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Carfì, David

University of Messina, University of California at Riverside

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OPTIMAL BOUNDARIES FOR DECISIONS

DAVID CARFÌ

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ABSTRACT. In this paper we state and prove some new results on the optimal boundaries. These boundaries (called Pareto boundaries too) are of increasing importance in the applications to Decision Theory. First of all the Pareto boundaries are the first and most important generalization of the concept of optimum; on the other hand, if f is a real functional defined on a non empty set X and K is a part of X, the determination of the optimal boundaries of the part K with respect to some preorder $\leq$ of X for which f is strictly increasing permits to reduce the optimization problem $(f, K$,inf) (or $(f, K$,sup)) to the problem $(f$,min$^P(K)$,inf) (resp. $(f$,max$^P(K)$,sup)), where by min$^P(K)$ we denoted the minimal boundary of K (that in general is greatly smoller than K).

1. Introduction

In section 2 we introduce Pareto optima in the most general context of preordered spaces. Unfortunately, the theory of optimal boundaries in the literature (see [1], [2] and [3]) is developed in ordered spaces and not in the more general context of preordered spaces. The results and definitions of the paper must be considered original also in those cases in which they are, apparently, simple generalizations of the existing theoretical developments, since the extensions are often not obvious. In the section we prove the characterization of maximal element by indifference (in the mathematical treatises this relation is often ignored, just because the theory is developed in ordered spaces). In section 3 we show some basic relations among Pareto optima. In section 4 we show the elementary relation among optima and Pareto optima. In section 5 we recall the concept of cofinal part (see [1]), it is often underestimated in Decision Theory, we note his relevance in this direction and we introduce (the definition is original) the smallest cofinal part of a preordered space. In section 6 we present one of the goals of the paper: the characterization of the maximal boundaries by cofinal parts and smallest cofinal part in ordered spaces. Section 7 is devoted to the extension of the above characterization to preordered spaces that are not ordered. Section 8 contains another goal of the paper: the cofinality of the maximal boundaries of the compact subsets of the euclidean space with respect to the usual order of this space. In section 9 we give a significant generalization to general preordered spaces of the basic theorem presented in section 8. In section 10 we show when a maximal boundary is contained in the topological boundary; some interesting examples follow. In section 11 we present some application to Optimization: a theorem about the weakly increasing functional is proved and the shadow minimum of the Cournot duopoly is determined.

2. Pareto optima

In Decision theory, when a problem has no upper optima (feasible choices weakly greater than each other) it is necessary to look for choices which are not strictly lower than any other choice: the so called optimal choices, introduced in the following definition.

Definition (of maximal element). *Let $(X, \leq)$ be a preordered space, S be a part of X and let $x_0 \in S$. The element x_0 is called* ***maximal*** *or* ***Pareto-maximum of*** S ***with respect to the preorder*** $\leq$ *if there is no $y \in S$ strictly preferred to x_0, i.e., such that $x_0 < y$. Analogously, are defined the* ***Pareto-minima of*** S*, those points x_0 of S such that there is no y of S strictly lower than x_0. The set of Pareto-maxima of S is denoted by $\max_{\leq}^{P}(S)$; the set of Pareto-minima is denoted by $\min_{\leq}^{P}(S)$. These two sets are called, respectively,* ***maximal Pareto-boundary of*** S *and* ***minimal Pareto-boundary****.*

Memento. Let $(X, \leq)$ be a preordered set and let x_0 and y be points of X. We remember that the relation $x_0 < y$ can be read "y is strictly preferred to x_0" and, by definition, it means that y is weakly preferred to x_0 and x_0 is not weakly preferred to y, in other terms, the relation $x_0 \leq y$ holds, but the opposite relation $x_0 \geq y$ does not. Analogously, the relation $x_0 > y$ means that the relation $x_0 \geq y$ holds but the opposite relation $x_0 \leq y$ does not. We observe that a strategy x is strictly preferred to another strategy y if it is weakly preferred to y but it is not indifferent to y.

Example 1. Let the plane $\mathbb{R}^2$ be endowed with its usual order $\leq$ (product of the usual lower order of the real line with itself). The Pareto maxima of the triangle with vertices the origin 0_2, the first canonical vector $e_1 = (1, 0)$ and the second canonical vector $e_2 = (0, 2)$, are the elements of the segment $[e_1, e_2]$. Note that the Pareto maxima are not pairwise comparable. The maximal boundary of the union U of the two segment $[e_1, 0_2]$ and $[0_2, e_2]$ is the set of the two canonical vectors e_1 and e_2; this maximal boundary is a totally disconnected subset of the union U.

Example 2. In the plane $\mathbb{R}^2$, endowed with the usual order, the maximal boundary of a closed circle centered at the origin is the part of the boundary contained in the first quadrant of the plane; on the contrary, the minimal boundary is the part of the topological boundary contained in the third quadrant. Note that the maximal (minimal) points are not comparable among them.

The maximal elements can be characterized as it follows.

Theorem 1 (characterization of the maximal elements). *An element x_0 of a preordered space is a maximal element of the space if and only if all the elements weakly greater than x_0 are indifferent to x_0.*

Proof. Let x_0 be maximal and let x be weakly preferred to x_0; $x \geq x_0$ means that x is strictly preferred to x_0 or x is indifferent to x_0, but x cannot be strictly preferred to x_0, and then x is indifferent to x_0. Conversely, suppose that every strategy weakly preferred to x_0 is indifferent to x_0; by contradiction, if a strategy x is strictly preferred to x_0, it is a fortiori weakly preferred to x_0, and, by assumption, it follows that x is indifferent to x_0, but this is an absurd because the strict preference cannot imply the indifference. ■

3. Relations among Pareto optima

In the preceding examples we noted that maximal elements was not comparable among them, this circumstance is completely explained by the following result.

Theorem 2 (not-comparability and indifference of maximal elements). *Let S be a part of a preordered set $(X, \leq)$. Then,*

i) two Pareto maxima of S, not-indifferent among them, are not comparable;

ii) if $(X, \leq)$ is an ordered space, two Pareto maxima of S, not equals, are not comparable;

iii) if $(X, \leq)$ is a totally preordered space, every Pareto maximum is a maximum of S;

iv) if $(X, \leq)$ is a totally ordered set, a Pareto maximum of S, if it exists, is unique and it is the (unique) maximum of S.

Proof. (i) Let assume, arguing by contradiction, that two strategies of Pareto x and y be not indifferent and comparable; consequently, one and only one of the two relations $x < y$ or $y < x$ must hold (indifference is excluded by assumption); but this deduction goes against the optimality of x or the optimality of y. (ii) is a direct consequence of the preceding (i). (iii) If $(X, \leq)$ is a totally preordered set, the not comparability is impossible; hence, if x_0 is a Pareto maximum, for every x in S, the only possible relation is $x \leq x_0$; thereby, x_0 is a maximum of S. (iv) follows by (iii). ■

4. Relations among maxima and maximal

If a preordered set of strategies has a maximum boundary (the set of all maxima), are Pareto maxima different (and then misleading) solutions to the decision problem? Fortunately the concept of maximal element is a good generalization of the concept of maximum, as shows the following result.

Theorem 3 (on the relations among maxima and maximal). *Let S be a part of a preordered space $(X, \leq)$. Then,*

i) every maximum of S is a maximal element of S;

ii) if S has a maximum, an element is a maximal element of S if and only if it is a maxima of S, and consequently two maximal elements are indifferent among them;

iii) if the space $(X, \leq)$ is ordered and S has the maximum, there exists a unique Pareto maximum (coinciding with the maximum).

Proof. i) Let M be a maximum of S; by definition, M dominates weakly every elements of S; to prove that M is a Pareto maximum we have to prove that every element weakly dominating M is indifferent to M. Let x be an element weakly dominating M, by definition of indifference, it must be indifferent to the maximum M. ii) If M is a maximum of S, M weakly dominates, in particular, all the Pareto maxima and then, for the characterization of the maximal elements, is indifferent to them. iii) In the ordered spaces the indifference turns into equality. ■

Remark (about Pareto optima). To verify that a strategy x_0 is a Pareto maximum of a part S of a preordered space, we have to consider the set of upper bounds of x_0; if such set meets S at points not indifferent with x_0, then such points are strict upper bounds of x_0, and thus x_0 is not a Pareto maximum; if, on the contrary, the intersection of the set of upper bounds with the part S contains only x_0 or points indifferent with x_0, then x_0 is Pareto maximum. Analogously one proceeds for Pareto minima.

Remark (about maxima). To verify that a strategy x_0 is a maximum of a part S of a preordered set, we have to consider the set of lower bounds of x_0; if such set contains S, then x_0 is a maximum. Analogously one proceeds for minima.

5. Cofinal parts and the smollest cofinal part

Of a certain interest is, in Decision Theory, the concept of cofinal part: it generalizes, in another direction (with respect to the maximal boundary), the concept of maximum.

Definition (of cofinal part). *Let $(X, \leq)$ be a preordered space and C be a part of X. The part C is defined* ***cofinal part of X with respect to the preorder*** $\leq$ *if, for every element $x \in X$, there exists an element c of C weakly preferred to x, i.e., such that $x \leq c$. A cofinal part of a subset S of the preordered space is the cofinal part of the subspace $(S, \leq_S)$ (where $\leq_S$ is the preorder induced by the space on S).*

Example 3 (of cofinal part). In the plane $\mathbb{R}^2$, the maximal boundary of the triangle with vertices the origin 0_2 and the two canonical vectors e_1, e_2, with respect to the usual order, is the segment $[e_1, e_2]$; it is a cofinal part of the triangle.

Remark (decisional). A preordered space is a cofinal part of itself (it is said *the trivial cofinal part* of the space). From a point of view of decision Theory, a cofinal part of a preordered space the smaller is the more is good. For instance, in an ordered space, a subset S of the space has a maximum if and only if there exists a cofinal part of S reduced to one point. More generally, a preordered space has a maximum (or a maxima boundary) if and only if there exists a cofinal part whose elements are in the same class of indifference.

The above remark justifies the following definition.

Definition (of smallest cofinal part). *Let $(X, \leq)$ be a preordered space and let C be a part of X. The part C is defined* ***the smallest cofinal part of X with respect to the preorder*** $\leq$ *if it is cofinal and it is contained in each cofinal parts of the space. In other words, the smallest cofinal part of the space is, if it exists, the minimum of the family of cofinal parts of the space with respect to the set-inclusion order.*

Obviously a space can be lacking in the smallest cofinal part: for instance an open ball in the n-space or the real line endowed with their natural order. But if the smallest cofinal part of a space there is, then it is unique (the set-inclusion is an order). Moreover we have the following result.

Theorem 4. *If the smallest cofinal part of the space there is, then it coincides with the intersection of all the cofinal parts of the space.*

Proof. It's trivial because the intersection of the family of the cofinal part of the space is the infimum of the family with respect to the set-inclusion, and if the minimum there exists it must be the infimum. Alternatively, if C_0 is the smallest cofinal part it is contained in the intersection, being contained in every element of the family; conversely, C_0 is cofinal and then it must contain the intersection of cofinal part. ■

6. Relations among cofinal subsets and the maximal boundary

Let us see the relations among the cofinal parts of an ordered space and the maximal boundary of the space itself.

Theorem 5. *Each cofinal part of an ordered space contains the maximal boundary of the space.*

Proof. Let C be a cofinal part of an ordered space and let x be a maximal element of the space itself; since the part C is cofinal there exists an element c_x of C weakly preferred to x; but this means (by definition of maximal element) that x coincides with the element c_x, and thus it belongs to C. ■

Actually, we can say more.

Theorem 6 (cofinal characterization of the maximal boundary of an ordered space). *The intersection of all the cofinal parts of an ordered space coincides with the maximal boundary of the space, and hence, if there is a maximal element of the space, such intersection is non-empty. In other words, the maximal boundary of an ordered space $(X, \leq)$ is the infimum of the set of cofinal parts of the ordered space with respect to the set-inclusion in the power-set $\mathcal{P}(X)$.*

Proof. The preceding theorem assures that the maximal boundary is contained in the intersection of all the cofinal parts of the space. Let us prove the converse, i.e., that the intersection of the cofinal parts is contained in the maximal boundary. If the intersection is void we have nothing to prove; on the contrary, let c be an element of such intersection and let C be a cofinal part of the space containing c. By contradiction, assume c be not maximal, then there is an element x of the space strictly greater than c. Consider the part C_x of the space obtained from C replacing c with x (observe that the element x, being strictly greater than c, is a fortiori different from c, so the two parts C and C_x are different). Such new part C_x is cofinal too; indeed, if y is any element of the space, there is an element c_y of the part C weakly preferred to y; such c_y is different from c, and in this case it belongs also to C_x, or else c_y is the element c, and in this case we have $y \leq c < x$; in every case y is weakly dominated by an element of C_x, which is, thus, a cofinal part of the space. But this conclusion is an absurd, since the part C_x does not contains c, on the contrary, c must belong to every cofinal part of the space. From the absurd it follows that c must be necessarily maximal. ■

Another way to state the above characterization is the following one.

Corollary. *The maximal boundary of an ordered space is the greatest part of the space contained in each cofinal part of the space.*

Note that, in general, the intersection of two cofinal parts is not necessarily cofinal and can be empty too.

Example 4. In the ordered space of natural numbers the set of even numbers and the set of odd numbers are cofinal parts and they are disjoints. In the ordered space of negative real numbers the sets $\{-1/2n\}_{n\in\mathbb{N}}$ and $\{-1/\left(2n+1\right)\}_{n\in\mathbb{N}}$ are cofinals and disjoints.

Corollary. *Assume there is the smallest cofinal part of an ordered space, then it coincides with the maximal boundary of the space. Moreover, an ordered space has the smallest cofinal part if and only if the maximal boundary of the space is cofinal.*

7. Saturated parts with respect to an equivalence relation

A certain interest has, in Decision Theory, the concept of saturated part with respect to an equivalence relation. In particular, it help us in the solution of the problem exposed in the following example, where the space is preordered but not ordered.

Example 5 (cofinal part not containing the maximal boundary). Consider the plane endowed with the preorder induced by the real function defined by $f(x,y)=xy$, for every point (x,y) of the plane. The interval $[0_2,(1,1)]_{\leq_f}$ has the singleton $\{(1,1)\}$ as cofinal part (by definition of interval, $(1,1)$ is a maximum of the interval); nevertheless, such cofinal part not contains the maximal boundary of the interval I (which is the entire level 1 of the function f).

Thus, in a preordered space which is not ordered, the cofinal parts not necessarily contain the maximal boundary; the saturated parts allow us to delineate again the context.

Definition (of saturated part with respect to an equivalence). *Let X be a non-empty set, $\mathcal{E}$ be an equivalence relation on X and let S be a part of X. The part S is called* ***saturated part of X with respect to the equivalence $\mathcal{E}$*** *if, for each element $s\in S$, all the elements of X, which are equivalent to s modulo $\mathcal{E}$, belong to S. In other terms, a part is saturated with respect to the equivalence relation if contains the classes of equivalence generated by all its own elements.*

Definition (of saturated part in preordered spaces). *In a preordered set a part is said to be* ***saturated*** *if it is saturated with respect to the indifference of the space.*

First of all we note the following theorem.

Theorem 7. *The maximal boundary of a preordered space is saturated.*

Proof. If x is a point of the space indifferent to a maximal element x_0 and if, by contradiction, y is a point of the space strictly dominating x, then y must be (by transitive) strictly dominating x_0, and this is impossible. So x belongs to the maximal boundary. ■

Theorem 8. *Every saturated cofinal part of a preordered space contains the maximal boundary of the space.*

Proof. Let C be a cofinal saturated part of the space and let x be a maximal element of the space itself; since C is cofinal there is an element c_x of C weakly greater than x, but this means (by definition of maximal element) that x is indifferent to the element c_x and hence, being C saturated, that x belongs to C. ■

Theorem 9 (cofinal characterization of the maximal boundary of a preordered space). *The intersection of all the saturated cofinal parts coincides with the maximal boundary of the space, and hence, if there is a maximal element of the space, such intersection is non-empty.*

Corollary. *Assume there is the smallest saturated cofinal part of a preordered space, then it coincides with the maximal boundary of the space. Moreover, a preordered space has the smallest saturated cofinal part if and only if the maximal boundary of the space is cofinal. The maximal boundary of a preordered space is the greatest part of the space contained in each saturated cofinal part of the space.*

8. Cofinality of the maximal boundaries of compacts in $\mathbb{R}^n$

In this section we shall conclude the study of the Pareto boundaries of compacts in $\mathbb{R}^n$ with respect to the usual order. We recall preliminarily the following important result

Theorem 10. *A strictly increasing real functional defined on a preordered space attains its maximum only on the maximal boundary of the space.*

Theorem 11 (cofinality of the maximal boundary of a compact in $\mathbb{R}^n$). *Let K be a compact part of $\mathbb{R}^n$. Then, the maximal boundary of K, with respect to the usual order of $\mathbb{R}^n$, is non-empty and cofinal for K with respect to the usual order of $\mathbb{R}^n$. Consequently, there exists the smallest cofinal part of K (and it coincides with the maximal boundary of K).*

Proof. Existence of Pareto maxima. Let f be a real linear functional strictly increasing on the space $\mathbb{R}^n$; since f is continuous, for the Weierstrass theorem, it admits a point of maximum in the compact K; since f is strictly increasing, every point in which f assumes his maximum shall be Pareto maximum of K; consequently, the Pareto maximal boundary contains all the maximum-points of every strictly increasing liner functional on the compact K. *Cofinality.* Let x_0 be a point of K and let K' be the intersection of the set of upper bounds of x_0 with K, in other terms the set of upper bounds of x_0 belonging to K. The part K' is non-empty (it contains x_0) and compact, since it is the intersection of the cone of upper bounds of x_0 (which is closed) and of the compact K. Thereby, by the preceding result of existence, K' has a maximal element p; such element p is a Pareto maximum also of K; indeed, if a point x of K dominates weakly p, hence it weakly dominates x_0 too (since p is an upper bound of x_0); then x, dominating x_0 (which belongs to K), belongs to K', and then it must be indifferent to the maximal element p; concluding, each point x_0 of K is dominated by a Pareto maximum of K. ■

Concerning the above theorem, if the part S is not compact, then the maximal boundary is not necessarily be cofinal; as shows the following example.

Example 6 (parts not compact with maximal boundary not cofinal). A part of the plane is compact if and only if it is closed and bounded (by the Borel-Lebesgue theorem). We shall show that none of the two assumptions can be omitted. *Bounded but not closed part.* Consider, in $\mathbb{R}^2$, the triangle T convex envelope of the origin and of the two canonical vectors e_1 and e_2, lacking of the segment $[e_2, (1/2, 1/2)]$: the part T is bounded but not closed. The maximal boundary, with respect to the usual order, of the part T is the segment $[e_1, (1/2, 1/2)[$; it is not cofinal, indeed, the point $(0, 1/2)$ belongs to T but it is not dominated by any maximal element. *Closed but not bounded part (upper).* Let S be the part of the plane union of the triangle convex envelope of the origin and of the two canonical vectors e_1 and e_2 with the strip $\mathbb{R}_{\geq} \times [0, 1/2]$: S is closed but not upper bounded. The maximal boundary of S is the segment $[e_2, (1/2, 1/2)[$; it is not cofinal, indeed, the point e_1 is not dominated by any maximal element.

Actually, if the part S is not compact, the maximal boundary can be the void set, as shows the following example.

Example 7 (of parts non compact with void maximal boundary). A part of the plane is compact if and only if it is closed and bounded. We shall show, as in the above example, that none of the assumptions can be omitted. *Bounded but not closed part.* Consider, in $\mathbb{R}^2$, the triangle T convex envelope of the origin and of the two canonical vectors e_1 and e_2, lacking of the segment $[e_2, e_1]$: the part T is bounded but not closed. The maximal boundary, with respect to the usual order, of the part T is void. *Closed but not (upper) bounded part.* Let S be the first quadrant of the plane; S is closed but not upper bounded and the maximal boundary of S is void.

Theorem 12 (about maximum points of functionals weakly increasing in $\mathbb{R}^n$). *A real upper semicontinuous weakly increasing functional defined on a compact K of $\mathbb{R}^n$, assumes its maximum in every cofinal part of the compact K and on the maximal boundary of K.*

Proof. Each real functional satisfying the assumptions attains his maximum, since it is upper semicontinuous on the compact K (Weierstrass theorem). Moreover, let x_0 be a maximum point of f and let C cofinal in K, since C is cofinal there is in C an upper bound x of x_0, and being f weakly increasing, f attains its maximum also in x. Since the maximal boundary of S (by the above theorem) is a cofinal part of S, the functional must attain its maximum on the maximal boundary too. ■

In general, a weakly increasing real functional can attain its maximum out of the maximal boundary of its domain, as the following example shows.

Example 8. Consider the canonical 2-simplex $\mathrm{conv}(0_2, e_1, e_2)$ and consider its interior with the point e_1, say S this union, the maximal boundary of S is $\{e_1\}$; well, now consider the real function f on the plane such that $f(x, y) = 1$ on the set of upper bounds of the point $(1/4, 1/2)$ and 0 elsewhere; the function f is upper semicontinuous (verify) and weakly increasing, but it does not attain his maximum in the point e_1.

9. Topological preordered space and Pareto boundaries

Definition (of topological preordered space). *A triple $(X, \leq, \mathcal{T})$ is said* ***topological preordered space*** *if X is a non void set, $\leq$ is a preorder on X, $\mathcal{T}$ is a topology on X, and the sets of upper bounds and of lower bounds of every element of X, with respect to the preorder $\leq$, are closed in the topology $\mathcal{T}$. If the sets of upper (lower) bounds of each element of X, with respect to the preorder $\leq$, are closed in the topology $\mathcal{T}$ the preorder $\leq$ is said* ***upper (lower) compatible with the topology*** *$\mathcal{T}$.*

Example 9. The plane endowed with its usual order and with its usual topology is a preordered topological space. A topological space endowed with the preorder induced by a continuous real function defined on itself is a preordered topological space, since the sublevels and the upper-levels of a continuous function are closed.

Theorem 13. *Let $(X, \leq)$ be a preordered space endowed with a topology upper compatible with the preorder; assume there is on X at least a real upper semicontinuous strictly increasing functional. Then, the maximal boundary of a compact S is non-empty and cofinal for S.*

We conclude this section with some counter-example. The maximal boundary of a closed part of a topological preordered space is not necessarily closed, neither if the part is connected by arcs. Moreover, if the part is connected by arcs its Pareto boundaries are not necessarily connected. The following example will explain the cases.

Example 10 (about the maximal boundary of a closed part). Let S be the part of the plane union of the two segments $[e_2, e_1]$ and $[e_1, 2e_1]$; the part S is closed and connected, with respect to the natural topology of the plane; the maximal boundary of S is the segment $[e_2, e_1[$ with the point $2e_1$, it is not closed and not connected.

If a part of a preordered space is upper bounded its maximal boundary is obviously upper bounded, but it is not necessarily lower bounded, as shows the following example.

Example 11 (about the maximal boundary of an upper bounded part). If a part of the plane is upper bounded but not lower bounded, with respect to the usual order of the plane, its maximal boundary is not necessarily bounded. Indeed, let S be the ipograph of the real function f defined on the interval $]-\infty, 0[$ of the real line by $f(x) = 1/x$. The maximal boundary of S is the graph of the function f, which is upper bounded but not lower bounded with respect to the usual order of the plane.

10. Topological boundaries and Pareto boundaries

We prove a property concerning the relation among Pareto and topological boundaries.

Theorem 14. *Let $(X, \leq)$ be a preordered space and let $\mathcal{T}$ be a topology on X. Assume that every point of X belongs to the $\mathcal{T}$-closure of the set of its own strict upper (lower)* *bounds. Then, the maximal (minimal) boundary of a subset S of X is contained in the (topological) $\mathcal{T}$-boundary of S.*

Proof. Let x be a maximal element of S and let U be a neighborhood of x, we must prove that U meets S and its complement. The neighborhood U meets S since both U and S contains x. On the other hand, by assumption, there exists a strict upper bound of x, say y, belonging to U (since every neighborhood of x must intersect the set of strict upper bounds of x). Assume, by contradiction, that U is included in S, then the strict upper bound y must belong to S, and this goes against the maximality of x in S. ■

Example 12. The plane endowed with its usual order enjoys the assumption of the above theorem. More generally, the plane endowed with the conjunction of the preorders induced by two different linear functionals on the plane enjoys the assumption of the above theorem.

On the contrary, we have the following counter-example.

Example 13. Consider the function f defined on the plane by $f(p) = -|p|^2$, for every point p of the plane, where $|\cdot|$ is the euclidean norm. The induced preorder $\leq_f$ does not enjoy the assumption; in fact, the set of strict upper bounds of the origin $(0,0)$ is void, and consequently the origin cannot belong to the closure of this set. Note, that the maximal boundary of the square $[0,1]^2$ with respect to this preorder is reduced to the origin, which is not in the boundary of the square.

Example 14. Let S be the set of natural numbers endowed with its canonical order and its canonical topology. The point n of S is not in the closure of the set $]n, \rightarrow[_S$ that is the interval $[n+1, \rightarrow[_S$.

11. Application to optimization problems

This section is devoted to the possible application of the study of Pareto boundary to the solution of optimization problems.

First of all we give the following result (see [4]).

Theorem 15. *Let f be a real differentiable functional on the space $\mathbb{R}^n$. Let U be an open connected subset of $\mathbb{R}^n$ with compact cloasure such that*

i) for every x in U, the gradient of f in x has each component different from zero;

ii) for every index i, the sign of the partial derivative $\partial_i f$ is constant on the open U.

Then we have:

1) the functional f is strictly increasing on U with respect to the preorder induced by the differential $df(x_0)$ (which is a linear real functional) where x_0 is any point of U;

2) the functional f is weakly increasing on the cloasure of U;

3) attains its maximum at least on the maximal boundary of the compact $\mathrm{cl}(U)$ and only on the boundary of U.

In the below example we follow the definitions of [5] and [6].

Example 15 (the Cournot duopoly). We shall determine the shadow minimum in the Cournot duopoly with biloss function $f : [0,1]^2 \to \mathbb{R}^2$ such that

$$f(x,y) := (x(x+y-1), y(x+y-1)),$$

for every bistrategy (x,y). Let us see the minimum of the first component of f. We study the gradient of f_1 to determine the part of the domain of f_1 where f_1 is strictly increasing (with respect to appropriate preorders), we have

$$\nabla f_1(x,y) = (2x+y-1, x),$$

$\partial_1 f_1$ it is strictly positive in the part of its domain with $y > 1-2x$, and strictly negative in the part of the domain with $y < 1-2x$; $\partial_2 f_1$ is strictly positive in the part of the domain with $x > 0$. Consequently, the function f_1 is strictly increasing on the convex envelope

$$K_> = \mathrm{conv}(e_1, e_2, (1,1), e_1/2),$$

with respect to the usual preorder and weakly increasing on the convex envelope

$$K_< = \mathrm{conv}(0_2, e_2, e_1/2),$$

with respect to the costs-benefit preorder and stricly increasing on $K_< \setminus [0_2, e_2]$. Concluding the minimum-point of the function f_1 is the point $e_1/2$, since the minimal boundary of the convex envelope $K_>$ with respect to the usual preorder is the segment $S = [e_2, e_1/2]$ and the minimum point is necessarily on the topological boundary of the square (the gradient of f_1 never vanishes on the interior of the bistrategy square). Then the two candidates for minimum-point are the end-points of the segment S. The minimal boundary of the convex envelope $K_<$ is the point $e_1/2$ with respect to the costs-benefits order and then the f_1 attains its minimum in the bistrategy $e_1/2$ on $K_<$. Concluding the minimum of f_1 on K is the value $f_1(e_1/2) = -1/4$ and $e_1/2$ is the only minimum-point of f_1 in K. Analogously we can procede for f_2 and so the shadow minimum shall be the pair $(-1/4, -1/4)$.

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David Carfì
Università degli Studi di Messina
Dipartimento "Vilfredo Pareto"
Via Tommaso Cannizzaro, 278
98122 Messina, Italy
E-mail: davidcarfi71@yahoo.it

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