Non-reactive strategies in decision-form games

Carfì, David and Ricciardello, Angela

University of Messina, University of California at Riverside

2009

DOI: 10.1478/C1A0902002

Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. LXXXVII, No. 2, C1A0902002 (2009)

NON-REACTIVE STRATEGIES IN DECISION-FORM GAMES

DAVID CARFÌ [a*] AND ANGELA RICCIARDELLO [b]

ABSTRACT. In this paper we propose a concept of rationalizable solution for two-player decision-form games: the solution by iterated elimination of non-reactive strategies. Several original theorems are proved about this kind of solution. We study the relations between solutions by iterated elimination of non reactive strategies and game equilibria. We present an existence theorem for bistrategies surviving the iterated elimination and an existence theorem for solution by iterated elimination in contracting games. We, also, show that an equilibrium of a game survives iterated elimination of non reative strategies. At the end we prove a characterization of solvability by iterated elimination of non-reactive strategies.

1. Introduction

The concept of solution by iterated elimination of non-reactive strategies, for two-player decision-form games was conceived by one of the authors and presented in [1], where two-player decision-form games were introduced. In this paper decision rules are used in the sense introduced by J. P. Aubin in [2] and [3], and they represent the *action-rationality*, the *behavioural way* itself, of each player in front of the competitive situation represented by the game. For different concepts of rationalizable solution, for instance solutions obtained by elimination of dominated strategies, the reader can see in [4], [5] and [6].

2. Preliminaries, notation and terminology

The context. We deal with *two-player games*. We shall consider two non-void sets E and F, viewed as the respective sets of strategies at disposal of two players. The aim is to form ordered pairs of strategies $(x, y) \in E \times F$, called *strategy profiles* or *bistrategies*, via the (individual or collective) selection of their components x and y, done by the two players in the sets E and F, respectively, in order that the strategy x of the first player is a good reaction to the strategic behavior y of the second player and vice versa.

Let us formalize our starting point.

Definition (strategy base and bistrategy space). *Let (E, F) be a pair of non-empty sets, we call it* ***strategy base of a two-player game****. The first set E is said* **the first player's** ***strategy set****; the second set F is said* **the second player's** ***strategy set****. Any element x of E*

*is said a **first player's strategy** and any element y in F is said a **second player's strategy**. Every pair of strategies $(x, y) \in E \times F$ is said **a bistrategy of the strategy base** (E, F) and the cartesian product $E \times F$ is said **the bistrategy space of the base** (E, F).*

Interpretation and terminology. We call the two players of a game Emil and Frances: Emil, simply, stands for "first player"; Frances stands for "second player". Emil's aim is to choose a strategy x in the set E , Frances' aim is to choose a strategy y in F, in order to form a bistrategy (x, y) such that the strategy x is an Emil's good response to the Frances' strategy y and vice versa.

Definition (decision rule). *Let (E, F) be a strategy base of a two-player game. An **Emil's decision rule on the base** (E, F) is a correspondence from F to E, say $e : F \to E$. Symmetrically, a **Frances' decision rule on the base** (E, F) is a correspondence from E to F, say $f : E \to F$.*

Let us formalize the basic concept of our discourse.

Definition (decision-form game). *Let (E, F) be a strategy base of a two-player game. A **two-player decision-form game on the base** (E, F) is a pair (e, f) of decision rules of the players Emil and Frances, respectively, on the strategy base (E, F).*

Definition (of possible reaction and of capability of reaction). *Let (e, f) be a decision-form game. Let y be a Frances' strategy, the elements of the image of y by the correspondence e (that is, the elements of the set $e(y)$), i.e., the direct corresponding strategies of y by the rule e, are called **Emil's possible responses**, or* **Emil's** ***possible reactions, to the Frances' strategy** y. Analogously, let x be an Emil's strategy, the elements of the image of x by the decision rule f (that is, the elements of the set $f(x)$), i.e. the direct corresponding strategies of x by the rule f, are said* **Frances'** ***possible responses**, or* **Frances'** ***possible reactions, to the Emil's strategy** x. The set of Emil's possible reactions (responses) to the Frances' strategy y is said the **Emil's reaction set to the Frances' strategy** y. Finally, we say that **Emil can react to the Frances' strategy** y if the corresponding reaction set $e(y)$ is non-void.*

Definition (of a disarming strategy). *Let (e, f) be a game. The Emil's strategies x to which Frances cannot react, i.e. such that the image $f(x)$ is empty, are called **Emil's disarming strategies (for Frances)**. The Frances' strategies y to which Emil cannot react, namely such that the reaction set $e(y)$ is empty, are called **Frances' disarming strategies (for Emil)**.*

We now introduce another fundamental notion, that of subgame.

Definition (of subgame). *Let $G = (e, f)$ be a decision-form game with strategy base (E, F) and let (E', F') be a **subbase** of (E, F), namely a pair of subsets of E and F, respectively. We call **subgame of G with strategy base** (E', F') the pair (e', f') of the restrictions of the decision rules e and f to the pairs of sets (F', E') and (E', F'), respectively. It is important to remember that e' is the correspondence from F' to E' which associates with every strategy y' in F' the part $e(y') \cap E'$. In other words, it sends every* *strategy y' of F' into the corresponding Emil's reaction strategies to y' which belong to E'. We also call the subgame (e', f') the **restriction of the game G to the strategy pair** (E', F').*

3. Reactive strategies

In a decision-form game, if a certain player's strategy s does not react to any strategy of the other one, this strategy s can't be a reasonable action of the first player. For this reason, we are motivated to formalize, in the below definition, the concept of non-reactive strategy.

Definition (of a reactive strategy). *Let (e, f) be a two-player decision-form game. Let y_0 be a Frances' strategy, we call it **reactive (with respect to the decision rule f)** if it is a possible reaction to some Emil's strategy. In other words, a Frances' strategy y_0 is called reactive (with respect to f), if it belongs to the set $f(x)$, for some Emil's strategy x. A Frances' strategy is called **non-reactive** if it is not reactive. Analogously, let x_0 be an Emil's strategy, we call it **reactive (with respect to the decision rule e)** if it is a possible reaction to some Frances' strategy. In other words, an Emil's strategy x_0 is called reactive (with respect to e), if it belongs to the set $e(y)$, for some Frances' strategy y. An Emil's strategy is called **non-reactive** if it is not reactive.*

Remark (on the sets of reactive strategies). Emil's and Frances' sets of respective reactive strategies are the two unions $\cup e := \bigcup_{y \in F} e(y)$ and $\cup f := \bigcup_{x \in E} f(x)$, i.e., the images of the correspondences e and f, respectively. Note that, for example, with the correspondence $e : F \to E$ it is, in a standard way, associated with the mapping $M_e : F \to \mathcal{P}(E)$, sending any Frances' strategy y into the reaction set $e(y)$. The mapping M_e is, therefore, a family of subsets of E indexed by the set F. Analogously, for the correspondence f, we can consider the family $M_f = (f(x))_{x \in E}$. So, the above two unions are the unions of the families M_e and M_f, respectively.

Example (of reactive and non-reactive strategies). Let (e, f) be a two-player decision-form game, let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of the players and let the decision rules $e : F \to E$ and $f : E \to F$ be defined by

$$e(y) = \begin{cases} \{-1\} & if \ \ y < 0 \\ E & if \ \ y = 0 \\ \varnothing & if \ \ y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & if \ \ x < 1 \\ \varnothing & if \ \ x = 1 \\ \{1\} & if \ \ x > 1 \end{cases},$$

for each bistrategy (x, y) of the game. All of Emil's strategies are reactive, since $\cup e = E$. Otherwise, only the Frances' strategies -1 and 1 are reactive, since $\cup f = \{-1, 1\}$.

4. Reduced games by elimination of non-reactive strategies

Definition (of a reduced game by elimination of non reactive strategies). *A game (e, f) is called **reduced by elimination of non-reactive strategies** if the images of the decision rules e and f are the strategy sets E and F, respectively. In other words, the game is reduced if the decision rules of the two players are onto.*

Example (of a non reduced game). Let (e, f) be a decision-form game, let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of the two players and let the decision rules

$e : F \to E$ and $f : E \to F$ be defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ \{-1, 2\} & if \;\; y = 0 \\ \{2\} & if \;\; y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \{0\} & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases},$$

for every bistrategy (x, y). The images of the rules e and f are the sets $\{-1, 2\}$ and $\{-1, 0, 1\}$; so, the game is not reduced by elimination of non-reactive strategies.

5. Elimination of non-reactive strategies

In a game, a rational behavior of the players is to use only reactive strategies, eliminating the non-reactive ones. So, they will play a subgame of the previous one, that we call reduction of the game by elimination of non-reactive strategies.

Before defining the reduction of a game we recall that, if $F : X \to Y$ is a correspondence and if X' and Y' are subset of X and Y, respectively, the restriction to the pair (X', Y') of F is the correspondence $F_{|(X',Y')}$ whose graph is $gr(F) \cap (X', Y')$.

Definition (the reduction of a game by elimination of non-reactive strategies). *Let (e, f) be a decision-form game on the strategy base (E, F). We call **(first) reduction of the game** (e, f) **by elimination of non-reactive strategies** the subgame (e', f') on the subbase $(e(F), f(E))$, pair of images of the decision rules e and f, respectively. In other words, the **(first) reduction of the game** (e, f) **by elimination of non-reactive strategies** is the game whose decision rules are the restrictions $e_{|(F',E')}$ and $f_{|(E',F')}$, where E' and F' are the images of the rules e and f.*

Example (of reduction). Let (e, f) be the game, on the base $E = [-1, 2]$ and $F = [-1, 1]$, with decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ \{-1, 2\} & if \;\; y = 0 \\ \{2\} & if \;\; y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \{-1, 0, 1\} & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases},$$

for every bistrategy (x, y) of the game. The images of the rules e and f are the sets $E_1 = \{-1, 2\}$ and $F_1 = \{-1, 0, 1\}$; so, the game is not reduced, since they don't overlap the spaces E and F, respectively. The (first) reduction of the game (e, f), by elimination of non reactive strategies, is the game whose decision rules $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ are defined by

$$e_1(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \{-1, 2\} & if \;\; y = 0 \\ \{2\} & if \;\; y = 1 \end{cases}, \quad f_1(x) = \begin{cases} -1 & if \;\; x = -1 \\ 1 & if \;\; x = 2 \end{cases}.$$

Note that the subgame (e_1, f_1) is not reduced (since f_1 is not onto). The second reduction of the game (that is, the reduction of the first reduction), has the rules

$$e_2(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \{2\} & if \;\; y = 1 \end{cases}, \quad f_2(x) = \begin{cases} -1 & if \;\; x = -1 \\ 1 & if \;\; x = 2 \end{cases},$$

on the base (E_2, F_2), where $E_2 = \{-1, 2\}$ and $F_2 = \{-1, 1\}$. In this case, both rules are onto and, so, the subgame $G_2 = (e_2, f_2)$ is reduced by elimination of non-reactive strategies.

6. Iterated elimination of non-reactive strategies

As we saw, the first reduction of a game can be non-reduced, so, we can consider the successive reductions to find a reduced subgame.

Definition (of k-th reduction by elimination of non-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0) and let k be a natural number. We define (recursively) the **k-th reduction, or reduction of order k, by elimination of non-reactive strategies of the game** G_0 as follows: the same game G_0, if $k = 0$; the subgame $G_k = (e_k, f_k)$ on the base (E_k, F_k), pair of the images of the decision rules of the $(k-1)$ reduction, i.e., pair of the sets $e_{k-1}(F_{k-1})$ and $f_{k-1}(E_{k-1})$, if $k \geq 1$. In other words, if $k \geq 1$, the decision rules e_k and f_k are the restrictions to the pairs (F_k, E_k) and (E_k, F_k) of the decision rules e_{k-1} and f_{k-1}, respectively. We say that a strategy $x_0 \in E$ survives the k-th elimination of non-reactive strategies if it belongs to E_k.*

Theorem (on the values of the reduced decision rules). *In the conditions of the above definition, for each strategy s of a player, which survived the k-th elimination of non-reactive strategies, the reaction set of the other player remains unchanged. In particular, if the game G_0 has not disarming strategies, all the reductions G_k has not disarming strategies.*

Proof. The first reduction (e_1, f_1) has strategy base $(e_0(F_0), f_0(E_0))$, so

$$e_1(y) = e_0(y) \cap e_0(F_0) = e_0(y),$$

for all Frances' strategy $y \in f_0(E_0)$. By induction we have

$$e_k(y) = e_0(y) \quad \text{AND} \quad f_k(x) = f_0(x),$$

for all k and for all bistrategy (x, y) in $E_k \times F_k$. ■

Definition (reducing sequence by elimination of non-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0). We define **reducing sequence by elimination of non-reactive strategies of the game** G_0 the sequence of reduced subgames $G = (G_k)_{k=0}^{\infty}$. In other words, it is the sequence with first term the game G_0 itself and with k-th term the k-th reduction of the game G_0.*

7. Solvability by iterated elimination

The reducing sequence allows us to introduce the concept of solvability and solution by iterated elimination of non-reactive strategies.

Definition (of solvability by iterated elimination of non-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a decision-form game and let G be its reducing sequence by elimination of non-reactive strategies. The game G_0 is called **solvable by iterated elimination of non-reactive strategies** if there exists only one bistrategy common to all the subgames of the sequence G. In this case, that bistrategy is called **solution of the game** G_0 **by iterated elimination of non-reactive strategies**.*

Remark. The definition of solvability by iterated elimination of non-reactive strategies means that the intersection of the bistrategy spaces of all the subgames forming the reducing sequence, that is the intersection

$$\bigcap_{k=0}^{\infty} E_k \times F_k,$$

has one and only one element, which we said the solution of the game.

Remark. If the game G_0 is finite, it is solvable by iterated elimination of non-reactive strategies if and only if there exists a subgame of the sequence G with a unique bistrategy. In this case, that bistrategy is the solution, by iterated elimination of non-reactive strategies, of the game G_0.

8. Example of resolution

In the following example we present a simple resolution by iterated elimination of non-reactive strategies of an infinite game.

Example (solution by elimination of non-reactive strategies). Let $E = [-1, 2]$ and $F = [-1, 1]$, and let $e : F \to E$ and $f : E \to F$ be the decision rules defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ E & if \;\; y = 0 \\ \varnothing & if \;\; y > 0 \end{cases} \quad , \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} .$$

By elimination of non-reactive strategies, we obtain the subgame G_1, with strategy sets $E_1 = E$ and $F_1 = \{-1, 1\}$ and multifunctions $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ defined by

$$e_1(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \varnothing & if \;\; y = 1 \end{cases} \quad , \quad f_1(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} .$$

In the new game only the Emil's strategy -1 is reactive. Deleting all the others, we obtain an other subgame with strategy sets $E_2 = \{-1\}$ and $F_2 = F_1$ and multifunctions $e_2 : F_2 \to E_2$ and $f_2 : E_2 \to F_2$ defined by

$$e_2(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \varnothing & if \;\; y = 1 \end{cases} \quad , \quad f_2(-1) = \{-1\} .$$

At last, Frances strategy 1 is, now, non-reactive, so, we have the trivial subgame with strategy sets $E_3 = F_3 = \{-1\}$ and multifunctions $e_3 : F_3 \to E_3$ and $f_3 : E_3 \to F_3$ defined by

$$e_3(-1) = \{-1\} , \quad f_3(-1) = \{-1\} .$$

We solved the game by iterated elimination of non-reactive strategies, and the solution is the unique survived bistrategy: the bistrategy $(-1, 1)$.

9. Iterated elimination in Cournot game

The game. Let $G_0 = (e_0, f_0)$ be the Cournot decision-form game with bistrategy space the square $[0, 1]^2$ and (univocal) decision rules defined, for every $x, y \in [0, 1]$, by $e_0(y) = (1-y)/2$ and $f_0(x) = (1-x)/2$. Set $x' = 1-x$ and $y' = 1-y$, the complement to 1 of the production strategies, for all bistrategy (x, y) of the game; briefly, we have $e_0(y) = y'/2$ and $f_0(x) = x'/2$.

The reducing sequence. Let $G = (G_k)_{k=0}^{\infty}$ be the reducing sequence by elimination of non-reactive strategies of the game G_0. The sequence G has as starting element the game G_0 itself. Let E_k and F_k be the strategy spaces of the k-th game $G_k = (e_k, f_k)$, for all natural k. The base of the game G_{k+1} is, by definition, the pair of images $e_k(F_k)$ and $f_k(E_k)$.

Reduction of the strategy spaces. The function e_k is strictly decreasing and continuous, so, the image of a real interval $[a, b]$ is the compact interval $[e_k(b), e_k(a)]$. The initial strategy spaces E_0 and F_0 are intervals, then, by induction, all the spaces $E_k = F_k$ are intervals. Let $[a_k, b_k]$ be the k-th strategy space E_k, then, concerning the $(k+1)$-th, we have

$$F_{k+1} = e_k([a_k, b_k]) = [e_k(b_k), e_k(a_k)] = [e_0(b_k), e_0(a_k)] = [b'_k/2, a'_k/2] \, .$$

The interval $[b'_k/2, a'_k/2]$ does not coincide with the space $F_k = E_k$: the reduction G_k is not reduced.

Iterated reduction. We now study the sequence of the initial end-points of the interval family $(a_k, b_k)_{k=1}^{\infty}$. We have

$$a_{k+1} = e(b_k) = e(f(a_{k-1})).$$

The composite function $e \circ f$ is increasing and, moreover, the following inequalities $a_0 \leq a_1 \leq a_2$ hold true. So, the sequence a is increasing, and, then, it is convergent (as it is upper bounded by b_0). Moreover, being

$$a_{k+1} = \frac{1 - b_k}{2} = \frac{1 - a'_{k-1}/2}{2} = \frac{2 - a'_{k-1}}{4} = \frac{1 + a_{k-1}}{4},$$

and putting $a^* := \lim(a)$, we deduce

$$4a^* = 1 + a^*,$$

which gives $a^* = 1/3$. Similarly, we prove that b is a decreasing sequence and it converges to $1/3$.

Solution. Concluding, the unique point common to all the strategy intervals is the strategy $1/3$, in other terms we have

$$\bigcap_{k=0}^{\infty} [a_k, b_k] = \{1/3\} \, .$$

Then, *the game is solvable by elimination of non-reactive strategies and the solution is the bistrategy* $(1/3, 1/3)$.

10. Iterated elimination survival

In this section, we deal with the relations between solutions by iterated elimination of non-reactive strategies and game equilibria.

We introduce some definitions.

Definition (of survival the iterated elimination). *Let $G = (e, f)$ be a decision-form game. We say that a bistrategy* ***survives the iterated elimination of non-reactive strategies*** *if it belongs to all the bistrategy spaces of the reducing sequence of the game G.*

Terminology. We say that a base-game (E, F) is compact if E and F are both compact.

Theorem (existence of bistrategies surviving the iterated elimination). *Let $G = (e, f)$ be a game on a strategy base (E, F). Assume that (ε, φ) is a pair of topologies on the strategy sets of the base (E, F) and assume the game with closed-graph decision-rules* *and that there is at least a compact base of the reducing sequence of the game. Then, there exists at least one bistrategy surviving the iterated elimination of non-reactive strategies.*

Proof. Assume the subbase (E_k, F_k) be compact. Then, the images $F_{k+1} = f_k(E_k)$ and $E_{k+1} = e_k(F_k)$ are compacts, since e and f are with closed graph. By induction, every subbase (E_j, F_j) is compact, for $j > k$. So, the sequence of bistrategy space $(E_j \times F_j)_{j>k}$ is a sequence of compact sets with the finite intersection property; indeed, for every finite subset H of the set $\mathbb{N}(> k)$, that is the interval $]k, \to[_{\mathbb{N}}$, setting $h^* := \max H$, we have

$$\bigcap_{h \in H} E_h \times F_h = E_{h^*} \times F_{h^*}.$$

So, since the bistrategy space $E_k \times F_k$ is compact, that sequence has a non-void intersection. ■

Theorem (existence of a solution by iterated elimination). *Let $G = (e, f)$ be a game on a strategy base (E, F). Assume that (ε, φ) is a pair of complete metrics on the strategy sets of the base (E, F) and assume all the bistrategy spaces of the game are closed and with the sequence of their diameters vanishing. Then, there exists one and only one bistrategy surviving the iterated elimination of non-reactive strategies. So, under these assumptions, the game is solvable by elimination of non reactive strategies.*

Proof. It is a direct consequence of the nested closed principle in complete metric spaces (see, for instance, Kolmogorov - Fomin, Functional Analysis). ■

Corollary (existence of a solution, by iterated elimination, for sequentially continuous games). *Let $G = (e, f)$ be a sequentially continuous game on a compact base (E, F), with respect to a pair of complete metrics (ε, φ). Then, if the sequence of the diameters of the reduced bistrategy spaces is vanishing, the game is solvable by iterated elimination of non-reactive strategies.*

Proof. Strategy spaces of the game are compact and the decision rules are sequentially continuous, so, all the reduced bases are compact and, then, we can apply the previous theorem. ■

11. Survival of equilibria

Memento. Let $G = (e, f)$ be a decision-form game. Recall that a bistrategy (x^*, y^*) is called an equilibrium of the game G if $x^* \in e(y^*)$ and $y^* \in f(x^*)$.

The following theorem gives the first relation between solutions by iterated elimination of non-reactive strategies and game equilibria.

Theorem (survival of equilibria). *Let (x, y) be an equilibrium of a game (e_0, f_0). Then it survives iterated elimination of non-reactive strategies.*

Proof. By equilibrium definition, $x \in e_0(y)$ and $y \in f_0(x)$, that is, $(x, y) \in E_1 \times F_1$. Moreover, if $x \in e_k(y)$, then $x \in e_k(F_k) = E_{k+1}$, for every natural number k; analogously, if $y \in f_k(x)$, then $y \in f_k(E_k) = F_{k+1}$. By induction, we deduce that $(x, y) \in E_k \times F_k$, for each $k \in \mathbb{N}$. ■

Before the next theorem we need a lemma.

Lemma (characterization of Lipschitz continuity via bounded subsets). *Let (X, d) be a metric space, let $f : X \to X$ be a function and let L be a positive real number. Then, the following conditions are equivalent:*

1) f is Lipschitz continuous with a Lipschitz-constant L;
2) for any bounded subset B of the space, the following inequality holds

$$^{d}f(B) \leq L\,^{d}B;$$

3) for any natural n and for any bounded subset B of the space, it is

$$^{d}f^{n}(B) \leq L^{n}\,^{d}B.$$

Consequently, if the function f is a contraction, the sequence of diameters $(^{d}f^{n}(B))_{n=0}^{\infty}$ is vanishing.

Proof. 1) $\Rightarrow$ 2). It's easy to see, by induction, that, for every non-negative integer n,

$$^{d}f^{n}(B) \leq L^{n}\,^{d}(B)\,.$$

Indeed, let us prove the inequality for $n = 1$. Let y and y' lie in $f(B)$, then there are two points x and x' in B such that $y = f(x)$ and $y' = f(x')$. By Lipschitz continuity, we have

$$\begin{aligned} d(y, y') &= d(f(x), f(x')) \leq \\ &\leq Ld(x, x') \leq \\ &\leq L\,^{d}B. \end{aligned}$$

Since the preceding inequality holds for all y and y' in $f(B)$, we deduce

$$^{d}(f(B)) \leq L\,^{d}(B),$$

as we desired. If the result is true for $n - 1$, we have

$$\begin{aligned} ^{d}(f^{n}(B)) &= \;^{d}(f(f^{n-1}(B))) \leq \\ &\leq L\,^{d}(f^{n-1}(B)) \leq \\ &\leq LL^{n-1}\,^{d}(B) \leq \\ &\leq L^{n}\,^{d}(B)\,. \end{aligned}$$

If $L < 1$, from the above inequality, since $^{d}(B) < \infty$, the result follows immediately. *3) $\Rightarrow$ 2).* Obvious. *2) $\Rightarrow$ 1).* Let x, y be two points of the space and let $B = \{x, y\}$. The set B is bounded and its diameter is the distance $d(x, y)$. The image $f(B)$ is the pair $\{f(x), f(y)\}$, whose diameter is the distance $d(f(x), f(y))$. From 2) it follows the Lipschitz -inequality $d(f(x), f(y)) \leq Ld(x, y)$. The Lemma is completely proved. ■

Theorem (existence of solution in contracting games). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0). Assume (ε, φ) be a pair of complete metrics on the base (E_0, F_0) and let the game be univocal, contracting and with a bounded base (bounded with respect to the metrics). Then, there exists a unique bistrategy surviving the iterated elimination of non-reactive strategies, and it is the pair of the fixed points of the game rules.*

Proof. Existence of a survivor. Note that the intersection I of all the bistrategy spaces of the reduction sequences is non-void. Indeed, let $d : E \times F \to E \times F$ be the function defined by

$$d(x, y) = (e(y), f(x)),$$

for each ordered pair (x, y) in $E \times F$; the unique fixed point of the function d, that is the point (x^*, y^*) such that $d(x^*, y^*) = (x^*, y^*)$, belongs to the intersection I, thanks to the above theorem. *Uniqueness of the survivor.* We have just to prove that there is only one bistrategy surviving the iterated elimination. For, we claim that the sequence of diameters of the sequences E and F of the strategy spaces of the reduction sequence are vanishing. It is simple to prove that, for every natural n, we have

$$E_{2n} = (e \circ f)^n(E_0) \quad , \quad E_{2n+1} = (e \circ f)^n(E_1).$$

Since the strategy base (E_0, F_0) is bounded, by the preceding lemma, the two subsequences $(E_{2n})_{n=1}^{\infty}$ and $(E_{2n+1})_{n=1}^{\infty}$ have the corresponding sequences of diameters vanishing. So, the intersection $\cap E$ can contain at most one point. Analogously, we can proceed for the sequence F, and the theorem is proved. ■

12. Nested compacts lemma

The following lemma will allow us to provide sufficient conditions in order that a solution of a game, by iterated elimination, is an equilibrium.

Notations and terminology. Recall that:

- A sequence of sets is said to be nested if each term of the sequence contains (widely) the following one.

Lemma (nested compacts lemma). *Let $F = (F_n)_{n=1}^{\infty}$ be a sequence of nested compact subsets of a metric space (F_0, φ) whose intersection contains one and only one point y_*. Then, the sequence of diameters of the family F is vanishing, that is*

$$\lim_{k\to\infty} {}^{\varphi}(F_k) = 0.$$

Consequently, each sequence y in the set F_0 such that $y_n \in F_n$, for each positive integer n, converges to the point y_*.

Proof. Let d be the sequence of diameters of the family F. Since the family F is nested, the sequence d is (widely) decreasing and it is bounded below by 0, so, it converges to its infimum d_*, that is to say to

$$d_* = \inf_{n\in\mathbb{N}} \sup_{y,z\in F_n} \varphi(y, z).$$

Since the set F_n is compact, for every index n, and since the metric φ is continuous, by Weierstrass theorem, there exist two sequences s and t in the compact F_0 such that $s_n, t_n \in F_n$, for all natural n, and such that

$$d_* = \inf_{n\in\mathbb{N}} \varphi(s_n, t_n).$$

Let us prove that the real d_* is zero. For all natural k, it is $\varphi(s_k, t_k) \geq d_*$. Moreover, since F_1 is compact, there exist two subsequences s' and t', extracted from the sequences s and t, respectively, converging in F_1 to points s_* and t_*, respectively. These subsequences are eventually contained in any closed F_k, and then, their limits are in any closed F_k, that is, in their intersection $\cap F = \{y_*\}$. This circumstance implies $s_* = t_* = y_*$, from which it must be

$$0 \leq d_* \leq \lim_{k\to\infty} \varphi(s'_k, t'_k) = \varphi(s_*, t_*) = \varphi(y_*, y_*) = 0.$$

Now, let y be a sequence in F_0 such that $y_k \in F_k$, for all k. For all k, then we have

$$\varphi(y_k, y^*) \leq {}^{\varphi}(F_k),$$

because both y_* and y_k belong to F_k. By the squeeze theorem, the sequence $(\varphi(y_k, y_*))_{k=1}^{\infty}$ is vanishing, then $y \to^{\varphi} y_*$, so, the lemma is proved. ■

Remark. The hypothesis of compactness is not avoidable. First of all we prove that the assumption of boundness is unavoidable. *Case of closed but not bounded sets.* Let F_0 be the real line $\mathbb{R}$ and $F_k = \{0\} \cup [k, \to[$, for all natural k. The sequence $(F_k)_{k=1}^{\infty}$ is a sequence of nested closed sets whose intersection is $\{0\}$, but all of the closed sets are not bounded. Then the diameter sequence must be vanishing. Now, we prove that the closedness needs, too. *Case of open bounded sets.* Consider the union

$$A_k =]0, 1/k[\cup B(1, 1/k),$$

for all natural $k \geq 1$, where $B(1, 1/k)$ is the open ball centered at 1 and with radius $1/k$. The sequence $A = (A_k)_{k=1}^{\infty}$ is a sequence of nested bounded open sets, whose intersection is (evidently) $\{1\}$. But, the diameter sequence converges to 1 and not to 0. Indeed, we have

$$^{d}(A_k) = 1 + 1/k,$$

for every natural integer k.

13. Solution by elimination of non-reactive strategy and game equilibrium

Notations and terminology. Let $G = (e, f)$ be a game.

- If (X, μ) is a metric space and if S is a subset of X, we call diameter of S, with respect to μ, the following extremum

$$^{\mu}(S) := \sup_{x,y \in S} \mu(x, y).$$

- If the base of G is the pair (E, F) and (ε, φ) is a pair of metric on E and F, respectively, G is called a sequentially continuous game on the compact base (E, F) with respect to the pair of metrics if its rules are sequentially continuous, or if the graphs are closed in the product topology on the base $F \times E$ and $E \times F$, respectively.

Theorem (characterization of solvability). *Let (E_0, ε) and (F_0, φ) be compact metric spaces and let $G_0 = (e_0, f_0)$ be a decision-form game upon the base (E_0, F_0), without disarming strategies and sequentially continuous with respect to the pair of metrics (ε, φ). Then, the game G_0 is solvable by iterated elimination of non-reactive strategies if and only if the two diameter sequences of the strategy spaces of the reduced games are vanishing. Moreover, if the game is solvable, its solution is a game equilibrium.*

Proof. Let E and F be the sequences of the strategy spaces of the reducing sequence G of the game G_0 and let (x^*, y^*) be the solution by iterated elimination of non-reactive strategies. By definition of solution, it is

$$\bigcap_{k=1}^{\infty} (E_k \times F_k) = \{(x^*, y^*)\},$$

then, we have $\bigcap_{k=0}^{\infty} E_{k+1} = \{x^*\}$ and $\bigcap_{k=0}^{\infty} F_{k+1} = \{y^*\}$, or, in other terms

$$\bigcap_{k=0}^{\infty} f_k(E_k) = \{y^*\} \quad \text{et} \quad \bigcap_{k=0}^{\infty} e_k(F_k) = \{x^*\}.$$

Consequently, for all natural k, there exist a strategy $y_k \in F_k$ and a strategy $x_k \in E_k$ such that $x^* \in e_k(y_k)$ and $y^* \in f_k(x_k)$, that means $x^* \in e_0(y_k)$ e $y^* \in f_0(x_k)$. Since the correspondences e_0 and f_0 are sequentially continuous, all of their restrictions e_k and f_k are sequentially continuous. Moreover, since G_0 has not disarming strategies and all the subgames of reducing sequence G are the restrictions to the images of the decisional rules of the previous game, these subgames have not disarming strategies. Therefore, the decision rules of the subgames are all sequentially continuous and with nonempty values and, because the initial strategy spaces E_0 and F_0 are compact, all the images of those decision rules are compact. Then, the sequences $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ converges to x^* and y^* respectively, thanks to the nested compacts lemma. Finally, because e_0 and f_0 are correspondences with closed graph, we have $x^* \in e_0(y^*)$ e $y^* \in f_0(x^*)$, ending the proof of the theorem. ■

References

[1] D. Carfì, *Foundation of Game Theory* (Il Gabbiano, 2008).
[2] J. P. Aubin, *Mathematical Methods of Game and Economic Theory* (North-Holland, 1982).
[3] J. P. Aubin, *Optima and Equilibria* (Springer Verlag, 1998).
[4] M. J. Osborne and A. Rubinstein, *A Course in Game Theory* (MIT Press, 1994).
[5] G. Owen, *Game Theory* (Academic Press, 1995).
[6] R. B. Myerson, *Game Theory: Analysis of Conflict* (Harvard University Press, 1991).

---

[a] Università degli Studi di Messina
Facoltà di Economia
Via dei Verdi
98122, Messina, Italy

[b] Università degli Studi di Messina
Dipartimento di Matematica
Contrada Papardo,
98166, Messina, Italy

* To whom correspondence should be addressed. E-mail: davidcarfi71@yahoo.it

Presented: 27 November 2008; published online: 24 July 2009.
