Switching to a sustainable efficient extraction path

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Abstract The economy depends on the essential nonrenewable resource and the path of extraction is nondecreasing and inefficient. At some point the government gradually switches to a sustainable (in sense of nondecreasing consumption over time) pattern of the resource use. Technical restrictions do not allow to switch to the efficient extraction instantly. Transition curves calibrated to the current pattern of world oil production are used as the extraction paths in the "intermediate" period. However, there is no solution in finite time for the "smooth" switching from the optimal "transition" to the optimal efficient path, constructed with respect to the same welfare criterion. We analyze numerically two approaches for the approximate solution: "epsilon-smooth" switching and "epsilon-optimal" transition curve with smooth switching. Both cases give the unexpected result: the consumption path along the "inefficient" transition curve is always superior to the constant which we obtain after switching to the "efficient" Hartwick's curve. The result implies that for the correct switching to the efficient curve in finite time the saving rule must be adjusted. We estimate the importance of following the efficient path by comparing the consumption along the plausible transition path and the efficient pattern of the resource use. For simplicity we use in our examples the constant per capita consumption as a welfare criterion and the Hartwick rule as the benchmark of investment rule.

1 Introduction

A sustainable extraction of a nonrenewable resource must be decreasing in the long run if the resource is limited and essential for the economy even if we do not take into account the environmental externalities. The requirement of efficient extraction in terms of consumption (Dasgupta and Heal, 1979) implies the fulfillment of the Hotelling rule which in our case also trucks ever-declining resource use. Empirical tests of the Hotelling rule have yielded mixed results and some researchers have examined possible wedges between the theoretical price behavior (exponentially growing) and historical data (U-shaped or almost constant prices, see a broad review in Krautkraemer, 1998). For example, Davis and Cairns (1999) introduce the assumption that the rate of change in the oil prices is less than the rate of interest. This assumption implies relatively rapid extraction (restricted only by regulations and decreasing well pressure) and yields a reconciliation between the theoretical and observed time paths of extraction.

There is also a literature on the design of government interventions for realizing sustainable resource use via price changes[1] and extraction activity directly (using regulations like in (Davis and Cairns, 1999) or affecting the households' demand with environmental policy like in (Grimaud and Rouge, 2005) and (Pezzey, 2002)).

In this paper we postulate that society must depart from the inefficient and unsustainable pattern of extraction and we will concentrate on some normative and technical problems which can arise during the switching in finite time to the path with desirable properties. For example, it turns out that if we use the mechanisms of influencing the extractive path in a discontinuous

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[1]For example, Karp and Livernois, (1992) examine the efficiency-inducing taxation for a monopolist. A recent review on sustainability and environmental policies can be found in (Pezzey, 2002).

("regime-shifting") way, then it can be possible that the consumption path along the efficient extraction curve, which we construct, is inferior to the path of the "sustainable" but inefficient "transition" curve linked to historical path. The pattern of saving must also be adjusted in order for us to be able to compare the consumption paths along the efficient and the transition curves. We report on a numerical example based on data of world oil extraction in order to estimate the gap between the consumption paths along the efficient and the transition curves.

We assume that the economy can not be set in motion initially with current oil extraction per capita, declining. We focus on the process of switching from the increasing extraction to the decreasing extraction and efficient path. We assume that technical restrictions prevent the economy from changing the pattern of extraction instantly from growing to the sustainable and efficient. We split our problem into two periods. In the first (transition) period we construct a path which is "intermediate" between the nondecreasing and the efficient (in the second period) patterns of the resource use. Both the transition and the efficient curves are constructed as optimal with respect to the same welfare criterion. For simplicity we use the constant per capita consumption over time as a welfare criterion and the Hartwick rule (Hartwick, 1977) as the saving rule.

We review first the Solow (1974) model with the Cobb-Douglas technology. For simplicity we consider the case with zero population growth[2] and so all the paths of our economy such as output $q(t)$, consumption $c(t)$, capital $k(t)$ and so on are defined below in per capita units. For the case with no capital depreciation, no technological progress, and zero extraction cost, we have output $q = f(k, r) = k^{\alpha} r^{\beta}$ where k is produced capital, r - current resource use, $r = -\dot{s}$,

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[2]In fact, numerous literature on sustainable development starting T. Malthus work in 1798 and some recent papers, e.g., (Brander, 2007) consider the population growth as the main threat to sustainability. The debates on this problem are concentrating around the estimate of the constant which could be the limit to the population growth. Hence, we can assume that the population is already stabilized on this limit.

s - per capita resource stock ($\dot{s} = ds/dt$), α, $\beta \in (0,1)$ are constants. Prices of capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$ where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$. The Hartwick saving rule implies $c = q - rf_r$ or, substituting for f_r, we have $c = q(1 - \beta)$, which means that instead of $\dot{c} = 0$ we can check $\dot{q} = 0$.

The efficient path of extraction can be derived from the Hotelling rule $\dot{f_r}/f_r = f_k$ which implies $\alpha\beta q/k + \dot{r}(\beta - 1)/r = f_k = \alpha q/k$ or

$$\dot{r}/r = -\alpha q/k. \tag{1}$$

Then

$$\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \beta(\alpha q/k + \dot{r}/r) = 0, \tag{2}$$

which means that we really have $\dot{q} = \dot{c} = 0$ or $q = const$. Then $rf_r = \beta q = const$ and we have $\dot{k} = \beta q = const$ for deriving $k(t)$ and (1) for deriving $r(t)$. We can find two constants of integration k_0 for $k(t) = k_0 + \beta qt$ and the constant of equation $\dot{r}/r = -1/(k_0/\alpha q + \beta t/\alpha)$ using initial conditions $r(0) = r_0$ and $s(0) = s_0$, where s_0 is the given resource stock which must be used for production over infinite time: $s_0 = \int_0^\infty r(t)dt$. Then we have

$$r(t) = r_0 \left[1 + r_0\beta t/s_0(\alpha - \beta)\right]^{-\alpha/\beta}, \tag{3}$$

where $\alpha > \beta$ (Solow condition) and

$$\dot{r}(t) = -\ddot{s}(t) = -\alpha r_0^2/s_0(\alpha - \beta)\left[1 + r_0\beta t/s_0(\alpha - \beta)\right]^{-(\alpha+\beta)/\beta}. \tag{4}$$

Since we assume that our economy depends on the resource essentially, we obtain path $r(t)$ (Hartwick curve (3)), asymptotically approaching zero (dotted line on Fig. 1 is $R_{Hart}(t)-$ in
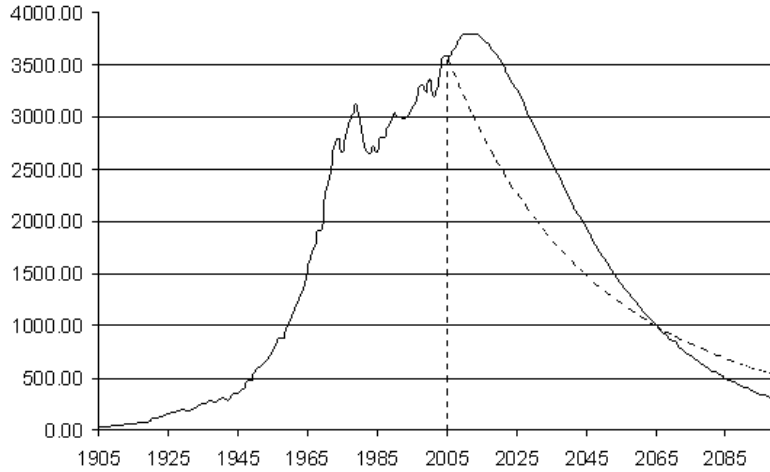
5

Figure 1: World oil extraction: historical data (before 2006); Hartwick curve (dotted); transition curve (solid)

absolute units) and the path of extraction changes $\dot{r}(t)$ (or negative acceleration of stock $s(t)$ diminishing, dotted line on Fig. 2) also approaching zero, but starting from negative value

$$\dot{r}_0 = -\frac{\alpha r_0^2}{s_0(\alpha - \beta)}. \tag{5}$$

However, according to our assumptions we are not able to realize technically the efficient Hartwick curve and we must switch to a sustainable path along some "smooth continuation" (solid line on Fig. 1 after 2006). Our definitions in the next section reflect these technical restrictions.

2 Feasibility, efficiency, and technical restrictions

The constant per capita consumption over time in our case is the result of

1) total investment of oil rent in capital ($\dot{k} = r f_r$) and

6

2) fulfillment of the Hotelling rule ($\dot{f}_r / f_r = f_k$).

We analyze the case when technical restrictions prevent us from starting the extraction using an efficient path and so we must find the optimal path among inefficient curves. We set down these assumptions below in the definitions 1 - 4, and the Propositions 1 and 2.

Definition 1 An *intertemporal program* $\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}$ is a set of paths $f(t)$, $c(t)$, $k(t)$, $r(t)$, $t \geq 0$ such that $f(t) = f[k(t), r(t)]$ and $c(t) = f(t) - \dot{k}(t)$.

Definition 2 For positive initial stock of capital and resource $(k_0, s_0) \gg 0^3$ the set of the programs $F = \{\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}\}$ is *a feasible sheaf* at $t = 0$ and each of the paths $f(t)$, $c(t)$, $k(t)$, $r(t)$ is *a feasible path* if any program $\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}$ from F for all $t \geq 0$ satisfies the conditions:

1) $(f(t), c(t), k(t), r(t)) \gg 0$;

2) $r(t), k(t), c(t)$ are continuously differentiable and $\sup_t |\dot{r}(t)| \leq \dot{r}_{\max} < \infty$;

3) $f(t)$ is twice continuously differentiable;

4) $\int_t^{\infty} r(t)dt \leq s(t)$;

5) $k(0) = k_0, c(0) = c_0, r(0) = r_0, \dot{r}(0) = a_0 \leq \dot{r}_{\max}$.

Definitions 1 and 2 are based on the definition of the interior feasible path in (Asheim et al., 2007). The differences reflect our assumptions: a) population is constant; b) the speed of change of the extraction rate $\dot{r}$ is limited and continuous for all t including $t = 0$ (technical restrictions). Henceforth, a "program" and a "path" will refer to a feasible program and a feasible path.

Definition 3 (Dasgupta and Heal, 1979, p. 214) A feasible program $\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}$ from F is *intertemporally inefficient* if there exists a program $\langle \overline{f}(t), \overline{c}(t), \overline{k}(t), \overline{r}(t) \rangle_{t=0}^{\infty}$ from F

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[3] $(x_1, \ldots x_n) \gg 0$ if $x_i > 0$ for all $i = \overline{1, n}$.

such that $\overline{c}(t) \geq c(t)$ for all $t \geq 0$ and $\overline{c}(t) > c(t)$ for some t.

Definition 4 (Dasgupta and Heal, 1979, p. 214) A set of feasible programs $E = \{\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}\}$ is a *set of efficient programs* if all the programs $\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty}$ from E are not inefficient.

Proposition 1 *If $\dot{f}_r(0)/f_r(0) \neq f_k(0)$ then $F \cap E = \emptyset$ or all the feasible paths are inefficient.*

Proof. Since $f(t)$ is twice continuously differentiable at $t = 0$, then there exists $\varepsilon > 0$ such that for any $t \in [0, \varepsilon)$ and for any feasible program $\langle f(t), c(t), k(t), r(t) \rangle_{t=0}^{\infty} \in F$ the Hotelling rule is not satisfied: $\dot{f}_r(t)/f_r(t) \neq f_k(t)$. Necessity of the Hotelling rule for the efficiency of a program (see, e.g., Asheim et al., 2007, Dasgupta and Heal, 1979) follows the assertion of the Proposition.

Now we will show that in our assumptions (zero extraction cost) all the growing paths of extraction are inefficient.

Proposition 2 *For an economy with technology $q = k^{\alpha} r^{\beta}$ where $\alpha, \beta \in (0, 1)$; $k(t), r(t) > 0$ and $\dot{k}(t) < q(t)$ for all t, the path of extraction is inefficient if there is $\overline{t} \geq 0$ such that $\dot{r}(\overline{t}) > 0$.*

Proof. Since the Hotelling rule is a necessary condition for efficiency, it is enough to show that it does not hold for the growing rate of extraction. Indeed, we can write the Hotelling rule $\dot{f}_r(t)/f_r(t) = f_k(t)$ as $\dot{f}_r/f_r = r\beta \left[\alpha \dot{k}/kr + \beta q \dot{r}/r^2\right]/(\beta q) - \dot{r}/r = \alpha \dot{k}/k - (1-\beta)\dot{r}/r = \alpha q/k$ (since $f_k = \alpha q/k$). Then we have $\alpha \dot{k}/k + (\beta - 1)\dot{r}/r = \alpha q/k$ or $(\beta - 1)\dot{r}/r = \left(q - \dot{k}\right)\alpha/k$. The right hand side of the last equation is always positive and the left hand side can be positive only for $\dot{r} < 0$ for any $t \geq 0$ (since $(\beta - 1) < 0$ and $r > 0$).

According to our formulation of the problem and the definition of the feasible path of extraction, we have the restriction on changes in extractions: $\sup_t |\dot{r}(t)| \leq \dot{r}_{\max} < \infty$. This
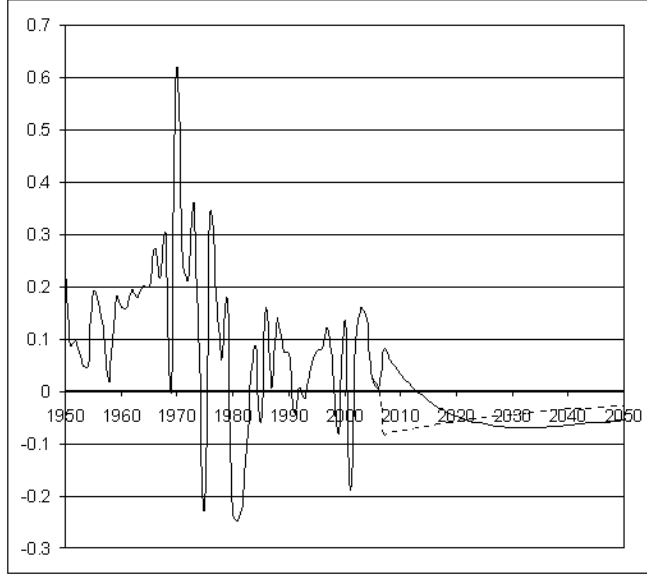
Figure 2: Per capita extraction accelerations: historical data (before 2006); Hartwick curve (dotted); transition curve (solid)

condition means that the extraction can be reduced without losing consumption only with the rate not exceeding $\dot{r}_{\max}$ which is defined by the rate of introducing the substitute technology. For our numerical examples we can estimate $\dot{r}_{\max}$ from historical data (Fig. 2)[4]. Note that $\dot{r}$ oscillated around 0.2 before 1980. As a result of energy crises in 1973 and 1979-80 it was a period of introducing new technologies. Then after 1980 per capita accelerations oscillate already around zero. But these energy crises followed by declines in output and consumption. Hence, since we consider the problem of switching to sustainable path without losing consumption we

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[4]The methodology of estimation of the accelerations for historical extraction data is described in (Bazhanov, 2006b). It is shown in this paper, that there is empirical evidence that the Hamilton variation principle holds in economics of nonrenewable resources. Then the accelerations of the resource extraction can be estimated as follows: $a_i = 2\frac{s_i - r_i(t_{i+1} - t_i) - s_{i+1}}{(t_{i+1} - t_i)^2}$ where $s_i, s_{i+1}-$ reserves at t_i and t_{i+1}; r_i- rate of extraction at t_i; $[t_i, t_{i+1})$ - the period when the sum of all the reasons influencing the resource extraction can be considered as a constant. The reserve at initial point of extraction was considered as the sum of the final historical reserve estimate and the sum of all historical extractions. Since acceleration is proportional to the generalized force (reason of changes), the values of a_i can be considered as the indices of the resource market. As a coefficient of proportionality (inertia coefficient) we can use the inverted price elasticity of the resource demand (the less is the elasticity, the more efforts must be applied to change the pattern of the resource use).

can take as a reasonable estimate for our numerical examples $\dot{r}_{\max} = 0.1$.

3 Transition curves

The transition path can be constructed in the same class of rational functions as the Hartwick curve (3). The difference is in the numerator, which in the expression for acceleration $a = \dot{r}$ must depend on t with a negative coefficient to control "smooth breaking" in the neighborhood of $t = 0$. Namely, we will find $a(t)$ in the form of

$$a(t, b, c, d) = \dot{r}(t) = \frac{a_0 + bt}{(1 + ct)^d}, \qquad (6)$$

where $b < 0, c > 0, d > 1$ (for convergence $a(t) \to -0$ with $t \to \infty$). Corresponding to (6) $r(t)$ has a dependence on $b, c,$ and d in

$$r(t) = \left\{ -\left[a_0 + b/[c(d-2)]\right] / [c(d-1)] + bt/[c(2-d)] \right\} / (1 + ct)^{d-1}.$$

Note, that a constant of integration for $\dot{r}(t) = a(t)$ must be zero for the convergence of $\int_0^\infty r(t)dt$, and also for the convergence, d actually must be greater than 3. We have $r_0 = -\left[a_0 + b/[c(d-2)]\right] / [c(d-1)]$ to express b :

$$b(c, d) = -c(d-2)\left[r_0 c(d-1) + a_0\right]. \qquad (7)$$

Then the transition curve has a dependence on c and d in

$$r(t) = \frac{r_0 \left(1 + b_r t\right)}{(1 + ct)^{d-1}} \qquad (8)$$

where $b_r = c(d-1) + a_0/r_0$. Coefficient c can be expressed from the condition that resource is finite $s_0 = \int_0^\infty r(t)dt$:

$$
\begin{aligned}
s_0/r_0 &= \int_0^\infty (1 + ct)^{1-d}dt + [c(d-1) + a_0/r_0] \int_0^\infty t/(1 + ct)^{d-1}dt \\
&= \left[1 + \{r_0 c(d-1) + a_0\} / \{r_0 c(d-3)\}\right] / [c(d-2)],
\end{aligned}
$$

10

which means that c is a solution of quadratic equation

$$c^2 s_0/r_0 - 2c/(d-3) - a_0/[r_0(d-3)(d-2)] = 0.$$

The only relevant root (because we are looking for $c > 0$) is

$$c(d) = \left[r_0/(d-3) + \left\{ r_0^2/(d-3)^2 + s_0 a_0/[(d-3)(d-2)] \right\}^{0.5} \right] /s_0. \qquad (9)$$

Hence, we have a single independent parameter d which defines the shape of the curve (including its peak) and we can use this parameter as a control variable in some selected optimization problem

$$F\left[r(t, d)\right] \rightarrow \max_d$$

which can be connected with the short- or long-run policy in output or in consumption. In our numerical examples we used $a_0 = 0.08$ and as world oil reserves and extraction on January 1, 2007 (Oil & Gas J., 2006, 104, 47: 20-23.): $R_0 = 72,486.5$ [1,000 bbl/day] $\times 365 = 26,457,572$ [1,000 bbl/year] (or 3.6243 bln t/year); $S_0 = 1,317,447,415$ [1,000 bbl] (or 180.47 bln t). We use coefficient 1 ton of crude oil $= 7.3$ barrel.

4   Consumption along transition curves

Transition path (8) is not efficient in our formulation of the problem (extraction grows in a neighborhood of $t = 0$) unlike the Hartwick curve (3) which is derived from the Hotelling rule and so satisfies it identically. Hence, to examine the consumption behavior in our case along some path we should check the fulfillment of the Hotelling rule along this curve. For the general case $\dot{q} = f_k \dot{k} + f_r \dot{r}$. Then $\dot{f}_r = \beta d \left(q/r\right)/dt = \beta \left[f_k \dot{k}/r + f_r \dot{r}/r \right] - \beta \dot{r} q/r^2$. Dividing on $f_r = \beta q/r$

11

we have $\dot{f}_r/f_r = r\beta\left[\alpha\dot{k}/kr + \beta q\dot{r}/r^2\right]/(\beta q) - \dot{r}/r = \alpha\dot{k}/k - (1-\beta)\dot{r}/r$. Since $f_k = \alpha q/k$ we have $\dot{f}_r/f_r = f_k\left[\dot{k}/q - (1-\beta)k\dot{r}/(\alpha qr)\right]$ and substitution for $\dot{k}$ the saving rule $\dot{k} = \beta q$ gives us

$$\dot{f}_r/f_r = f_k\left[\beta - (1-\beta)k\dot{r}/(\alpha qr)\right]. \tag{10}$$

Just to check, we can see, that for the Hartwick curve $[\cdot] \equiv 1$, because the Hotelling rule implies $\dot{r}/r = -\alpha q/k$.

Hence, if $[\cdot] < 1$, then $\dot{q} > 0$, because $\dot{f}_r/f_r < f_k$,[5] which follows $-\dot{r}/r < \alpha q/k$ or $\alpha q/k + \dot{r}/r > 0$. And the latter, using expression in the left hand side of (2), means $\dot{q} > 0$. In the same way, $[\cdot] > 1$ follows $\dot{q} < 0$ and, in general, $\operatorname{sgn}\dot{q} = \operatorname{sgn}\{1 - [\cdot]\}$. So, to examine long-run consumption $c = (1-\beta)q$ along the LA curve, we can check asymptotic behavior of $[\cdot]$ in (10).

Proposition 3 *If an economy with technology $q = k^\alpha r^\beta$ is such that $\alpha, \beta \in (0,1)$; $\beta < \alpha$ and*

1) resource rent is completely invested in capital;

2) there is no time lag between the moment of investment and the corresponding increase in capital;

3) rate of extraction $r(t)$ is such that

$$\dot{r}(t) = (a_0 + bt)/(1 + ct)^d, \ b < 0, \ c > 0, \ d > 3,$$

then the asymptotic behavior in output q for different β is:

$$\lim_{t\to\infty} \operatorname{sgn}\dot{q}(t) = \begin{cases} -1, & \beta(d-2) \geq 1, \\ \operatorname{sgn}L(d,\alpha,\beta), & \beta(d-2) < 1, \end{cases} \tag{11}$$

where

$$L(d,\alpha,\beta) = \frac{[\alpha - \beta(d-2)]}{[\alpha - \alpha\beta(d-2)]}.$$

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[5]In our case the nature of inequality $\dot{f}_r/f_r < f_k$ is stipulated by the Hartwick investment rule and the fast introduction of the substitute technologies which implies fast decrease in demand for the resource and corresponding decrease in rates of extraction. So, inequality $\dot{q} > 0$ here is a sign of sustainable growth.

Proof of the Proposition is in (Bazhanov, 2006a, Appendix).[6]

Corollary 1. Under the assumption of the Proposition 3 the consumption $c(t)$ is

1) asymptotically decreasing if $d > \alpha/\beta + 2$;

2) asymptotically constant if $d = \alpha/\beta + 2$;

3) asymptotically growing if $3 < d < \alpha/\beta + 2$.

Proof. Note that for $\beta(d-2) < 1$ or $d < 1/\beta + 2$ denominator of $L(d, \alpha, \beta)$ is positive. Then the sign of $L(d, \alpha, \beta)$ is defined by nominator. Since $c = (1 - \beta)q$ and sgn $\dot{c}$ =sgn $\dot{q}$ then substituting the expressions for d into $L(d, \alpha, \beta)$ in (11) we obtain the assertion of the Corollary. In the case when $d \geq 1/\beta + 2$ or $\beta(d-2) \geq 1$ we define the sign of $\dot{c}$ by the first line in (11) which is included in the first case of the Corollary.

5   Switching to an efficient path

We define the moment of shifting to the second period $\widetilde{t}_0$ (the period of "efficient extraction") as a solution of the "smooth switching" problem. Namely, the economy enters the efficient path when the acceleration $\dot{r}$ along the transition curve is equal to the initial acceleration of the efficient curve which is being constructed at the each current moment. In our case the efficient curve (3) is being dynamically constructed with the use of "floating" initial conditions $\widetilde{r}_0(t), \dot{\widetilde{r}}_0(t), \widetilde{s}_0(t)$ which are being calculated along the transition path. Equations (5) and (6) for the accelerations imply that $\widetilde{t}_0$ must be a solution of the equation

$$\frac{a_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{12}$$

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[6]The simplified expression for $L(d, \alpha, \beta)$ was obtained by direct substitution of expressions for b, c and ρ.

where $r(\widetilde{t}_0) = \widetilde{r}_0(\widetilde{t}_0)$ is defined by equation (8) and the rest of resource $\widetilde{s}_0$ at $\widetilde{t}_0$ is $\widetilde{s}_0(\widetilde{t}_0) = s_0 - \int_0^{\widetilde{t}_0} r(t)dt$. Since our efficient curve (3) with the Hartwick investment rule give us constant consumption over time, it is natural to construct the transition curve (8) which is consistent with the same welfare criterion. Namely, according to the Corollary 1, the rational curve (8) with $d = \alpha/\beta + 2$ and with the Hartwick saving rule implies asymptotically constant consumption.[7] We will examine the existence of the solution of equation (12) in the following lemma and proposition.

Lemma 1 *The rational curve of extraction (8) is such that*

a) $s_0 = \frac{r_0 p_0}{c(d-2)}$;

b) the rest of the resource $s(\overline{t})$ along this curve at $\overline{t} \geq 0$ is

$$s(\overline{t}) = s_0 - \int_0^{\overline{t}} r(t)dt = \frac{r_0}{c(d-2)} \frac{p_0 + p_1\overline{t}}{\left(1 + c\overline{t}\right)^{d-2}} = s_0 \frac{1 + \frac{p_1}{p_0}\overline{t}}{\left(1 + c\overline{t}\right)^{d-2}}$$

where $p_0 = 1 + \frac{b_r}{c(d-3)}$, $p_1 = \frac{b_r(d-2)}{d-3}$;

$r_0 = r(0)-$ *initial rate of extraction,* s_0- *initial stock;*

$b_r = b_r(d)$, $c = c(d)$, *and* d *are the parameters of the curve (8).*

(Proof is in Appendix 1).

Proposition 4 *Equation (12) has real roots if and only if parameter d of the rational curve (8) is such that*

$$d \leq \frac{\alpha}{\beta} + 2. \tag{13}$$

There are two real roots if inequality (13) is strict and one real root if it holds as an equality.

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[7]Rational path with $d = \alpha/\beta + 2$ is optimal in the class of rational functions (8), e.g., with respect to the following criterion, consistent with constant consumption over time: $F(d) = \min_d \max_t |c_{\max} - c(t)|$, where $c_{\max}-$ asymptote for the path with asymptotically constant consumption. For any d_1 and d_3 such that $d_1 < d_2 = \alpha/\beta + 2 < d_3$ we have according to Corollary 1: $F(d_1) = \infty > F(d_3) = c_{\max} \geq F(d_2) = c_{\max} - c_0$.
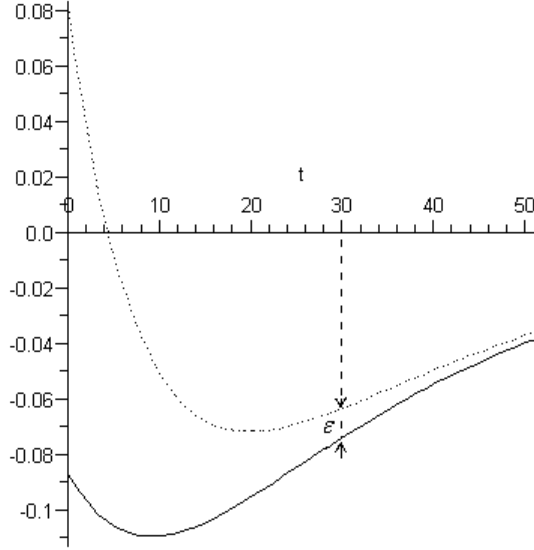
Figure 3: Accelerations for the transition curve with $d = \frac{\alpha}{\beta} + 2$ (dotted line) and the initial accelerations for the Hartwick's curve (solid line), constructed along the transition curve.

(Proof is in Appendix 2).

Proposition 5 *Equation (12) has only one real finite positive root if and only if $d < \frac{\alpha}{\beta} + 2$.*

(Proof is in Appendix 3).

For the example with world oil extraction data, the accelerations of the transition curve (8) with $d = \frac{\alpha}{\beta} + 2$ (left hand side of equation (12)) and dynamically constructed initial accelerations of the efficient curve (right hand side of equation (12)) are shown on Fig. 3. It can be seen that the residual of equation (12) approaches zero only asymptotically which means that our problem of "smooth switching" has no solution.

15

6 Approximations of the "smooth switching" problem

We will consider two approaches to the approximate solution of the problem (12).

(a) *"ε−smooth switching"* which means that the economy will enter the efficient curve of extraction in a "regime shifting" way when the residual of equation (12) is small enough. For example, using our historical data estimate for $\dot{r}_{\max}$ we can define this moment as $\bar{t}_0$ such that

$$|\dot{r}_{trans} - \dot{r}_{Hart}| = \left| \frac{a_0 + b\bar{t}_0}{(1 + c\bar{t}_0)^d} + \frac{\alpha r^2(\bar{t}_0)}{\widetilde{s}_0(\bar{t}_0)(\alpha - \beta)} \right| \leq \varepsilon = 0.1\dot{r}_{\max} = 0.01.$$

As an approximate solution of this problem we can take $\bar{t}_0 = 30$ (see Fig. 3).

(b) *"ε−optimal transition curve"* which means that using the result of Proposition 5 the economy will follow some ε−optimal (with respect to constant consumption over time) transition curve with $d < \frac{\alpha}{\beta} + 2$ for which equation (12) has a single finite positive root. For the comparison between consumption paths in cases (a) and (b) we will take $d = 5.875$ (given $\alpha = 0.3$ and $\beta = 0.05$)[8]. For this value of d we have the same moment of switching $\bar{t}_0 = 30$ (see Fig. 4 and Fig. 13 in the Appendix 3). The difference between two cases is that in case (a) we must apply some "additional efforts" at the moment $\bar{t}_0$ to make discontinuous switch to the efficient path while in case (b) realization of the transition path with $d < \frac{\alpha}{\beta} + 2$ needs more efforts during all transition period (substitute technologies must be introduced faster).

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[8]For $\alpha = 0.2$ and $\beta = 0.05$ estimated in (Nordhaus and Tobin, 1972) the prospects for growth along the rational paths are less optimistic. The peak of oil extraction for the "borderline"-transition path with $d = \frac{\alpha}{\beta} + 2$ must be closer which implies that the substitute technologies must be introduced faster, and the level of asymptote for consumption in the case with $\alpha = 0.2$ is less than for $\alpha = 0.3$ (see Bazhanov, 2007a).
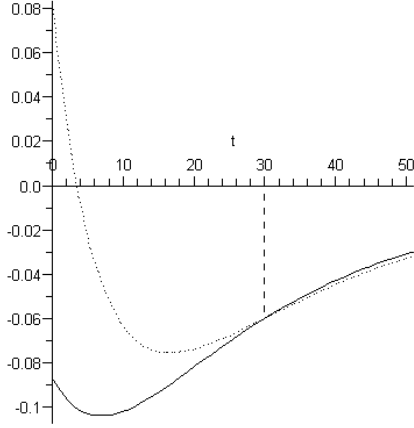
Figure 4: Accelerations for the $\varepsilon-$optimal transition curve with $d = 5.875$ (dotted line) and the initial accelerations for the Hartwick's curve (solid line), constructed along the transition curve.

7 Consumption along the "approximate switching" scenarios of extraction

The Hartwick saving rule which we use in our economy implies that the consumption path is $c = q - \dot{k} = (1 - \beta)q = (1 - \beta)k^{\alpha}r^{\beta}$ where $r(t)$ is a known combination of the transition and efficient paths and $k(t)$ is an unknown path of capital. We can calculate $k(t)$ from the equation for the saving rule $\dot{k} = \beta k^{\alpha}r^{\beta}$ assuming that we have estimation of k_0. From (2) we have $\dot{q}/q = \beta(\alpha q/k + \dot{r}/r)$ which implies the expression for k_0, given $r_0, \dot{r}_0$, and output percent change $(\dot{q}/q)_0$:

$$k_0 = \left\{ \left[\left(\frac{\dot{q}}{q}\right)_0 \frac{1}{\beta} - \frac{\dot{r}_0}{r_0} \right] / \left(\alpha r_0^{\beta}\right) \right\}^{\frac{1}{\alpha - 1}}. \tag{14}$$

Using $\left(\frac{\dot{q}}{q}\right)_0 = 0.04$ and estimates of $\dot{r}_0, r_0$ for world oil extraction we have $k_0 = 0.2810456$ and $c_0 = 0.692337$ which gives us the paths of consumption along the transition curves. In order to

17

construct the consumption path along the Hartwick's curve (3) we must assume that we manage not only to change instantly acceleration of the extraction at the moment of switching $\bar{t}_0$ but also to stop the growth of our economy. The last requirement connected with condition $\dot{q} = 0$ along the Hartwick's curve including the initial point $\bar{t}_0$.[9] Substituting for $\dot{r}_{0Hart} = -\frac{\alpha r_0^2(\bar{t}_0)}{\hat{s}_0(\bar{t}_0)(\alpha-\beta)}$ and $\dot{q} = 0$ into (14) we have the expression of capital in "different units": $k_0 = 30.47656$. Since "physically" capital is the same at this moment,[10] we must adjust its value using scale factor in order to obtain paths of consumption expressed in the "same units".

For our numerical example the process of switching from the transition path with $d = \alpha/\beta + 2$ to the efficient curve (case (a)) is depicted on the Fig. 5. Consumption paths are on the Fig. 6. The dash line is the limit ($c_{\max} = 2.480$) for the growth of consumption along the transition path. The process of switching from the $\varepsilon-$optimal transition path with $d < \alpha/\beta + 2$ to the efficient curve (case (b)) is on Fig. 7 and the corresponding consumption paths are on Fig. 8. Note that according to the Corollary 1 consumption along the rational curve with $d < \alpha/\beta + 2$ grows with no limit. We can see on Fig. 8 that consumption along this path exceeds the limit for the path with $d = \alpha/\beta + 2$ (dash line).

Hence, in both cases (a) and (b) our attempts to switch to the efficient sustainable path of extraction gave us unexpected and seemingly paradoxical results. Consumption along the efficient path of extraction (circled lines on Fig. 6 and Fig. 8) is inferior to the consumption along the inefficient transition path in all moments of time except the point of switching $\bar{t}_0 = 30$ where they are equal. At first glance the example contradicts the definition of inefficient

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[9] At the moment of switching to the Hartwick's curve ($\bar{t}_0 = 30$) we have output growth at rate $\left(\frac{\dot{q}}{q}\right)_{\bar{t}_0} = 0.00886$ for the case (a) and $\left(\frac{\dot{q}}{q}\right)_{\bar{t}_0} = 0.00881$ for the case (b).

[10] By the time $\bar{t}_0 = 30$ for our example the value of capital along the transition curve with $d = 5.875$ is $k(\bar{t}_0) = 1.8206$ and for the path with $d = 8$ it is 1.8251.
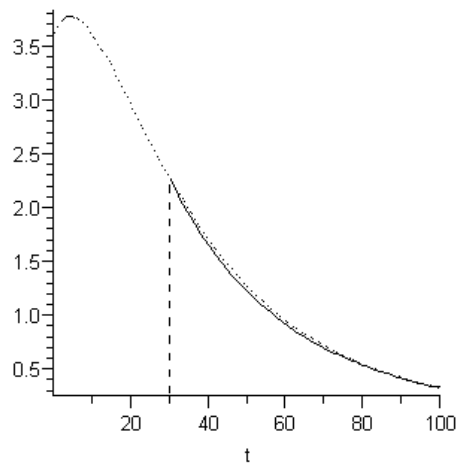
Figure 5: Case (a): "$\varepsilon-$smooth switching" from the transition curve (dotted line) to the efficient curve (solid line).
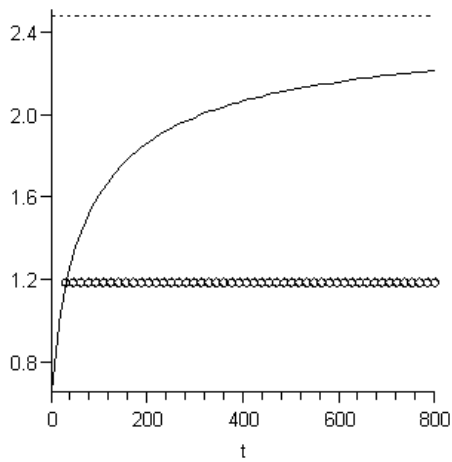

Figure 6: Consumption along the transition curve (solid) and the Hartwick's curve (circles) for switching in case (a).

19

Figure 7: Case (b): switching from the "ε−optimal transition curve" (dotted line) to the efficient curve (solid line).

Figure 8: Consumption along the transition curve (solid) and the Hartwick's curve (circles) for switching in case (b).
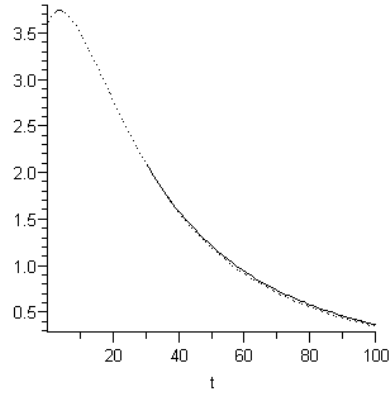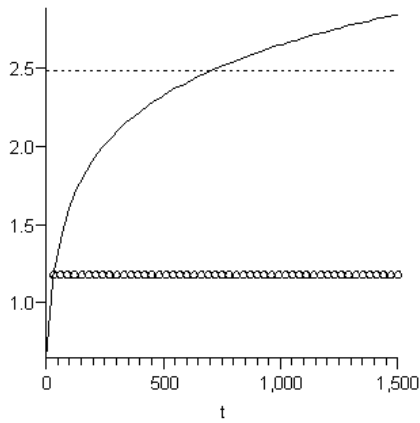
curve (definition 3) according to which everything must be exactly vice versa. However this definition works only for the feasible paths $r(t), k(t), c(t)$ which according to definition 2 must be continuously differentiable and $f(t)$ must be twice continuously differentiable. This implies the continuity of the output percent change $\frac{\dot{q}}{q}$ but in our "approximate solutions" we violated this requirement assuming that we will manage to stop the growth of economy at the moment of switching to the efficient path.[11] So, if it is really possible to change the economy in a "regime shifting" manner as a result of some political actions or natural disaster, then we can not be sure that the continuation of the inefficient program from the "previous life" would be inferior to our efficient program which we have managed to realize.

For our economy with technology $q = k^{\alpha} r^{\beta}$ and the Hartwick investment rule output can be only growing ($\dot{q} > 0$) for all t when $\dot{r} > 0$. This implies that for our model consumption must exhibit an infinite growth[12] along the sustainable[13] patterns of the resource use (limited growth as in case (a) along the transition curve or unlimited as in case (b)). Otherwise, if we discontinuously switch our economy into "different world" which is inferior with respect to future levels of consumption, the comparison of consumption along the paths from these "different worlds" will be incorrect.

In order to estimate the amount of consumption which we lose due to the inefficient extraction, we must compare correctly the consumption behavior along the transition and the efficient paths. To draw this comparison we will construct a saving rule for the transition path which implies $\dot{q}(\bar{t}) = 0$ at the moment of switching to the Hartwick's curve and which also is "close"

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[11]This violation explains also the big differences in consumption along the transition and the efficient paths (Fig. 6 and Fig. 8) despite very small residual in extractions (Fig. 5 and Fig. 7).

[12]We assume that our economy has technological progress compensating for the capital depreciation (see Bazhanov, 2007b) which allows to have an infinite growth.

[13]We consider the simplest sustainability criterion (in a weak sense) meaning nondecreasing consumption over time.

asymptotically to the Hartwick rule. We will use this saving rule in the second period only as an artificial tool for correct comparison of the consumption along our paths. This means that the efficient Hartwick's curve will be used in the second period with another saving rule which will lead to the consumption behavior different from the constant over time. We consider this case in the following section.

8 Constant output at the moment of switching

Technical restrictions (definition 2) imply, that given $\dot{q}(0) > 0$, there is no saving rule which will give us $\dot{q}(t) = 0$ for all t in the transition period including the moment of switching $\bar{t}$. Then we will construct a saving rule for which $\dot{q}(\bar{t}) = 0$ and $\dot{q}(t)$ has arbitrary (feasible) values at all other moments t in the transition period $t \in [0, \bar{t})$. Another requirement for this saving rule is that it must have a feasible continuation for the second period of efficient extraction $(t \in [\bar{t}, \infty))$ in order to draw the correct comparison of the consumption paths for this rule. Note that in our formulation we can not find this saving rule in the class of the rules with constant saving rates $\dot{k} = \delta q$ because we will obtain qualitatively the same behavior of consumption which will vary only in parameters. For example, for the transition path with $d = \alpha/\beta + 2$, which implies asymptotically constant consumption we will have different levels of asymptote $c_{\max}$ for different δ with monotonically growing consumption c and output q. So we will construct a feasible function $\delta(t)$ which gives us $\dot{q}(\bar{t}) = 0$. Since $\delta(t)$ has some level of arbitrariness, we can construct it in such a way that $q(t)$ is nonmonotonic in transition period and $\delta(t)$ asymptotically approaches β. Then our saving rule asymptotically approaches the Hartwick rule and the consumption paths will have to be asymptotically constant. Thus, for our numerical example we can find $\delta(t)$, for example, in the following form (Fig. 9):
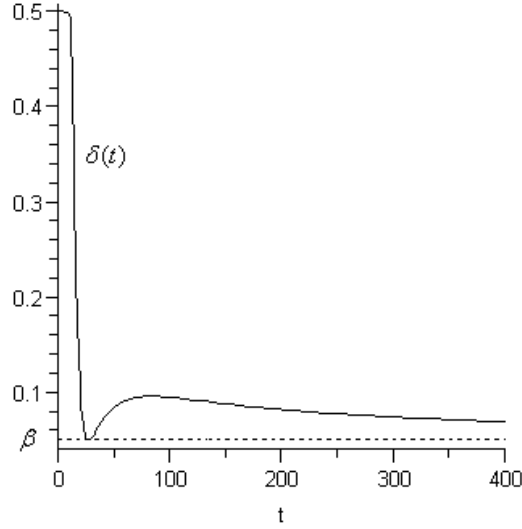
Figure 9: An example of saving rate $\delta(t)$ in the transition period (solid line) and the Hartwick saving rate (dotted line).

$$\delta(t) = \delta_0 - (\delta_0 - \beta)e^{-\frac{\nu(t-t^*)^2}{(1+t)^3}}$$

with parameters $\delta_0 = 0.5$, $\nu = 20$, and t^* defined from the condition $\dot{q}(\bar{t}) = 0$ using an iterative numerical procedure. The condition $\dot{q}(\bar{t}) = 0$ implies that the expression $\alpha k^{\alpha-1} r^{\beta} \dot{k} + \beta r^{\beta-1} k^{\alpha} \dot{r}$ or (substituting for $\dot{k} = \delta q$ and expressing k)

$$k(\bar{t}) - \left[-\frac{\beta}{\alpha\delta(\bar{t})} \frac{\dot{r}(\bar{t})}{r(\bar{t})^{\beta+1}} \right]^{\frac{1}{\alpha-1}} \tag{15}$$

must be equal to zero. Then for defining t^* we can use the following procedure:

1) set t_0^*, iterative parameter $i = 0$, and define $\delta^i(t) = \delta_0 - (\delta_0 - \beta)e^{-\frac{\nu(t-t_i^*)^2}{(1+t)^3}}$;

2) calculate (from equation 2)[14]

$$k_0^i = \left\{ \frac{1}{\delta^i(t)\alpha r_0^{\beta}} \left[\left(\frac{\dot{q}}{q} \right)_0 - \beta\frac{\dot{r}_0}{r_0} \right] \right\}^{\frac{1}{\alpha-1}};$$

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[14]For $\delta^i(t) \equiv \beta$ this formula coincides with (14).
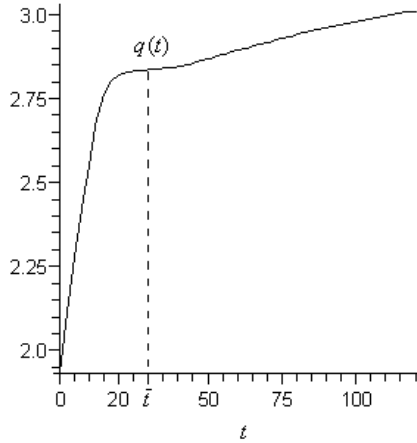
23

Figure 10: Output $q(t)$ along the transition curve (for $d = \frac{\alpha}{\beta} + 2$) with the saving rule $\dot{k} = \delta(t)q$; $\bar{t}$- the moment of switching to the efficient path.

3) given k_0^i solve differential equation $\dot{k} = \delta^i k^\alpha r^\beta$ for $k(t)$;

4) if the expression in (15) is small enough then $t^* = t_i^*$ and our saving rule is constructed; else change t_i^* to reduce the residual in (15), $i := i + 1$, and go to 2).

Since $\delta(t)$ can be chosen in such a way that output q is nonmonotonic along $\delta(t)$ (see Fig. 10) and since points with $\dot{q}(t) = 0$ depend on parameter t^*, it can be shown that the procedure converges. For our numerical example we obtained $t^* = \bar{t} - 3.48946$ which gave us the difference (15) equal to $7.8 \cdot 10^{-7}$.

Now, given the saving rate $\delta(t)$ which implies $\dot{q}(\bar{t}) = 0$, we can correctly switch at $\bar{t}$ to the efficient Hartwick's curve and compare the levels of consumption (Fig. 11). Note that when $\dot{q}(\bar{t}) = 0$, the estimates for the capital value at the moment of switching coincide for the transition and the efficient paths and we have no problem of scaling for correct comparison.

We can see from Fig. 11 that consumption along the efficient curve is always superior to
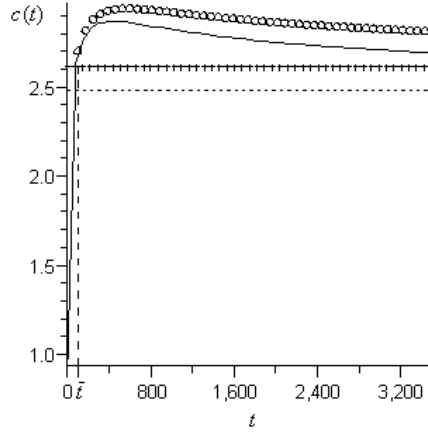
24

Figure 11: Consumption with the saving rate $\delta(t)$ along the Hartwick's curve (circled) and the transition curve with $d = \frac{\alpha}{\beta} + 2$ (solid); the line in crosses is the asymptote for the Hartwick's path, dotted line - asymptote for the transition path.

the consumption along the transition path except the point of switching $\bar{t}$ where they are the same. The asymptote for the efficient path (crosses) $c_{\max Hart} = 2.6145$ is also higher then the one for the transition path (dotted) $c_{\max trans} = 2.4802$. Hence, we can conclude that it makes sense to control the efficiency of the extraction path because, as we can see, the economy in our example is losing more than 5% of consumption at each moment of time in the long run along the sustainable but inefficient path of extraction.

An interesting source for contemplation is the example for "correct switching" in case (b), when we use "$\varepsilon-$optimal" transition path with $d = 5.875 < \alpha/\beta + 2$. Using the described above procedure we obtained that in this case the consumption path along the efficient curve is also superior but only in the short run (Fig. 12 a). Then in accord with Corollary 1 consumption is growing along the transition path with no limit while along the efficient curve it is decreasing
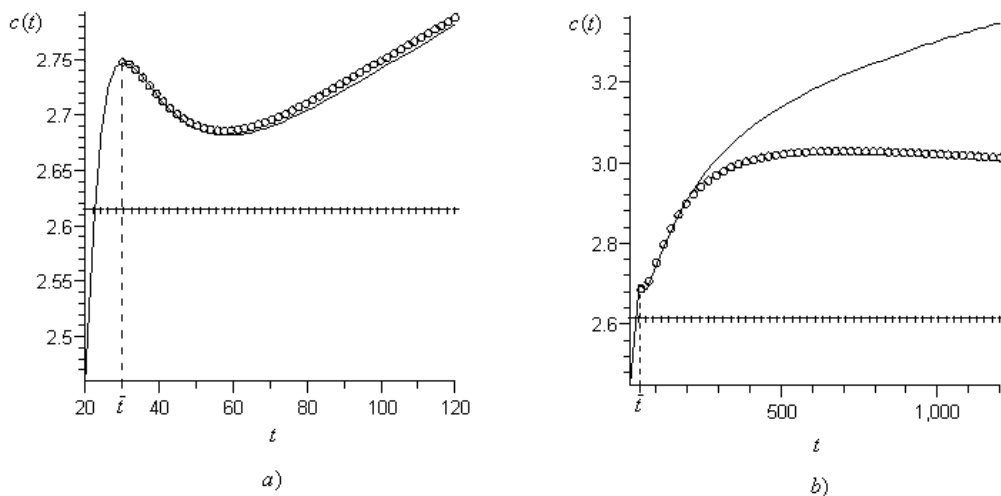
Figure 12: Consumption with the saving rate $\delta(t)$ along the Hartwick's curve (circled) and the transition curve with $d = 5.875$ (solid); the line in crosses is the asymptote for the Hartwick's path (a - short run; b - long run).

to the same asymptote depicted with crosses (Fig. 12 b).

Of course, our comparison of the satisfactoriness of the extraction paths in this case is problematic because the transition curve is optimal with respect to a different welfare criterion, one which implies unlimited growth of consumption. But the example is interesting from the point of view of selecting a criterion. We can see how small sacrifices of consumption in the short run yield large future benefits even for the case when an "almost superior" but inefficient path of consumption is constructed.

9 Concluding remarks

We considered the economy with a restricted rate of substitution between the nonrenewable essential resource and man-made capital. We think that this restriction is plausible when the

man-made capital is represented by new technologies (rather than financial capital in some fund) which substitute for the resource in production, keeping intact the structure and amount of output. The new technologies (e.g., solar plants) diminish the demand for resource implying a declining trajectory of resource extraction. The restriction on the rate of substitution is expressed by the limiting the rate of change in resource extraction: $\dot{r} \leq \dot{r}_{\max}$. In our numerical examples we estimate this bound from historical data.

The restrictions preclude the economy starting resource extraction in an efficient way (i.e. with a decreasing rate). It also prevents the economy from switching instantly at some moment to the efficient and sustainable path. Therefore we considered the problem of switching in finite time as a two-period problem. It turned out that in order to switch "correctly" in finite time to the desirable path, the economy must "smoothly" adjust during the transition period not only the path of extraction but also the saving behavior. Violation of this "smooth" process by discontinuous regime-switching can lead to the seemingly paradoxical result that consumption along the efficient path, realized by the discontinuous switch, is inferior to the consumption along the continuation of the sustainable but inefficient path of extraction during the transition period (Fig. 6 and Fig. 8). In our examples we obtained this interesting result as a consequence of our "artificial" cessation of the growth of output.

When we stuck with the sustainable saving rule we observed that consumption along the efficient path is more than 5% higher (Fig. 11) than consumption along the inefficient transition path which is optimal with respect to the same welfare criterion.

An interesting result was obtained in our comparison of the consumption along the "$\varepsilon-$optimal" transition path and the efficient curve (Fig. 12). The consumption along the efficient path is really superior in the short run (about 130 years) to the consumption along the transition curve

but to a very small extent (less than 0.2% at any moment during this period). But then the "transition consumption" always exceeds the efficient one and the difference between them goes to infinity. Of course, for the valid comparison with this transition curve we must construct a sui generis saving rule and the efficient extraction path consistent with an alternative welfare criterion, one which implies the unlimited growth of consumption. For example, we can use a variant of the Generalized Rawlsian Criterion (Bazhanov, 2006a) in a form of $c^w \dot{c}^{1-w} = \gamma = const$ which implies quasi-arithmetic growth $c(t) = c_0(1 + \mu t)^\varphi$ where $\mu = \frac{1}{\varphi} \left(\frac{\gamma}{c_0} \right)^{\frac{1}{\varphi}}$ and $\varphi = 1 - w$. We think that this problem deserves a separate investigation.

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10   Appendix 1

Lemma 1 *The rational curve of extraction (8) is such that*

a) $s_0 = \frac{r_0 p_0}{c(d-2)}$;

b) the rest of the resource $s(\bar{t})$ along this curve at $\bar{t} \geq 0$ is

$$s(\bar{t}) = s_0 - \int_0^{\bar{t}} r(t)dt = \frac{r_0}{c(d-2)} \frac{p_0 + p_1 \bar{t}}{\left(1 + c\bar{t}\right)^{d-2}} = s_0 \frac{1 + \frac{p_1}{p_0}\bar{t}}{\left(1 + c\bar{t}\right)^{d-2}}$$

where $p_0 = 1 + \frac{b_r}{c(d-3)}$, $p_1 = \frac{b_r(d-2)}{d-3}$;

$r_0 = r(0) -$ initial rate of extraction, $s_0 -$ initial stock;

$b_r = b_r(d)$, $c = c(d)$, and d are the parameters of the curve (8).

Proof. a) By the construction of $r(t)$ and since $d > 3$ we have

$$\frac{s_0}{r_0} = \int_0^\infty (1 + ct)^{1-d}dt + b_r \int_0^\infty t(1 + ct)^{1-d}dt = \frac{1}{c(d-2)}\left[1 + \frac{b_r}{c(d-3)}\right] = \frac{p_0}{c(d-2)}.$$

b) By direct calculations we have

$$s(\bar{t}) = s_0 - \int_0^{\bar{t}} r(t)dt = s(\bar{t}) = s_0 - r_0\left[\frac{1}{c(d-2)}\left\{1 - \left(1 + c\bar{t}\right)^{2-d}\right\} + b_r I(\bar{t})\right] \qquad (16)$$

where

$$
\begin{aligned}
I(\bar{t}) &= \frac{1}{c^2}\left\{\frac{1}{d-3}\left[1 - \left(1 + c\bar{t}\right)^{3-d}\right] - \frac{1}{d-2}\left[1 - \left(1 + c\bar{t}\right)^{2-d}\right]\right\} \\
&= \frac{1}{c^2(d-2)(d-3)}\left\{(d-2)\left[1 - \left(1 + c\bar{t}\right)\left(1 + c\bar{t}\right)^{2-d}\right] - (d-3)\left[1 - \left(1 + c\bar{t}\right)^{2-d}\right]\right\} \\
&= \frac{1}{c^2(d-2)(d-3)}\left\{\left(1 + c\bar{t}\right)^{2-d}\left[(d-3) - (d-2)\left(1 + c\bar{t}\right)\right] + (d-2) - (d-3)\right\} \\
&= \frac{1}{c^2(d-2)(d-3)}\left\{1 - \frac{1 + (d-2)c\bar{t}}{\left(1 + c\bar{t}\right)^{d-2}}\right\}.
\end{aligned}
$$

31

Then the bracket $[\cdot]$ in (16) is

$$
\begin{aligned}
[\cdot] &= \frac{1}{c(d-2)} \left\{ \frac{\left(1+c\bar{t}\right)^{d-2}-1}{\left(1+c\bar{t}\right)^{d-2}} \right\} + \frac{b_r}{c^2\,(d-2)\,(d-3)} \left\{ \frac{\left(1+c\bar{t}\right)^{d-2}-1-(d-2)\,c\bar{t}}{\left(1+c\bar{t}\right)^{d-2}} \right\} \\
&= \frac{1}{c\,(d-2)\left(1+c\bar{t}\right)^{d-2}} \left\{ \left(1+\frac{b_r}{c\,(d-3)}\right)\left(1+c\bar{t}\right)^{d-2} - \left(1+\frac{b_r}{c\,(d-3)}\right) - \frac{b_r\,(d-2)}{(d-3)}\bar{t} \right\} \\
&= \frac{1}{c\,(d-2)} \left\{ p_0 - \frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}} \right\}.
\end{aligned}
$$

Then (16) can be rewritten as follows

$$
s(\bar{t}) = s_0 - \frac{r_0}{c\,(d-2)} \left\{ p_0 - \frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}} \right\}.
$$

Using the result of the case a) we have

$$
s(\bar{t}) = \frac{r_0}{c\,(d-2)} \frac{p_0+p_1\bar{t}}{\left(1+c\bar{t}\right)^{d-2}} = s_0 \frac{1+\frac{p_1}{p_0}\bar{t}}{\left(1+c\bar{t}\right)^{d-2}}
$$

or the assertion of the case b).

11 Appendix 2

Proof of Proposition 4. We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the efficient curve

$$\frac{a_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{17}$$

has real roots if and only if parameter d of the rational curve (8) is such that

$$d \leq \frac{\alpha}{\beta} + 2$$

and that there are two real roots if the last inequality is strict and one real root if it holds as an equality.

Substituting for $r(\widetilde{t}_0)$ and multiplying both sides of (17) by $\left(1 + c\widetilde{t}_0\right)^d$ we have

$$a_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + c\widetilde{t}_0\right)^{d-2}}.$$

Applying assertion b) of Lemma 1 it can be written as

$$a_0 + b\widetilde{t}_0 = -\frac{\alpha r_0^2}{s_0(\alpha - \beta)} \frac{\left(1 + b_r\widetilde{t}_0\right)^2}{\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right)}$$

which means that the moment of "smooth switching" $\widetilde{t}_0$ is a solution of quadratic equation

$$\left(a_0 + b\widetilde{t}_0\right)\left(1 + \frac{p_1}{p_0}\widetilde{t}_0\right) + \frac{\alpha r_0^2}{s_0(\alpha - \beta)}\left(1 + b_r\widetilde{t}_0\right)^2 = 0$$

or

$$\lambda_2\widetilde{t}_0^2 + \lambda_1\widetilde{t}_0 + \lambda_0 = 0 \tag{18}$$

where $\lambda_2 = b\frac{p_1}{p_0} + \frac{b_r^2\alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_1 = \frac{p_1}{p_0}a_0 + b + \frac{2b_r\alpha r_0^2}{s_0(\alpha-\beta)}$, $\lambda_0 = a_0 + \frac{\alpha r_0^2}{s_0(\alpha-\beta)}$. This equation has at least

one real root (two if inequality is strict) if and only if $D = \lambda_1^2 - 4\lambda_2\lambda_0 \geq 0$ where

$$\lambda_1^2 = \frac{1}{s_0^2(\alpha-\beta)^2}\left\{\left(\frac{p_1}{p_0}a_0 + b\right)^2 s_0^2(\alpha-\beta)^2 + 4b_r\alpha r_0^2\left(\frac{p_1}{p_0}a_0 + b\right)s_0(\alpha-\beta) + 4b_r^2\alpha^2 r_0^4\right\}$$

$$\lambda_2\lambda_0 = \frac{1}{s_0^2(\alpha-\beta)^2}\left\{b\frac{p_1}{p_0}a_0 s_0^2(\alpha-\beta)^2 + s_0(\alpha-\beta)\left[b_r^2\alpha r_0^2 a_0 + b\frac{p_1}{p_0}\alpha r_0^2\right] + b_r^2\alpha^2 r_0^4\right\}$$

Cancelling like terms and multiplying by $s_0(\alpha-\beta) > 0$ we can write our condition as $\widetilde{D} \geq 0$

where

$$\widetilde{D} = s_0(\alpha-\beta)\left[\left(\frac{p_1}{p_0}a_0 + b\right)^2 - 4b\frac{p_1}{p_0}a_0\right] + 4\left[b_r\alpha r_0^2\left(\frac{p_1}{p_0}a_0 + b\right) - b_r^2\alpha r_0^2 a_0 - b\frac{p_1}{p_0}\alpha r_0^2\right]$$

Note that the first bracket $[\cdot]$ in this expression is

$$\left(\frac{p_1}{p_0}a_0 + b\right)^2 - 4b\frac{p_1}{p_0}a_0 = \left(\frac{p_1}{p_0}a_0 - b\right)^2$$

and the second bracket is

$$\left[b_r\alpha r_0^2\left(\frac{p_1}{p_0}a_0 + b\right) - b_r^2\alpha r_0^2 a_0 - b\frac{p_1}{p_0}\alpha r_0^2\right] = \alpha r_0^2\left[b_r a_0\left(\frac{p_1}{p_0} - b_r\right) + b\left(b_r - \frac{p_1}{p_0}\right)\right]$$

$$= \alpha r_0^2\left(\frac{p_1}{p_0} - b_r\right)(b_r a_0 - b).$$

Then the condition of the root existence is

$$\widetilde{D} = s_0(\alpha-\beta)\left(\frac{p_1}{p_0}a_0 - b\right)^2 + 4\alpha r_0^2\left(\frac{p_1}{p_0} - b_r\right)(b_r a_0 - b) \geq 0 \tag{19}$$

where

$$\frac{p_1}{p_0}a_0 - b = c(d-2)r_0 b_r + a_0\frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3) + b_r} = b_r c(d-2)\left[r_0 + \frac{a_0}{c(d-3) + b_r}\right]$$

$$= b_r c(d-2)r_0\left[\frac{c(d-3) + b_r + \frac{a_0}{r_0}}{c(d-3) + b_r}\right] = b_r c(d-2)r_0\left[\frac{c(d-3) + c(d-1) + \frac{a_0}{r_0} + \frac{a_0}{r_0}}{c(d-3) + c(d-1) + \frac{a_0}{r_0}}\right]$$

$$= 2b_r c(d-2)r_0\left[\frac{c(d-2) + \frac{a_0}{r_0}}{2c(d-2) + \frac{a_0}{r_0}}\right],$$

34

$$\begin{aligned}
\frac{p_1}{p_0} - b_r &= \frac{b_r(d-2)}{d-3}\frac{c(d-3)}{c(d-3)+b_r} - b_r = b_r\left[\frac{c(d-2)}{c(d-3)+b_r} - 1\right] \\
&= b_r\left[\frac{c(d-2)-c(d-3)-b_r}{2c(d-2)+\frac{a_0}{r_0}}\right] = -b_r\left[\frac{c(d-2)+\frac{a_0}{r_0}}{2c(d-2)+\frac{a_0}{r_0}}\right],
\end{aligned}$$

$$b_r a_0 - b = b_r a_0 + b_r c(d-2)r_0 = b_r r_0\left[c(d-2)+\frac{a_0}{r_0}\right].$$

Substituting for these expressions in (19) we obtain

$$\widetilde{D} = s_0(\alpha-\beta)4b_r^2 c^2(d-2)^2 r_0^2\left[\frac{c(d-2)+\frac{a_0}{r_0}}{2c(d-2)+\frac{a_0}{r_0}}\right]^2 \geq 4\alpha r_0^3 b_r^2\left[\frac{c(d-2)+\frac{a_0}{r_0}}{2c(d-2)+\frac{a_0}{r_0}}\right]\left[c(d-2)+\frac{a_0}{r_0}\right]$$

or

$$\frac{s_0(\alpha-\beta)c^2(d-2)^2}{2c(d-2)+\frac{a_0}{r_0}} \geq \alpha r_0.$$

Substituting for s_0 (Lemma 1, a)) into the LHS we have

$$\frac{p_0 c(d-2)}{2c(d-2)+\frac{a_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

and substituting for p_0 we obtain

$$\frac{(d-2)}{(d-3)}\frac{2c(d-2)+\frac{a_0}{r_0}}{2c(d-2)+\frac{a_0}{r_0}} \geq \frac{\alpha}{\alpha-\beta}$$

or

$$1 - \frac{\beta}{\alpha} \geq 1 - \frac{1}{d-2}.$$

The last expression gives us $\frac{1}{d-2} \geq \frac{\beta}{\alpha}$ or $d \leq \frac{\alpha}{\beta} + 2$.

12   Appendix 3

Proof of Proposition 5. We will show that the equation defining the moment $\widetilde{t}_0$ of "smooth switching" to the efficient curve

$$\frac{a_0 + b\widetilde{t}_0}{(1 + c\widetilde{t}_0)^d} = -\frac{\alpha r^2(\widetilde{t}_0)}{\widetilde{s}_0(\widetilde{t}_0)(\alpha - \beta)} \tag{20}$$

has only one real finite positive root if and only if $d < \frac{\alpha}{\beta} + 2$.

It was shown in the Appendix 2 that equation (20) is equivalent to the quadratic equation (18) which using lemma 1 is equivalent to equation

$$\mu_2 \widetilde{t}_0^2 + \mu_1 \widetilde{t}_0 + \mu_0 = 0 \tag{21}$$

where $\mu_2 = bp_1 + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha - \beta}$, $\mu_1 = p_1 a_0 + bp_0 + \frac{2b_r r_0 \alpha c(d-2)}{\alpha - \beta}$, $\mu_0 = a_0 p_0 + \frac{r_0 \alpha c(d-2)}{\alpha - \beta}$.

Substituting for b, p_0, p_1, and reorganizing we have

$$
\begin{aligned}
\mu_2 &= -b_r c(d-2) r_0 \frac{b_r(d-2)}{d-3} + \frac{b_r^2 r_0 \alpha c(d-2)}{\alpha - \beta} = b_r^2 r_0 c(d-2) \left[\frac{\alpha}{\alpha - \beta} - \frac{d-2}{d-3} \right] \\
&= \frac{b_r^2 r_0 c(d-2)}{(\alpha - \beta)(d-3)} \left[\beta(d-2) - \alpha \right].
\end{aligned}
$$

Note that in our formulation of the problem the multiplier $\frac{b_r^2 r_0 c(d-2)}{(\alpha-\beta)(d-3)}$ in the last formula is always positive since $d > 3$, $\alpha > \beta$, $r_0 > 0$, $a_0 > 0$ and it follows $c > 0$. Then the sign of μ_2 is defined by the sign of $\beta(d-2) - \alpha$. Namely, μ_2 is negative when $d < \frac{\alpha}{\beta} + 2$, positive when $d > \frac{\alpha}{\beta} + 2$, and zero when $d = \frac{\alpha}{\beta} + 2$.

Coefficient μ_1 is

$$
\begin{aligned}
\mu_1 &= \frac{b_r(d-2)}{d-3} a_0 - b_r c(d-2) r_0 \left(1 + \frac{b_r}{c(d-3)} \right) + \frac{2b_r r_0 \alpha c(d-2)}{\alpha - \beta} \\
&= b_r(d-2) \left[\frac{a_0}{d-3} - r_0 \frac{2c(d-2) + \frac{a_0}{r_0}}{d-3} + \frac{2r_0 \alpha c}{\alpha - \beta} \right] \\
&= b_r(d-2) \left[\frac{a_0 - 2r_0 c(d-2) + a_0}{d-3} + \frac{2r_0 c\alpha}{\alpha - \beta} \right].
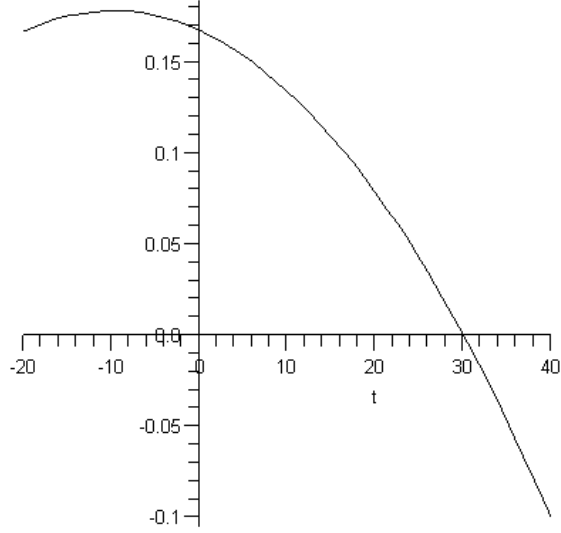\end{aligned}
$$

36

Figure 13: The root of equation (12) for $d = 5.875$.

Finally we have

$$\mu_1 = 2b_r r_0 c(d-2) \left[\frac{\alpha}{\alpha - \beta} - \frac{d-2}{d-3} \right].$$

Note that b_r is also positive in our formulation (because of the growing rate of extraction in the neighborhood of $t = 0$). Then the sign of μ_1 like the sign of μ_2 is completely defined by the same expression $\beta(d-2) - \alpha$. It can be shown that $\mu_0 > 0$ for $a_0 > 0$. The peak of parabola (21) is defined by equation

$$t^* = -\frac{\mu_1}{2\mu_2} = -\frac{2b_r r_0 c(d-2)\left[\beta(d-2) - \alpha\right]}{2b_r^2 r_0 c(d-2)\left[\beta(d-2) - \alpha\right]} = -\frac{1}{b_r} < 0.$$

Hence our parabola is convex for $d < \frac{\alpha}{\beta} + 2$ and has only one positive finite root (Fig. 13).

With $d \to \frac{\alpha}{\beta} + 2 - 0$ parabola degenerates into a positive constant and the root goes to infinity.

37