/or national origin perspectives).

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[1] The conclusions from this conference are summarized in a document available at the OECD web page.

There is a controversial debate against affirmative action policies as some agents believe that they are discriminatory practices. As an illustrative example, we can mention that[2]

> During the November 5, 1996 election, California voters voted 54% to 46% to amend the California Constitution through an initiative commonly known as Proposition 209, or the California Civil Rights Initiative. The proposition has been incorporated into the California Constitution under Article 1, Section 31. Although the constitutionality of the initiative was legally challenged, the U.S. Supreme Court denied further appeal and let stand the new California law on November 3, 1997. [...]
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> Students and state lawmakers had urged the repeal, arguing that the ban on "affirmative action" had caused the university to be perceived as inhospitable to minority students. Proponents of the repeal cited a sharp drop in the number of in-state black and Hispanic first-year students and the hiring rates of women and underrepresented minority faculty members.

The aim of the present paper is to shed some light onto the debate on affirmative action measures versus non-discriminatory practices. Our point is that, in agreement with the 1964 Civil Rights Act, and also the Supreme Court's opinion,[3]

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[2]This quotation has been borrowed from the University of California at Irvine, Office of Equal Opportunity and Diversity (OEOD). Website: http://www.oeod.uci.edu.

[3]See the arguments by Justice O'Connor on Grutter v. Bollinger, at the end of this Introduction.

these policies should cease to exist when the effects of 'historic discrimination practices' are dissipated. Therefore, it is important to concentrate on how these 'everlasting' inequalities could be reduced. In this sense, we believe that the educational training of 'future generations' is one of the most relevant variables to be taken into account, as whenever children have unequal opportunities to achieve a comparable educational level, in the broadest sense, they will not be able to compete fairly in the job market. Moreover, since

$(a)$ there is empirical evidence to show the positive correlation between school quality and housing prices in their 'influence areas' (see, for instance Kane et al., 2006 or Dougherty et al., 2009), and

$(b)$ one of the two main factors for a student to be attached to a certain school is to reside in its 'influence area',

the main conclusion is that the non-application of affirmative action measures, in favor of certain population groups, will lead to a more inequitable income distribution in the long run.

This is why we concentrate on the analysis of school allocation procedures as a way to reduce certain inter-group inequalities. We propose to introduce integration policies into the processes used for school allocation at elementary level. To reach our objective we introduce a two-step procedure. For the first step, we suggest a tentative allocation procedure with a minimal segregation level.[4] The

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[4]Frankel and Volij (2011) propose an exhaustive analysis for segregation indexes. Our approach differs from that of Frankel and Volij (2011) as our aim does not consider the different districts into which each municipality is divided, but rather each municipality as a whole.

second step adopts the idea of trading places, introduced by Alcalde and Romero-Medina (2011b) to attain efficiency in the allocation process.

Concerning the first step, we proceed as follows. Each school is allocated a number of available places, commonly known as its quota. We distribute each school quota among the different groups of agents[5] in such a way that (unless it is impossible) each group has a number of 'reserved places' and, if we analyze how many places have been assigned to each group, the share is equitable between groups. Note that such a division allows one *isolated* school allocation problem to be considered for each group. We then propose an initial distribution of *its schools places* for each group satisfying 'internal non-justified envy'.[6] This objective is reached by applying (a modified version of) the classical 'deferred-acceptance algorithm' introduced by Gale and Shapley (1962), where offers are made by students. This algorithm is applied to each group separately.

Once this first step is concluded, we can proceed to an 'integrate' second step, whose main objective is to achieve improvements, in terms of efficiency, related to the previous distribution. In this sense, we consider two different scenarios, related to the Municipal School Board, MSB henceforth, objectives.

$(a)$ Minimizing segregation is the main objective of the MSB. By this we mean that the MSB would prefer to implement an inefficient allocation than 'to

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[5]Although our model, in Section III, deals with two-group problems, our results are straightforwardly extended to any number of population groups. What it is important is that the number of available places at each school is high enough to distribute them among the groups by guaranteeing some diversity representation at each school.

[6]The term *non justified envy* was coined by Abdulkadiroğlu and Sönmez (2003) to re-interpret, in the framework of school allocation, the classical notion of stability in *matching problems*.

sacrifice' the level of integration.

(b) The MSB objective is to obtain an efficient allocation; whilst minimizing segregation is a secondary although relevant objective.

The two scenarios described above highlight the existence of some trade-offs which are difficult to solve. The first incompatibility, referred to in situation (a) above, concerns a trade-off between *efficiency* and *non justified envy* from an intra-group perspective. This was pointed out by Alcalde and Romero-Medina (2011b), showing that no procedure always proposes an allocation system that satisfies the two properties above. Their solution was to propose an allocation procedure satisfying non justified envy, in a weaker sense, and efficiency. The second trade-off, proposed in statement (b) above, marks the incompatibility of *global* efficiency and minimal segregation. Our suggestion to counteract this fact is to concentrate on efficiency as the main objective and, then, select the efficient solution that minimizes the segregation level. This approach is related to a controversy on individual versus collective rights. It seems to be clear that, from a social point of view, segregation must be prevented. Sociologists and lawyers study the benefits of integration from a social perspective[7]. Nevertheless, there are some *popular* comments suggesting that certain groups are against integration. In this respect van Buren (1995) says, on pg. 179,

Jews resist, for example any form of quotas for admission to pro-

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[7]From a Sociological perspective, see the paper by Oliver (1985) for an old debate, or Curran et al. (2006) for an analysis of migration and integration. Legal aspects are studied among others in Seitles (1998) and Garda (2009).

fessional schools, because they remember their own experience of having been excluded by quotas in the past. Blacks, on the other hand, demand "afirmative action," knowing from their own past that "open admission" in a white society has always meant black exclusion. So blacks conclude that Jews are racists, and Jews conclude that blacks are anti-Judaic. Their common experience of suffering only serves to set them at odds.

Therefore, what we propose in affirmation $(b)$ can be seen as a test to contrast whether this 'popular belief' about *others' aversion to be integrated* is empirically sustained or not.

A further question that we wish to highlight in our proposal is *legal implementability*. In this sense, Gajendragadkar (2006) says:

[...] racial balancing provisions, as governmental uses of race, must survive the strictest judicial scrutiny to accord with the Equal Protection Clause of the Fourteenth Amendment.

Moreover, bearing in mind the case of Grutter v. Bollinger, and relative to the controversial (long run) affirmative action policies versus (short run) non-discriminatory practices, Justice O'Connor argued that race-conscious admissions policies must be limited in time:

We take the Law School at its word that it would "like nothing better than to find a race-neutral admissions formula" and will terminate its race-conscious admissions program as soon as practicable.

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[...] It has been 25 years since Justice Powell first approved the use of race to further an interest in student body diversity in the context of public higher education. Since that time, the number of minority applicants with high grades and test scores has indeed increased. We expect that 25 years from now, the use of racial preferences will no longer be necessary to further the interest approved today.

At this point, in order to avoid any misinterpretation, we want to stress that by *integration policies* we are not necessarily meaning *racial policies* for integration. This term should be understood as widely as possible. For instance, some policymakers could be interested in *gender integration* by reducing the *structural gap* favoring men over women; others might well be interested in *income-based integration* by helping children from low-income families to access *high-income district* schools. What is important in this paper is that, once the policymakers have decided which specific integration policy they are interested in, the population is partitioned into mutually exclusive groups; and therefore, no agent is free to decide which group she belongs to. In this way we avoid certain types of strategic behavior.

## II.   Related Literature and Overview

The problem that we study here can be viewed as a many-to-one, two-sided matching problem, following the original model by Gale and Shapley (1962). There is a large tradition on this family of problems, as pointed out in the overview by Roth

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and Sotomayor (1990). These models have been used to study, both from a positive and a normative point of view, how several specific markets work or would be redesigned. For instance, Roth and Xing (1994) identify the general class of market failures due to the unraveling of appointment dates; Roth and Peranson (1999) analyze some modifications introduced into the National Resident Matching Program; and Niederle et al. (2008) overview some of the recent literature in that matter.

For the case of *school allocation*, some recent papers explore the former Boston system and propose modifications to reach a better allocative process (Abdulkadiroğlu et al., 2005b, Abdulkadiroğlu et al., 2006 or Ergin and Sönmez, 2006, among others), whilst others suggest improvements to the current Boston system (Alcalde and Romero-Medina, 2011b, or Kesten, 2010); the New York City system for high schools has also been explored in some papers (see, for instance, Abdulkadiroğlu et al. (2005a) or Abdulkadiroğlu et al., 2009).

Nevertheless, as far as we know, not many papers explore the school admission problem taking into account the possibility of affirmative action *ingredients*. Abdulkadiroğlu (2005) deals with college admissions, and considers that the colleges are the only agents that decide how their affirmative action policy should be implemented, i.e. each school decides how to distribute its quota among the different groups of agents. Then, he proposes a strategic analysis for the *student optimal stable matching mechanism*. To the contrary, in the present paper, we consider that schools do not have any capacity to influence the way in which their places are ex-ante distributed among the different groups of agents. Our interest

is not about *global* stability, as in Abdulkadiroğlu (2005), but *internal* stability plus *inter-group* fairness, which is not mentioned by this author. Moreover, our analysis in the present paper does not concentrate on agents' strategic behavior. Recently, Kojima (2010) explores the possibility of proposing *affirmative action policies* that do not hurt the *disadvantaged students* (in terms of efficiency). What he finds is general impossibility of reaching such an objective. Meanwhile, Ehlers (2010) studies a model closer to the one proposed in this paper. The main difference between the model in Ehlers (2010) and ours is that he considers that schools places are ex-ante distributed among the two groups of agents and that this restriction should be satisfied ex-post. However, our proposal does not require ex-post distribution to satisfy any pre-established condition.

The rest of the paper is organized as follows. Section III introduces the basic model and discusses the notions of stability and efficiency from an intra-group perspective. It also proposes a way to formalize the notion of the *degree of external envy*. Section IV introduces an *algorithmic* procedure yielding a matching combining *internal stability* and *external fairness*. Moreover, the matching that our procedure suggests is the unique *second-best* efficient allocation, restricted to the two properties above. Section V explores two ways to improve the allocation suggested in the previous section, in terms of efficiency. Each of these possibilities for improvement follows one of the MSB objectives proposed in the Introduction. The main objective for Section VI is to propose a value for the *fairness parameter*. We also suggest a method to evaluate the fairness degree of any matching. Section VII introduces our main conclusions. Finally, the proofs are gathered in

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the Appendices.

### III. School Allocation Problem and Diversity. A Model

We consider a city with $n$ schools. Let $q_i$ denote the set of available places for new students that school $c_i$ has available, $c_i \in \mathcal{C} = \{c_1, \ldots, c_i, \ldots, c_n\}$. $Q = (q_1, \ldots, q_i, \ldots, q_n)$ is called the vector of quotas.

We also consider a group of $m$ new students, $m \leq \sum_{i=1}^{n} q_i = q$,[8] each of them belonging to one of the two mutually exclusive classes $\mathcal{F}$ and $\mathcal{D}$, having $f$ and $d$ individuals, respectively. For interpretative purposes, $\mathcal{D}$ is considered the group of *disadvantaged* students (i.e., the set of agents that eventually benefit from *affirmative action* policies), whereas $\mathcal{F}$ is referred as the group of *favored* students.

Each student $s_j \in \mathcal{S} = \mathcal{F} \cup \mathcal{D}$ has a preference relation defined on the set of schools. Let linear preorder $P_j$ denote student $s_j$'s preferences, and $R_i$ the induced *weak preferences*.[9] Similarly, we assume that each school has a linear preorder on $\mathcal{S}$ that is interpreted as its priority list. Let $\Pi_i$ denote the priority list for school $c_i$.[10] Vector $\mathbb{P} = \left( \{\Pi_i\}_{i=1}^{n} ; \{P_j\}_{j=1}^{m} \right)$ summarizing both school priorities and student preferences will be called a profile of preferences.

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[8]This assumption implies that there are enough places to allocate each student a place in some school. Note that this requirement is not necessary to define a matching problem, and will not be used in the description provided for some of our allocation procedures. Therefore, the only reason we introduce such a consideration is the capacity for providing *realistic* interpretations in agreement with the fact that compulsory schooling is established by law.

[9]i.e., given a student $s_j$ and two schools, $c_i$ and $c_h$, $c_i R_j c_h$ is interpreted as either $c_i P_j c_h$ or $c_i = c_h$.

[10]We do not discuss here how these lists are (or should be) built. The paper by Alcalde and Romero-Medina (2011a) gives some insights on this matter.

The problem that we would like to solve is how to distribute the students among the available school places. A first approach to answer such a question is given by the notion of *matching*.

**Definition 1** *Let* $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ *be a* School Allocation Problem. *A matching for* $\mathcal{SAP}$ *is a correspondence* $\mu : \mathcal{C} \cup \mathcal{S} \twoheadrightarrow \mathcal{C} \cup \mathcal{S}$ *such that*

$(a)$ *For each* $c_i \in \mathcal{C}$, $\mu(c_i) \subseteq \mathcal{S}$;

$(b)$ *For each* $s_j \in \mathcal{S}$, $\mu(s_j) \in \mathcal{C} \cup \{s_j\}$;

$(c)$ $|\mu(c_i)| \leq q_i$[11] *for each school* $c_i$; *and*

$(d)$ *For each school-student pair,* $(c_i, s_j)$, $s_j \in \mu(c_i)$ *if, and only if,* $c_i = \mu(s_j)$.

Among the set of matchings that a $\mathcal{SAP}$ admits, there are two particular requirements that have been commonly asked to be satisfied by a solution in order to be considered a satisfactory allocation. The first, called *efficiency*, implies that students are unable to obtain a place each in a preferred school. The second, called *stability*, establishes that student envies in the allocation process, if any, are based on the way in which schools prioritize the students.[12] Therefore, as far as the MSB builds priority lists based on equity criteria, these envies could be considered as unjustified. Definitions 2 and 3 formalize these ideas.

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[11]Given a set A, $|A|$ denotes its cardinal.

[12]The paper by Abdulkadiroğlu and Sönmez (2003) proposes an interpretation of stability in terms of absence of *justified envy*.

**Definition 2** *Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a* School Allocation Problem*, and $\mu$ a matching for $\mathcal{SAP}$. We say that $\mu$ is* efficient *if for any other matching $\mu' \neq \mu$ there is a student, say $s_j$, such that[13] $\mu(s_j) P_j \mu'(s_j)$.*

**Definition 3** *Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a* School Allocation Problem*, and $\mu$ a matching for $\mathcal{SAP}$. We say that $\mu$ is* stable *if for each student $s_j$ and school $c_i$ such that $c_i P_j \mu(s_j)$, we have that*

*(a) $|\mu(c_i)| = q_i$; and*

*(b) $s_h \Pi_i s_j$ for each $s_h \in \mu(c_i)$.*

Any $\mathcal{SAP}$ whose students are divided into two groups, $\mathcal{F}$ and $\mathcal{D}$, allows us to consider two school allocation problems, namely $\mathcal{SAP}^{\mathcal{F}}$ and $\mathcal{SAP}^{\mathcal{D}}$ separately. There can be described easily, once we have been able to disaggregate school quotas by describing two vectors, $Q^{\mathcal{F}}$ and $Q^{\mathcal{D}}$ such that

(a) $Q = Q^{\mathcal{F}} + Q^{\mathcal{D}}$; and

(b) $\sum_{i=1}^{n} q_i^{\mathcal{F}} \geq f$, $\sum_{i=1}^{n} q_i^{\mathcal{D}} \geq d$, where $q_i^{\mathcal{F}}$ ($q_i^{\mathcal{D}}$, resp.) denotes the number of places that school $c_i$ reserves for students in $\mathcal{F}$ (in $\mathcal{D}$, resp.).

Given $Q^{\mathcal{F}}$ and $Q^{\mathcal{D}}$ we define $\mathcal{SAP}^{\mathcal{F}} = \{\mathcal{C}, \mathcal{F}, \mathbb{P}^{\mathcal{F}}, Q^{\mathcal{F}}\}$ where the preference profile $\mathbb{P}^{\mathcal{F}} = \left( \{\Pi_i^{\mathcal{F}}\}_{c_i \in \mathcal{C}}, \{P_j^{\mathcal{F}}\}_{s_j \in \mathcal{F}} \right)$ agrees with $\mathbb{P}$ in the new set of agents, i.e.

(a) for each $c_i \in \mathcal{C}$, and any two students in $\mathcal{F}$, say $s_j$ and $s_h$, $s_j \Pi_i^{\mathcal{F}} s_h$ if and only if $s_j \Pi_i s_h$; and

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[13]For the sake of simplicity, we consider in this paper that the *to be unmatched option* is worse than to be matched to any school, i.e. $c_i P_j s_j$ for each student $s_j$ and school $c_i$.

$(b)$ for each $s_j \in \mathcal{F}$, and any two schools, say $c_i$ and $c_k$, $c_i P_j^{\mathcal{F}} c_k$ if and only if $c_k$,

$c_i P_j c_k$.

$\mathcal{SAP}^{\mathcal{D}}$, the school allocation problem that $Q^{\mathcal{D}}$ induces, involving schools and students in $\mathcal{D}$, can be described in a similar way.

Therefore, given a $\mathcal{SAP}$ and a matching $\mu$, we can study how students in $\mathcal{F}$ and $\mathcal{D}$ are located by $\mu$, from an *intra-group* point of view, when considering only students in any of the two classes. To reach our objective, for $\mu$ given, we consider the disaggregate school quotas by defining for each school $c_i$ such that $q_i^{\mathcal{D}} = |\mu(c_i) \cap \mathcal{D}|$.[14]

Given vectors $Q^{\mathcal{F}}$ and $Q^{\mathcal{D}}$ described above, we denote by $[\mu]^{\mathcal{F}}$ and $[\mu]^{\mathcal{D}}$ the matchings for $\mathcal{SAP}^{\mathcal{F}}$ and $\mathcal{SAP}^{\mathcal{D}}$ respectively that agree with $\mu$, i.e. for each $c_i \in \mathcal{C}$, $[\mu]^{\mathcal{F}}(c_i) = \mu(c_i) \cap \mathcal{F}$; and, similarly $[\mu]^{\mathcal{D}}(c_i) = \mu(c_i) \cap \mathcal{D}$.

The above considerations are useful for introducing certain properties that a matching might exhibit from an *intra-group* perspective.

**Definition 4** *Let* $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ *be a* School Allocation Problem*, and $\mu$ a matching for* $\mathcal{SAP}$*. We say that*

$(a.1)$ *$\mu$ is $\mathcal{F}$-stable if $[\mu]^{\mathcal{F}}$ is stable for $\mathcal{SAP}^{\mathcal{F}}$;*

$(a.2)$ *$\mu$ is $\mathcal{D}$-stable if $[\mu]^{\mathcal{D}}$ is stable for $\mathcal{SAP}^{\mathcal{D}}$;*

$(b)$ *$\mu$ is $\mathcal{G}$-stable if it is both $\mathcal{F}$-stable and $\mathcal{D}$-stable;*

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[14]Provided that agents in $\mathcal{D}$ benefit from the affirmative action policy, and that some of our definitions are *sensitive* to the way in which *excess places* are distributed among both groups of students, we might consider that these places are (ex-post) attributed to agents in $\mathcal{F}$ as a way to compensate the effect of such a policy.

$(c.1)$ $\mu$ is $\mathcal{F}$-efficient if $[\mu]^{\mathcal{F}}$ is efficient for $\mathcal{SAP}^{\mathcal{F}}$;

$(c.2)$ $\mu$ is $\mathcal{D}$-efficient if $[\mu]^{\mathcal{D}}$ is efficient for $\mathcal{SAP}^{\mathcal{D}}$;

$(d)$ $\mu$ is $\mathcal{G}$-efficient if it is both $\mathcal{F}$-efficient and $\mathcal{D}$-efficient.

The next result, whose straightforward proof is omitted, reveals a close relationship between *stability* and $\mathcal{G}$-*stability*; and a similar connection between *efficiency* and $\mathcal{G}$-*efficiency*.

**Proposition 5** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a *School Allocation Problem*, and $\mu$ a matching for such a problem. If $\mu$ is *stable* (resp. efficient), then it is $\mathcal{G}$-*stable* (resp. $\mathcal{G}$-*efficient*). The converse is, in general, not true; i.e. $\mathcal{G}$-*stability* does not imply *stability* and $\mathcal{G}$-*efficiency* does not guarantee *efficiency* either.

Note that the notions introduced in Definition 4 deal with internal stability and/or efficiency for agents in any of the two groups, $\mathcal{F}$ and $\mathcal{D}$. Nevertheless, they do not consider any comparison involving students belonging to the two classes. What we next propose is a way to evaluate external fairness, from the point of view of the agents belonging to the same class. Related to this point, let us consider that agents in a certain class assume that a proportion of places, say $\alpha$, has to be *reserved* for the students in the other category. Taking this premise as a fact, the possibilities that a student might have to object to an allocation (based on the priority lists) are reduced, related to what stability suggests.

**Definition 6** *Let* $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ *be a* School Allocation Problem*, and* $\alpha$ *and* $\beta$, $0 \leq \alpha, \beta \leq 1$ *two real numbers. We say that matching* $\mu$ *is*

(*a*) $\alpha^{\mathcal{F}}$*-fair, or* $\alpha$*-fair from* $\mathcal{F}$*'s point of view if, and only if, for each school* $c_i \in \mathcal{C}$,

$$\textit{if } |\mathcal{D} \cap \mu(c_i)| > [\alpha q_i]^+, \textit{ then } \{s_h \in \mathcal{F} \colon c_i P_h \mu(s_h)\} = \emptyset.$$

(*b*) $\beta^{\mathcal{D}}$*-fair, or* $\beta$*-fair from* $\mathcal{D}$*'s point of view if, and only if, for each school* $c_i \in \mathcal{C}$,

$$\textit{if } |\mathcal{F} \cap \mu(c_i)| > [\beta q_i]^-, \textit{ then } \{s_h \in \mathcal{D} \colon c_i P_h \mu(s_h)\} = \emptyset.$$

(*c*) $\alpha$*-fair if it is* $\alpha^{\mathcal{F}}$*-fair and* $(1 - \alpha)$*-fair from* $\mathcal{D}$*'s point of view;*

*where for a real number* $r$, $[r]^+$ *is the integer* $v$ *such that* $v - 1 < r \leq v$*; and* $[r]^-$ *is the integer* $z$ *such that* $z \leq r < z + 1$.

Note that Definition 6 implicitly contains a flavor of 'affirmative action policy' benefiting agents in $\mathcal{D}$. The first requirement that $\alpha$-*fairness* suggests is that agents in both sets agree how these places should be distributed, in relative terms. The second assumption is that the fact that places are *indivisible* should not hurt students in $\mathcal{D}$. Just to illustrate this, let us consider two schools with 25 and 23 places respectively to be distributed. Let us assume that there is a *consensus* establishing that 70.83 % should be assigned to agents in $\mathcal{F}$ and the remaining 29.17 % correspond to students in $\mathcal{D}$. Therefore, and provided that school places are indivisible, students in $\mathcal{D}$ would be prioritized in 7 or 8 places at the first school and 6 or 7 at the second. Since our aim is to promote affirmative action for these students, we should consider that they have priority in 8 places at the first school and 7 at the second.

Continuing with the above illustrative example, we can proceed as follows. Let us consider two *independent* $\mathcal{SAP}$s, one involving the students in $\mathcal{F}$, with school quotas $(17, 16)$, and the other including students in $\mathcal{D}$, with school quotas $(8, 7)$, as previously argued. Now, let us assume that $f = 34$ and $d = 14$. Note that any solution that we could propose should imply that one student in $\mathcal{F}$ must have no school place and simultaneously one school has to have one unassigned place. This proposal is hardly justifiable as a solution for our problem. The next section deals with how to divide school quotas among the two groups of agents in such a way that each student might be allocated a place.

## IV. $\mathcal{G}$-Stability and Inter-Group Fairness

The aim of this section is to propose a formal procedure showing the existence of $\mathcal{G}$-stable allocations. What we introduce here is a family of sequential procedures, one for each parameter $\alpha$ used to distribute the school places among both groups of students.

Just to introduce our next proposal, let $\Omega$ be the set of values for $\alpha$ guaranteeing that each student in $\mathcal{D}$ would be allocated a school place, i.e. $\Omega = \left\{ \alpha \in [0, 1] : d \leq \sum_{i=1}^{n} [\alpha q_i]^+ \right\}$

**Definition 7** *For $\alpha$ given, $\alpha \in \Omega$, we define the $\alpha$-fair deferred-acceptance procedure, that associates to each $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ the matching $\mu^{\alpha}$, to be called its $\alpha$-fair matching, where*

*(a) for each $s_j \in \mathcal{D}$, $\mu^{\alpha}(s_j)$ is $s_j$'s mate at the student optimal stable matching*

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*for $\{\mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}}\}$, where the quota for school $c_i$ is $q_i^{\mathcal{D}} = [\alpha q_i]^+$; and*

(b) *for each $s_h \in \mathcal{F}$, $\mu^{\alpha}(s_h)$ is $s_h$'s mate at the student optimal stable matching*

*for $\{\mathcal{C}, \mathcal{F}, \mathbb{P}^{\mathcal{F}}, Q^{\mathcal{F}}\}$, with $q_i^{\mathcal{F}} = q_i - |\{s_j \in \mathcal{D} : \mu^{\alpha}(s_j) = c_i\}|$.* ⊡

What the $\alpha$-*fair deferred-acceptance procedure* suggests is the following. First, let us consider the sub-problem involving *disadvantaged* students and a number of school places guaranteeing that each $\mathcal{D}$-student would be allocated a place at a school. These places should be uniformly distributed across the schools. For this sub-problem, let us apply the deferred-acceptance algorithm introduced by Gale and Shapley (1962) in which students made the proposals. Then a new sub-problem emerges. In such a case, the set of students coincides with $\mathcal{F}$, and each school still has available all the places that have been reserved for the *favored* students, or no student in $\mathcal{D}$ previously applied for such a place. Now, let us apply again the student optimal algorithm proposed by Gale and Shapley (1962) to this second problem. Since no student belongs to both categories, i.e. $\mathcal{F}$ and $\mathcal{D}$ do not intersect, the above description on how to allocate students among schools defines a matching for the original problem.

The next result points out some of the properties satisfied by the matching obtained when the $\alpha$-*fair deferred-acceptance procedure* is applied.

**Theorem 8** Let $\mathcal{SAP}$ be a school allocation problem and, for $\alpha$ given, $\alpha \in \Omega$, let $\mu^{\alpha}$ be the matching obtained by applying the $\alpha$-fair deferred-acceptance procedure. Then $\mu^{\alpha}$ satisfies the following properties

(a) $\mu^{\alpha}$ is $\mathcal{G}$-stable;

($b$) Each student is attached to a school, i.e. for each student $s_j$, $\mu^\alpha(s_j) \in \mathcal{C}$;

($c$) $\mu^\alpha$ is $\alpha$-fair; and

($d$) No student has any incentive to misreport her preferences.

The above result shows that the outcome for the $\alpha$-fair deferred-acceptance procedure satisfies some interesting properties. Nevertheless, it can be far from efficient, as the following situation illustrates. Let us consider a problem with $d = 50$ and $f = 100$. Now let us imagine that there are 8 schools with 21 vacancies each, and fix $\alpha = 1/3$. If $c_1$ is the best school for all the students in $\mathcal{D}$ and, similarly, it is the worst school for all the students in $\mathcal{F}$, only 7 students in $\mathcal{D}$ will obtain a place at school $c_1$ and its remaining 14 places will stay vacant as no student in $\mathcal{F}$ will apply for a place in such a school. Therefore, what this example shows is that the matching proposed by the $\alpha$-fair deferred-acceptance procedure might be inefficient; and such an inefficiency, which is not present when students are not segregated into two classes, can be reduced by reallocating the unassigned school places.

A way to reduce the inefficiency of the allocation procedure is by introducing what we call the *multi-stage $\alpha$-fair procedure*. It can be described straightforwardly by iteratively applying the procedure proposed in Definition 7. Let us consider a given $\mathcal{SAP}$. As a first *tentative* matching, we propose its $\alpha$-fair matching. We now investigate to determine whether there is any school with vacancy at $\mu^\alpha$ that has been over-demanded by agents in $\mathcal{D}$. If there is no such school, the process is over and $\mu^\alpha$ is implemented. Otherwise, let us reconsider the share of

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school quotas among the two groups, by increasing the quotas for agents in $\mathcal{D}$, and proceed in a way similar to the one proposed in Definition 7. This process ends up when no school being under-demanded was over-demanded by disadvantaged students.

**Definition 9** *For $\alpha$ given, $0 \leq \alpha \leq 1$, we define the* multi-stage $\alpha$-fair *procedure which associates to each $\mathcal{SAP}$ the matching $\mu^{ms\alpha}$ obtained throughout the following algorithm.*

(**Step** 1) *Let $\mu^1$ be the matching described as follows*

    (a) *For each $s_j \in \mathcal{D}$, $\mu^1(s_j)$ is her* student optimal stable matching *for $\{\mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}(1)}\}$, where the quota for school $c_i$ is $q_i^{\mathcal{D}(1)} = [\alpha q_i]^+$; and*

    (b) *For each $s_h \in \mathcal{F}$, $\mu^1(s_h)$ is her* student optimal stable matching *for $\{\mathcal{C}, \mathcal{F}, \mathbb{P}^{\mathcal{F}}, Q^{\mathcal{F}(1)}\}$, with $q_i^{\mathcal{F}(1)} = q_i - |\{s_j \in \mathcal{D} : \mu^1(s_j) = c_i\}|$.*

*If, for each school $c_i$, $|\{s_j \in \mathcal{D} : \mu^1(s_j) = c_i\}| < q_i^{\mathcal{D}(1)}$, then the algorithm stops proposing $\mu^{ms\alpha} = \mu^1$. Otherwise, go to Step 2.*

(**Step** $t$) *Let $\mu^t$ be the matching described as follows*

    (a) *For each $s_j \in \mathcal{D}$, $\mu^t(s_j)$ is her* student optimal stable matching *for $\{\mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}(t)}\}$, with $q_i^{\mathcal{D}(t)} = q_i - |\{s_j \in \mathcal{F} : \mu^{t-1}(s_j) = c_i\}|$; and*

    (b) *For each $s_h \in \mathcal{F}$, $\mu^t(s_h)$ is her* student optimal stable matching *for $\{\mathcal{C}, \mathcal{F}, \mathbb{P}^{\mathcal{F}}, Q^{\mathcal{F}(t)}\}$, with $q_i^{\mathcal{F}(t)} = q_i - |\{s_j \in \mathcal{D} : \mu^t(s_j) = c_i\}|$.*

*If $\mu^t \equiv \mu^{t-1}$, then the algorithm stops proposing $\mu^{ms\alpha} = \mu^t$. Otherwise, go to Step $t + 1$.* □

We would like to point out that Definition 7 assumes that $\alpha \in \Omega$, whereas Definition 9 does not impose a lower bound for $\alpha$. The only reason for introducing such a *formal difference* is that this assumption is both necessary and sufficient for the former to guarantee that each student will be assigned a place in a school, provided that there are enough places; nevertheless this lower bound is not necessary for the latter to reach such an objective.

Let us observe that this algorithm is *student monotonic*, i.e. for two given steps $t$ and $t'$, $t < t'$, and each student $s_j$ we have that $\mu^{t'}(s_j) R_j \mu^t(s_j)$. Therefore, a natural question arises regarding the study of the *restricted* efficiency relative to matching $\mu^{ms\alpha}$. Definition 10 introduces the notion of *sub-optimality* that we are going to explore, and Theorem 11 points out that by applying the above mechanism we always reach a sub-optimal allocation.

**Definition 10** *Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem, and $\mu$ a matching for such an instance. Given $\alpha$, $0 \leq \alpha \leq 1$, we say that $\mu$ is $\mathcal{G}$-Stable $\alpha$-fair optimal for $\mathcal{SAP}$ whenever*

(1) *$\mu$ is $\mathcal{G}$-Stable;*

(2) *$\mu$ is $\alpha$-fair; and*

(3) *for any $\mathcal{G}$-Stable matching $\mu' \neq \mu$, such that $|\mu'(c_i) \cap \mathcal{D}| \geq |\mu(c_i) \cap \mathcal{D}|$ for each school $c_i$, there is a student, say $s_j$, such that $\mu(s_j) P_j \mu'(s_j)$.*

What the above notion for *restricted optimality* proposes is that we restrict our analysis to matchings satisfying both $\mathcal{G}$-stability and $\alpha$-fairness. Moreover, since integration is a relevant feature for the MSB policies, any alternative matching to be considered should respect the integration level that the one proposed has. This is the aim of the Condition $(3)$ in Definition 10, which states that the presence of students in $\mathcal{D}$ should be not reduced at any school.

**Theorem 11** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem and, for $\alpha$ given, $\mu^{ms\alpha}$ be the outcome of the multi-stage $\alpha$-fair procedure for $\mathcal{SAP}$. Then $\mu^{ms\alpha}$ is a $\mathcal{G}$-*Stable* $\alpha$-*fair optimal* matching for $\mathcal{SAP}$.

The next corollary points out that the *complex* sequentiality introduced in Definition 9 is not present when the number of exceeding places is low enough.

**Corollary 12** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem, with $m = q$, and let $\alpha$ be a distribution parameter, $\alpha \in \Omega$. Then $\mu^{\alpha}$ is a $\mathcal{G}$-*Stable* $\alpha$-*fair optimal* matching for $\mathcal{SAP}$.

## V. Towards Efficiency

The main aim of this section is to explore how the MSB could act to improve the allocation proposed by the multi-stage $\alpha$-fair procedure, in terms of efficiency. As anticipated in the Introduction, we distinguish two scenarios, depending on the MSB objectives. Therefore, each possible objective for the MSB is analyzed in an independent subsection.

The solutions for both policies can be summarized in a simple way. Given a $\mathcal{SAP}$, and fixed $\alpha$, let us compute its multi-stage $\alpha$-fair matching, $\mu^{ms\alpha}$. An interpretation of what this matching proposes can be established in terms of students having *usufruct right* on their places, and thus they are free to exchange the use of their places.[15] Consequently, the allocation that $\mu^{ms\alpha}$ proposes to each student can be understood as her *initial endowment* to be exchanged (if she wished to) in a *market for school places*. Based on this approach, what we propose here is to apply Gale's *top trading cycle algorithm*, introduced in Shapley and Scarf (1974). For the MSB to reach its objective, it is important to decide how to use the information about the students' declared preferences for exchanging places. To formalize this, we need to introduce additional notation. From now on, we assume that each student, say $s_j$, is endowed with a linear preorder $E_j$ on $\mathcal{S}$. For student $s_j$, $s_h E_j s_k$ might be interpreted as, *ceteris paribus*, $s_j$ would prefer to exchange her place with student $s_h$ rather than with $s_k$.

The MSB role in this section can be summarized as follows: Given matching $\mu$ for $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ and students' preferences for exchanging $E = \{E_j\}_{j=1}^{m}$, the MSB determines, for each student $s_j$, her *MSB-agreeing preferences for exchange*. Let $\succ_j^{\mathcal{P}}$ be the linear preorder in $\mathcal{S}$ that MSB associates to student $s_j$ when, given the above primitives, it wants to implement policy $\mathcal{P}$.

Given subset of students $\widehat{\mathcal{S}}$, and $s_j$, let $\mathcal{B}\left(s_j; \widehat{\mathcal{S}}\right)$ be the only $\succ_j^{\mathcal{P}}$-maximal in

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[15]Alcalde and Romero-Medina (2011b) propose a legal justification for such an interpretation.

$\widehat{\mathcal{S}}$. Similarly, for any two sets of students, say $\bar{\mathcal{S}}$ and $\widehat{\mathcal{S}}$, we denote

$$\mathcal{B}\left(\bar{\mathcal{S}};\widehat{\mathcal{S}}\right) = \bigcup_{s_j \in \bar{\mathcal{S}}} \mathcal{B}\left(s_j;\widehat{\mathcal{S}}\right).$$

Before introducing matching $\mu^{ttc}$, in Definition 14, we need to formalize when students are said to be involved in a *minimal cycle*.

**Definition 13** *Let $\bar{\mathcal{S}}$ and $\widehat{\mathcal{S}}$ be two groups of students, with $\emptyset \neq \bar{\mathcal{S}} \subseteq \widehat{\mathcal{S}}$. We say that $\bar{\mathcal{S}}$ forms a* minimal cycle *on $\widehat{\mathcal{S}}$ according to $\{\succ_j^{\mathcal{P}}\}_{j=1}^m$ if, and only if,*

*(a)* $\mathcal{B}\left(\bar{\mathcal{S}};\widehat{\mathcal{S}}\right) = \bar{\mathcal{S}}$, *and*

*(b)* $\mathcal{B}\left(\mathcal{S}';\widehat{\mathcal{S}}\right) \neq \mathcal{S}'$, *for each non-empty $\mathcal{S}' \subsetneq \bar{\mathcal{S}}$.*

*Similarly, for a given set of students $\widehat{\mathcal{S}}$, we say that $s_j$ is a $\widehat{\mathcal{S}}$-cycle student if there exists $\mathcal{S}'$, $s_j \in \mathcal{S}' \subseteq \widehat{\mathcal{S}}$, such that $\mathcal{S}'$ forms a* minimal cycle *on $\widehat{\mathcal{S}}$ according to $\{\succ_j^{\mathcal{P}}\}_{j=1}^m$.*

**Definition 14** *Let $\mathcal{SAP}$ be a school allocation problem, and $\mu$ a matching for it. Let us assume that each student is endowed with a linear preorder $\succ_j^{\mathcal{P}}$ in $\mathcal{S}$, we define its* top trading cycle *matching, denoted by $\mu^{ttc}$, as the solution for the next iterative process.*

(**Step 1**) *Let $\mathcal{S}^{(C1)} \subseteq \mathcal{S}$ be the set of $\mathcal{S}$-cycle students. Then, for each $s_j \in \mathcal{S}^{(C1)}$ set $\mu^{ttc}(s_j) = \mu(\mathcal{B}(s_j;\mathcal{S}))$. If $\mathcal{S}^{(C1)} = \mathcal{S}$, matching $\mu^{ttc}$ is implemented and the process stops. Otherwise, let set $\mathcal{S}^1 = \mathcal{S} \setminus \mathcal{S}^{(C1)}$ and go to Step 2.*

(**Step** $t$) *Let $\mathcal{S}^{(Ct)} \subseteq \mathcal{S}^{t-1}$ be the set of $\mathcal{S}^{t-1}$-cycle students. Then, for each $s_j \in \mathcal{S}^{(Ct)}$ set $\mu^{ttc}(s_j) = \mu(\mathcal{B}(s_j; \mathcal{S}^t))$. If $\mathcal{S}^{(Ct)} = \mathcal{S}^t$, matching $\mu^{ttc}$, as described trough steps 1 to t, is implemented and the process stops. Otherwise, let set $\mathcal{S}^t = \mathcal{S}^{t-1} \setminus \mathcal{S}^{(Ct)}$ and go to Step $t + 1$.* ∎

## A. Towards Partial Efficiency, without Increasing Segregation

This subsection deals with policy $\mathcal{P}_1$, the first possible objective for the MSB, establishing that minimizing segregation is its main target. Therefore, once a matching $\mu$ is *socially supported* because a distribution parameter $\alpha$ has been agreed, any further modification that the MSB might propose should exhibit the segregation level that $\mu$ does. To reach this objective, a simple way to proceed is by restricting any alternative suggestion, say $\mu'$, by imposing that, for each school $c_i$,

$$|\mu(c_i) \cap \mathcal{F}| = |\mu'(c_i) \cap \mathcal{F}|, \text{ and } |\mu(c_i) \cap \mathcal{D}| = |\mu'(c_i) \cap \mathcal{D}|.$$

What we propose is to apply the next procedure. Given $\alpha$, let us associate each $\mathcal{SAP}$ with its multi-stage $\alpha$-fair matching. Next, given $\mu^{ms\alpha}$ and $\{E_j\}_{j=1}^m$, let us consider MSB-agreeing preferences for exchange satisfying that,[16] for each $s_j \in \mathcal{D}$, and any two students $s_h$ and $s_k$,

($a$) If $s_h \in \mathcal{F}$, then $s_j \succ_j^{\mathcal{P}_1} s_h$,

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[16] Note that, since we do not impose how students in $\mathcal{F}$ are compared under $\succ_j^{\mathcal{P}_1}$, our conditions do not fully determine a specific preorder. In fact it is not necessary for our purposes.

$(b)$ If $s_h$ and $s_k$ are in $\mathcal{D}$,

$(i)$ If $\mu^{ms\alpha}(s_h) \neq \mu^{ms\alpha}(s_k)$ then $s_h \succ_j^{\mathcal{P}_1} s_k$ if $\mu^{ms\alpha}(s_h) P_j \mu^{ms\alpha}(s_k)$;
and

$(ii)$ If $\mu^{ms\alpha}(s_h) = \mu^{ms\alpha}(s_k)$ then $s_h \succ_j^{\mathcal{P}_1} s_k$ whenever $s_h E_j s_k$.

The description for students in $\mathcal{F}$ is made in a similar way, just by exchanging the roles for $\mathcal{F}$ and $\mathcal{D}$.

The main properties satisfied by this procedure might be summarized as follows.

**Theorem 15** Given a school allocation problem, $\mathcal{SAP}$, a distribution parameter, $\alpha$, and $\{E_j\}_{j=1}^m$, the students' preferences for exchanging, let $\mu^{ms\alpha}$ and $\mu^{ttc_1}$ be the matchings obtained by applying the multi-stage $\alpha$-fair procedure and the top trading cycle algorithm, according to preferences $\{\succ_j^{\mathcal{P}_1}\}_{j=1}^m$, respectively. Then,

$(a)$ $\mu^{ttc_1}$ is $\mathcal{G}$-efficient;

$(b)$ for each school $c_i$, $|\mu^{ms\alpha}(c_i) \cap \mathcal{F}| = |\mu^{ttc_1}(c_i) \cap \mathcal{F}|$;

$(c)$ for each school $c_i$, $|\mu^{ms\alpha}(c_i) \cap \mathcal{D}| = |\mu^{ttc_1}(c_i) \cap \mathcal{D}|$; and

$(d)$ for each student $s_j$, $\mu^{ttc_1}(s_j) R_j \mu^{ms\alpha}(s_j)$.

Let us note that statements $(a)$ and $(d)$ inform us that $\mu^{ttc_1}$ is not only efficient from each group point of view, but also that it is obtained by Pareto improving matching $\mu^{ms\alpha}$. On the other hand, $(b)$ and $(c)$ establish that the transition from

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$\mu^{ms\alpha}$ to $\mu^{ttc_1}$ does not require any modification on the segregation level, measured from the distribution of $\mathcal{D}$-students among the schools.

## B. Towards Global Efficiency and the Level of Segregation

This subsection deals with policy $\mathcal{P}_2$, the second possible objective proposed for the MSB. It consists of obtaining an efficient allocation in which the level of segregation is as low as possible.

To reach our aim we proceed following a similar structure to the one shown in the previous subsection. Therefore, taking $\mu^{ms\alpha}$ as given, we propose a way to build MSB-agreeing preferences for exchanging that capture what policy $\mathcal{P}_2$ suggests. Then, once $\left\{\succ_j^{\mathcal{P}_2}\right\}_{j=1}^m$ has been specified, we apply Gale's *top trading cycle algorithm* to obtain what the MSB would propose, namely $\mu^{ttc_2}$.

Given student $s_j \in \mathcal{D}$, we define $\succ_j^{\mathcal{P}_2}$ as the preorder satisfying that, for each two students $s_h$ and $s_k$,

(a) If $\mu^{ms\alpha}(s_h) P_j \mu^{ms\alpha}(s_k)$ then $s_h \succ_j^{\mathcal{P}_2} s_k$; and

(b) Otherwise, i.e. if $\mu^{ms\alpha}(s_h) = \mu^{ms\alpha}(s_k)$, then

    (i) If $s_h \in \mathcal{D}$ and $s_k \in \mathcal{F}$, then $s_h \succ_j^{\mathcal{P}_2} s_k$;

    (ii) If both $s_h$ and $s_k$ belong to the same group, $\mathcal{D}$ or $\mathcal{F}$, and $s_h E_j s_k$, then $s_h \succ_j^{\mathcal{P}_2} s_k$.

For each student in $\mathcal{F}$, say $s_j$, preorder $\succ_j^{\mathcal{P}_2}$ is described in the same way as above, just exchanging the roles of $\mathcal{D}$ and $\mathcal{F}$.

Our first result in this framework deals with the *global stability* for the allocation suggested by the MSB.

**Theorem 16** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem, $\alpha$ a distribution parameter, and $\{E_j\}_{j=1}^m$ the students' preferences for exchanging. If $\mu^{ttc_2}$ is the matching that results from applying policy $\mathcal{P}_2$, then it is globally efficient, i.e. for any other matching $\mu' \neq \mu^{ttc_2}$ it should be $s_j \in \mathcal{S}$ such that $\mu^{ttc_2}(s_j) P_j \mu'(s_j)$.

We next deal with the analysis of situations where segregation can be a *natural* property. To reach our objective, we need to introduce some additional concepts.

**Definition 17** *Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. We say that $\mathcal{C}^{\mathcal{D}}$, $\mathcal{C}^{\mathcal{D}} \subset \mathcal{C}$, is a group of $\mathcal{D}$-schools if*

$(\mathcal{D}.a)$ *For each $s_j \in \mathcal{D}$, and any two schools $c_i \in \mathcal{C}^{\mathcal{D}}$ and $c_k \notin \mathcal{C}^{\mathcal{D}}$, $c_i P_j c_k$, and*

$(\mathcal{D}.b)$ *For each $s_h \in \mathcal{F}$, and any two schools $c_i \in \mathcal{C}^{\mathcal{D}}$ and $c_k \notin \mathcal{C}^{\mathcal{D}}$, $c_k P_h c_i$.*

*Similarly, we say that $\mathcal{C}^{\mathcal{F}}$, $\mathcal{C}^{\mathcal{F}} \subset \mathcal{C}$, is a group of $\mathcal{F}$-schools if*

$(\mathcal{F}.a)$ *For each $s_j \in \mathcal{F}$, and any two schools $c_i \in \mathcal{C}^{\mathcal{F}}$ and $c_k \notin \mathcal{C}^{\mathcal{F}}$, $c_i P_j c_k$, and*

$(\mathcal{F}.b)$ *For each $s_h \in \mathcal{D}$, and any two schools $c_i \in \mathcal{C}^{\mathcal{F}}$ and $c_k \notin \mathcal{C}^{\mathcal{F}}$, $c_k P_h c_i$.*

**Lemma 18** $[Segregation\ Lemma]$
Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. Let us assume that there is a group of $\mathcal{D}$-schools, say $\mathcal{C}^{\mathcal{D}}$, such that

$$\sum_{c_i \in \mathcal{C}^{\mathcal{D}}} q_i \geq d,$$

then for each distribution parameter $\alpha$ and any students' preferences for exchanging, $\{E_j\}_{j=1}^m$,

$$\mu^{ttc_2}(s_j) \in \mathcal{C}^{\mathcal{D}},$$

for each $\mathcal{D}$-student $s_j$.

What the above *Segregation Lemma* points out can be seen as a first approach to sufficient conditions guaranteeing that segregation is a *natural consequence* of student characteristics. Let us observe that the above result can also be established in terms of $\mathcal{F}$-students, related to conditions for a group of $\mathcal{F}$-schools, as assumed in Lemma 19.

**Lemma 19** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. Let us assume that there is a group of $\mathcal{F}$-schools, say $\mathcal{C}^{\mathcal{F}}$, such that

$$\sum_{c_i \in \mathcal{C}^{\mathcal{F}}} q_i \geq f,$$

then for each distribution parameter $\alpha$ and any student preferences for exchanging, $\{E_j\}_{j=1}^m$,

$$\mu^{ttc_2}(s_j) \in \mathcal{C}^{\mathcal{F}},$$

for each student $s_j \in \mathcal{F}$.

As a direct consequence of Theorem 16 and Lemma 18, we can find a suffi-

cient condition for segregation.

**Theorem 20** [*Segregation Theorem*]

Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. Let us assume that there is a group of $\mathcal{D}$-schools, say $\mathcal{C}^{\mathcal{D}}$, such that

$$\sum_{c_i \in \mathcal{C}^{\mathcal{D}}} q_i \geq d, \text{ and}$$

$$q - \sum_{c_i \in \mathcal{C}^{\mathcal{D}}} q_i \geq f,$$

then for each distribution parameter $\alpha$ and any student preferences for exchanging, $\{E_j\}_{j=1}^{m}$,

$$\mu^{ttc_2}(s_j) \in \mathcal{C}^{\mathcal{D}},$$

for each $\mathcal{D}$-student $s_j$, and

$$\mu^{ttc_2}(s_h) \notin \mathcal{C}^{\mathcal{D}}$$

for any $s_h \in \mathcal{F}$.

Let us observe that, as long as students in $\mathcal{D}$ and $\mathcal{F}$ are treated symmetrically, we can also establish a result similar to the *Segregation Theorem* 20, when conditions are imposed on a group of $\mathcal{F}$-schools.

**Theorem 21** Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. Let us assume that there is a group of $\mathcal{F}$-schools, say $\mathcal{C}^{\mathcal{F}}$, such that

$$\sum_{c_i \in \mathcal{C}^{\mathcal{F}}} q_i \geq f, \text{ and}$$

$$q - \sum_{c_i \in \mathcal{C}^{\mathcal{F}}} q_i \geq d,$$

then for each distribution parameter $\alpha$ and any student preferences for exchanging, $\{E_j\}_{j=1}^m$,

$$\mu^{ttc_2}(s_j) \in \mathcal{C}^{\mathcal{F}},$$

for each $\mathcal{F}$-student $s_j$, and

$$\mu^{ttc_2}(s_h) \notin \mathcal{C}^{\mathcal{F}}$$

for any $s_h \in \mathcal{D}$.

We now proceed to give some insight on a condition guaranteeing the existence of a unique school, to be called a *border school*, where integration occurs.

**Definition 22** *Let $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$ be a school allocation problem. We say that $c_i$ is a* border school *if there are two non-empty sets of schools, $\mathcal{S}^{\mathcal{F}}$ and $\mathcal{S}^{\mathcal{D}}$ such that*

($a$) $\mathcal{C}^{\mathcal{D}}$ is a group of $\mathcal{D}$-schools,

($b$) $\mathcal{C}^{\mathcal{F}}$ is a group of $\mathcal{F}$-schools, and

($c$) $\mathcal{C}^{\mathcal{F}} \cap \mathcal{C}^{\mathcal{D}} = \{c_i\}$

The next result provides sufficient conditions to have a unique school where both types of students are allocated.

**Theorem 23** Let $\mathcal{SAP}$ be a school allocation problem. Let us assume that $c_i$ is a border school satisfying

$$q^1 = \sum_{c_k \in \mathcal{C}^1} q_k < d < q_i + q^1, \text{ and } q^2 = \sum_{c_k \in \mathcal{C}^2} q_k < f < q_i + q^2,$$

where $\mathcal{C}^1$ (resp. $\mathcal{C}^2$) is the set of schools preferred by all the students in $\mathcal{D}$ (resp. $\mathcal{F}$), rather than $c_i$. Then, for each $\alpha$ and any $\{E_j\}_{j=1}^m$, the matching $\mu^{ttc_2}$ satisfies

($a$) For each $c_k \in \mathcal{C}^1$, $\mu(c_k) \cap \mathcal{F} = \emptyset$;

($b$) For each $c_k \in \mathcal{C}^2$, $\mu(c_k) \subseteq \mathcal{F}$; and

($c$) $|\mu(c_i) \cap \mathcal{F}| = f - q^2$, and $|\mu(c_i) \cap \mathcal{D}| = d - q^1$.

## VI.   On $\alpha$ and the Degree of External Fairness

The aim of this section is to provide some insight on the determination of a *reasonable range* for the value of $\alpha$. As a first approach, let us suppose that $q = f + d$.

In such a case it seems reasonable to select

$$\alpha = \frac{d}{f + d} \, ,$$

as, with no *historic* discrimination, it could be considered that students would be uniformly distributed around the city. Therefore, it might be expected that each district has the same proportion of $\mathcal{D}$-students, yielding our previous proposal.

Now, for a more general setting, when $q \geq f + d$, and taking into consideration that the dispersion of students among the districts is a consequence of previous discrimination practices, students belonging to each group would propose

$(a)$ From the point of view of $\mathcal{F}$-students, agents in $\mathcal{D}$ should have priority as regards, at most, $d$ places, therefore the value for $\alpha$ should be

$$\alpha_{\mathcal{F}} = \frac{d}{q} \, ,$$

$(b)$ From the $\mathcal{D}$ students viewpoint, there should be, at most, $f$ places in which students in $\mathcal{F}$ would have priority. Therefore, and in agreement with this opinion, the value for $\alpha$ would be

$$\alpha_{\mathcal{D}} = \frac{q - f}{q} \, ,$$

Therefore, provided that there is a conflict of interest on how the *extra-places* should be distributed between the students in both groups, it could be assumed

that each $\alpha \in [\alpha_{\mathcal{F}}, \alpha_{\mathcal{D}}]$ can somehow be justified.

When evoking segregation in a particular school, what is argued is that the ratio of $\mathcal{D}$-students in that school, related to the total number of accepted students, differs from an *ideal ratio*. In our opinion, there are some relevant variables that are not taken into account, such as the (potential) number of places, some of which may be unoccupied. This is why we propose distinguishing two types of segregation, one for each reason that causes an unbalanced ratio

$(a)$ $\mathcal{F}$-**segregation**. This is related to the case in which the number of places allocated to $\mathcal{F}$-students is too large and thus the $\mathcal{D}$ students cannot be integrated; and

$(b)$ $\mathcal{D}$-**segregation**. This occurs when the number of $\mathcal{D}$ students with a place is too large enough and prevents integration for agents in $\mathcal{F}$.

**Definition 24** *Let $\mathcal{SAP}$ be a school allocation problem, and $\mu$ a matching for such an instance. We say that $\mu$ exhibits*

$(i)$ *$\mathcal{F}$-segregation at school $c_i$ if*

$$q_i - |\mu(c_i) \cap \mathcal{F}| < [\alpha_{\mathcal{F}} q_i]^+ ;$$

$(ii)$ *$\mathcal{D}$-segregation at school $c_i$ if*

$$|\mu(c_i) \cap \mathcal{D}| > [\alpha_{\mathcal{D}} q_i]^+ ; \text{ and}$$

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$(iii)$ *Segregation at school $c_i$ if it exhibits $\mathcal{F}$-segregation or $\mathcal{D}$-segregation.*

In accordance with Definition 24, given a school allocation problem, we can describe the *degree of segregation* at school $c_i$, associated to matching $\mu$, as

$$\Delta\left(c_i;\mu\right) = \begin{cases} \frac{|\mu(c_i)\cap\mathcal{F}|-[(1-\alpha_\mathcal{F})q_i]^-}{q_i} & \text{if } |\mu\left(c_i\right)\cap\mathcal{F}| \geq [(1-\alpha_\mathcal{F})\,q_i]^- \\ \frac{|\mu(c_i)\cap\mathcal{D}|-[\alpha_\mathcal{D}q_i]^+}{q_i} & \text{if } |\mu\left(c_i\right)\cap\mathcal{D}| > [\alpha_\mathcal{D}q_i]^+ \\ 0 & \text{otherwise} \end{cases}$$

The considerations above are useful to introduce a way to measure the *degree of segregation* associated to a matching $\mu$,

**Definition 25** *Let $\mathcal{SAP}$ be a school allocation problem, and $\mu$ a matching for $\mathcal{SAP}$. We define the degree of segregation associated to $\mu$ as the expected segregation level for the set of schools,*

$$\Psi\left(\mu\right) = \sum_{i=1}^{n} \frac{q_i}{q}\Delta\left(c_i;\mu\right).$$

Our next result points out that the presence of a high segregation level can be a consequence of some of the following factors

(1) The selection of an inappropriate value for $\alpha$;

(2) The existence of a high number of *exceeding places*;

(3) The selected matching process.

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**Theorem 26** Let $\mathcal{SAP}$ be a school allocation problem, with $m = q$, and let us assume that there exists $\hat{\alpha}$ such that $\sum_{i=1}^{n} [\hat{\alpha} q_i]^+ = d$. Then, the matching obtained by applying the multi-stage $\hat{\alpha}$-fair procedure, satisfies

$$\Psi\left(\mu^{ms\hat{\alpha}}\right) = 0.$$

**Corollary 27** Let $\mathcal{SAP}$ be a school allocation problem, with $m = q$, and let us assume that there exists $\hat{\alpha}$ such that $\sum_{i=1}^{n} [\hat{\alpha} q_i]^+ = d$. Then, for any $\{E_j\}_{j=1}^{m}$, the matching that results by implementing Policy $\mathcal{P}_1$, for distribution parameter $\hat{\alpha}$, satisfies

$$\Psi\left(\mu^{ttc_1}\right) = 0.$$

To conclude this section, let us point out the *general difficulty* of guaranteeing the possibility of always reaching a null value for the *segregation level*. As it is usual in the (*real life*) description of school quotas, let us consider that there is a number of places associated to each classroom, which is the same for all schools. Let us assume that this number is 20 and, for the sake of simplicity, let us imagine that there are 150 classrooms equally distributed in 50 schools, i.e. there are 50 schools with a quota of $q_i = 60$ each. This means that 3000 new students can be accepted.

Now, let us assume that we have exactly 3000 new students so that, as assumed in Theorem 26, $m = q$, and they are distributed among the two categories where $d = 483$ and $f = 2517$, i.e. 16.1% of students are in $\mathcal{D}$ and the remaining 83.9%

belong to $\mathcal{F}$. In such a case, there should not be $\hat{\alpha}$ satisfying the condition assumed in Theorem 26. This is because the number of schools is even, and thus for each $\alpha$, $\sum_{i=1}^{n} [\alpha q_i]^+$ must be even too. Nevertheless, $d$ is odd.

Provided that we must treat all schools in a symmetrical way, we should reserve (at least) 10 places for $\mathcal{D}$-students in each school to give them the possibility of having a place each. Let us observe that, for $\alpha' = 0.161$, $[50\alpha']^+ = 10$. Now, to evaluate the maximum possible value of $\Psi$, let us assume that all the students show the *same* preferences, which satisfy

$$c_1 \ P_j \ c_2 \ \ldots \ P_j \ c_i \ P_j \ c_{i+1} \ \ldots P_j \ c_{50} \ P_j \ s_j$$

In such a case matching $\mu^{ms\alpha'}$ satisfies:

- $\left| \mu^{ms\alpha'}(c_i) \cap \mathcal{D} \right| = 10$ for each school $c_i \in \mathcal{C} \setminus \{c_{49}, c_{50}\}$;

- $\left| \mu^{ms\alpha'}(c_{49}) \cap \mathcal{D} \right| = 3$;

- $\left| \mu^{ms\alpha'}(c_{50}) \cap \mathcal{D} \right| = 0$; and

- $\left| \mu^{ms\alpha'}(c_i) \cap \mathcal{F} \right| = 60 - \left| \mu^{ms\alpha'}(c_i) \cap \mathcal{D} \right|$, for each school $c_i$.

Let us observe that this implies that $\Delta\left(c_i; \mu^{ms\alpha'}\right) = 0$ for each $c_i \notin \{c_{49}, c_{50}\}$; $\Delta\left(c_{49}; \mu^{ms\alpha'}\right) = 0.117$; and $\Delta\left(c_{50}; \mu^{ms\alpha'}\right) = 0.167$. Therefore, $\Psi(\mu') = 0.006$, which is the minimal value for $\Psi$ associated to this problem, whereas its maximal value is $0.272$.

## VII. Conclusions and Policy Suggestions

This paper aims to contribute to solving a *social debate*, namely the design of policies against segregation (or promoting integration). We concentrate on proposing procedures for the distribution of primary school places, helping to reduce the *observed* segregation level. This is the aim of the so-called *multi-stage $\alpha$-fair procedure*, introduced in Section IV. We then deal with the possibility of improvements, in terms of efficiency, for the above-mentioned allocation process. For this purpose, we consider two different frameworks, each of them associated with a different policy. The first policy deals with the case in which the level of integration cannot be waived. Therefore, the notion of efficiency should be adjusted to each group of students separately. For the second policy, the efficiency level should be global. Therefore, the (previously suggested) tentative matching allows the *disadvantaged students* a chance to obtain a better school place, in comparison with what they could obtain if the *integration process* was not implemented.

Finally, Section VI introduces a debate on how the integration level should be evaluated. For this, according to Aristotle's idea of *potency*, we consider that all the places that have not been assigned to the *favored students* can be distributed amongst the *disadvantaged* students as they wish. Therefore, in our opinion, the relevant variable to be considered for evaluating whether a particular matching procedure *induces segregation* is not the final outcome, based on the students' desires. For us, it is the possibilities that the procedure itself provides to the *disadvantaged students*, if they wished to be integrated, that should be taken into

account.

To conclude we wish to layout some specific recommendations, related to a question that the MSB should solve: How many places should be offered? One could be tempted to answer by suggesting that *as many as possible*. Note that, by setting $q_i = m$ for each school $c_i$, we can guarantee that each student obtains a place at her preferred school. Nevertheless, this option is neither possible, due the large number of students, nor reasonable, due to the degree of overcrowding it could induce. Imagine a small number of schools being the preferred option for all the students! On the contrary, our recommendation is highly related to avoiding an excessive offer of places, i.e. we propose to adjust, as far as possible, the number of offered places to the students' needs, thus equalizing school quotas,[17] so that for each school $c_i$,

$$q_i = \left\lceil \frac{m}{n} \right\rceil^+ \tag{1}$$

Then, since the number of excess places is low enough, when $\alpha$ is high enough and students do not act strategically, the outcomes for the $\alpha$-fair and the multi-stage $\alpha$-fair deferred-acceptance procedures should coincide. Therefore, given the above school quotas, let us choose $\alpha = \alpha_{\mathcal{D}}$ and apply the $\alpha_{\mathcal{D}}$-fair deferred-acceptance procedure. By Theorem 8 we know that students are expected to report their true preferences. And, Theorem 26 informs us that the matching so implemented

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[17]Our recommendation is made under the assumption that all the schools have the same number of classrooms for the new students. Otherwise, expression (1) should be reconsidered by replacing $n$ for the total number of classrooms. In such a case, the ratio proposed in expression (1) would reflect the quota for each classroom.

induces maximal integration, i.e. $\Psi\left(\mu^{\alpha_\mathcal{D}}\right)$ is close to zero. In a further step, the MSB might be interested in applying either of the policies $\mathcal{P}_1$ or $\mathcal{P}_2$. Note that, in such a case, the strategy-proof property might be lost. Moreover, if $\mathcal{P}_2$ is implemented, the search for a higher efficiency could induce a lower integration level, i.e. $\Psi\left(\mu^{ttc_2}\right)$ might be very high.

A further problem that has not been explored in this paper is related to the need for me MSB to implement policies $P_1$ or $P_2$. Note that when $q \simeq m$ the *multi-stage $\alpha$-fair* and the *$\alpha$-fair deferred-acceptance* procedures coincide, for $\alpha \in \Omega$. This implies that the multi-stage $\alpha$-fair procedure satisfies, in such a case, the properties proposed in theorems 8 and 11. In particular, students have no reason to misreport their preferences. The introduction of policies $P_1$ or $P_2$ may provide students with an incentive to play strategically, and thus strategy-proofness might be lost. A way to avoid such a problem can be found in how schools build their priority lists. In particular, and following Proposition 5 in Alcalde and Romero-Medina (2011a), we know that when schools rankings satisfy the so called *Common Priorities Condition*, there exists a unique matching which is stable and Pareto efficient. This allows policies for the MSB to be reconsidered by redesigning how school rankings may be built, in order to avoid the implementation of policy $P_1$ or $P_2$ to reach optimality.

### References

Abdulkadiroğlu, A., 2005. College Admissions with Affirmative Action. International Journal of Game Theory 33, 535–549.

Abdulkadiroğlu, A., Pathak, P.A., Roth, A.E., 2005a. The New York City High School Match. American Economic Review, Papers and Proceedings 95, 364–367.

Abdulkadiroğlu, A., Pathak, P.A., Roth, A.E., 2009. Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match. American Economic Review 99, 1954–1978.

Abdulkadiroğlu, A., Pathak, P.A., Roth, A.E., Sönmez, T., 2005b. The Boston Public School Match. American Economic Review, Papers and Proceedings 95, 368–371.

Abdulkadiroğlu, A., Pathak, P.A., Roth, A.E., Sönmez, T., 2006. Changing the Boston School Choice Mechanism: Strategy-Proofness as Equal Access. Mimeographed. Harvard University.

Abdulkadiroğlu, A., Sönmez, T., 2003. School Choice: A Mechanism Design Approach. American Economic Review 93, 729–747.

Alcalde, J., Romero-Medina, A., 2011a. On Stability and Efficiency in Scool Choice Problems. SSRN WP 1760592.

Alcalde, J., Romero-Medina, A., 2011b. Re-reforming the Bostonian System: A Novel Approach to the Schooling Problem. SSRN WP 1743082.

Alcalde-Unzu, J., Molis, E., 2011. Exchange of Indivisible Goods and Indifferences: The Top Trading Absorbing Sets Mechanisms. Games and Economic Behavior, forthcoming.

van Buren, P.M., 1995. A Theology of the Jewish-Christian Reality: A Christian Theology of the People Israel. University Press of America, Lanham, MD.

Curran, S.R., Shafer, S., Donato, K.M., Garip, F., 2006. Mapping Gender and Migration in Sociological Scholarship: Is It Segregation or Integration? International Migration Review 40, 199–223.

Dougherty, J., Harrelson, J., Maloney, L., Murphy, D., Smith, R., Snow, M., Zannoni, D., 2009. School Choice in Suburbia: Test Scores, Race, and Housing Markets. American Journal of Education 115, 523–548.

Ehlers, L., 2010. School Choice with Control. Mimeographed. Université de Montréal.

Ergin, H., Sönmez, T., 2006. Games of School Choice under the Boston Mechanism. Journal of Public Economics 90, 215 – 237.

Frankel, D.M., Volij, O., 2011. Measuring School Segregation. Journal of Economic Theory 146, 1–38.

Gajendragadkar, S.S., 2006. The Constitutionality of Racial Balancing in Charter Schools. Columbia Law Review 106, 144–181.

Gale, D., Shapley, L.S., 1962. College Admissions and the Stability of Marriage. American Mathematical Monthly 69, 9–15.

Garda, R.A., 2009. The White Interest in School Integration. Mimeographed. New Orleans College of Law, Loyola University.

Kane, T.J., Riegg, S.K., Staiger, D.O., 2006. School Quality, Neighborhoods and Housing Prices. American Law and Economics Review 8, 183–212.

Kesten, O., 2010. School Choice with Consent. Quarterly Journal of Economics 125, 1297–1348.

Kojima, F., 2010. School Choice: Impossibilities for Affirmative Action. Mimeographed. Stanford University.

Niederle, M., Roth, A.E., Sönmez, T., 2008. Matching and Market Design, in: Durlauf, S.N., Blume, L.E. (Eds.), The New Palgrave Dictionary of Economics. Palgrave Macmillan, Basingstoke.

Oliver, M., 1985. The Integration–Segregation Debate: Some Sociological Considerations. British Journal of Sociology of Education 6, 75–92.

Roth, A.E., 1982. The Economics of Matching: Stability and Incentives. Mathematics of Operations Research 7, 617–628.

Roth, A.E., Peranson, E., 1999. The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design. American Economic Review 89, 748–780.

Roth, A.E., Sotomayor, M.O., 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge University Press, New York.

Roth, A.E., Xing, X., 1994. Jumping the Gun: Imperfections and Institutions

Related to the Timing of Market Transactions. American Economic Review 84, 992–1044.

Seitles, M., 1998. The Perpetuation of Residential Racial Segregation in America: Historical Discrimination, Modern Forms of Exclusion, and Inclusionary Remedies. Journal of Land Use and Environmental Law 14, 1–38.

Shapley, L.S., Scarf, H., 1974. On Cores and Indivisibility. Journal of Mathematical Economics 1, 23–37.

# APPENDIX

## A. Proving Theorem 8

The aim of this appendix is to propose formal proof for Theorem 8. Let us remember that this statement informs about certain properties satisfied by the matching $\mu^\alpha$, obtained by applying the $\alpha$-fair deferred-acceptance procedure.

Let us observe that what this allocation procedure proposes is the following. First, given a school allocation problem, $\mathcal{SAP} = \{\mathcal{C}, \mathcal{S}, \mathbb{P}, Q\}$, let us associate to each school $c_i$ a number of *reserved places* for students in $\mathcal{D}$, say $d_i \leq q_i$, such that[18]

$$d \leq \sum_{i=1}^{n} d_i \tag{A1}$$

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[18]The assumption that $\alpha \in \Omega$ induces the next condition, which is sufficient for all the properties satisfied by $\mu^\alpha$ except the one related to $\alpha$-fairness.

Then, we can consider the school allocation problem whose elements are

$(i)$ The set of schools is $\mathcal{C}$, where the quota for school $c_i$ is $d_i$; and its priority list is $\Pi_i^{\mathcal{D}}$; and

$(ii)$ The set of students is $\mathcal{D}$, where preferences for student $s_j \in \mathcal{D}$ are $P_j$.

Let us compute the student optimal stable matching for such a problem. Then, when the quotas for schools are induced by some $\alpha$, as assumed in Theorem 8, for each student in $\mathcal{D}$, its mate at $\mu^\alpha$ coincides with the student optimal matching for the above problem.

Since for each student $s_j \in \mathcal{D}$, and any college $c_i$, $c_i P_j s_j$; each student is admissible at any school; and, by condition $(A1)$, the number of available places is not lower than the number of students, we have that

$(a)$ $\mu^\alpha$ is $\mathcal{D}$-stable;

$(b)$ For each student $s_j \in \mathcal{D}$, $\mu^\alpha(s_j) \in \mathcal{C}$; and

$(c)$ By Theorem 5 in Roth (1982), we have that truthful revelation is a dominant strategy for all students in $\mathcal{D}$.

The second step in the description of the $\alpha$-fair deferred-acceptance procedure lies in agents in $\mathcal{F}$. In this step, given the previously determined allocation for the $\mathcal{D}$-students, $\{\mu^\alpha(s_h)\}_{s_h \in \mathcal{D}}$, the school allocation problem to be considered is the following

$(i)$ The set of schools is $\mathcal{C}$, where the quota for school $c_i$, namely $q_i^{\mathcal{F}}$, is

$$q_i^{\mathcal{F}} = q_i - |\mu^\alpha(c_i) \cap \mathcal{D}|;$$

and its priority list is $\Pi_i^{\mathcal{F}}$.

$(ii)$ The set of students is $\mathcal{F}$, where preferences for student $s_j \in \mathcal{F}$ are $P_j$.

To complete the $\alpha$-fair deferred-acceptance procedure, we apply the student optimal stable mechanism to the previous problem. Let us observe that

$(a)$ $\mu^\alpha$ is $\mathcal{F}$-stable;

$(b)$ $\sum_{i=1}^n q_i^{\mathcal{F}} = q - d \geq f$. Moreover, students in $\mathcal{F}$ consider that any school is preferred to not being allocated a place; and all schools consider that all students are acceptable. Therefore, for each student $s_j \in \mathcal{F}$, $\mu^\alpha(s_j) \in \mathcal{C}$; and

$(c)$ By Theorem 5 in Roth (1982), we have that truthful revelation is a dominant strategy for all students in $\mathcal{F}$.

Now, let us observe that, for each $\alpha$, $\alpha \in \Omega$, and any school $c_i$,

$$|\mu^\alpha(c_i) \cap \mathcal{D}| \leq [\alpha q_i]^+,$$

which implies that $\mu^\alpha$ is $\alpha^{\mathcal{F}}$-fair. To conclude, let us assume that, for some $\mathcal{SAP}$ there is $\alpha$ yielding a matching $\mu^\alpha$ which is not $(1-\alpha)$-fair from $\mathcal{D}$'s point of view.

This implies that there is a student $s_j \in \mathcal{D}$ and school $c_i$ such that $c_i P_j \mu^\alpha (s_j)$ and

$$|\mu^\alpha (c_i) \cap \mathcal{F}| > [(1 - \alpha) q_i]^-$$

which is equivalent to

$$|\mu^\alpha (c_i) \cap \mathcal{D}| < [\alpha q_i]^+ ,$$

contradicting $\mathcal{D}$-stability of $\mu^\alpha$. ∎

## B.    Proving Theorem 11

This appendix proposes formal proof for Theorem 11. To reach our objective, we proceed as follows. Let $\mathcal{SAP}$ be a school allocation problem, and $\mu^{ms\alpha}$ the realization of the algorithm introduced in Definition 9. First, we see, in Lemma A1, that $\mu^{ms\alpha}$ is $\mathcal{G}$-stable. Secondly, Lemma A2 establishes the $\alpha$-fairness of $\mu^{ms\alpha}$. Finally, through Lemma A3, we show that the *optimality condition* (3) in Definition 10 is satisfied by $\mu^{ms\alpha}$.

Let us remember that $\mu^t$ denotes the (tentative) matching obtained at the end of the $t$-th step of the multi-stage $\alpha$-fair algorithm introduced in Definition 9.

**Lemma A1** Let $\mathcal{SAP}$ be a school allocation problem and, for $\alpha \in [0, 1]$ given, $\mu^{ms\alpha}$ its multi-stage $\alpha$-fair matching. Then $\mu^{ms\alpha}$ is $\mathcal{G}$-stable for $\mathcal{SAP}$.

**Proof**

Let us observe that, by Theorem 8, $\mu^1$ is $\alpha$-fair and $\mathcal{G}$-stable. Moreover, by construction, at each step $t$, $\mu^t$ is also $\mathcal{G}$-stable. Since there is some finite $\hat{t}$, at which $\mu^{\hat{t}} = \mu^{ms\alpha}$, the result follows. ∎

**Lemma A2** Let $\mathcal{SAP}$ be a school allocation problem and, for $\alpha \in [0, 1]$ given, $\mu^{ms\alpha}$ its multi-stage $\alpha$-fair matching. Then $\mu^{ms\alpha}$ satisfies $\alpha$-fairness.

**Proof**

To see that $\mu^{ms\alpha}$ is $\alpha$-fair, let us assume that, contrary to our conclusion, there is a school allocation problem, $\mathcal{SAP}$, and an $\alpha \in [0, 1]$ such that its multi-stage $\alpha$-fair matching, say $\mu^{ms\alpha}$, fails to be $\alpha$-fair. This implies that there is a school $c_i$ and student $s_j$ such that $c_i P_j \mu^{ms\alpha}(s_j)$ and

$(a)$ If $s_j \in \mathcal{D}$, then

$$|\mu^{ms\alpha}(c_i) \cap \mathcal{F}| > [(1 - \alpha) q_i]^- ; \text{ or}$$

$(b)$ If $s_j \in \mathcal{F}$, then

$$|\mu^{ms\alpha}(c_i) \cap \mathcal{D}| > [\alpha q_i]^+ .$$

We next show that none of the above situations could occur.

Let us assume that $s_j$ above is a student in $\mathcal{F}$. Then $\mu^1$ should satisfy that

$$|\mu^1(c_i) \cap \mathcal{D}| \le [\alpha q_i]^+$$

Therefore, there should be one step, say $t$, such that

$$\left| \mu^t (c_i) \cap \mathcal{F} \right| < q_i - \left| \mu^t (c_i) \cap \mathcal{D} \right|. \tag{A2}$$

Let $t^0$ be the lowest value of $t$ for which condition (A2) holds.

Since the algorithm is student monotonic, this contradicts $\mathcal{F}$-stability of $\mu^{t^0}$. This is because

(1) Since $c_i$ has a vacancy at stage $t^0$, this would imply that $\mu^{t^0} (s_j) P_j c_i$;

(2) By student monotonicity, $\mu^{ms\alpha} (s_j) R_j \mu^{t^0} (s_j)$; then,

(3) By transitivity, $\mu^{ms\alpha} (s_j) P_j c_i$.

Therefore, if $\mu^{ms\alpha}$ is not $\alpha$-fair, then $s_j$ should be a student in $\mathcal{D}$. Then, since

(1) $c_i P_j \mu^{ms\alpha} (s_j)$, by hypothesis, and

(2) $\mu^{ms\alpha} (s_j) R_j \mu^1 (s_j)$, by student monotonicity,

we have that transitivity implies that $c_i P_j \mu^1 (s_j)$. Therefore, since $\mu^1$ is $\mathcal{G}$-stable, we have that

$$\left| \mu^1 (c_i) \cap \mathcal{F} \right| \leq [(1 - \alpha) q_i]^-.$$

Note that, since $\mu^{ms\alpha}$ is not $\alpha$-fair, there should be some step, say $t$, in which

$$\left| \mu^t (c_i) \cap \mathcal{F} \right| > [(1 - \alpha) q_i]^-. \tag{A3}$$

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Let $t_0$ be the lowest value of $t$ for which condition (A3) holds.

This implies that, at stage $t_0$,

$$\left|\mu^{t_0}\left(c_i\right) \cap \mathcal{D}\right| < \left[\alpha q_i\right]^+.$$

Then, $\mathcal{G}$-stability of $\mu^{t_0}$ implies that $\mu^{t_0}\left(s_j\right) P_j c_i$. A contradiction. ∎

**Lemma A3** Let $\mu'$ be a $\mathcal{G}$-stable matching for $\mathcal{SAP}$, $\mu' \neq \mu^{ms\alpha}$, such that for all $s_j \in \mathcal{S}$, $\mu'\left(s_j\right) R_j \mu^{ms\alpha}\left(s_j\right)$. Then, there exists $c_i \in \mathcal{C}$ such that

$$\left|\mu'\left(c_i\right) \cap \mathcal{D}\right| < \left|\mu^{ms\alpha}\left(c_i\right) \cap \mathcal{D}\right|. \tag{A4}$$

**Proof**

Let $\mu'$ be a matching satisfying the assumptions of Lemma A3. Let us assume that condition (A4) does not hold. Then, for each school $c_i$

$$\left|\mu'\left(c_i\right) \cap \mathcal{D}\right| \geq \left|\mu^{ms\alpha}\left(c_i\right) \cap \mathcal{D}\right|. \tag{A5}$$

Since each student $s_j$ will be allocated a place at some school, we have that

$$d = \sum_{c_i \in \mathcal{S}} \left|\mu^{ms\alpha}\left(c_i\right) \cap \mathcal{D}\right| \leq \sum_{c_i \in \mathcal{S}} \left|\mu'\left(c_i\right) \cap \mathcal{D}\right|. \tag{A6}$$

Note that (A5) and (A6) together imply that, for each $c_i$

$$\left|\mu'\left(c_i\right) \cap \mathcal{D}\right| = \left|\mu^{ms\alpha}\left(c_i\right) \cap \mathcal{D}\right|. \tag{A7}$$

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Now, let $\hat{t}$ be the last step in the multi-sequential $\alpha$-fair deferred-acceptance algorithm, i.e. $\hat{t}$ is such that $\mu^{\hat{t}} = \mu^{ms\alpha}$. In sub-step $(\hat{t}.a)$, when analyzing the problem $\left\{ \mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}(\hat{t})} \right\}$, we have that, for each school $c_i$,

$$ q_i^{\mathcal{D}(\hat{t})} \geq |\mu^{ms\alpha}(c_i) \cap \mathcal{D}| = |\mu'(c_i) \cap \mathcal{D}|. $$

Note that $\mathcal{G}$-stability of both $\mu^{ms\alpha}$ and $\mu'$ implies stability of $[\mu^{ms\alpha}]^{\mathcal{D}}$ and $[\mu']^{\mathcal{D}}$, respectively, for $\left\{ \mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}(\hat{t})} \right\}$.

Given that $[\mu^{ms\alpha}]^{\mathcal{D}}$ is the student-optimal stable matching for the problem $\left\{ \mathcal{C}, \mathcal{D}, \mathbb{P}^{\mathcal{D}}, Q^{\mathcal{D}(\hat{t})} \right\}$, and $[\mu']^{\mathcal{D}}$ Pareto dominates $[\mu^{ms\alpha}]^{\mathcal{D}}$, following Theorem 2 in Roth (1982), we should conclude that $[\mu^{ms\alpha}]^{\mathcal{D}}$ is $\mathcal{D}$-unstable. A contradiction. $\blacksquare$

## C.  Proving Results in Section V

The aim of this appendix is to provide formal proof for each of the main results in Section V. To reach our objective, we will highlight the close relationship between the processes described in this section, and ways to solve *housing problems*, introduced by Shapley and Scarf (1974). The situation described in these problems can be introduced as follows. A group of individuals, each owning an indivisible object, face the possibility of reallocating the objects according to their own interests. The primitives for such problems are

- A set of $m$ agents, $A = \{1, \ldots, j, \ldots, m\}$;

- A set of $m$ objects, $O = \{o_1, \ldots, o_j, \ldots, o_m\}$,[19] and

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[19]In this description object $o_j$ is usually interpreted as the *initial endowment* of agent $j$.

- For each agent, say $j$, a transitive and complete binary relation on $O$, $\succsim_j$. This is interpreted as $j$'s preferences on the set of objects.

According to our model, given a $\mathcal{SAP}$ and a matching $\mu$, $\mathcal{S}$ can represent the set of agents. Therefore, the set of objects can be identified with the set of *personalized school places*; i.e. for each student $s_j$, $o_j = \mu(s_j)$. The element which is still *momentarily imprecise* is how to describe students' preferences. In general, for a given student, say $s_j$, $\succsim_j$ might depend eventually on four factors, namely $(1)$ student preferences, $P_j$; $(2)$ the initial matching, $\mu$; $(3)$ student preferences for exchanging, $E_j$; and $(4)$ the policy that the MSB wish to implement.

**Proof of Theorem 15**

Let us observe that, when policy $\mathcal{P}_1$ is implemented, the MSB-agreeing preferences for exchange imply that, for any two students, $s_j \in \mathcal{D}$ and $s_h \in \mathcal{F}$, and any initial matching $\mu$,

$$\mu(s_j) \succ_j \mu(s_h), \text{ and } \mu(s_h) \succ_h \mu(s_j) \tag{A8}$$

Taking into account Alcalde-Unzu and Molis (2011), Theorem 4, we have that $\mu^{ttc_1}$ should be efficient regarding preferences $\left\{ \succ_j^{\mathcal{P}_1} \right\}_{j=1}^m$. Let us observe that efficiency and condition (A8) imply that

$(a)$ All the exchanges are intra-group; i.e., for each student $s_j$, if when applying the *top trading cycle* algorithm, $s_j \in \mathcal{S}^{(Ct)}$, then

$$\mathcal{B}\left(s_j; \mathcal{S}^t\right) \in \mathcal{D} \text{ if, and only if } s_j \in \mathcal{D};$$

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which implies that for each school $c_i$,

$$\left| \mu^{ms\alpha}(c_i) \cap \mathcal{D} \right| = \left| \mu^{ttc_1}(c_i) \cap \mathcal{D} \right|,$$

and

$$\left| \mu^{ms\alpha}(c_i) \cap \mathcal{F} \right| = \left| \mu^{ttc_1}(c_i) \cap \mathcal{F} \right|.$$

(b) $\left[ \mu^{ttc_1} \right]^{\mathcal{D}}$ is efficient for $\mathcal{SAP}^{\mathcal{D}}$;

(c) $\left[ \mu^{ttc_1} \right]^{\mathcal{F}}$ is efficient for $\mathcal{SAP}^{\mathcal{F}}$.

To conclude, let us assume that, for student $s_j$, $\mu^{ttc_1}(s_j)$ is determined at stage $t$. This implies that $\mu^{ttc_1}(s_j) = \mu^{ms\alpha}(\mathcal{B}(s_j; \mathcal{S}^t))$. Since $s_j \in \mathcal{S}^t$, we have that $\mu^{ttc_1}(s_j) R_i \mu^{ms\alpha}(s_j)$. ∎

**Proof of Theorem 16**

Let us observe that implementation of policy $\mathcal{P}_1$ by the MSB implies that the reallocation of places for the two groups of students might be considered as an *isolated problem*, as illustrated in condition (A8). Nevertheless, when policy $\mathcal{P}_2$ is implemented, it is not necessarily the case. Therefore, our result follows on from Theorem 4 in Alcalde-Unzu and Molis (2011). ∎

**Proof of Lemma 18**

Let us assume that there is a group of $\mathcal{D}$-schools, say $\mathcal{C}^{\mathcal{D}}$ satisfying

$$\sum_{c_i \in \mathcal{C}^{\mathcal{D}}} q_i \geq d.$$

Since $\mu^{ms\alpha}$ is $\mathcal{G}$-stable, as shown in Theorem 11, we have that, for each $s_j \in \mathcal{D}$, if $\mu^{ms\alpha}(s_j) \notin \mathcal{C}^{\mathcal{D}}$, then it should be the case that $|\mu^{ms\alpha}(c_i)| = q_i$ for each $c_i \in \mathcal{C}^{\mathcal{D}}$. Moreover, by Theorem 16 we know that $\mu^{ttc_2}$ must be efficient. This implies that $\mu(s_j) \in \mathcal{C}^{\mathcal{D}}$, for each $s_j \in \mathcal{D}$, yielding the desired result. ∎

**Proof of Theorem 20**

Note that, under our assumptions, by Lemma 18 we have that the students in $\mathcal{D}$ should get a place at a $\mathcal{D}$-school. Moreover, since

$$f \leq q - \sum_{c_i \in \mathcal{C}^{\mathcal{D}}} q_i,$$

we have that $\mathcal{C} \setminus \mathcal{C}^{\mathcal{D}}$ is a set of $\mathcal{F}$-schools with enough places to allocate to all the $\mathcal{F}$-students. Therefore, since $\mu^{ttc_2}$ has to be efficient, we have that

$$\mu^{ttc_2}(s_j) \in \mathcal{C} \setminus \mathcal{C}^{\mathcal{D}};$$

which is the desired result. ∎

**Proof of Theorem 23**

Let us observe that $\mathcal{C}^1 \cup \{c_i\}$ is a group of $\mathcal{D}$-schools. Therefore, by Lemma 18 we have that no $\mathcal{D}$-student would occupy a place in a school being in $\mathcal{C}^2$. Similarly,

we can see that $\mathcal{C}^2 \cup \{c_i\}$ constitutes a group of $\mathcal{F}$-schools. Thus, by Lemma 19, we have that $\mu^{ttc_2}(c_k) \subseteq \mathcal{F}$, for each $c_k \in \mathcal{C}^2 \cup \{c_i\}$.

On the other hand, since

$$\sum_{c_k \in \mathcal{C}^1} q_k < d < q_i + \sum_{c_k \in \mathcal{C}^1} q_k,$$

and $\mu^{ttc_2}$ is efficient, we have that for each school $c_k \in \mathcal{C}^1$,

$$\left| \mu^{ttc_2}(c_k) \cap \mathcal{D} \right| = c_k,$$

which implies that

$$\left| \mu^{ttc_2}(c_i) \cap \mathcal{D} \right| = d - \sum_{c_k \in \mathcal{C}^1} q_k.$$

A similar reasoning for students in $\mathcal{F}$, related to colleges belonging to $\mathcal{C}^2$, informs us that the number of $\mathcal{F}$-students getting a place at $c_i$ is $f - q^2$. ∎

## D. Proving Theorem 26

To prove Theorem 26, let us observe that the existence of $\hat{\alpha}$ such that

$$\sum_{i=1}^{n} [\hat{\alpha} q_i]^+ = d, \tag{A9}$$

and the fact that $m = q$ imply that $\mu^{ms\hat{\alpha}} = \mu^{\hat{\alpha}}$, i.e. the multi-stage $\hat{\alpha}$-fair algorithm stops at its first step. This implies that, for each school, say $c_i$,

$$\left| \mu^{ms\hat{\alpha}} (c_i) \cap \mathcal{D} \right| = [\hat{\alpha} q_i]^+ .$$

Moreover, since $m = q$, we have that, for each school, say $c_i$, $\left| \mu^{ms\hat{\alpha}} (c_i) \right| = q_i$. Therefore

$$q_i - \left| \mu^{ms\hat{\alpha}} (c_i) \cap \mathcal{F} \right| = \left| \mu^{ms\hat{\alpha}} (c_i) \cap \mathcal{D} \right| = [\hat{\alpha} q_i]^+ ;$$

which is equivalent to establishing

$$\left| \mu^{ms\hat{\alpha}} (c_i) \cap \mathcal{F} \right| = [(1 - \hat{\alpha}) q_i]^- ,$$

which implies that, for each $c_i \in \mathcal{C}$, $\Delta \left( c_i; \mu^{ms\hat{\alpha}} \right) = 0$. ∎