Differentiable game complete analysis for tourism firm decisions

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2009

DIFFERENTIABLE GAME COMPLETE ANALYSIS FOR TOURISM FIRM DECISIONS

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Abstract. In this paper we apply the *complete analysis of a differentiable game* (recently introduced by the author) to determine possible *suitable behaviors* (actions) of tourism firms during strategic interactions with other tourism firms, from both non-cooperative and cooperative point of view. To associate with a real strategic interaction among tourism firms a differentiable game any player's strategy-set must, for instance, be a part of a topological vector space, closure of an open subset of the space. The most frequent case is that in which the strategy-sets are compact intervals of the real line. On the other hand, very often, the actions at disposal of a player can form a finite set, and in this case a natural manner to construct a game representing the economic situation is the von Neumann convexification (also known as *canonical extension*) that leads to a differentiable game with probabilistic scenarios, and thus *even more suitable* for the purpose of represent real interactions. For what concerns the complete analysis of a differentiable game, its first goal is the precise knowledge of the Pareto boundaries (maximal and minimal) of the payoff space, this knowledge will allow us to evaluate the quality of the different Nash equilibria (by the distances from the Nash equilibria themselves to Pareto boundaries, with respect to appropriate metrics), in order to determine some "focal" equilibrium points *collectively more satisfactory* than each other. Moreover, the complete knowledge of the payoff-space will allow to develop *explicitly* the *cooperative phase* of the game and the various bargaining problems rising from the strategic interaction of the tourist firms (Nash bargaining problem, Kalai-Smorodinski bargaining problem and so on). In the paper we shall deal with some practical study cases.

Keywords. *Tourism firm, differentiable game, strategic interaction, non-cooperative behaviour, cooperative behavior, Pareto efficiency.*

Jel classification. C7, C71, C72, C78, D21.

AMS classification. 91Bxx *Mathematical economics.* 91Bxx, 91B38, 91Axx. 91A10, 91A12, 91A20 91A40, 91A44, 91A80.

Themes. *Competitiveness in Tourism Industry.*

Introduction

Game theory has proved a powerful tool to suggest the strategies that must be employed by *rational* tourism firms in competitive environments. Nevertheless, in the great part of current literature on the matter, the methodologies used are drawn only from the Finite Game Theory, and this precludes several more deep applications, studies and developments. In this paper, on the contrary, we concentrate our attention on infinite differentiable games, which are models more complex and can be greatly more adherent to the real economic situations. In order to illustrate an application of the *complete study of an infinite differentiable game*, introduced in [2] and [6] by the author, a case of pricing decisions and entry deterrence are considered, starting form a simple finite game (see, for similar finite games, [1]), we show how the complete study of its mixed extension gives many useful answers than the study of the finite one.

1. The complete study of a game

We recall that for the complete study of a game we shall follow the following points of investigation, we shall:

0a) classify the game (linear, symmetric, invertible, symmetric in the strategies, ...);
0b) find the critical zone of the game and its image by f;
0c) determine the biloss space im(f);
0d) determine inf and sup of the game and see if they are shadow optima;
0e) determine the Pareto boundaries of the bistrategy space and their images by f;
1a) specify the control of each player upon the boundaries;
1b) specify the noncooperative reachability and controllability of the Pareto boundaries;
1c) find the possible Pareto solutions and crosses;
1d) find devotion correspondences and devotion equilibria;
1e) specify the efficiency and noncooperative reachability of devotion equilibria;
2a) find best reply correspondences and Nash equilibria;
2b) study the existence of Nash equilibria (Brouwer and Kakutany);
2c) evaluate Nash equilibria: noncooperative reachability, position with respect to and, efficiency, devotion;
2d) find, if there are, dominant strategies;

2e) find strict and dominant equilibria, reduce the game by elimination of dominated strategies;
3a) find conservative values and worst loss functions of the players;
3b) find conservative strategies and crosses;
3c) find all the conservative parts of the game (in the bistrategy and biloss spaces);
3d) find core of the game and conservative knots;
3e) evaluate Nash equilibria by the core and the conservative bivalue;
4a) find the worst offensive correspondences and the offensive equilibria;
4b) specify noncooperative reachability of the offensive equilibria and their efficiency;
4c) find the worst offensive strategies of each player and the corresponding loss of the other;
4d) find the possible dominant offensive strategies;
4e) confront the possible non-cooperative solutions;
5a) find the elementary best compromises (Kalai-Smorodinsky solutions) and corresponding biloss;
5b) find the elementary core best compromise and corresponding biloss;
5c) find the Nash bargaining solutions and corresponding bilosses;
5d) find the solutions with closest bilosses to the shadow minimum;
5e) find the maximum collective utility solutions;
5f) confront the possible cooperative solutions.
6f) study the transferable utility case.

This kind of study has been proposed by D. Carfì in [2], and it springs from the bases given by J.P. Aubin in [3] and [4]. For a deep introduction to Pareto boundaries the reader can see [5].

2. The deterrence-pricing game

The economic situation. We consider two tourism firms. The second one is already in the market the first one is not. The possible strategies of the first are *Enter* (in the market) and *Not Enter* (in market). The strategies of the second one are *High prices* and *Low prices*. The payoff (losses) of the two firms are represented in the following table:

	High prices	*Low prices*
Enter	(*-4,-2*)	(*0,-3*)
Not enter	(*0,-3*)	(*0,-4*)

The finite game associated with the economic situation. The above table defines a bimatrix M and, consequently, a finite loss game ($M,<$). It is evident that the pair of strategies (*Enter, Low prices*) is a dominant Nash equilibrium of the game, in other terms it's a strict Nash equilibrium. This Nash equilibrium leads to the payoff ($0,-3$).

Problems. We have to do some considerations:

1) the second player could gain much more if the first does not enter;
2) the market offers a potential total gain 6, in correspondence to the bistrategy (*Enter, High prices*).

We have three questions:

a) is it possible for the second player to gain more than 3?
b) is it possible that the two firms collectively obtain the total amount offered by the market?
c) if the case (b) happens, what is a possible fair division of the total amount among the firms?

We shall answer to these questions during our study. To do so, we consider the mixed extension of the game ($M,<$).

3. The mixed extension

Scope of the mixed extension. We shall examine the von Neumann extension of the finite game ($M,<$) to find other possible realistic and applicable economic behaviours and solutions.

The Extension. We, firstly, have to imbed (canonically) the finite strategy spaces into the probabilistic canonical 1-simplex of the plane (since there are two strategies per firm). The strategy *Enter* (and *High prices*) shall be transformed into the first canonical vector e_1 of the plane and the strategy *Non Enter* (and *Low prices*) shall be transformed into the second canonical vector of the plane e_2. So our new bistrategy space is the Cartesian square of the canonical 1-simplex of the plane (wich is the convex envelop of the canonical base e of the plane): conv$(e)^2$. It is a 2-dimension bistrategy space, since the canonical 1-simplex is 1-dimensional. So we can imbed the simplex in the real line $\mathbf{R}$ and the bistrategy space into the Euclidean plane $\mathbf{R}^2$. To do this, roughly speaking, we consider the injection associating with the pure strategies *Enter* and *High prices* the probability 1 and to the pure strategies *Not Enter* and *Low prices* the probability 0.

In other (rigorous) terms, we associate to the mixed strategy $(x,1-x)$ the probability x of the interval $[0,1]$ and to the mixed strategy $(y,1-y)$ the probability y of the same interval. Moreover, the considered finite loss game $(M,<)$ is the translation by the loss vector $(0,-4)$ of the game $(M',<)$ represented in the following table:

	High prices	*Low prices*
Enter	$(-4,2)$	$(0,1)$
Not enter	$(0,1)$	$(0,0)$

The mixed extension of the game $(M,<)$ is, thus, the translation, by the same vector, of the extension of the game $(M',<)$, so we can study this latter extension.

4. The study of the mixed extension

Formal description of the game under study. The mixed extension of the finite game $(M',<)$ is the infinite differentiable loss-game $G = (f,<)$ with strategy sets $E = F = [0,1]$ and biloss (disutility) function f defined on the Cartesian square $[0,1]^2$ by

$$f(x,y) = (-4xy,x+y),$$

for every bistrategy (x,y) of the game. We shall denote the bistrategy space by S. The bistrategy space is the Cartesian square $[0,1]^2$, and we shall denote by A, B, C and D its four vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

Remark. The conservative bivalue of the finite bimatrix game M is the pair $(0,1)$.

Classification. The game is not linear, it is, utterly, bilinear. It is not symmetric with respect to the players, since the loss $f_1(x,y)$ is different from the loss $f_2(y,x)$, but it is symmetric with respect to the bistrategies, since $f_i(x,y)$ equals $f_i(y,x)$, for every player i. It is not invertible, since there are two different equivalent bistrategies, since the biloss $f(1,0)$ equals the biloss $f(0,1)$, which equals the biloss $(0,1)$.

5. The critical space of the mixed extension

Jacobian matrix. The Jacobian matrix of the function f at a point (x,y) of the bystrategy space S - denoted by $Jf(x,y)$ - is the matrix having as rows the gradients of the loss functions f_1 and f_2, respectively, that are grad $f_1(x,y) = (-4y,-4x)$ and grad $f_2(x,y) = (1,1)$, for every bistrategy (x,y). The Jacobian determinant, at the bistrategy (x,y), is det $Jf(x,y) = -4y+4x$, for every pair (x,y) in the bistrategy space S.

Critical space. The critical zone is the subset of the bistrategy space S of those bistrategies (x,y) at which the jacobian matrix is not invertible, that is verifying the relation $\det Jf(x,y) = 0$, i.e., $x = y$. In symbols, the critical zone is the set

$$C(f) = \{(x,y) \text{ in } S : x = y\} = [A,C].$$

Transformation of the critical space. We must determine the image $f([A,C])$. The segment $[A,C]$ is the set of real pairs (x,y) such that $x = y$ and y is in the interval $[0,1]$. The value of the biloss function on the generic point (y,y) of that segment is $f(y,y) = (-4y^2,2y)$. Setting, for any y in $[0,1]$, $X = -4y^2$ and $Y = 2y$, we have $y = Y/2$ and $X = -Y^2$, with Y in $[0,2]$. Thus, the image of the critical zone is the parabolic segment of equation $X = -Y^2$ with end points $A' = f(A) = (0,0)$ and $C' = f(C) = (-4,2)$.

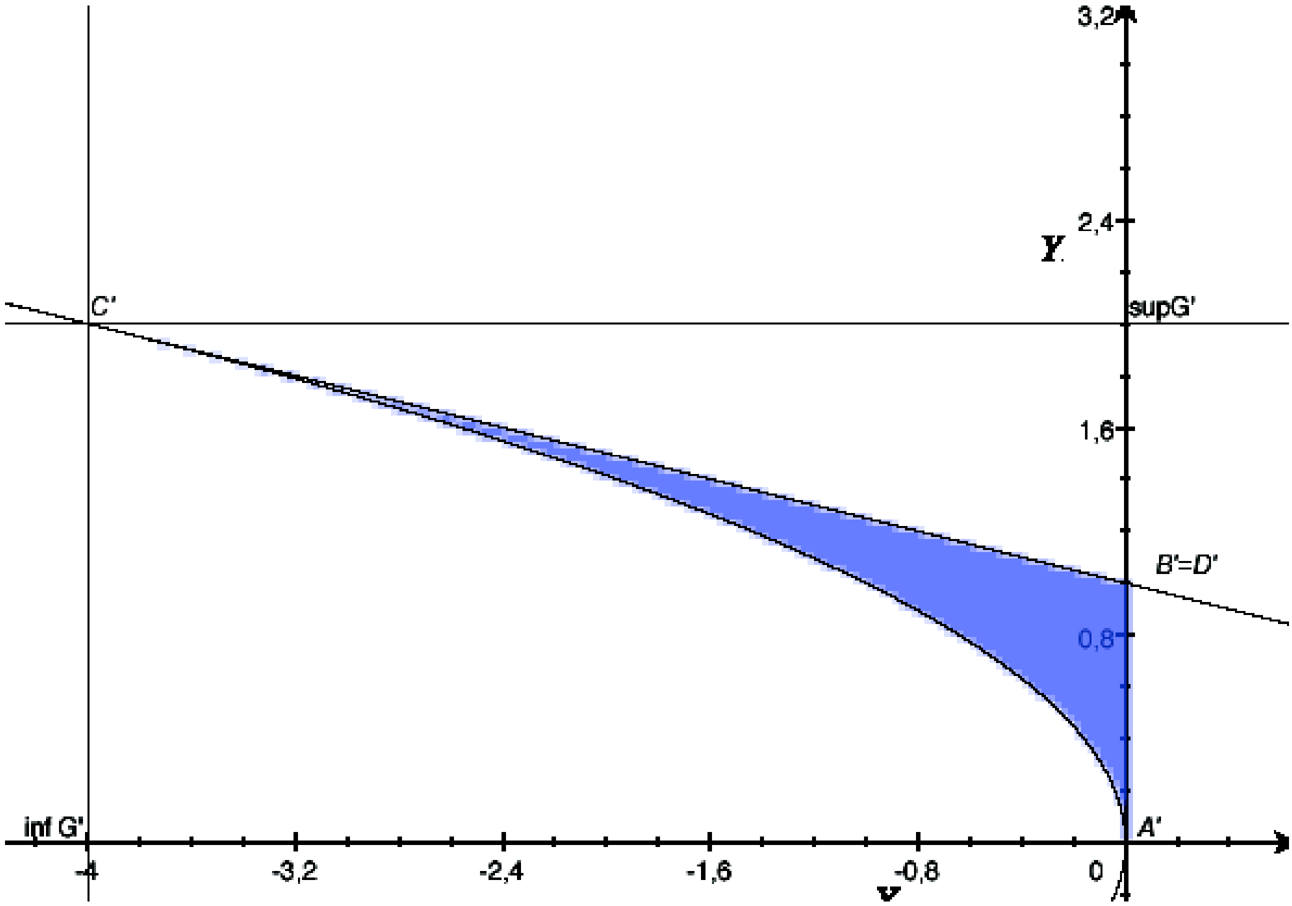
6. The biloss (disutility) space

Transformation of the topological boundary of the bistrategy space. We start from the image $f([A,B])$. The segment $[A,B]$ is the set of points (x,y) of the Euclidean plane such that the ordinate y equals 0 and the abscissa x lies in the interval $[0,1]$. The value of the biloss function upon the generic point $(x,0)$ of this segment is the biloss $f(x,0) = (0,x)$. Setting, for all x in E, $X = 0$ and $Y = x$, we have $X = 0$ and Y belonging to $[0,1]$. Thus the image of the segment $[A,B]$ is the segment of end points $A' = (0,0)$ and $B' = f(B) = (0,1)$. ***Image of the segment*** **$[D,C]$.** The segment $[D,C]$ is the set of points (x,y) such that $y = 1$ and x lies in $[0,1]$. The image of the generic point $(x,1)$ of this segment is therefore the biloss $f(x,1) = (-4x,x+1)$. Setting, for all x in E, $X = -4x$ and $Y = x+1$, we have $x = Y-1$ and $X = -4Y+4$, with Y in $[1,2]$. Thus the image is the segment of end points $D' = f(D) = (0,1)$ and $C' = (-4,2)$. ***Transformation of the segment*** **$[C,B]$.** The segment $[C,B]$ is the set of all points of the plane such that $x = 1$ and y is in $[0,1]$. The image of the generic bistrategy of this segment is the biloss $f(1,y) = (-4y,1+y)$. Setting, for any Frances' strategy y in F, $X = -4y$ and $Y = 1+y$, we obtain $y = Y-1$ and $X = -4Y+4$. Then, the image is the set of bilosses (X,Y) such that $X = 4-4Y$, with Y is in the interval $[-4,0]$. So the image is the segment of end points $C' = (-4,2)$ and $B' = (0,1)$. ***Transformation of the segment*** $[A,D]$. The segment $[A,D]$ is the set of bistrategies (x,y) such that $x = 0$ and y lies in the interval $[0,1]$. The image of the generic point of this segment is the biloss $f(0,y) = (0,y)$. Setting, for all y in F, $Y = y$, we obtain $X = 0$ and Y belonging to $[0,1]$. So the image is the segment of end points $A' = (0,0)$ and $D' = (0,1)$.

The biloss space. The biloss space $f(S)$ is simply the bounded part of the plane having for topological boundary the union of the four above images with the image of the critical part of the game.

Extrema of the game. The extrema of the game are the infimum $\inf G = (-4,0)$ - not belonging to the biloss space $f(S)$ of the game G – and the supremum $\sup G = (0,2)$ – also in this case, not belonging to the biloss space $f(S)$. They are, thus, two shadow extremes.

0e) Pareto boundaries. The Pareto minimal and maximal boundaries of the biloss space are, respectively: the image of the critical zone of the game, that is the parabolic arc with end points A' and C' we already considered, and the segment $[B',C']$. The Pareto minimal and maximal boundaries of the bistrategy space are, respectively, the segment $[A,C]$ and the union of two segments $[B,C]$ and $[D,C]$.



7. I phase: non-cooperative friendly phase

1a) Pareto-controls. Both Emil and Frances do not control the Pareto minimal boundary. On the contrary, Emil controls part of the Pareto maximal boundary, precisely the segment $[B,C]$ playing the control strategy 1. Frances controls part of the Pareto maximal boundary, precisely the segment $[D,C]$ playing the control strategy 1.

1b) Non-cooperative reachability of Pareto boundaries. Both players can reach non-cooperatively the Pareto maximal boundary playing the respective reaching-strategies 1 and 1, respectively. Neither Emil nor Frances can reach non-cooperatively the Pareto minimal boundary (very regrettably).

1c) Non-cooperative Pareto solutions. Do not exist non-cooperative minimal Pareto solutions. There is only a non-cooperative maximal Pareto solution, the bistrategy $C = (1,1)$, that is a control-cross.

1d) Devotion correspondences. The partial derivative of the Emil's loss function f with respect to the second argument is $D_2f_1(x,y) = -4x$, for every bistrategy (x,y) in the space S, then, concerning its sign, there are two cases: $x = 0$ and $x > 0$. If x is zero, then the partial function $f_1(x,.)$ is constant and then all Frances' strategies are devote to Emil' strategy 0. If

Emil's strategy x is strictly positive, then the partial loss function $f_1(x,.)$ is strictly decreasing, and then his minimum value is reached at the point 1. Concluding, the Frances' devotion correspondence is defined by $L_2(x) = F$ if $x = 0$ and $L_2(x) = 1$ if $x > 0$, for each Emil's strategy x in E. Concerning Emil's devotion, we have $D_1 f_2(x,y) = 1$, so the Frances' partial loss function $f_2(.,y)$ is strictly increasing, for every strategy y in F, and then it assumes its minimum at the strategy 0. Concluding the Emil's devotion correspondence is defined by $L_1(y) = 0$, for every strategy y in the Frances' strategy space F.

Devotion equilibria. The set of all devotion equilibria - intersection of the inverse graph of Emil's devotion rule with the graph of Frances' devotion rule - is the segment $[A,D]$ and it is an infinite set.

1e) About the devotion equilibria. The devotion equilibria are non-cooperatively reachable, playing Emil the reaching-strategy 0. Concerning the efficiency, the devotion equilibrium D is bad, since it lies upon the Pareto maximal boundary, on the contrary, the devotion equilibrium A is efficient, since it belongs to the Pareto minimal boundary.

8. II phase: properly non-cooperative (egoistic phase)

2a) Best reply correspondences. Concerning the partial derivative of Emil's loss function with respect to the first argument, we have $D_1 f_1(x,y) = -4y$, for every bistrategy (x,y) of S. So, for the sign of that derivative, there are two cases. *I case.* If $y = 0$, the partial loss function $f_1(.,y)$ is constant and then the Emil's best reply strategy to this Frances' action 0 is any strategy of the space E, in other terms $B_1(0) = E$. *II case.* If the Frances' action y is strictly positive, then the partial derivative $D_1 f_1(x,y)$ is strictly negative, so the section $f_1(.,y)$ is strictly decreasing and the minimum point of the section is the strategy 1, thus the best reply rule is defined by $B_1(y) = 1$, for every France's strategy y in $F\backslash\{0\}$. Resuming, the Emil's best-reply correspondence is defined by $B_1(y) = E$ if $y=0$ and $B_1(y) = 1$ if $y > 0$. Concerning the Frances' best reply rule, we have $D_2 f_2(x,y) = 1$, hence the partial loss function $f_2(x,.)$ is strictly increasing, for each Emil's strategy x, and the Frances' best reply rule is defined by $B_2(x) = 0$, for every x in E.

Nash equilibria. The intersection of the graph of the best reply correspondence B_2 with the reciprocal graph of the best reply rule B_1 is the segment $[A,B]$, so there are infinitely many Nash equilibria. All these equilibria are equivalent for Emil (Emil's loss function is constantly equal zero on the segment $[A,B]$) but not for Frances, so they are not collectively equivalent.

2b) Existence of Nash equilibria. Kakutany's fixed point theorem assures the existence of at least a Nash equilibrium, Brouwer's fixed point theorem does not, since the Emil's best reply rule is not a function.

2c) About Nash equilibria. The Nash equilibrium zone is reachable, playing Frances the reaching-strategy 0. The Nash equilibrium A is minimal, and it's the unique minimal equilibrium (good equilibrium), on the contrary, the Nash equilibrium B is maximal (very bad equilibrium) the other ones are neither minimal nor maximal. The Nash equilibrium A is also a devotion equilibrium.

2d) Dominant strategies. Frances has one (and only one) dominant strategy, the strategy 0. Emil has one (and only one) dominant strategy, the strategy 1. As in the finite bimatrix game.

2e) Dominant equilibria. The Nash equilibrium B is a dominant Nash equilibrium.

9. III phase: defensive phase

3a) Emil's conservative value. We have, with a simple calculation, $v^{\#}_1 = 0$.

Emil's worst loss function. By definition, the Emil's worst loss function w_1 is defined on the Emil's strategy space E by $f^{\#}_1(x) = \sup_F f_1(x,.)$, for every Emil's strategy x in E. Let's compute the function w_1: for the sign of the partial derivative of the Emil's loss function, i.e., the derivative $D_2 f_1(x,y) = -4x$, there are two cases. *I case.* If $x = 0$, then the partial derivative is constantly 0 and so the partial function $f_1(0,.)$ is constant on E. Then, the worst offensive reply to the Emil's strategy 0 is any Frances' strategy, in other words $O_2(0) = [0,1]$. *II case.* If the Emil's action x is strictly positive, then, the partial derivative is strictly decreasing, and so the function $f_1(x,.)$ is strictly decreasing, consequently $O_2(x) = 0$, for every non-zero x in E. The Emil's worst loss function is then defined by $f^{\#}_1(x) = 0$ if $x = 0$ and $f^{\#}_1(x) = 0$ if $x > 0$, for every x in E.

Frances' conservative value. We have $v^{\#}_2 = 1$.

Frances' worst loss function. It is very simple to deduce that the Emil's worst offensive multifunction is defined by $O_1(y) = 1$, for each y in F. The Frances' worst loss function is defined by $f^{\#}_2(y) = 1+y$, for each y in F.

Conservative bivalue. The conservative bivalue is the vector $v^{\#} = (0,1) = B'$.

3b) Players' conservative strategies. All the Emil's strategies are conservative, in other terms $E^{\#} = E$. The unique Frances' conservative strategy is 0.

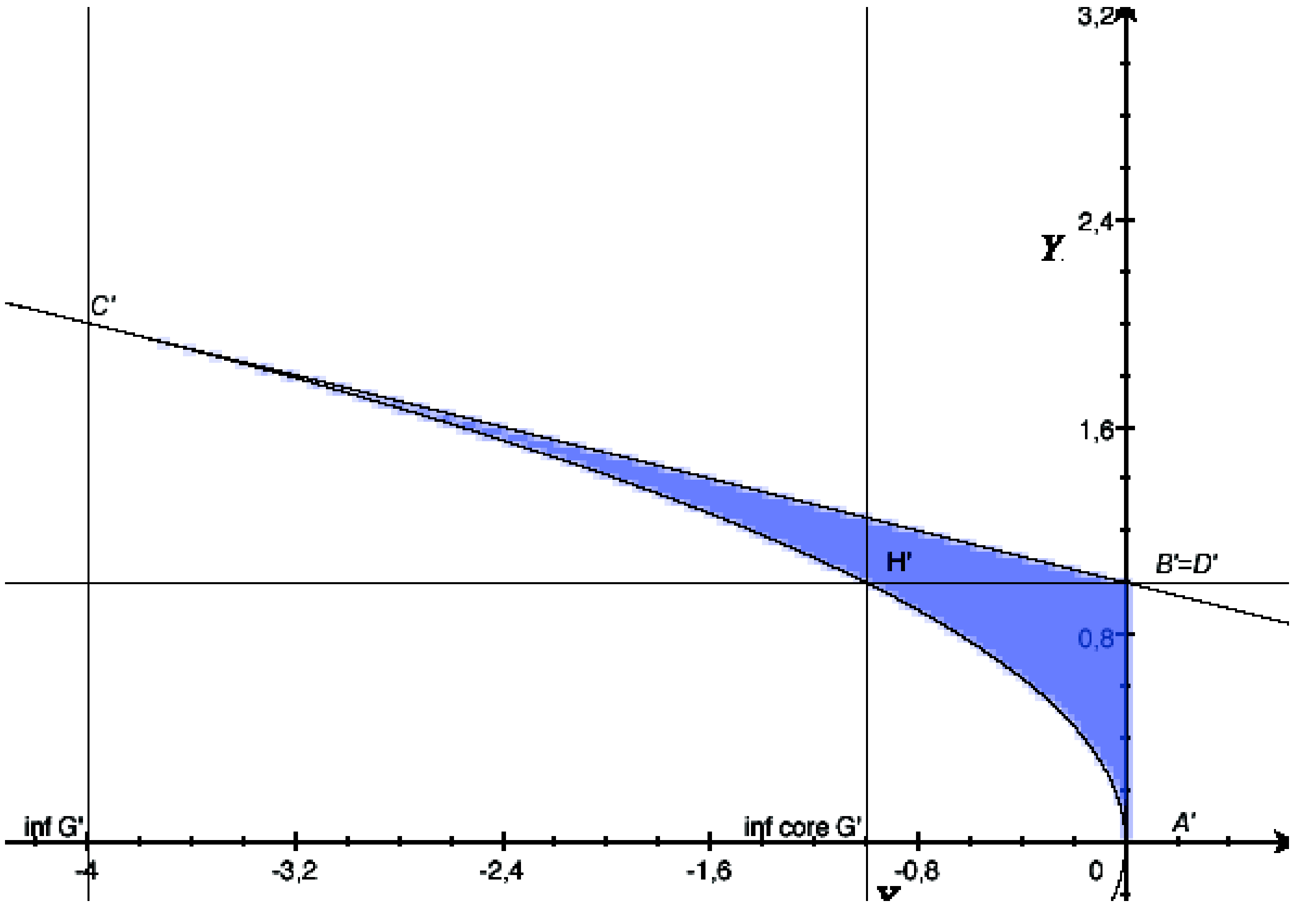
Conservative crosses. Finally, the set of all conservative crosses is the segment $[A,B]$, since any bistrategy $(x,0)$ is a conservative cross.

3c) Conservative parts. The conservative part of the biloss space is the subset of the biloss space $f(S)$ contained in the triangle equal to the convex envelope conv (K',B',A'). The Emil's conservative part $E^{\#}$ is the set of bistrategies (x,y) such that the loss $-4xy$ is non-positive (less or equal to $v^{\#}_1 = 0$), that is the set of any bistrategy (x,y) such that the product xy is non-negative, then all the bistrategies of the game are conservative for Emil. Concerning Frances, the conservative part $F^{\#}$ is the set of bistrategies such that the loss $x+y$ is less or equal to 1, consequently it coincides with the conservative part of the game.

3d) Core. The core of the biloss space is the arc-segment of Pareto minimal boundary with end points H' and A': where $H' = (-1,1)$. To determine the core of the game, in the bistrategy space, we have to find the retroimage of the biloss H'. A bistrategy $H = (x,y)$ such that $H' = f(H)$ verifies the two equalities $-4xy = -1$ *and* $x+y = 1$, the resolvent equation of this system of equations is $-4y+4y^2+1 = 0$, which gives us the following feasible solution $H = (1/2,1/2)$, then the core is the line-segment core$(G) = [A,K]$.

Conservative knots. A possible conservative knot $N^{\#}$ verifies the equality $v^{\#} = f(N^{\#})$, that is, the system $-4xy = 0$ and $x+y = 1$, which has the solutions $N^{\#}_1 = (0,1) = D$ and $N^{\#}_2 = (1,0) = B$.

3e) About Nash equilibria. There are infinitely many Nash equilibria, forming the segment $[A,B]$. All the equilibria are conservative but there is only one core equilibrium: the point A.



10. IV phase: offensive phase

4a) Offensive correspondences and equilibria. We already saw that the players's worst offensive correspondences are defined by $O_1(y) = 1$, for every strategy y in F, and $O_2(x) = 1$ if $x = 0$ and 0 if $x > 0$, respectively. The intersection of the graph of the correspondence O_2 with the reciprocal graph of O_1 is the unique offensive equilibrium B.

4b) About the offensive equilibrium. The unique offensive equilibrium is reachable non-cooperatively with the strategies 1 and 1, respectively. It is strongly-inefficient, since it lies on the Pareto maximal boundary.

4c) Confrontation of the equilibria. The unique Nash equilibrium that is a devotion equilibrium too is A. The unique Nash equilibrium that is an offensive equilibrium too is B.

4d) Dominant offensive strategies. Emil has the unique dominant offensive strategy 1. Frances has the unique dominant offensive strategy 0.

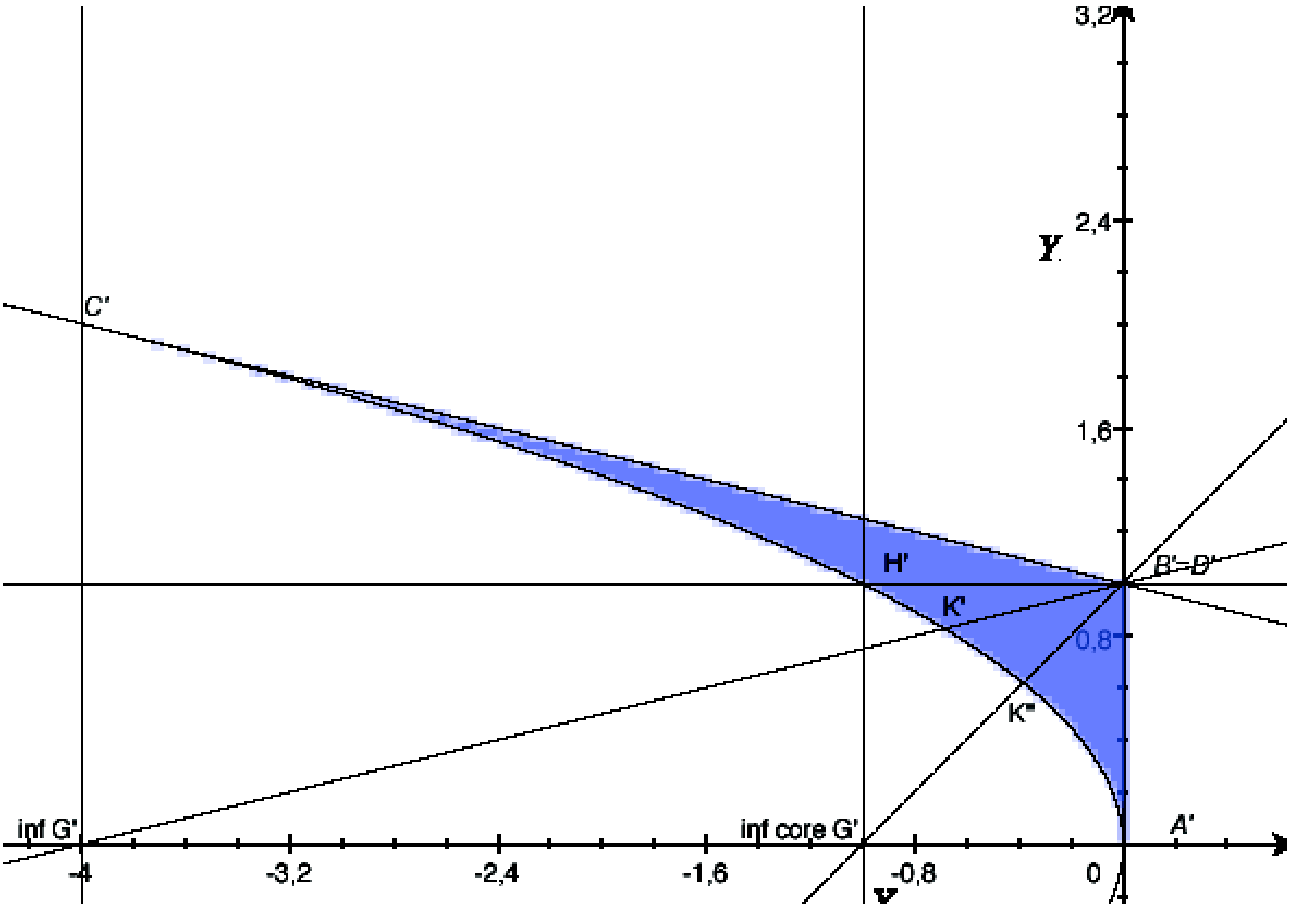
4e) About the noncooperative solution. The set of all Nash equilibria $[A,B]$ is controlled by Frances by the strategy 0. The equilibrium A is a focal point in the sense of Meyerson: it unique.

11. V phase (cooperative): Selection of Pareto bistrategies

We shall examine the most common cooperative solutions.

5a) Elementary best compromise. The elementary best compromise biloss is the intersection of the segment joining the threat biloss $v^{\#}$ with the infimum of the game, thus it satisfies the system $Y=(1/4)X+1$ and $X = -Y^2$ with X in $[-4,0]$ and Y in $[0,2]$, leading to the resolvent equation $X^2+24X+16 = 0$, its acceptable solution is $a = -12$-sqr(128), so the biloss $K' = (a,a/4 + 1)$ is the best compromise biloss. The Kalai Smorodinsky solution is the unique corresponding bistrategy solving the system $-4xy = a$ and $x+y = a+1$, i.e., the strategy profile $K = ((a+1)/2,(a+1)/2)$.

5b) Kalai Smorodinsky solution (core best compromise). The Kalai Smorodinsky biloss (or core best compromise biloss) is the intersection of the segment joining the threat biloss $v^{\#}$ with the infimum of the core, thus it is the biloss (X,Y) satisfying the relations $Y = X+1$ and $X = -Y^2$ and with X,Y in $[0,1]$. Putting $c = (-1+\text{sqr}(5))/2$ the solution is the biloss $K'' = (-c^2,c)$, it is the unique core best compromise biloss. The core best compromise solution, that is the reciprocal image (x,y) by f of the biloss K'', satisfies the relations $-4xy = -c^2$ and $x+y = c$, taking into account that this solution must belong to the core, we known also that $x = y$, and then $x = y = c/2$.



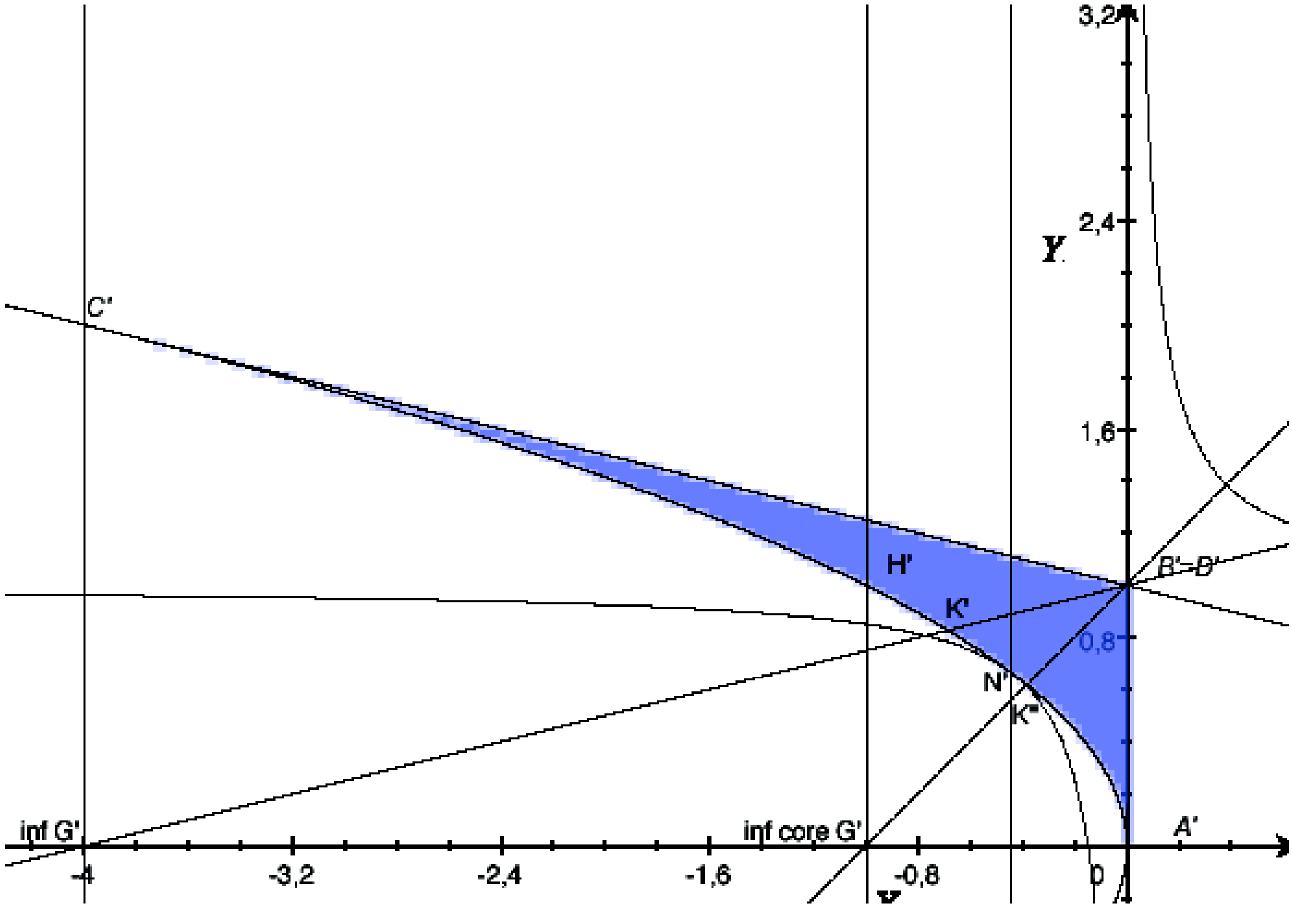
5c) Nash bargaining solution with $v^{\#}$ as disagreement point. The possible Nash bargaining bilosses, with disagreement point represented by the conservative bivalue $v^{\#}$, are the possible solutions of the following optimization problem:

$$\max (X-v^{\#}_{1})(Y-v^{\#}_{2}) = \max X(Y-1), \text{ sub } X = -Y^2.$$

The section of the Nash bargaining objective function upon the constraint is defined by

$$g(Y) = -Y^2 (Y-1) = -Y^3+Y^2,$$

for every Frances' loss Y. The derivative $g'(Y) = -3Y^2+2Y$ is non-negative when the product $Y(3Y-2)$ is non-positive, that is on the interval $[0,2/3]$, consequently the maximum point of the section g is the loss $Y = 2/3$, with corresponding Emil's loss $X = -4/9$ (obtained by the constraint). Concluding the biloss $N' = (-4/9,2/3)$ is the unique Nash bargaining biloss. The set of Nash bargaining solutions is the reciprocal image of this biloss by the function f.



5d) Bilosses closest to the shadow minimum. The closest biloss to the shadow minimum S' is the pair $(-7/2, \text{sqr}(7/2))$. In fact, we can follow two ways. ***1st way.*** Let us impose the orthogonality of the vector $\boldsymbol{v}$ joining the point $(-4,0)$ with the generic point $P = (-Y^2,Y)$ of the minimal Pareto boundary with a tangent vector, say $\boldsymbol{t}$, to the boundary at the same point P. The vector $\boldsymbol{v}$ is the pair difference between the point P and the shadow minimum $a = \inf G'$, that is the pair $(4-Y^2,Y)$; a tangent vector at P is the vector $(-2Y,1)$ (obtained by derivation of the parametrizazion of the Pareto boundary w defined by $w(s) = (-s^2,s)$, for every real s). The orthogonality imposes that the Euclidean scalar product

$$(4-Y^2,Y).(-2Y,1)$$

must be 0, that is

$$-2Y(4-Y^2) + Y = 0,$$

this equation has the immediate solution $Y = 0$ and leads to the simple equation $2Y^2-7 = 0$. ***2nd way.*** We can minimize the distance function $d(a,.)$ upon the Pareto boundary; this method leads to the same resolvent equation. Indeed, we have

$$h(Y) = d(a,(-Y^2,Y))^2 = (4-Y^2)^2 + Y^2,$$

for every Frances' loss Y. We must minimize the function h and we have

$$h'(Y) = 2(4-Y^2)(-2Y) + 2Y,$$

and this is 0 if and only if $2Y^2-7 = 0$, as we predicted.

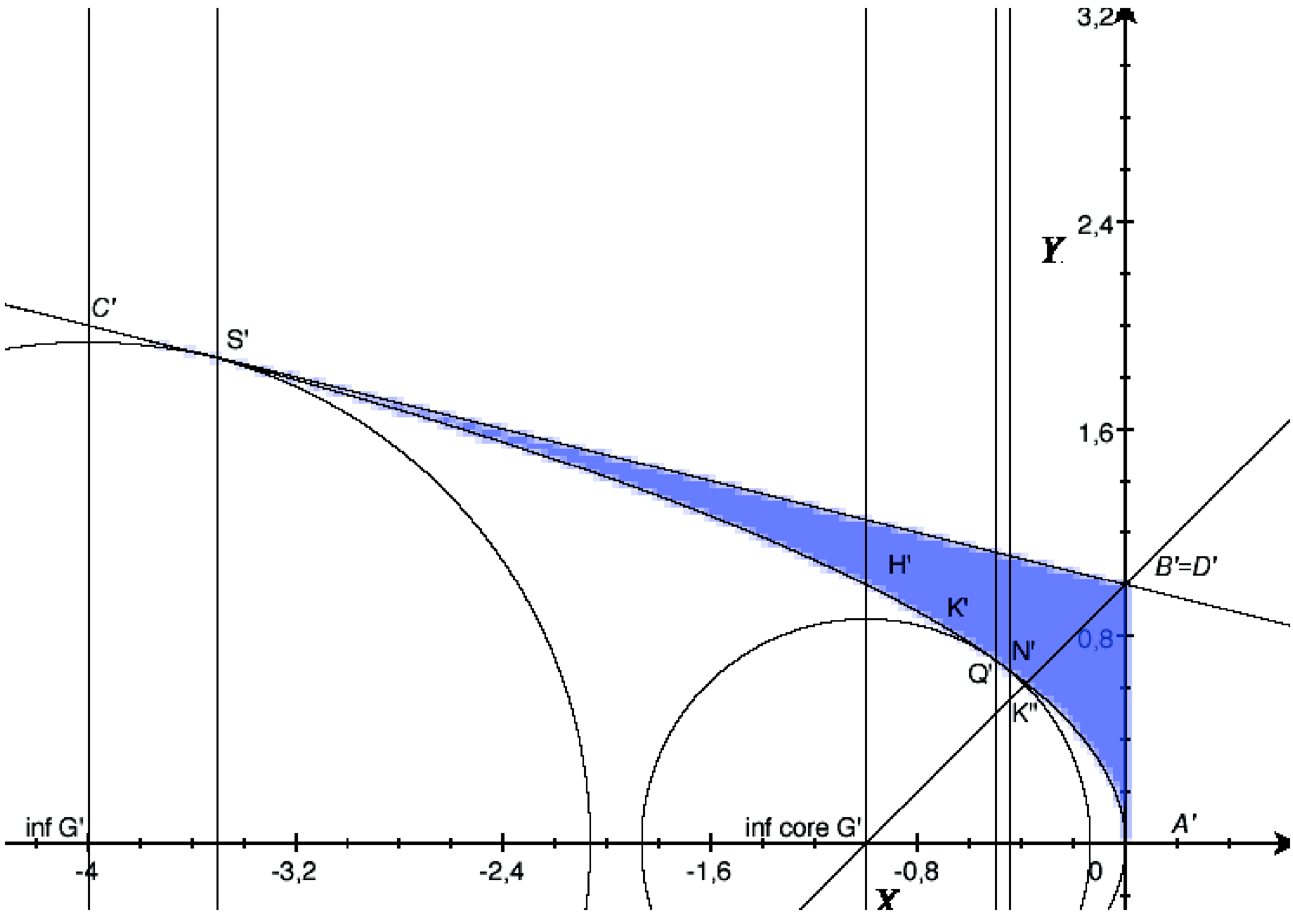
5d bis) Bilosses closest to the core shadow minimum. The closest biloss to the core shadow minimum is the core biloss $Q' = (-1/2, \mathrm{sqr}\,(1/2))$. ***1st way.*** Let us impose the orthogonality of the vector $\boldsymbol{v}$ joining the point $(-1,0)$ with the generic point $P = (-Y^2,Y)$ of the minimal Pareto boundary with a tangent vector, say $\boldsymbol{t}$, to the boundary at the same point P. The vector $\boldsymbol{v}$ is the pair difference between the point P and the shadow minimum inf coreG', that is the pair $(1-Y^2,Y)$; a tangent vector at P is the vector $(-2Y,1)$ (obtained by derivation of the parametrizazion w defined by $w(s) = (-s^2,s)$, for every real s). The orthogonality imposes that the Euclidean scalar product

$$(1-Y^2,Y) \cdot (-2Y,1)$$

must be 0, that is

$$-2Y\,(1-Y^2) + Y = 0,$$

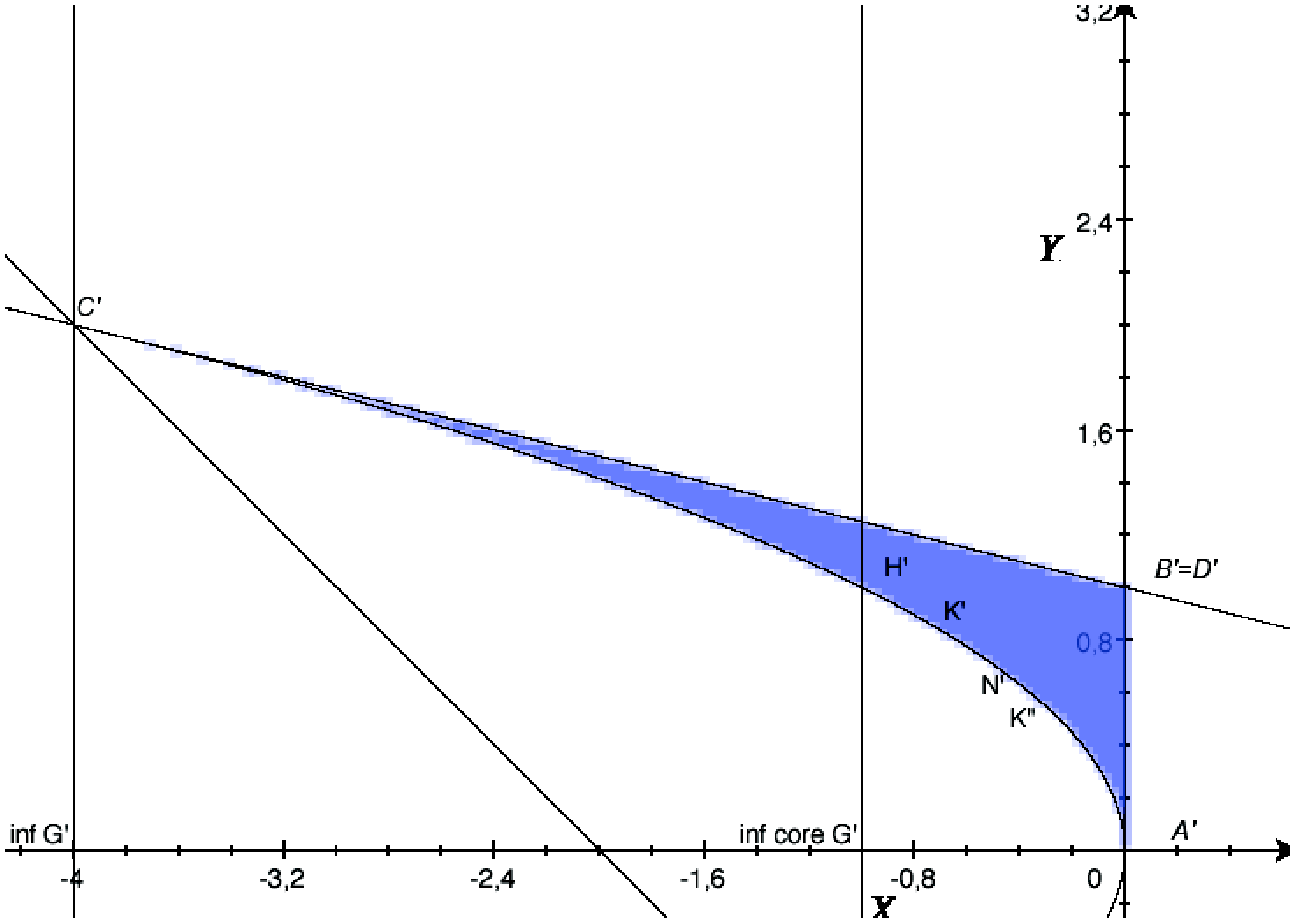
this equation has the immediate solution $Y = 0$ and leads to the simple equation $2Y^2-1 = 0$. ***2nd way.*** We can minimize the distance function $d(a,.)$ upon the core; this method leads to the same resolvent equation.



5e) Minimum aggregate loss (maximum collective utility). The possible bilosses with maximum collective utility are the possible solutions of the following optimization problem:

$$\min\,(X + Y) \text{ sub } X = -Y^2.$$

We immediately see that the unique biloss with these two properties is $C' = (-4,2)$, with collective utility 2. The unique maximum utility solution of the game is then the corresponding bistrategy C.



5f) About the cooperative solutions. The cooperative solutions we found are different and not equivalent each other.

6f) Transferable utility. Assuming the games with transferable utility, certainly the maximum utility solution is a good solution; but in this last case, the players must face the bargaining problem of fair division of the maximum collective utility.

Fair division of the total gain. It is easy to see that all the ***core*** compromise solutions on the minimal collective loss boundary are coincident with the Kalai Smorodinsky biloss in this new situation, i.e., the fair biloss (*-3/2,-1/2*).

References

[1] M. Thea Sinclair - Mike Stabler, *The economics of tourism*, Routledge 1997
[2] D. Carfì, *Topics in Game Theory*, Il gabbiano 2009
[3] Jean Pierre Aubin, *Mathematical methods of game and economic theory*, (Revised Edition) North-Holland.
[4] Jean Pierre Aubin, *Optima and equilibria*, Springer Verlag.
[5] David Carfì, *Optimal boundaries for decisions*, Atti dell'Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali Vol. LXXXVI, (2008)
[6] David Carfì, *Payoff space and Pareto boundaries of C1 Games*, APPS vol. 2009