# Not so cheap talk: Costly and discrete communication

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**Abstract**

We model an interaction between an informed sender and an uninformed receiver. As in the classic cheap talk setup, the informed player sends a message to an uninformed receiver who is to take an action which affects the payoffs of both players. However, in our model the sender can communicate only through the use of discrete messages. The messages are ordered by the cost incurred by the sender upon its transmission. We characterize the resulting equilibria under a weak out-of-equilibrium condition. We apply the stronger no incentive to seperate ($NITS$) condition to our model. We show that if the sender and receiver have aligned preferences regarding the action of the receiver then $NITS$ only admits the most informative equilibrium. When the preferences between players are not aligned, we show that $NITS$ does not guarantee uniqueness and we provide an example where an increase in communication costs can improve communication. As we show this improvement can occur to such an extent that an equilibrium outperforms the Goltsman et. al. (2009) upper bound for payoffs in mediated communication.

Keywords: information transmission, cheap talk, equilibrium selection
JEL: C72, D82, D83

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## 1 Introduction

A person will often use words to convey information about a complex and nuanced reality. Words are discrete objects in the sense that their properties are very different from that of real numbers. In this paper, we take the view that communication is necessarily discrete. We analyze the implications of such discrete communication in a strategic interaction between an informed sender and an uninformed receiver. In our model, the sender learns the state of the world on the unit interval and transmits a discrete message to the receiver. In particular, the sender incurs a cost in the transmission of the message. After observing the message, the receiver is to take an action which affects the payoffs of both sender and receiver.

We first consider a condition in which an out-of-equilibrium message does not induce an action which is not induced in equilibrium. We refer to this condition as *no new actions* ($NNA$). This condition roughly states that if a receiver ever observes an out-of-equilibrium message then the receiver believes that, among the messages sent with positive probability, that states are those associated with the most costly message. We characterize the equilibrium under $NNA$. One advantage of employing $NNA$ is that the multiplicity of equilibrium is analogous to the multiplicity of the original cheap talk paper of Crawford and Sobel (1982) (hereafter referred to as CS). Another advantage of employing $NNA$, is that the multiplicity is helpful in demonstrating the utility of our subsequent equilibrium refinement.

We also employ the *no incentive to separate* ($NITS$) condition of Chen et. al. (2008). This condition roughly states that if the receiver ever observes an out-of-equilibrium message then the receiver believes that the state is 0. We show that under $NITS$, if there is perfect alignment between the preferences regarding the receiver's action then the equilibria is the one most preferred by the receiver: the state space is partitioned into the largest number of possible elements. This result is analogous to that of Chen et. al. (2008) when $NITS$ is applied to the original cheap talk model.

If preferences regarding the receiver's action are not aligned, we show that $NITS$ does not guarantee a unique equilibrium and we show that an increase in communication costs improve communication. We also show that when preferences are not aligned there exists an equilibrium in which the receiver's payoffs outperform the Goltsman et. al. (2009) upper bound for payoffs in such communication games.

The balance of the paper is organized as follows. In Section 2, we review the related literature. In Section 3, we introduce the model and in Section 4, we offer some preliminary analysis. In Section 5, we characterize the equilibrium under $NNA$ and discuss the merits of doing so. In Section 6, we characterize the equilibrium under $NITS$ where there is perfect alignment of preferences and in Section 7, we examine $NITS$ in the case of imperfect alignment. In Section 8, we discuss the modeling choices specific to this paper and in Section 9, we conclude. In the appendix, we offer the proofs which were not presented in the body of the paper. Further, we present an example where there does not exist an equilibrium under an alternate, and arguably more reasonable, specification of $NITS$ and we offer a proof that an equilibrium under $NITS$ implies that it is also an equilibrium under $NNA$.

## 2 Related Literature

### 2.1 Cheap Talk and Related Models

The large strand of cheap talk literature was initiated by CS. The authors show that for mild differences in the preferences of receiver and sender, meaningful, albeit incomplete, communication can occur. CS shows that equilibrium always takes the form that the state space is partitioned and the messages are sent such that a unique action is induced within each element of the partition. Our equilibrium is analogous in that a unique action is induced on each partition element.

A number of papers have extended the original CS model. For instance, Morgan and Stocken (2003) extend the CS model to the case where there is uncertainty regarding the degree of divergence between the preferences of the sender and receiver. Fischer and Stocken (2001) model a situation where the sender has imperfect information about the state. Blume et. al. (2007) modify the CS setup where communication errors (or noise) can occur. In our view, the present paper shares the goal of these papers: to learn the significance of a particular assumption in the CS model. Here we seek to learn the importance of the assumption that messages are plentiful and equally costless.

Blume et. al. (2007) demonstrated that a small amount of noise can improve communication in the CS model. In particular, the authors show that there is an optimal amount of noise which maximizes the receiver's payoffs. Relatedly, Goltsman et. al. (2009) study general communication in the CS model. The authors study mediation, whereby a neutral third party (or mediator) will advise both of the players. Goltsman et. al. (2009) find that the payoffs in the equilibrium with the optimal amount of noise found by Blume et. al. (2007) is the the upper bound of payoffs for the receiver in any mediated communication in the CS setting. We provide an example of an equilibrium in which communication costs imply that the receiver can attain a payoff above this upper bound.

The original CS model exhibits a large number of possible equilibria. Specifically, CS shows that for a given difference in the preferences of the sender and receiver, if there is an equilibrium where the state space is partitioned into a finite number of partitions (say $n$) then there are equilibria which partition the state space into 1, 2, .., and $n-1$ elements. Like CS, our model under $NNA$ exhibits an analogous multiplicity.

As is often the case for multiple equilibria, researchers have sought to reduce the number of cheap talk equilibria through refinements.[1] A recent innovation in this regard is the *no incentive to separate* ($NITS$) condition of Chen et. al. (2008). $NITS$ restricts attention to equilibria in which it is not the case that the sender at the state $s = 0$ (with a state space of $[0, 1]$) prefers to perfectly reveal the state. In their Proposition 3, the authors show that if the monotonicity condition[2] holds in the CS model (as it does in the commonly used "uniform-quadratic" case) then $NITS$ selects a unique equilibrium which is the most informative, i.e. contains the largest possible number of partitions. We present a similar result, when there are

[1] For instance, see Banks and Sobel (1987), Cho and Kreps (1987), Farrell (1993), Kohlberg and Mertens (1987), Matthews et. al. (1991).

[2] In the literature, this is commonly refered to as Condition $M$.

communication costs and the preferences are aligned, $NITS$ admits only the most informative equilibria. We also show that when preferences are not aligned and there are communication costs, $NITS$ does not guarantee uniqueness.

## 2.2 Costly Communication

We are not the first to introduce costly communication into the CS model. Austen-Smith and Banks (2000) and Kartik (2007) investigate the effect of including both costly and costless messages in the original CS model.[3] In other words, these burning money papers ask, what happens if we include the option of sending costly messages, in addition to the cheap, plentiful messages of the CS model. By contrast, in our paper the message space is not uncountably infinite, but countably infinite. Further, due to their cost only a finite number of messages will be sent in equilibrium. In other words we ask, what happens if messages are relatively scarce and increasingly costly to send. The burning money papers find that the inclusion of these additional costly messages can expand the set of equilibria and that there can be regions of full separation. By contrast, we never find full separation and in general the presence of communication costs reduces the informativeness of communication. Given that communication costs in our model tends to degrade communication and that our setup is different from the burning money literature, it is rather surprising that we have identified equilibria which outperform the Goltsman et. al. (2009) upper bound. While we share the feature that we allow messages which cost the sender, roughly we investigate the case of a smaller message space rather than the burning money literature which investigates the case of a larger message space.

In Dewatripont and Tirole (2005) the sender incurs costs of effectively communicating information and the receiver incurs costs in absorbing information. In Dewatripont and Tirole, information is either understood or not.[4] By contrast, the states in our model are better characterized by the degree to which they are understood. Additionally, in Dewatripont and Tirole the sender and receiver necessarily have different preferences over the action of the receiver. By contrast, we examine both the cases where they are aligned and are unaligned.[5] Lastly, in our model the communication costs are exclusively incurred by the sender. We focus on this case for the following reasons. When communication is discrete, it is not obvious how to best model the cost associated with absorbing and processing messages. Even if a suitable formulation could be found, a higher cost incurred by the receiver would presumably induce a lower correlation between the state and the action. We suspect there exists a profile of communication costs borne exclusively by the sender which would yield an identical distribution of actions as in the model in which both sender and receiver incur communication costs.

[3] The cost of these messages are unrelated to the unknown state of the world. See Spence (1973) for the classic model of the case where the cost of transmitting a signal varies with the underlying state of the world. Also see Gossner et. al. (2006). Kartik et. al. (2007) investigate a model of costly lying, credulous receivers, and show that a separating equilibrium can emerge.

[4] See Austen-Smith (1994) for another costly communication paper in which information is either understood or not.

[5] Also note that we are not the first to model communication between a sender and receiver who have identical preferences over the receiver's action. For instance, Morris (2001) presents such a model in which, due to reputation effects, the sender might not truthfully reveal the state of the world. Also see Blume et. al. (2007), Blume and Board (2010b), Che and Kartik (2009), Cremer et. al. (2007) and Jager et. al. (2009).

To our knowledge, there are two costly communication papers in which there are shades of understanding. In Calvo-Armengol et. al. (2009) the sender transmits a necessarily noisy signal but can affect its precision by a incurring larger communication cost. In our view, this assumption is less appropriate when modeling discrete communication as it is not obvious how to model noise when messages are discrete. In Cremer et. al. (2007) a fixed number of partition elements are optimally arranged in order to minimize communication problems between an informed sender and an uninformed receiver who have identical preferences over the action of the receiver. Like Cremer et. al., we find that the equilibrium mapping from the state space to the message space is not uniform. In Cremer et. al. this is due to the differential likelihood of events and in our paper it is due to the differential cost of the available messages. Also note that in Cremer et. al. the size of the language is exogenously given however in our model the size of the language endogenously emerges due to the costs of communication.

# 3 Model

A sender ($S$) and receiver ($R$) play a communication game in a single period. Payoffs for both players depend on the receiver's action $a$, as well as the state of the world $s$. The state is an element of the closed interval $[0,1]$. The receiver's action space is $\mathbb{R}$. The receiver's utility is:

$$u^R(a,s) = -(a-s)^2.$$

The receiver has ex-ante beliefs that the state is uniformly distributed on $[0,1]$. The sender observes the state and can communicate some information about the state to $R$ by sending a message $m$ where $m \in \mathcal{M}$. Associated with each message $m^i$, there is a cost $c(i)$ which the sender incurs when it is transmitted. The cost of communication ($c : \mathbb{N} \Rightarrow \mathbb{R}$) is an increasing function of its index.[6] Further, we require that $c(i+1) - c(i) \geq \psi > 0$. We also assume that $c(0) = 0$. In a slight abuse of notation, we will refer to the case described above as $c > 0$, the case where there is no communication costs as $c = 0$ and the case of both communication costs and the absence of communication cost as $c \geq 0$.[7] The sender's utility is:

$$u^S(a, m^i, s) = -(a - s - b)^2 - c(i)$$

where $b \geq 0$.

The sender's strategy is $\mu : [0,1] \to \Delta\mathcal{M}$ and the receiver's strategy is $\alpha : \mathcal{M} \to \Delta\mathbb{R}$. We seek an equilibrium $(\mu^*, \alpha^*)$ such that $S$ chooses the optimal action, $R$ chooses the optimal action and $R$'s beliefs are derived from Bayes' Rule wherever possible. We denote $R$'s beliefs as $\beta(s|m)$.

**Definition 1** *For an equilibrium $(\mu^*, \alpha^*)$ we require:*

$$\textit{for each } s \in [0,1],\ m = \underset{m'}{argmax}\ u^S(\alpha^*(m'), m', s)$$

$$\textit{for each } m \in \mathcal{M},\ \alpha(m) = \underset{a'}{argmax} \int u^R(a', s)\beta(s|m)ds$$

[6] See Vartiainen (2009) for a related notion of communication costs.

[7] Of course, since there is no outside option, adding any amount to the function $c$ would not affect our results. We assume that $c(0) = 0$ in order to render meaningful our notation of $c = 0$ and $c > 0$.

*and that R's beliefs are derived from S's strategy using Bayes' Rule.*

# 4 Preliminaries

Before we offer a characterization of the equilibria, we introduce some notation and provide a necessary condition.

Although our equilibria share some of the familiar characteristics of the cheap talk literature, the additional results which emerge will require the flexibility provided by the notation which we now define. Like the CS equilibria, messages are sent on connected, nonoverlapping intervals.[8] Therefore, we may characterize an equilibrium by a set of cutoff states where we denote the number of messages used in equilibrium as $n$ by listing the order of the messages messages $m_1$, ..., $m_n$. The messages induce a set of cutoff states which we denote:

$$0 = s_1 \leq s_2 \leq ... \leq s_h \leq ... \leq s_n \leq 1 = s_{n+1}.$$

Equilibrium is such that $S$'s messages are sent on intervals of the state space:

$$m_h = \mu^*(s) \text{ for } s \in [s_h, s_{h+1})$$

and $R$ best responds in a straightforward manner:

$$\alpha^*(m_h) = \overline{a}(s_h, s_{h+1}) = \underset{a'}{\operatorname{argmax}} \int_h^{h+1} u^R(a', s)\beta(s|m_h)ds \tag{1}$$

where $\overline{a}(s_h, s_{h+1})$ is the best response of $R$ if the state is known to be between $s_h$ and $s_{h+1}$ and therefore:

$$\overline{a}(s_h, s_{h+1}) = \frac{s_h + s_{h+1}}{2}.$$

Definition 1 implies the arbitrage equation, also standard in the cheap talk literature. This expression characterizes the equilibrium set of cutoff states:

$$u^S(\overline{a}(s_h, s_{h+1}), m_h, s) = u^S(\overline{a}(s_{h+1}, s_{h+2}), m_{h+1}, s) \text{ for } h \in \{1, ..., n-1\}. \tag{2}$$

We define $\lambda_h$ to be the mass of states such that $m_h = \mu(s)$. Since the messages are sent on an interval of the state space and the states are distributed uniformly, $\lambda_h = s_{h+1} - s_h$ when $m_h = \mu(s)$ for $s \in [s_h, s_{h+1})$ and $m_h \neq \mu(s)$ for $s \notin [s_h, s_{h+1})$.

While subscripts refer to the order of the messages, we use superscripts to denote the cost index of the message. Therefore, we denote the lowest cost message as $m^0$, the next costly message as $m^1$ and so on. Correspondingly, we define $\lambda^j$ to be the mass of states associated with the message which has cost index $j$. An an equilibrium in which there are $n$ actions

[8] See the appendix for the proof of Lemma 6 which shows that the equilibrium strategy for $S$ entails sending a message for states which are conntected intervals and Lemma 7 which shows that the intervals are not partially overlapping.

induced will obviously require that:

$$\lambda_1 + \lambda_2 + ... + \lambda_n = 1 \text{ where } \lambda_h \geq 0 \text{ for every } h \in \{1, ..., n\} \tag{3}$$

We now provide a necessary condition for an equilibrium. Lemma 1 describes the relative the size of two adjacent intervals. As in the CS model, for $b > 0$ the interval size is increasing in its location on the state space. In other words, for $b > 0$ and $c = 0$, the intervals representing larger numbers on the state space are larger than intervals representing smaller numbers. The lemma also shows that the size of the interval is decreasing in the cost of the signal transmitted on that interval.

**Lemma 1** *For any equilibrium $(\mu^*, \alpha^*)$ where $b \geq 0$ and $c \geq 0$ in which there are $n$ actions induced, it must be that:*

$$\left(\lambda_{h+1}^j\right)^2 - \left(\lambda_h^i\right)^2 = 4b\left[\lambda_{h+1}^j + \lambda_h^i\right] + 4\left[c(i) - c(j)\right] \text{ for } h \in \{1, ..., n-1\}. \tag{4}$$

Note that more costly signals are conserved: the cost of a signal is negatively related to the size of the state space on which it is transmitted. Also note that when $b > 0$ and $c = 0$, we are essentially in the CS model because expression (4) easily reduces to expression (21) in CS. Therefore, when $b > 0$ and $c = 0$ the intuition behind the relationship between the interval size and its location on the state space is identical to that in CS. Also note that Lemma 1 together with the restriction that messages have a unique cost, implies that there are no equilibrium mixed strategies for the sender.

## 5 Equilibrium Characterization under $NNA$

In this section we characterize the equilibrium under the no new actions ($NNA$) condition. Under this mild restriction, we show that there will be a great deal of equilibria. From this viewpoint, we can best glean the insights from the model and highlight the utility of our subsequent refinement.

The $NNA$ condition specifies that, upon observing an out-of-equilibrium message, $R$ seeks to infer its origin among the messages which are sent with some probability. Among these candidate deviations, $R$ reasons that the deviation must have come from the states which are incurring the highest communication cost in the transmission of the message.

**No new actions:** Given a strategy pair $(\mu, \alpha)$, if $R$ observes $\widetilde{m}$ where $\widetilde{m} \notin \mu([0,1])$ then among the messages $\{m^i | m^i \in \mu([0,1])\}$, $R$ believes that the states are those associated with the most costly message $m^j$ so that $\beta([s_h^j, s_{h+1}] | \widetilde{m}) = 1$.

As the name of the condition suggests, under $NNA$ an out-of-equilibruim message does not induce an action $a$ which is not induced in equilibrium: there are no new actions induced by an out-of-equilibruim message.

We now characterize the equilibria under $NNA$. Our first result is that in equilibrium there will be no unused cheap messages. As a consequence of this, we will say that there will be *no holes* in equilibrium. Further, for any positive communication costs there will exist an uninformative equilibrium where the least costly message is sent on all states.

**Lemma 2** *Consider an equilibrium $(\mu^*, \alpha^*)$ in which $m^i$ which is transmitted. Under $NNA$ then every $m^j$ where $c(j) < c(i)$ is also used in equilibrium. Also for any $b \geq 0$ and $c > 0$, there will exist an equilibrium $(\mu^*, \alpha^*)$ such that $\mu^*(s) = m^0$ for all $s \in [0, 1]$.*

**Proof:** Consider an equilibrium $(\mu^*, \alpha^*)$ in which the message $m^i$ is sent on some states, however there are no states where message $m^j$ is sent such that $c(j) < c(i)$. If $m^i$ is the highest cost equilibrium message then $S$ could deviate from sending $m^i$ by sending $m^j$. This deviation to $m^j$ would induce the same action as $m^i$ but could be sent with a smaller cost. If $m^i$ is not the highest cost equilibrium message, then a profitable deviation still exists. Say message $m'$ is the highest cost equilibrium message, then $S$ could deviate from sending $m'$ by sending $m^j$. This deviation to $m^j$ would induce the same action as $m'$ but could be sent with a smaller cost. For both cases, there is a profitable deviation for $S$, and so $(\mu^*, \alpha^*)$ cannot be an equilibrium. Now consider strategy pair $(\mu, \alpha)$ such that $\mu(s) = m^0$ for all $s \in [0, 1]$. All costly messages would not induce a different action from the receiver and yet the sender would incur a cost of communication. Therefore, there is no profitable deviation from this completely uninformative strategy pair and so it is an equilibrium.■

Next, we offer a definition which summarizes the necessary conditions for an equilibrium under $NNA$.

**Definition 2** *A strategy pair $(\mu, \alpha)$ is feasible if it satisfies expressions (1), (3), (4) and additionally there does not exist an unused message $m^i$ and a used message $m^j$ such that $c(j) \leq c(i)$.*

We now describe a procedure for determining the strategy pairs $(\mu, \alpha)$ which are feasible. From the lemma above, we note that the completely uninformative equilibrium will always be feasible. Also by Lemma 2, the next step is to check whether a strategy pair including messages $m^0$ and $m^1$ is feasible. To accomplish this, we need to check both whether the strategy pair where $m^0$ is sent on smaller states is feasible and whether the strategy pair where $m^1$ is sent on smaller states is feasible. If at least one is feasible, we proceed to the next step, where we check whether strategy pairs including $m^0$, $m^1$ and $m^2$ are feasible. We need to check whether any of the 6 permutations of the three messages are feasible. If at least one of the 6 possible permutations are feasible, we proceed to the set of messages $m^0$, $m^1$, $m^2$ and $m^3$. We proceed in this manner until we arrive at a set of messages $m^0$, ..., $m^{k^*}$ such that none of the $k^*!$ permutations are feasible.

We are ready to characterize the equilibria under $NNA$. As the following proposition shows, each feasible strategy pair $(\mu, \alpha)$ will form an equilibrium.

**Proposition 1** *If $(\mu, \alpha)$ is feasible then it is an equilibrium under $NNA$.*

As the above proposition implies, there are many equilibria when the $NNA$ condition is applied. This abundance of equilibria stands in contrast to our results in the following section. There we show that for $b = 0$, the only equilibria which satisfies $NITS$ are, the equilibria which are most informative.

It is natural to ask, rather than $NNA$, why not use more neutral set of out-of-equilibrium beliefs for the initial analysis of the model? In other words, are there other preferable alternatives to $NNA$. There seem to be two primary options in this regard. One option is that out-of-equilibrium messages are simply ignored. In this case, an out-of-equilibrium message will induce $R$ to select an action of 0.5. Unfortunately, these beliefs imply some undesirable equilibrium properties. For instance, it is possible that this condition would not admit the informative equilibria which our subsequent equilibrium refinement admits and so this condition does not facilitate the exposition of our subsequent refinement efforts. In other words, although $NNA$ can admit equilibria which are not optimal, $NNA$ does not reject equilibria which are optimal. The same cannot be said of the beliefs which ignore out-of-equilibruim messages. Also, by ignoring out-of-equilibruim messages, $R$ makes no attempt to infer the origin of the message. Again in this regard, $NNA$ is superior because there is an attempt, albeit somewhat incomplete, to infer the origin of the out-of-equilibrium message. Therefore, the restriction that out-of-equilibrium messages are ignored neither facilitates the subsequent analysis nor has a rational appeal.

A second option is that an out-of-equilibruim message would induce beliefs that the message was sent on the states which are incurring the smallest communication costs. This second restriction is weaker than that of $NNA$, and much of the proceeding analysis would hold, with the exception of the no holes result. However, again this alternate condition would not facilitate the subsequent analysis because it admits equilibrium with holes. Further, the weaker condition has less rational appeal because it would seem reasonable that the most likely deviation would come from the states facing the highest, not lowest, communication costs.

In summary, there seems to be no natural and neutral position to take regarding the out-of-equilibrium messages. $NNA$ seems to be more reasonable than simply ignoring out-of-equilibruim messages or that the sender regards them as being sent on the states associated with the lowest cost message. Further, $NNA$ seems to provide the best setting from which to view the utility of $NITS$.

# 6 Alignment of Preferences under $NITS$

Here we focus on the implications of the *no incentive to separate* ($NITS$) condition for case where the preferences regarding the receiver's actions are perfectly aligned ($b = 0$). In this section, we show that $NITS$ selects the equilibrium most preferred by the receiver.

We begin by noting that when $b = 0$, the order of the signals does not matter. We can rewrite expression (4) in Lemma 1 for the case of $b = 0$ as:

$$(\lambda^j)^2 - (\lambda^i)^2 = 4\left[c(i) - c(j)\right]. \tag{5}$$

As expression (5) suggests, the interval size on which a message is sent is determined only by its communication cost and not by its location on the state space. As a result, expression (5) does not contain subscripts. This step somewhat simplifies the process of identifying equilibria under $NNA$. However, the task is still difficult because $NNA$ only implies that there are no holes. All other combinations of signals which are feasible are not ruled out. Now we show that $NITS$ greatly reduces the difficulty in identifying equilibria because only a unique class of strategy pairs can form an equilibrium.

If the incentives are aligned ($b = 0$) and there are $n$ actions induced in equilibrium then there are $n+1$ states in which the sender has the *most incentive* to deviate.[9] Therefore, as a matter of convention, we select one of these $n+1$ states. Hence, $NITS$ specifies that if an out-of-equilibrium message is observed then $R$ believes that the state is certain to be $s = 0$.

**No incentive to separate:** Given a strategy pair $(\mu, \alpha)$, if $R$ observes $\widetilde{m}$ where $\widetilde{m} \notin \mu([0,1])$ then $R$ believes that the state is certain to be $s = 0$, $\beta(0|\widetilde{m}) = 1$.

Note that our version of $NITS$ is not identical to the original specification of Chen et. al. (2008). Our specification focuses on the out-of-equilibrium beliefs which are implicit in the original specification. The authors motivate their condition by suggesting that, upon observing an out-of-equilibrium message, a natural place to expect such a deviation is from the *lowest* state.[10] Before the statement of $NITS$, we noted that if $b = 0$ and there are $n$ actions induced in equilibrium then there are $n+1$ states in which the sender has the most incentive to deviate. All of the results involving $b = 0$ would follow if we selected any of the other such $n$ states.

Before we state the main result regarding the equilibria under $NITS$, we provide the following lemmas which are used in the proof of the result. The first lemma which we present shows that, similar to $NNA$, $NITS$ does not admit equilibrium with holes. Specifically, if a message of a certain cost is used in equilibrium, it must be the case that all messages of smaller costs are also used in equilibrium.

**Lemma 3** *Consider an equilibrium $(\mu^*, \alpha^*)$ in which $m^i$ which is transmitted. Under $NITS$, if $b = 0$ then every $m^j$ where $c(j) < c(i)$ is also used in equilibrium.*

**Proof:** Suppose that there is an equilibrium $(\mu^*, \alpha^*)$ such that $\mu^*(s) = m^i$ with cost $c(i)$ however there does not exist an $s'$ such that $\mu^*(s') = m^j$ and $c(j) < c(i)$. If the signal $m^j$ is observed, $R$ believes that the state is certain to be $s = 0$. On the interval in the state space for which the $S$ sends message $m^i$, $S$'s payoff cannot be higher than $-c(i)$. By Lemma 8, $S$ has identical payoffs at each of the states for which expression (5) is satisfied, including the states 0 and 1. Therefore, at $s = 0$, the sender has a payoff of less than $-c(i)$ and a profitable deviation is then to send $m^j$. Therefore, $(\mu^*, \alpha^*)$ cannot constitute an equilibrium.■

We compare Lemma 2 with Lemma 3. Although the out-of-equilibrium beliefs are different, we are guaranteed to not have holes in the equilibrium under either $NNA$ or $NITS$. As a result, we employ the same definition of feasible for $NNA$ as $NITS$.

We now show that we are guaranteed a feasible strategy pair with a most costly message $m^k$ such that there does not exist a feasible strategy pair with a most costly message $m^{k'}$ where $k' > k$. If such a $k$ is found then we say that the feasible strategy pair with a most costly message $m^k$ is *maximal*. In the lemma below, we now show that we are guaranteed a maximal, feasible strategy pair. Specifically, if preferences are aligned then for any amount of communication costs there is a feasible strategy pair $(\mu, \alpha)$ which is maximal.

[9] See Lemma 8 in the appendix.

[10] The reader should consult Chen et. al. (2008) for further justification of the $NITS$ condition.

**Lemma 4** *If $b = 0$ then for any $c > 0$ there is always exists a maximal, feasible strategy pair $(\mu, \alpha)$ under $NITS$.*

**Proof:** To check whether a $(\mu, \alpha)$ is feasible we rewrite expressions (3) and (4). The message which costs $c(k)$ is sent on an interval of size $\lambda^k$. The message which costs $c(k-1)$ is sent on an interval of size $\lambda^{k-1} = \sqrt{4\left[c(k) - c(k-1)\right] + (\lambda^k)^2}$. The message which costs $c(k-2)$ is sent on an interval of size $\lambda^{k-2} = \sqrt{4\left[c(k) - c(k-2)\right] + (\lambda^k)^2}$. The message which costs $c(2)$ is sent on an interval of $\lambda^2 = \sqrt{4\left[c(k) - c(2)\right] + (\lambda^k)^2}$. The message which costs $c(1)$ is sent on an interval of size $\lambda^1 = \sqrt{4\left[c(k) - c(1)\right] + (\lambda^k)^2}$. Finally for the costless message, we write $\lambda^0 = \sqrt{4c(k) + (\lambda^k)^2}$. Therefore, we may write expression (3) as

$$\begin{aligned} &\sqrt{4c(k) + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(1)\right] + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(2)\right] + (\lambda^k)^2} + \\ &... + \sqrt{4\left[c(k) - c(k-2)\right] + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(k-1)\right] + (\lambda^k)^2} + \lambda^k = 1. \end{aligned} \tag{6}$$

When communication costs increase, $\lambda^k$ must decrease to zero in order for expression (6) to hold. Recall that we required that $c(i+1) - c(i) \geq \psi > 0$ for all $i \in \{0, ..., k\}$. Therefore, we can write the lower bound of each term in the left hand side of expression (6):

$$\sqrt{4k\psi} + \sqrt{4(k-1)\psi} + \sqrt{4(k-2)\psi} + ... + \sqrt{4(2)\psi} + \sqrt{4\psi} > 1. \tag{7}$$

For every $\psi$, there is a $k$ large enough so that expression (7) is satisfied. Therefore, we are guaranteed a maximal, feasible $(\mu, \alpha)$.■

Intuitively, Lemma 4 shows that, for any communication costs when there are aligned preferences, there exists an upper bound on the number of messages used in an equilibrium.[11] It should not come as a surprise that full communication is not feasible when $c > 0$. However, the straightforward characterization of equilibrium under $NITS$ is perhaps surprising, given the complicated nature of characterizing the equilibrium under $NNA$.

We are now ready for the main result of the section. Proposition 2 shows that we are guaranteed an equilibria under $NITS$. Further, the only equilibria admitted under $NITS$ are the ones which are maximal among the feasible strategy pairs.

**Proposition 2** *If $b = 0$ then under $NITS$ an equilibrium $(\mu^*, \alpha^*)$ exists and it is a member of the maximal, feasible class.*

Proposition 2 shows that under $NITS$, only the equilibria with the largest possible number of equilibrium messages does not have a profitable deviation. The proposition uses the language *class* because when $b = 0$, the ordering of the messages does not matter. Also, one can see the full force of $NITS$ by noting the difference between the multiplicity of equilibria in Propositon 1 and the uniqueness in Proposition 2.

[11] Note that a variant of Lemma 4 would hold for the case of $b > 0$. However, it is not necessary for our present purposes and it would require a slightly different proof, therefore we do not provide it.

Our Proposition 2 is reminiscent of Proposition 3 in Chen et. al. (2008). The authors show that in the CS model where monotonicity holds, $NITS$ admits only the most informative equilibrium. In the notation of our model, Chen et. al. (2008) show that for $b > 0$ and $c = 0$ in the uniform-quadratic case, NITS uniquely selects the most informative equilibrium. Our Proposition 2 and Proposition 3 of Chen et. al. (2008) becomes more surprising when we provide an example which demonstrates that we are not guaranteed uniqueness when $b > 0$ and $c > 0$.

### 6.1 Simple Characterization

Here we focus on the case where preferences are perfectly aligned ($b = 0$) and communication costs are linear in the index of the message. In other words, we assume that $c(k) = ck$ where $c > 0$. One benefit of this exercise is that, for general communication costs, it is difficult to characterize the threshold level of costs which render a strategy pair $(\mu, \alpha)$ feasible. However, in the linear case the characterization is simple. If $c \leq c^*(k)$ then a strategy pair $(\mu, \alpha)$ which employs a most costly message $m^k$ is feasible and if $c > c^*(k)$ then such a $(\mu, \alpha)$ is not feasible.

**Lemma 5** *If $c(k) = ck$ and $c > 0$ then the cutoff cost for a strategy pair involving message $m^k$ is:*

$$c^*(k) = \left( \frac{1}{2 \sum_{j=1}^{k} \sqrt{j}} \right)^2 .$$

**Proof:** At the largest $c$ such that the strategy pair involving message $m^k$ is feasible, it must be that $\left(\lambda^k\right)^2 = 0$. By expression (5) it must be that, $\left(\lambda^{k-1}\right)^2 = 4c$, $\left(\lambda^{k-2}\right)^2 = 8c$, ..., $\left(\lambda^1\right)^2 = 4(k-1)c$, $\left(\lambda^0\right)^2 = 4kc$. Therefore, we may write expression (6) in the case of linear costs as

$$2\sqrt{c(k)} + 2\sqrt{c(k-1)} + 2\sqrt{c(k-2)} + ... + 2\sqrt{c(2)} + 2\sqrt{c(1)} = 1.$$

and so the lemma is proved.■

In order to provide some intuition for the characterization to this point, we provide the following example. Given linear communication costs, the example illustrates that the equilibria under $NNA$ possesses a large amount of multiplicity however $NITS$ admits only the most informative class of equilibria. The example also illustrates the utility of Lemma 5.

**Example 1** *Consider the case where $c(i) = 0.01i$ and $b = 0$. Note that:*

$$c^*(4) = 0.00662 < 0.01 < c^*(3) = 0.0145.$$

*Under $NNA$ there are four classes of equilibria where there are 1, 2, 3, and 4 messages used. There are no feasible strategy pairs $(\mu, \alpha)$ for the case of more than four messages. For the first equilibrium, $m^0$ is sent on all states. For the second, there are two equilibria. There is an equilibrium where $m^0$ is sent on states $[0, 0.52)$ and $m^1$ on states $[0.52, 1]$. There is another equilibrium where $m^1$ is sent on states $[0, 0.48)$ and $m^0$ on states $[0.48, 1]$. Note that in each of these equilibria $\lambda^0 = 0.52$ and $\lambda^1 = 0.48$. For the third case, there are six equilibria. There is a monotonic equilibria where $m^0$ is sent on states $[0, 0.392)$, $m^1$ on states $[0.392, 0.729)$*

*and $m^2$ on $[0.729, 1]$. The remaining 5 equilibria require that $\lambda^0 = 0.392$, $\lambda^1 = 0.337$, and $\lambda^2 = 0.271$. For the fourth case, there are 24 equilibria. There is a monotonic equilibria where $m^0$ is sent on states $[0, 0.363)$, $m^1$ on states $[0.363, 0.665)$, $m^2$ on $[0.665, 0.892)$ and $m^3$ on $[0.892, 1]$. The remaining 23 equilibria require that $\lambda^0 = 0.363$, $\lambda^1 = 0.302$, $\lambda^2 = 0.227$ and $\lambda^3 = 0.108$. Only the 24 equilibria in which four messages are sent, are admitted under $NITS$.*

# 7 Imperfect Alignement of Preferences under $NITS$

Recall Proposition 2 which demonstrated that the only equilibria admitted under $NITS$ are maximal and feasible. Similarly, Proposition 3 in Chen et. al. (2008) shows that in the CS model where monotonicity holds, $NITS$ admits only the most informative equilibrium. In the notation of our model, Chen et. al. (2008) show that for $b > 0$ and $c = 0$ in the uniform-quadratic case that $NITS$ uniquely selects the most informative equilibrium. However, when $b > 0$ and $c > 0$ we are not guaranteed uniqueness.

For the case of $b = 0$ and $c > 0$, the order of the messages does not matter as long as their size is governed by expression (5). For the case of $b > 0$ and $c = 0$, the order of the signals themselves does matter, but it does matter that the size of the intervals are increasing along the state space. However, when $b > 0$ and $c > 0$ there is an interaction between these two effects, which might cause the nonuniqueness which we now describe.

The nonuniqueness can manifest itself in two distinct ways. First, there could exist several equilibria with a given set of equilibrium messages, however these equilibria differ in their informativeness. Second, there can exist equilibria which differ in the set of equilibrium messages. The following example demonstrates this second aspect and the subsequent example demonstrates the first.

**Example 2** *Suppose that $b = 0.245$ and communication costs are $c(i) = 0.01i$. First, there exists an equilibrium $(\mu^*, \alpha^*)$ where two messages are used. Message $m^0$ is sent on $s \in [0, 0.03)$ and the $m^1$ is sent on $s \in [0.03, 1]$. The sender's $s = 0$ equilibrium payoffs are $-(0.015 - 0.245)^2 = -0.0529$, which is greater than deviation payoffs of $-(0.245)^2 - 0.02 = -0.080$. There also exists an equilibrium where $m^0$ is sent for all states. The sender's $s = 0$ equilibrium payoffs are $-(0.5 - 0.245)^2 = -0.065$, which is greater than deviation payoffs of $-(0.245)^2 - 0.01 = -0.070$.*

The example above shows that when $b > 0$ there can exist equilibria with a different set of messages. Our next example shows that when $b > 0$, there exist equilibria with identical sets of messages yet differ in their informativeness. Also note that the following example shows that when $b > 0$ there exists equilibria where an increase in communication costs will improve communication.

**Example 3** *First, consider the costless communication case. When $b = 0.2$, and $c(i) = 0$, there is only one outcome equivalent equilibria of the following form: a single action is induced on $s \in [0, 0.1)$ and a single action is induced on $s \in [0.1, 1]$. Message $m_0$ induces $a = 0.05$ and message $m_1$ induces $a = 0.55$. In this case, $E[-(a - s)^2] = -0.0608$. However, when $b = 0.2$,* *and $c(i) = 0.01i$, there are two non-outcome equivalent equilibria. In the first equilibria, $m^0$ is sent on $s \in [0, 0.12)$ and $m^1$ on $s \in [0.12, 1]$. In the second equilibria, $m^1$ is sent on $s \in [0, 0.08)$ and $m^0$ on $s \in [0.08, 1]$. In the first equilibria, $E[-(a-s)^2] = -0.0569$ and in the second, $E[-(a-s)^2] = -0.0649$. If the cost of communication is increased to $c(i) = 0.02i$ then in the first equilibria $m^0$ is sent on $s \in [0, 0.14)$ and $m^1$ on $s \in [0.14, 1]$, implying $E[-(a-s)^2] = -0.0532$. By way of comparison note that the equilibrium in which there is no communication implies $E[-(a-s)^2] = -0.0833$.*

The above provides an example where an increase in communication costs can lead to an improvement in communication. Also note that Example 3 contained an instance of two distinct equilibria, which share the set of equilibrium messages yet differ in their informativeness. Finally, note that the last equilibrium described in Example 3 outperforms the upper bound for payoffs as found by Goltsman et. al. (2009).[12] These authors find that the upper bound for the expected payoffs of the receiver in mediated communication is:

$$E[-(a-s)^2] = -\frac{1}{3}b(1-b) = -0.0533.$$

What is the intuition behind the equilibrium which outperforms the Goltsman et. al. (2009) upper bound? Note that there are two effects at work. When $b > 0$, the sender increases the intervals at the upper end of the state space, which reduces the expected payoff to the receiver. However, the communication costs induce the sender to decrease the interval sizes on which the costly signal is sent. In the relevant equilibrium, the costly message is in the upper end of the state space. Therefore, these effects work in opposite directions, thereby achieving an expected payoff above that of the upper bound for the case where communication is not costly.

## 8 Discussion of Modeling Choices

Before we proceed to the conclusion, we discuss some of our modeling choices. Our state space is designed to be richer than our message space[13] as the state space is uncountably infinite and there are only a finite number of messages which can be transmitted with a finite cost. We believe that this captures an important aspect of reality: it is impossible to completely communicate the complexity of the real world, one may only increase the precision of communication by expending more costly effort. Also note that the size of the language used in equilibrium arises endogenously. In our view, this captures another important feature of reality: the precision of communication is determined by the costs incurred by the sender.

We assumed that there is only a single message associated with a particular communication cost. This assumption yields several benefits. First, there is no need to restrict attention to pure strategies. In the case where there are several messages of a particular cost, obviously the receiver would not employ mixed strategies in equilibrium, however this is not the case for the sender. In this case, there exists equilibria in which the sender would use mixed strategies, and $NITS$ would not exclusively admit the most informative equilibrium. Further, even when

[12] This possibility was first suggested by Andreas Blume.

[13] This assumption also appears in Jager et. al. (2009) and Lipman (2009). Blume and Board (2010a) examine the opposite case where the message space is much larger than the state space.

restricting attention to pure strategies, Proposition 2 would not hold, as we would need an additional restriction to guarantee selection of the most informative equilibria.

Perhaps a natural questions is, why not model communication which is necessarily noisy where the sender incurs a communication cost which is decreasing in the variance of the possible messages. Within this possibility, there arise some features which we find unappealing.

First, suppose that the sender would costlessly specify the preferred action of the receiver and the preferred amount of communication costs to a third party who would then add the appropriate amount of noise to the message. As Blume et. al. (2007) showed, the quality of communication is not monotonic in the amount of noise and communication is always enhanced by a small amount of noise. It would seem to violate the spirit of the model that, even if communication is costless, the sender would prefer to transmit a message with noise. Further, this problem is not avoided if the noise is determined by the amount of effort expended by the sender. Therefore, we do not view this possibility as an adequate substitute for our modeling choices.

As a second option, suppose that the sender would specify the upper and lower bound of the possible states and incur a cost which is decreasing in the size of this interval. In this case, we would have to assume that the receiver is unsophisticated. For instance, if the sender wished to communicate the state, $s = 0.315789215$, the sender could send the message leading to the possible interval $[0.315789215, 1]$ and the sophisticated receiver would know that the state is certain to be 0.315789215. To avoid these types of problems, we would either have to model the receiver as unsophisticated or to model communication as we do here.

In both of the above options, the communication does not, in our view, resemble communication which is costly and discrete. Most notably the resulting equilibrium would be a fully separating equilibrium whereby each state would induce a unique action by the sender. By contrast, the equilibrium in our model is a pooling equilibrium in that several states induce identical actions by the sender. This seems to be more consistent with our intuition regarding communication.

# 9 Conclusions

We have modeled an interaction between an informed sender and an uninformed receiver where communication is costly and discrete. We have characterized the equilibria under the permissive no new actions ($NNA$) out-of-equilibrium restriction. When sender and receiver have aligned preferences over the action of the receiver, we have demonstrated that the no incentive to separate ($NITS$) out-of-equilibrium condition admits only the most informative class of equilibria. This result is analogous to the application of $NITS$ to the uniform-quadric version of Crawford and Sobel (1982). Finally, for the case that preferences are not aligned, we that $NITS$ does not identify a unique equilibrium and that an increase in communication costs might improve communication. Further, we show that this improvement can be large enough so that it outperforms the Goltsman et. al. (2009) upper bound on costless communication.

There remain interesting questions which are unanswered. For instance, we have modeled the interaction as a single repetition. However, we are interested to learn the equilibrium behavior where the interaction is repeated. There are three possibilities as the relationship is potentially finitely repeated, infinitely repeated or is repeated until the communication attains some threshold. There exists an additional issue, which arises only in the repeated version of the game: presumably there is a relationship between some publicly observable signal and the optimal action for the receiver and also that the sender wishes to teach the receiver this relationship. It would seem interesting to explore this learning. Additionally, we are eager to learn the significance of our assumption of quadratic preferences and a uniform probability distribution. Finally, we are interested to know whether an environment with several heterogenous senders and receivers, would produce a novel matching problem.

We are currently working on a version of our model where the sender imperfectly observes the state. Our preliminary results, consistent with Blume et. al. (2007), suggest that a small amount of this noise can improve communication.

Finally, Duffy et. al. (2011) tests our model in an experimental setting. Like most communication games, our equilibrium is quite complicated and this fact makes experimental investigation difficult. However, using the example of other such papers[14] the authors test a simplified version of the theoretical model presented above. Duffy et. al. (2011) finds that the size of the language arises endogenously, as it does in our paper. This suggests that further study of costly and discrete communication could prove fruitful.

[14] For instance, Cai and Wang (2006) and Kawagoe and Takizawa (2009). Also see Blume et. al. (1998) and Blume et. al. (2001).

# 10 Appendix

The appendix is organized as follows. First we prove a few results about the nature of the equilibria. Then we prove the results which appear in the body of the paper. Lemma 6 now shows that the intervals must be connected. Lemma 7 shows that the intervals cannot partially overlap. Then, we prove Lemma 1.

**Lemma 6** *In any equilibria it cannot be the case that* $m \in \mu^*(\underline{s}) = \mu^*(\overline{s})$*,* $m' \notin \mu^*(\underline{s}) = \mu^*(\overline{s})$*,* $m' \in \mu^*(s')$ *and* $m \notin \mu^*(s')$ *where* $\underline{s} < s' < \overline{s}$*.*

**Proof**: Suppose there exists $m$ such that $(\mu^*)^{-1}(m) = (s_1, s_2) \cup (s_3, s_4)$ with $(s_1, s_2) \cap (s_3, s_4) = \emptyset$ and $(\mu^*)^{-1}(m') = (s_2, s_3)$.

If $\overline{a}(m) = \overline{a}(m')$ then there exists a profitable deviation for $S$ in choosing the cheaper message. Now suppose that $\overline{a}(m) \neq \overline{a}(m')$. If $\overline{a}(m) < \overline{a}(m')$ and $m' \in \mu^*(s)$ for $s \in (s_2, s_3)$ as

$$-(\overline{a}(m) - s - b)^2 - c(m) < -(\overline{a}(m') - s - b)^2 - c(m') \text{ for } s \in (s_2, s_3)$$

then $m' \in \mu^*(s)$ for $s \in (s_3, s_4)$ as

$$-(\overline{a}(m) - s - b)^2 - c(m) < -(\overline{a}(m') - s - b)^2 - c(m') \text{ for } s \in (s_3, s_4).$$

If $\overline{a}(m) < \overline{a}(m')$ and $m \in \mu^*(s)$ for $s \in (s_3, s_4)$ as

$$-(\overline{a}(m) - s - b)^2 - c(m) > -(\overline{a}(m') - s - b)^2 - c(m') \text{ for } s \in (s_3, s_4)$$

then $m \in \mu^*(s)$ for $s \in (s_2, s_3)$ as

$$-(\overline{a}(m) - s - b)^2 - c(m) > -(\overline{a}(m') - s - b)^2 - c(m') \text{ for } s \in (s_2, s_3).$$

The proof for the case of $\overline{a}(m) > \overline{a}(m')$ follows in the analogous manner.■

**Lemma 7** *In any equilibria it cannot be the case that* $m' \in \mu^*(s')$ *where* $s' \in [s_1, s_3)$ *and* $m'' \in \mu^*(s'')$ *where* $s'' = [s_2, s_4)$ *where* $s_2 < s_3$*.*

**Proof**: Suppose that there was such an equilibrium. The message $m'$ induces action $a'$ and message $m''$ induced action $a''$. Therefore the payoff from sending $m'$ is

$$U^R(m') = -(a' - s - b)^2 - c(m')$$

and the payoff from sending message $m''$ is

$$U^R(m'') = -(a'' - s - b)^2 - c(m').$$

For $a' \neq a''$ there is only a single state for which

$$U^R(m') = U^R(m'')$$

and therefore this cannot be the case. For $a' = a''$ because $s_2 < s_1 < s_3 < s_4$ and $c(m') < c(m'')$ then there exists a profitable deviation by the sender to select the cheaper message. Therefore there cannot exist such an equilibrium.■

**Proof of Lemma 1**: If there are $n+1$ distinct actions induced by the sender then it must be that there are $n$ equations in expression (2). If this was not the case then Definition 1 would not hold. A typical such expression would be the cutoff state between intervals such that $m_h^i \in \mu^*(s')$ for $s' \in [s_h, s_{h+1})$, $m_{h+1}^j \in \mu^*(s'')$ for $s'' \in [s_{h+1}, s_{h+2})$:

$$-\left(\frac{s_h + s_{h+1}}{2} - s_{h+1} - b\right)^2 - c(i) = -\left(\frac{s_{h+1} + s_{h+2}}{2} - s_{h+1} - b\right)^2 - c(j).$$

Which we rewrite as:

$$\begin{aligned} -\left(\frac{s_h - s_{h+1}}{2} - b\right)^2 - c(i) &= -\left(\frac{s_{h+2} - s_{h+1}}{2} - b\right)^2 - c(j) \\ -\left(\frac{-\lambda_h^i}{2} - b\right)^2 &= -\left(\frac{\lambda_{h+1}^j}{2} - b\right)^2 + c(i) - c(j) \end{aligned}$$

so that

$$(\lambda_{h+1}^j)^2 - (\lambda_h^i)^2 = 4\left[c(i) - c(j)\right] + 4b\left(\lambda_h^i + \lambda_{h+1}^j\right).$$

■

Lemma 6 showed that the intervals must be connected. Lemma 7 showed that the equilibria cannot be partially overlapping. Lemma 1 showed the relative size of the intervals as a function of their position on the state space and the cost of message. Also note that Lemma 1 together with the assumption that a unique cost is associated with each message implies that the sender will not employ a mixed strategy.

**Proof of Proposition 1:** Suppose that strategy pair $(\mu, \alpha)$ is feasible. Given $\alpha$, there does not exist a profitable deviation from $\mu$ regarding the messages used in equilibrium since it satisfies expression (4). There does not exist a profitable deviation for $S$ by sending an out-of-equilibrium message since there are no new actions induced by such a message. Given $\mu$ which satisfies Lemma 1 there is no profitable deviation for $R$ from $\alpha$ since it satisfies expression (1). Therefore, the strategy pair $(\mu, \alpha)$ is an equilibrium under $NNA$.■

**Lemma 8** *If $b = 0$ and there are $n$ actions induced then there are $n+1$ solutions to* $\max_s(\overline{a}(m') - s')^2 + c(m')$.

**Proof:** Suppose that $U^S(\overline{a}, \widehat{m}, \underline{s}) > U^S(\overline{a}, \widehat{m}, \overline{s})$ where $\mu([\underline{s}, \overline{s})) = \widehat{m}$. As the distribution is uniform, $\overline{a} = \frac{\underline{s}+\overline{s}}{2}$. This implies that $\left(\frac{\underline{s}+\overline{s}}{2} - \underline{s}\right)^2 > \left(\frac{\underline{s}+\overline{s}}{2} - \overline{s}\right)^2$, which cannot be the case. Combined with expression (2), we have $n+1$ such solutions.■

Hence, if $b = 0$ and there are $n$ actions induced there are $n+1$ states with the worst ex-post payoff. Naturally these are candidates for reasonable beliefs in the event of an out-of-equilibrium message. Further, any of these $n+1$ states would be sufficient for the results under $NITS$ to hold.

**Proof Proposition 2:** First we show that an equilibrium under $NITS$ exists. As Lemma 4 shows, there will always be a maximal, feasible strategy pair $(\mu, \alpha)$. Suppose that $m^k$ where $k \in \mathbb{N}$ is the most costly message in this maximal, feasible strategy pair $(\mu, \alpha)$. We need to check that it is not profitable for the sender at $s = 0$, to transmit a message more costly than $m^k$. Because $(\mu, \alpha)$ satisfies expression (6) it must be that $\lambda^0 = \sqrt{4c(k) + \left(\lambda^k\right)^2}$. By Lemma 8, the equilibrium payoffs for the $S$ who received signal $s = 0$ is:

$$-\left(\frac{\lambda^0}{2} - 0\right)^2 - c(0) = -\left(\frac{\lambda^1}{2} - 0\right)^2 - c(1) = ... = -\left(\frac{\lambda^k}{2} - 0\right)^2 - c(k).$$

All of the messages used in equilibrium will not provide a profitable deviation, therefore we must use an out-of-equilibrium message to find a deviation. Any deviation accomplished by message $m^{k+x}$ where $x > 1$ can be accomplished by sending message $m^{k+1}$. Therefore, the cheapest (and therefore best candidate) out-of-equilibruim message is the message $m^{k+1}$. If such a message is sent, $R$ would have beliefs that the message was sent by state $s = 0$. Sending this message yields a payoff of $-c(k+1)$. Therefore, the signal will be profitable when:

$$-c(k+1) > -\left(\frac{\lambda^k}{2} - 0\right)^2 - c(k)$$

which we rewrite as:

$$\left(\lambda^k\right)^2 > 4\left[c(k+1) - c(k)\right].$$

If a strategy pair involving a most costly message of $m^{k+1}$ implies that $\widetilde{\lambda}^{k+1} = 0$, it must also be that $\widetilde{\lambda}^k = 4\left[c(k+1) - c(k)\right]$. However, when a strategy pair involving $m^{k+1}$ is not feasible, it must be that $\left(\lambda^k\right)^2 < 4[c(k+1) - c(k)]$ and so there is no profitable deviation. Therefore, when a strategy pair with a most costly message $m^k$ is feasible but the strategy pair with a most costly message $m^{k+1}$ is not, there is no profitable deviation with a signal more costly than $m^k$. Therefore, there does not exist a deviation from the maximal, feasible strategy pair where $m^k$ is the most costly message.

Now we will show that if $b = 0$ and there is an equilibrium with a most costly message $m^k$ then $NITS$ does not admit an equilibrium with a cheaper most costly message. For the case of $k > 0$, suppose that a strategy pair with a most costly message $m^k$ is feasible and a strategy pair with a most costly message $m^{k+1}$ is not.[15] Consider a candidate strategy pair involving a most costly message of $m^{k'}$ where $k < k'$. This candidate equilibrium is characterized by:

$$\left(\widehat{\lambda}^j\right)^2 - \left(\widehat{\lambda}^i\right)^2 = 4(c(i) - c(j)) \text{ for } i, j \in \{0, .., k'\}$$

$$\widehat{\lambda}^{k'} > 0$$

$$\widehat{\lambda}^0 + \widehat{\lambda}^1 + ... + \widehat{\lambda}^{k'} = 1$$

[15] For the case that $k = 0$, then there is no possible equilibria in which a less costly signal is used.

Each of the intervals in the candidate equilibrium are larger than their corresponding intervals in the original equilibrium. Namely that $\widehat{\lambda}^i > \lambda^i$ for $i \in \{0, ..., k'\}$. To see this, note that the difference between the size of the intervals on which messages $m^i$ for $i \in \{0, ..., k'\}$ are sent are identical in the original and candidate equilibria. However, for the original equilibria there are additional intervals to accommodate and so each of the intervals in the original equilibria must be smaller than their counterpart in the candidate equilibrium.

For our original equilibrium is must be that:

$$\left(\lambda^0\right)^2 > 4[c(k) - c(0)] + \left(\lambda^k\right)^2.$$

However, since $c(0) = 0$ and $\lambda^k \geq 0$ it must be that $(\lambda^0)^2 > 4c(k)$. Because corresponding intervals are larger in the candidate equilibria it also must be that $\left(\widehat{\lambda}^0\right)^2 > 4c(k)$. So we can write the equilibrium payoffs as:

$$U^S = -\left(\frac{\left(\widehat{\lambda}^0\right)^2}{2} - 0\right)^2 < -c(k)$$

Deviation payoffs are $-c(k)$, therefore equilibrium payoffs are less than deviation payoffs and so an equilibrium involving a most costly message of $m^{k'}$ cannot exist where there exists an equilibrium with a most costly message of $m^k$.

To see that each feasible strategy pair involving a most costly message $m^k$ uniquely determines the values of $\lambda$, we can rewrite expression (3) as:

$$\begin{aligned}&\sqrt{4c(k) + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(1)] + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(2)] + \left(\lambda^k\right)^2} + ... \\ &+\sqrt{4[c(k) - c(k-2)] + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(k-1)] + \left(\lambda^k\right)^2} + \lambda^k = 1\end{aligned} \tag{8}$$

The left hand side of expression (8) is strictly increasing in $\lambda^k$ and therefore must only hold for a single value of $\lambda^k$. And so the proposition is proved.■

### 10.1 Example of non-existence of equilibrium under an alternate specification of $NITS$

Here we provide an example where there does not exist an equilibrium for an alternate, and arguably more reasonable, specification of $NITS$. Recall that upon observing an out-of-equilibrium message, the receiver believes that the state is $s = 0$. A common justification for these beliefs is that $s = 0$ is the *lowest* state. However, in general that state does not yield the lowest ex-ante payoffs. Specifically, the state $s = 0$ shares with other the states the distinction of the smallest ex-ante payoffs for the case of $b = 0$. However, this is not true for the case of $b > 0$. When $b > 0$, the sender at state $s = 1$ has the lowest ex-ante payoffs, and this state would therefore seem to be the best candidate for beliefs upon observing an out-of-equilibrium message. Although these beliefs appear to be reasonable, as the following example shows, under these beliefs we are not guaranteed an equilibrium.

**Example 4** *Suppose that $c = 0$ and $b = 0.2$. Upon observing an out-of-equilibrium message,* *we assume that the receiver believes that the state is $s = 1$. Consider the strategy pair $(\mu, \alpha)$ in which one message is sent on all states. This induces an optimal action of $R$ of $a = 0.5$. The payoff of the sender at state $s = 1$ is $-0.49$, whereas the payoff to the sender at state $s = 0$ is $-0.09$. If the sender at state $s = 1$ transmits an out-of-equilibrium message then a payoff of $-0.04$ can be attained and so a profitable deviation exists. Therefore, the uninformative equilibrium cannot exist. Consider the strategy pair $(\mu, \alpha)$ in which two messages are sent. According to expression (4) one message is sent on states $[0, 0, 1)$ and the other is sent on states $[0.1, 1]$. The payoff of the sender at state $s = 1$, is $-0.4225$ and the payoff of the sender at state $s = 0$ is $-0.0225$. If the sender at state $s = 1$ transmits an out-of-equilibrium message then a payoff of $-0.04$ can be attained and so a profitable deviation exists. Note that in both the strategy pair in which one message is transmitted and the strategy pair in which two messages are transmitted, the $S$ at the state $s = 1$ obtains the smallest payoffs, and in this sense is an appropriate candidate for the origin of an out-of-equilibrium message. On the other hand, there does not exist an equilibrium under this alternate specification of $NITS$ when either one or two messages are transmitted. Additionally, a strategy pair $(\mu, \alpha)$ involving three messages cannot satisfy expressions (1), (3) and (4). Hence, there cannot exist an equilibrium with in which three or more messages are used. Therefore, there does not exist an equilibrium under this alternate specification of $NITS$.*

Although for $b > 0$ we are not guaranteed an equilibrium under this alternate specification of $NITS$, we are guaranteed an equilibrium for the case of $b = 0$.

### 10.2 Equilibrim under $NITS$ implies equilibrium under $NNA$

Although the following proposition is somewhat obvious, we offer it for the sake of completeness.

**Proposition 3** *For $b = 0$, if a strategy pair $(\mu, \alpha)$ is an equilibrium under $NITS$ then it is also an equilibrium under $NNA$.*

**Proof:** Suppose that strategy pair $(\mu, \alpha)$ is an equilibrium under $NITS$. Therefore, $(\mu, \alpha)$ is feasible and there is no profitable deviation for the sender at $s = 0$. When we apply the $NNA$ beliefs to the strategy pair $(\mu, \alpha)$, as there is no new actions induced, there cannot be a profitable deviation because there are no holes and therefore $(\mu, \alpha)$ is also an equilibrium under $NNA$.

■

## 11 References

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