A Review on Liao's Dissertation Entitled "The Solutions on Multi-choice Games" and Related Publications

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Abstract. In 2007, Liao finished his Ph.d. dissertation[18](Liao 2007) entitled "The Solutions on Multi-choice Games". Chapter 1 of the dissertation mainly worked on two special cases of the H&R multi-choice Shapley value. One assumes that the weight function $w(j)$ is a positive constant function for all $j \neq 0$ with $w(0) = 0$ and the other one assumes that the weight function $w(j) = j$ for all j. If $w(j)$'s are equal for all $j \neq 0$ then the formula of H&R multi-choice Shapley value can be significantly simplified to the original formula of the traditional Shapley value for the traditional games. Therefore, as a matter of fact, Definitions 1 and 2 in Chapter 1 of the dissertation [18] are simply the traditional Shapley value. Hence, in most part of Chapter 1, Liao was just writing "new results" of traditional games in terms of the notations of multi-choice games. Furthermore, the dissertation [18] did not cited [7](1994), [8](1995a) and [10](1996) which held the original ideas of its main part of chapter 1.

Keywords and Phrases: Multi-choice TU games, Shapley value, potential, w-consistency.

1 Introduction

Motivated by calculating the power indices of players in different levels of joint military actions, in [5](1992) and [6](1993), Hsiao and Raghavan extended the traditional cooperative game to a multi-choice cooperative game and extended the traditional Shapley value to a multi-choice Shapley value. Other researchers call the multi-choice Shapley value the H&R Shapley value.

In [6](1993), Hsiao and Raghavan give weights(discriminations) to action levels instead of players. The H&R Shapley value is symmetric among players and asymmetric among actions, therefore, the H&R Shapley value is an extention of both the symmetric and the asymmetric Shapley values.

In [3](1989), Hart and Mas-Colell were the first to introduce the potential approach to TU games. In consequence, they proved that the Shapley value (1953) can result as the vector of marginal contributions of a potential. The potential approach is also shown to yield a characterization for the Shapley value, particularly in terms of an internal consistency property.

The H&R Shapley value is monotone, transferable utility invariant, dummy free and independent of non-essential players, please see [5](1992), [6](1993) and [8](1995a) for details. In 1991, when Hsiao and Raghavan presented [6] in the 2rd International Conference on Game Theory at Stony-Brook, Shapley suggested that we should study the consistent property of the H&R Shapley value.

The property of consistency is essentially equivalent to the existence of a potential function. Following Shapley's advice, in [7](Hsiao, Yeh and Mo 1994), Hsiao, defined the potential function for multi-choice TU games and found an explicit formula of the potential function. Moreover, Hsiao defined the w-reduced games with respect to an action vector and a solution of multi-choice TU games. Also, Hsiao showed that the H&R Shapley value is w-consistent and showed the coincidence of the H&R Shapley value and the vector of marginal contributions of a potential. However, the definitions of the reduced game and related consistency were incomplete in [7]. As a matter of fact, Hsiao took full responsibility for whole of [7], Yeh and Mo were just using Möbius inversion formula to double check the explicit formula of potential function. As some referees said, there was no evidence of the existence of [7]. Fortunately, Hsiao had the potential functions in [10](1996) for his grant, NSC 85-2121-M-031-006(1995-1996), Taiwan. The technical report [10](1996) holds the very original explicit formula of the potential function for multi-choice games.

Since some definitions in [7] and [10] were incomplete, of course, Hsiao did not characterize the $H\&R$ Shapley value in terms of consistency. In the Master thesis[17](1999), under Hsiao's supervision, Liao tried to provide an axiomatization which is the parallel of Hart and Mas-Colell's (1989) axiomatization of the Shapley value by applying the w-consistency property. However, Liao failed to finish the job.

In 2007 Liao finished his Ph.D. dissertation [18] entitled "The Solutions on Multi-choice Games" under Hwang's supervision. In the dissertation (2007), Liao essentially repeated the results of [7],[10] and [17] for two special cases of H&R multi-choice Shapley value. But in the dissertation, Liao did not cite the original idea and the explicit formula of the potential function which appeared in [7], [10] and [17].

We would rather believe that Hwang knew nothing about [7] $\sim$ [10] and [17]. Please note that Liao had submitted his Master thesis [17] to Dong-Hwa university when he applied for admission to the Ph.d. program, and [8](1995a) was published in Games and Economic Behavior.

Chapter 1 of Liao's dissertation mainly worked on two special cases of the H&R multi-choice Shapley value. W.L.O.G, one assumes that $w(j) = 1$ for all j and the other one assumes that $w(j) = j$ for all j. When $w(j) = 1$ for all j, then the formula of H&R multi-choice Shapley value can be significantly simplified to the formula of the traditional Shapley value. Therefore, as a matter of fact, Definition 1 and 2 in Liao's Chapter 1 is just a traditional Shapley value. Hence, in most part of Chapter 1, Liao was just rewriting "new results" of traditional games in terms of the notations of multi-choice games.

2 Definitions and Notations

Traditional Cooperative Games and The Shapley Value We first review the traditional cooperative games and the traditional Shapley value. Following [24](Shapley 1953), we have the following definitions and notations. Let $N = \{1, 2, ..., n\}$ be the set of *players*. The collection of *coalitions* (subsets) in N is denoted by $2^N = \{S : S \subseteq N\}$.

The coalition N is called the grand coalition. The number of players in coalition S is denoted by $|S|$.

A cooperative n-person game in characteristic function form is the pair (N, v) defined by: $v : 2^N \to R$ with $v(\emptyset) = 0$. We can identify the set of all cooperative games by: $G \simeq R^{2^n-1}$.

The very original Shapley value satisfied three axioms, please see [24], for player i on G is well-known as the function $\phi_i : G \to R$ such that

$$\phi_i(v) = \sum_{\substack{i \in S \\ S \subseteq N}} \frac{(|S|-1)!(n-|S|)!}{n!} \, [v(S) - v(S - \{i\})] \tag{T1}$$

Multi-choice Games and Multi-choice Shapley Value
The very original mathematical setup of multi-choice games in [4],[5] and [6] matches the traditional mathematical symbols and notations. For example, a vector is denoted by a bold face lower-case letter $\mathbf{x}$ in most of mathematics text books.

Since the dissertation [18](Liao 2007) uses a different mathematical setup, we compromise with his notations, except bold face vector $\mathbf{x}$, as following.

Let U be the universe of players. Let $N \subseteq U$ be a set of players and let $\mathbf{m} = (m_i)_{i \in N}$ be the vector that describes the number of activity levels for each player, at which he can actively participate. For $i \in U$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N. Denote $\mathbf{0}_N$ the zero vector in $\mathbb{R}^N$.

Note O-1: In [5], [6], we emphasized that the action space is a well-ordered set $\{\sigma_0, \sigma_1, ...\sigma_m\}$, we denote the action space by $\{0, 1, ..., m\}$ just for notational convenience

A **multi-choice TU game** is a triple $(N, \mathbf{m}, v)$, where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\mathbf{x} = (x_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $v(\mathbf{0}_N) = 0$. If no confusion can arise, a game $(N, \mathbf{m}, v)$ will sometimes be denoted by its characteristic function v. Denote the class of all multi-choice TU games by MC. Given $(N, \mathbf{m}, v) \in MC$ and $\mathbf{x} \in M^N$, we write $(N, \mathbf{x}, v)$ for the multi-choice TU subgame obtained by restricting v to $\{\mathbf{y} \in M^N \mid y_i \leq x_i \ \forall i \in N\}$ only.

Given $(N, \mathbf{m}, v) \in MC$, let $L^{N,\mathbf{m}} = \{(i,j) \mid i \in N, j \in M_i^+\}$. Let $w : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ be a non-negative function such that $w(0) = 0$ and for all $j \leq l$, $w(0) < w(j) \leq w(l)$, then w is called a weight function. Given $(N, \mathbf{m}, v) \in MC$ and a weight function w for the actions, a solution on MC is a map ψ^w assigning to each $(N, \mathbf{m}, v) \in MC$ an element

$$\psi^w(N, \mathbf{m}, v) = \Big(\psi^w_{i,j}(N, \mathbf{m}, v)\Big)_{(i,j) \in L^{N,\mathbf{m}}} \in \mathbb{R}^{L^{N,\mathbf{m}}}.$$

Here $\psi^w_{i,j}(N, \mathbf{m}, v)$ is the power index or the value of the player i when he takes action j to play game v. For convenience, given a $(N, \mathbf{m}, v) \in MC$ and a solution ψ on MC, we define $\psi_{i,0}(N, \mathbf{m}, v) = 0$ for all $i \in N$.

An Important Note to readers: In [1] and [4]$\sim$[10], we denote $\psi^w_{i,j}(N, \mathbf{m}, v)$ as the power index or the value of the player "j" when he takes action" i" to play game v. That matches the notation of traditional matrix. However, in this article, we compromise with their notations.

To state the H&R Shapley value, some more notations will be needed. Given $S \subseteq N$, let $|S|$ be the number of elements in S, $S^c = N \setminus S$ and let $\mathbf{e}^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in S \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that if no confusion can arise $e_i^S(N)$ will be denoted by e_i^S.

Given $(N, \mathbf{m}, v) \in MC$ and a weight function w, for any $x \in M^N$ and $i \in N$, we define $\|\mathbf{x}\|_w = \sum_{i \in N} w(x_i)$, $\|\mathbf{x}\| = \sum_{i \in N} x_i$ and $M_i(\mathbf{x}; \mathbf{m}) = \{i \mid x_i \neq m_i, i \neq j\}$.

In [5](Hsiao and Raghavan 1992), the H&R Shapley value γ^w is obtained by

$$\gamma^w_{i,j}(N, \mathbf{m}, v) = \sum_{k=1}^{j} \sum_{\substack{x_i = k, \mathbf{x} \neq \mathbf{0}_N \\ x \in M^N}} \Big[\sum_{T \subseteq M_i(\mathbf{x}; \mathbf{m})} (-1)^{|T|} \frac{w(x_i)}{\|x\|_w + \sum\limits_{r \in T} [w(x_r + 1) - w(x_r)]} \Big] \cdot \Big[v(x) - v(x - e^{\{i\}}) \Big] \tag{A1}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we say $\mathbf{y} \leq \mathbf{x}$ if $y_i \leq x_i$ for all $i \in N$. In [4],[5] and [6] the analogue of unanimity games for multi-choice games are **minimal effort games** (N, m, u_N^x), where $\mathbf{x} \in M^N$, $\mathbf{x} \neq 0_N$, defined by

$$u_N^{\mathbf{x}}(y) = \begin{cases} 1 & \text{if } y \geq x \ ; \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in M^N$.

Note O-2 In Theorem 1 of [5](1992), Hsiao and Raghavan showed that for all $(N, \mathbf{m}, v) \in MC$, it holds that $v = \sum_{\substack{\mathbf{x} \in M^N \\ \mathbf{x} \neq \mathbf{0}_N}} a^{\mathbf{x}}(v)\ u_N^{\mathbf{x}}$, where $a^{\mathbf{x}}(v) = \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|}\ v(\mathbf{x} - \mathbf{e}^S)$. But in page 8 of the dissertation [18](2007), Liao used that Theorem to give Definitions 1, 2, and 3 and did not mention where the original $a^{\mathbf{x}}(v)$ came from.

Following [7] and [10], given $i \in N$ and $v(\mathbf{x})$, we define $d_i v(\mathbf{x}) = v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}^{\{i\}}$ then d_i is associative, i.e. $d_k(\ d_i) = d_i(\ d_k)$. For convenience, we denote $d_i\ d_k = d_{ik}$, $d_{ijk} = d_i\ d_j\ d_k$,...,etc. We also denote $d_{\ i_1,\ i_2, ...,\ i_t} = d_S$ whenever $\{\ i_1,\ i_2, ...,\ i_t\} = S$. Furthermore, we denote $d_{S(\mathbf{x})}$ by $d_{\mathbf{x}}$.

Now, it is a trivial homework for students in master program to show the following Homework.

Homework 1 Please check that $d_{\mathbf{x}} v(\mathbf{x}) = a^{\mathbf{x}}(v) = \sum_{S \subseteq S(\mathbf{x})} (-1)^{|S|}\ v(\mathbf{x} - \mathbf{e}^S)$.

Readers may down-load the master thesis [17](Liao 1999) form the following web-side. http://163.14.136.79/ETD-db/ETD-search/view_etd?URN=etd-0123107-172344-1493 Please click the button "etd-0123107-172344-1493.pdf" to see the master thesis. The 6th line of page 6 is exactly the above homework.

Now, in the proof of Theorem 1 of [8], the first two equations in page 428, we have a very trivial "reformulation" of the H&R multi-choice shapley value (A1) as follows.

$$\begin{aligned} \gamma_{i,j}^w(N, \mathbf{m}, v) &= \sum_{\substack{0 < x_i \leq j, \mathbf{x} \neq \mathbf{0}_N \\ \mathbf{x} \in M^N}} \Big[\sum_{S \subseteq S(x)} (-1)^{|S|}\ v(\mathbf{x} - \mathbf{e}^S) \Big] \cdot \frac{w(x_i)}{\|\mathbf{x}\|_w} \\ &= \sum_{\substack{0 < x_i \leq j, \mathbf{x} \neq \mathbf{0}_N \\ \mathbf{x} \in M^N}} w(x_i) \cdot \frac{a^{\mathbf{x}}(v)}{\|\mathbf{x}\|_w} \end{aligned} \tag{A2}$$

Comment 1 **[18]:** In the footnote on page 8 of the Ph.d. dissertation [18](2007), Liao declared that they defined H&R Shapley value in terms of dividend, i.e. they regarded (A2) as **"their definition".** But (A2) was in the proof of Theorem 1 in page 428 of [8](Hsiao 1995a) for a long time, and Liao did not cite it. Liao knew all the results in [4]-[10] while [17] is under our supervision.

[19]: Please see also the footnote on page 2 of [19](2007), Liao declared"∗ We define the H&R Shapley value in terms of the dividends. Hsiao and Raghavan (1993) provided an alternative formula of the H&R Shapley value." In other words, Liao declared that (A2) is "their definition".

[16]: Moreover, in page 600 of [16](Hwang, Liao 2009), at the first line below " their" Definition 1, they declared that Hwang and Liao provided a representation of the H&R Shapley value by dividends. But the so called representation which they provided is (A2) which has been in [8] since 1995. Please note that [8](1995a) is published in Games and Economics Behavior.

As a matter of fact, the special case of H&R Shapley value in Definition 1 of [18] can be significantly simplified to the traditional Shapley value for a traditional TU game. Apparently, Hwang and Liao were writing the traditional game in terms of multi-choice game. Many researchers have the same myth.

Now, we check the special cases of (A1)=(A2). W.L.O.G, replaced $w(j)$ by 1 for all $j > 0$ in (A2), we can easily see Definition 1 and Definition 2 of Liao's dissertation [18]. To make this article self-contained we copy "their" definition 1 and 2 as follows.

Definition 1 ([18])Peters and Zank (2005) proposed a multi-choice Shapley value, the P&Z Shapley value. We denote the P&Z Shapley value by Γ. Formally, the P&Z Shapley value is the solution on MC which associates with each game $(N, \mathbf{m}, v)$ and each $(i, j) \in L^{N,\mathbf{m}}$the value

$$\Gamma_{i,j}(\mathbf{m}, v) = \sum_{\substack{\mathbf{x} \in M^N(\mathbf{m}) \\ x_i = j}} \frac{a^{\mathbf{x}}(v)}{|S(\mathbf{x})|}.$$

Definition 2 ([18]) Hsiao and Raghavan (1992,1993) proposed a multi-choice Shapley value, the H&R Shapley value. We denote the symmetric form of the H&R Shapley value by γ. Formally, the H&R Shapley value is the solution on MC which associates with each game $(N, \mathbf{m}, v)$ and each $(i, j) \in L^{N,\mathbf{m}}$the value

$$\gamma_{i,j}(\mathbf{m}, v) = \sum_{\substack{\mathbf{x} \in M^N(\mathbf{m}) \\ x_i \leq j}} \frac{a^{\mathbf{x}}(v)}{|S(\mathbf{x})|}.$$

Comment 2 The P&Z value Γ_{ij} is just a subdivision of the so called symmetric form of the H&R Shapley value γ_{ij}. Please note that γ_{ij} is nothing but a special case of (A2)=(A1) where $w(j)$ are all equal for $j \neq 0$. W.L.O.G. γ_{ij} is a special case of (A1) where $w(j) = 1$ for all $j \neq 0$.

By Step 1 of the proof in Theorem 5.1 of [4](1991), in page 31, we used to give our students the following homework.

Homework 2 Please show that

$$\sum_{T \subseteq M_i(\mathbf{x};\mathbf{m})} (-1)^{|T|} \frac{w(x_i)}{\|x\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} = 0$$

whenever there exists $[w(x_r + 1) - w(x_r)] = 0$.

By Homework 2, (A1) can be significantly simplified to (B1) as follows. We used to tell our students to do the calculations.

$$\gamma_{i,j}(N, \mathbf{m}, v) = \sum_{S \subseteq N - \{i\}} \frac{(|S|!)(n - |S| + 1)!}{n!} \cdot \Big[v((j_i, \mathbf{m}^S, \mathbf{0}^{N-S-\{i\}})) - v((0_i, \mathbf{m}^S, \mathbf{0}^{N-S-\{i\}}) \Big], \quad \text{(B1)}$$

where the action vector $(j_i, \mathbf{m}^S, \mathbf{0}^{N-S-\{i\}})$ is the action vectors that player i takes action of level j, each player in S takes his highest-level-action and the other players in $N-S-\{i\}$ do nothing. Moreover, the action vector $(0_i, \mathbf{m}^S, \mathbf{0}^{N-S-\{i\}})$ is the action vectors that player i do nothing, each player in S take his highest-level-action and the other players do nothing.

Define a traditional game v^t such that

$$v^t(S) = \begin{cases} v((j_i, \mathbf{m}^{S-\{i\}}, \mathbf{0}^{N-S})) & \text{if } i \in S \\ v((0_i, \mathbf{m}^S, \mathbf{0}^{N-S-\{i\}})) & \text{if } i \notin S, \end{cases}$$

Then, (B1) is the traditional Shapley value for the traditional cooperative game v^t.

Now, Since $\gamma_{ij}(v) = (B1)$, whatever Liao do in his dissertation [18] for Γ_{ij} and γ_{ij} is simply working with the traditional Shapley value (B1) for traditional binary choice games, i.e. the dissertation is writing the traditional Shapley value in terms of the notations of the multi-choice Shapley value.

Please note that $\Gamma_{i,j}(v) = \gamma_{i,j}(v) - \gamma_{i,j-1}(v)$, therefore the Definition 1, P&Z Shapley value, is actually a traditional Shapley value too.

We used to let our students know: there are two ways to make the H&R Shapley value $\gamma^w_{ij}(v)$ become a traditional Shapley value. One is restricting v to a traditional binary choice game, the other one is assuming that $w(j)$'s are equal for all $j \neq 0$ i.e., restricting the H&R Shapley value to so call symmetric form. In [23](2005), Peters and Zank called $\Gamma_{ij}(v)$ egalitarian solution, actually they also found (B1) in [23] form different way.

In addition to integrity, the most important part of a research is the motivation, the motivation of the multi-choice games and the multi-choice Shapley value is to deal with the case that a player may have different choices, and a player might need different efforts(weights) to execute different levels of actions. If all the weights(efforts) are equal, then we see no reason why modeling a multi-choice game, several traditional games is enough for Definition 1 and 2. **Definition 1 and 2 make the multi-choice Shapley value lose its value!**

Definition 3([18]) Derks and Peters(1993) proposed a multi-choice Shapley value, the D&P Shapley value. We denote the D&P Shapley value by Θ. Formally, the D&P Shapley value is the solution on MC which associates with each game $(N, \mathbf{m}, v)$ and each $(i, j) \in L^{N,\mathbf{m}}$the value

$$\Theta_{i,j}(N, \mathbf{m}, v) = \sum_{\substack{\mathbf{x} \in M^N(\mathbf{m}) \\ x_i \geq j}} \frac{a^{\mathbf{x}}(v)}{\|\mathbf{x}\|}.$$

Let $w(j) = j$ for all j, denoted this kind of weight function by w^1, then by (A2) we have

$$\gamma^{w^1}_{i,j}(N,\mathbf{m},v) = \sum_{\substack{0<x_i\le j\\ \mathbf{x}\in M^N}} x_i \cdot \frac{a^{\mathbf{x}}(v)}{\|\mathbf{x}\|_w}$$
$$= \sum_{k=1}^{k=j} \sum_{\substack{x_i=k\\ \mathbf{x}\in M^N}} k \cdot \frac{a^{\mathbf{x}}(v)}{\|\mathbf{x}\|}$$

Then

$$\frac{1}{j}\cdot\left[\gamma^{w^1}_{i,j}(N,\mathbf{m},v) - \gamma^{w^1}_{i,j-1}(N,\mathbf{m},v)\right] = \sum_{\substack{x_i=j\\ \mathbf{x}\in M^N}} \frac{a^{\mathbf{x}}(v)}{\|\mathbf{x}\|}$$

Therefore

$$\Theta_{i,j}(N,\mathbf{m},v) = \sum_{k\ge j}\frac{1}{k}\cdot\left[\gamma^{w^1}_{i,k}(N,\mathbf{m},v) - \gamma^{w^1}_{i,k-1}(N,\mathbf{m},v)\right]$$

[20]: Now the D&P Shapley value is just a linear combination of H&R Shapley with the special weight w^1, the potential function in [7], [10] and [17] may also be applied to the P&D Shapley value. Liao should cite [7], [10] or [17], even if he just works on the potential function related to the P&D Shapley value. The publication [20](Liao 2009) did not tell the readers where explicit formula of potential function came from.

After, Hsiao and Raghavan extended the traditional cooperative games to the multi-choice cooperative games, researchers may try to extend any result of a traditional game to a multi-choice game. However, if we do not have a reasonable real-world example to justify the value of extending the result to a multi-choice game, then what we have done might be just rewriting the result in terms of the notations of multi-choice games. Similarly, researchers may rewrite the results of traditional games in terms of so called fuzzy games such as [15](Hwang, Liao 2009).

[15]: There are 19 Examples in the dissertation [18], none of them has a real-world interpretation. Reviewing the real world example in Remark 2 of [15](Hwang, Liao 2009), we find that it is essentially Example 1 in [9](Hsiao, 1995b).

3 Potential

Following [7], [10] and [17] and using the mathematical setup of [18] we see the following definitions and notations.

For $\mathbf{x} \in \mathbb{R}^N$, we write $\mathbf{x}_S$ to be the restriction of x at S for each $S \subseteq N$. Given a $(N,\mathbf{m},v) \in MC$ and $\mathbf{x} \in M^N$, let $i \in N$ and $j \in M_i$, for convenience we introduce the substitution notations x_{-i} to stand for $x_{N\setminus\{i\}}$. Moreover, $(\mathbf{x}_{-i}, j) = \mathbf{y} \in \mathbb{R}^N$ be defined by $\mathbf{y}_{-i} = \mathbf{x}_{-i}$ and $y_i = j$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we say $\mathbf{y} \leq \mathbf{x}$ if $y_i \leq x_i$ for all $i \in N$.

Note O-3: It is a well-known notation in reliability theory that $(j_i, \mathbf{x})$ denotes a vector that the ith component is replaced by j. As a matter of fact, multi-choice games are quite similar to multi-state coherent systems in reliability. We see no reason why the other researchers do not make use of the well-known knowledge from reliability theory.

Following [7], [10] and [17], given $(N, \mathbf{m}, v) \in MC$ and a weight function w, we define a function $P_w : MC \longrightarrow I\!R$ which associates a real number $P_w(N, \mathbf{m}, v)$. Subsequently, we define the following operators :

$$D^{i,j} P_w(N, \mathbf{m}, v) = w(j) \cdot \left[P_w\big(N, (\mathbf{m}_{-i}, j), v\big) - P_w\big(N, (\mathbf{m}_{-i}, j-1), v\big) \right]$$

and

$$H_{i,x_i} = \sum_{l=1}^{x_i} D^{i,l}.$$

Definition O-1([7], [10], [17]) A function $P_w : MC \longrightarrow I\!R$ with $P_w(N, 0_N, v) = 0$ is called w-**potential function** if it satisfies the following condition :
Given $(N, \mathbf{m}, v) \in MC$ and a weight function w,

$$\sum_{i \in S(\mathbf{m})} H_{i,m_i} P_w(N, \mathbf{m}, v) = v(\mathbf{m}).$$

Theorem O-1 ([7],[10],[17])The potential of a multi-choice cooperative game is unique. Furthermore, given a weight function w and $(N, \mathbf{m}, v) \in MC$, the H&R Shapley value γ^w and the w-potential P_w have the following relationship. For all $(i, j) \in L^{N,\mathbf{m}}$,

$$\gamma^w_{i,j}(N, \mathbf{m}, v) = H_{i,j} P_w(N, \mathbf{m}, v).$$

A Very Important Note: In [7] and [10](1996), page 5, Theorem 2.1, (see also [17](1999), page 6, Theorem 2.1), we have the following explicit closed form of the w-potential function which is the key to prove this theorem.

Given $(N, \mathbf{m}, v) \in MC$ and a weight function w. Hsiao, Yeh and Mo[7] (1994) proved that the w-potential of a multi-choice cooperative game is unique, and

$$\begin{aligned} P_w(N, \mathbf{m}, v) &= \sum_{\substack{\mathbf{y} \leq \mathbf{m}, \\ \mathbf{y} \neq \mathbf{0}_N}} \frac{1}{\|\mathbf{y}\|_w} a^{\mathbf{y}}(v) \\ &= \sum_{\substack{\mathbf{y} \leq \mathbf{m}, \\ \mathbf{y} \neq \mathbf{0}_N}} \frac{1}{\|\mathbf{y}\|_w} d_{\mathbf{y}} v(\mathbf{y}) \end{aligned} \tag{A3}$$

By formula (A3), one can prove this theorem by some combinatorial calculation, please see[7] or [17](Liao 1999), for the calculation.

Comment 3 Because the formula (A1) of the H&R Shapley value is complicated and the formula (A2) of the H&R Shapley value γ_{ij} has a variable j and a factor $w(j)$, it is not easy to find the explicit closed form (A3) of the potential function $P_w(N, \mathbf{m}, v)$ by observing (A1) or (A2). By try and error, Hsiao found the formula (A3) in [7] where Hsiao took full responsibility for it. Liao finished [17] under our supervision. Therefore, Liao knew that [7] and [10] held the original idea of (A3) of the potential function. But Liao used special cases of the explicit formula (A3) in his dissertation [18](2007) and did not cite any one of [7], [10] or [17], even after we reminded him to cite the papers. **The publications of Liao related to the explicit formula (A3) of the potential function and its special cases are all doubtful.**

Comment 4 [13]: There is a typo in [13](2008b), Hwang and Liao use formula (A3) in page 596 of [13] as equation (4.3) of the potential function, but the equation (4.3) in [13] has a big typo. We leave it to the readers to find out the big typo. We asked Liao to cite [7], but they simply put [7] in the list of the references of [13], did not tell the readers where (4.3) came from.

4 w-Consistency Property

Hart's incomplete definitions First, we copy the very original definitions and notations in [3](Hart 1989) for the traditional cooperative games. Then show that the reduced game defined by Hart and Mas-Colell in 1989 was not well-defined. Therefore, the consistent property based on the reduced game was not well-defined either. Hence a characterization of the Shapley value proposed by Hart and Mas-Colell was incomplete. **This is the main reason why we did not send [7] or [10] for publication.**

Following [3], we have the following definitions and notations. Let N be a finite set of players and $|N|$ denote the number of players in N.

A cooperative game with side payments - in short, *a game* - consists of a pair (N, v), where N is a finite set of players and $v : 2^N \to R$ is the *characteristic function* satisfying

$$v(\emptyset) = 0.$$

A subset $S \subset N$ is called a *coalition*.

Let $\mathbf{G}$ denote the set of all games. Formally, a solution function ϕ is a function defined on $\mathbf{G}$ that associated to every $(N, v) \in \mathbf{G}$ a payoff vector $\phi(N, v) = (\phi^i(N, v))_{i \in N} \in R^n$.

Given a solution function ϕ, a game (N, v) and a coalition $T \subset N$, the reduced game is defined by

$$v_T^{\phi}(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi^i(S \cup T^c, v)$$

for all $S \subset T$, where $T^c = N \backslash T$. The solution function ϕ is *consistent* if

$$\phi^j(T, v_T^\phi) = \phi^j(N, v)$$

for every game (N, v), every coalition $T \subset N$ and all $j \in T$.

Remark 1 Before we recognize v_T^ϕ as a game, we have to provide that

$$v_T^\phi(\emptyset) = v(T^c) - \sum_{i \in T^c} \phi^i(T^c, v) = 0$$

That is

$$v(T^c) = \sum_{i \in T^c} \phi^i(T^c, v).$$

In other words, ϕ is efficient for (T^c, v).

But, in the beginning of the definition, we did not provide that ϕ is efficient, i.e. we did not provide the sufficient condition which makes v_T^ϕ a game. In particular, we even did not provide that

$$\phi^1(\{1\}, v) = 0, \tag{***}$$

for the trivial one-person game $(\{1\}, v)$ where $v(\{1\}) = v(\emptyset) = 0$. Therefore, given a two-person game $(\{1, i\}, v)$ such $v(\{1\}) = v(\emptyset) = 0$ and $v(\{1, i\}) = v(\{i\}) \neq 0$, for $T = \{i\}$ and ϕ, we can not say that the reduced game v_T^ϕ is a game before we provide (***). Since ϕ is defined on the set of all games, if the reduced game v_T^ϕ is not a game then $\phi(v_T^\phi)$ is not defined, then the consistent property is not well-defined. To make the definition of reduced game well-defined, Hart must ether assume that ϕ is efficient or assume that $v_T^\phi(\emptyset) = 0$ for all $T \subset N$. Then the value of [3] is lessened.

To make this article self-contained, we copy the definition of *standard for two-person games*, Theorem B and part of its proof, form page 598 and page 599 in [3](Hart 1989) as follows.

A solution is standard for two-person games if

$$\phi^i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}[v(\{i, j\}) - v(\{i\}) - v(\{j\})] \tag{1.1}$$

for all $i \neq j$ and all v. Thus, the "surplus " $[v(\{i, j\}) - v(\{i\}) - v(\{j\})]$ is equally divided among the two players. Most solutions satisfy this requirement, in particular, the Shapley and the nucleolus.

Theorem B Let ϕ be a solution function. Then ϕ is (i)consistent and (ii) standard for two-person games, if only if ϕ is the Shapley value.

We now copy, from [1], the proof that if ϕ satisfies (i) and (ii) then ϕ is efficient as follows. **Proof** Assume ϕ satisfy (i) and (ii). We claim first that ϕ is efficient, i.e.,

$$\sum_{i \in N} \phi^i(N, v) = v(N) \tag{1.2}$$

for all (N, v). This indeed holds for $|N| = 2$ by (1.1). Let $n \geq 3$, and assume (1.2) holds for all games with less than n players. For a game (N, v) with $|N| = n$, let $i \in N$; by consistency

$$\sum_{j \in N} \phi^j(N, v) = \sum_{j \in N \backslash \{i\}} \phi^j(N \backslash \{i\}, v_{-i}) + \phi^i(N, v)$$

where $v_{-i} \equiv v^{\phi}_{N \backslash \{i\}}$. By assumption, ϕ is efficient for games with $n-1$ players, thus

$$= v_{-i}(N \backslash \{i\}) + \phi^i(N, v) = v(N)$$

(by definition of v_{-i}). Therefore ϕ is efficient for all $n \geq 2$.

Finally, for $|N| = 1$, we have to show that $\phi^i(\{i\}, v) = v(\{i\})$. Indeed, let $v(\{i\}) = c$, and consider the game $(\{i, j\}, \bar{v})$(for some $j \neq i$), with $\bar{v}(\{i\}) = \bar{v}(\{i, j\}) = c$, $\bar{v}(\{j\}) = 0$. By (ii), $\phi^i(\{i, j\}, \bar{v}) = c$ and $\phi^j(\{i, j\}, \bar{v}) = 0$; hence $\bar{v}_{-j}(\{i\}) = c - 0 = c = v(\{i\})$, and $c = \phi^i(\{i, j\}, \bar{v}) = \phi^i(\{i\}, \bar{v}_{-j}) = \phi^i(\{i\}, v)$ by consistency. This concludes the proof of the efficiency of ϕ.

Note 1 The above proof, by Hart and Mas-Colell, of the efficiency of ϕ is incomplete, or say, has an error. Let's check the final statement of the proof:

$$c = \phi^i(\{i, j\}, \bar{v}) = \phi^i(\{i\}, \bar{v}_{-j}) = \phi^i(\{i\}, v). \tag{1.3}$$

We need to prove that $\bar{v}_{-j} \equiv v$ before we claim $\phi^i(\{i\}, \bar{v}_{-j}) = \phi^i(\{i\}, v)$, i.e. we have to prove

$$\bar{v}_{-j}(\emptyset) = 0 = v(\emptyset) \tag{1.4}$$

and

$$\bar{v}_{-j}(\{i\}) = c - 0 = c = v(\{i\}). \tag{1.5}$$

Now, (1.4) holds if and only if $\bar{v}_{-j}(\emptyset) = \bar{v}^{\phi}_{\{i\}}(\emptyset) = 0$, i.e., $\bar{v}^{\phi}_{\{i\}}(\emptyset) = \bar{v}(\{j\}) - \phi^j(\{j\}, \bar{v}) = 0$. Therefore, (1.4) holds if and only if $\bar{v}(\{j\}) = \phi^j(\{j\}, \bar{v})$

That is, we have to provide that ϕ is efficient for the one-person game $(\{j\}, \bar{v})$ before we claim that (1.4) hold. Please note that no matter if j is dummy or not, ϕ is efficient for $(\{j\}, \bar{v})$ if and only if $\bar{v}(\{j\}) = \phi^j(\{j\}, \bar{v})$.

In other words, let player j in the above proof be the player 1 in (***), we find that without (***), we can not reduce the two-person game $(\{i, j\}, v)$ to one person game$(\{i\}, v^{\phi}_{\{i\}})$. Therefore, using (i) and (ii) by adding a dummy player to show that ϕ is efficient for $|N| = 1$ is incorrect.

Comment 5 In the 18th International Conference on Game Theory at Stony Brook University, USA, July 9-13 2007, we told Hart that his definitions of reduced games and the related consistency was incomplete and he admitted it. Accordingly, the definitions of multi-choice reduced game and related consistency in [7],[10] and [17] were all incomplete.

The credit of [3](Hart 1989) is that paper characterizes the traditional Shapley value by just two axioms, two-person-standard and consistency, however, to make the reduced game well-defined, we have to impose some extra assumptions on the reduced game, then the credit of [3] is lessened.

Finally, in [1](2010),Chiou and Hsiao fixed the error by partially consistency and extended Hart's ideas to a well-defined reduced game of a multi-choice game and its solution.

However, in July 2007, we had informed Liao that the definitions of reduced multi-choice game and related consistency in [7],[10] and [17] were all incomplete. But, after that, they kept sending papers concerning the multi-choice reduced game and consistency defined in [7], [10] and [17] for publication.

Hsiao's incomplete definitions In [7], [10] and [17], we have the following definitions which were not well-defined.

Definition O-2 Given $(N, \mathbf{m}, v) \in MC$, a weight function w and its solution,

$$\psi^w(N, \mathbf{m}, v) = (\psi^w_{i,j}(N, \mathbf{m}, v))_{(i,j)\in L^{N,\mathbf{m}}}.$$

For each $\mathbf{z} \in M^N$, we define an action vector $\mathbf{z}^* = (z^*_i)_{i\in N}$ where

$$\begin{cases} z^*_i = m_i & \text{if } \; z_i < m_i \\ z^*_i = 0 & \text{if } \; z_i = m_i. \end{cases}$$

Furthermore, We define a new game $v^{\psi^w}_{\mathbf{z}}$ such that

$$v^{\psi^w}_{\mathbf{z}}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{k\in S(\mathbf{z}^*)} \psi^w_{k,m_k}(N, (\mathbf{y} \vee \mathbf{z}^*), v) \;\text{ for all } \mathbf{y} \le \mathbf{z}. \tag{A4}$$

We call $(N, \mathbf{z}, v^{\psi^w}_{\mathbf{z}})$ a w-reduced game of v with respect to $\mathbf{z}$ and the solution ψ^w, where $(y \vee z^*)_i = \max\{y_i, z^*_i\}$ for all $i \in N$.

Comment 6 The reduced game (A4) is not well-defined. In order to fix (A4), we must either assume the efficiency of ψ^w or impose $v^{\psi^w}_{\mathbf{z}}(\emptyset) = 0$ for all $\mathbf{x} \neq \mathbf{0}$ to (A4).

Definition O-3([7], [10], [17]) Given a weight function w. A solution $\psi^w \in MC$ is w-consistent if for all $(N, m, v) \in MC$,

$$\psi^w_{i,j}(N, \mathbf{m}, v) = \psi^w_{i,j}(N, z, v^{\psi^w}_z) \quad \text{for all } i \in N\backslash S(\mathbf{z}^*) \text{ and for all } j \le z_i.$$

Comment 7 In [14], [16] and [18], Hwang Liao defined a reduced game only for the $H\&R$ Shapley value with symmetric form as following : For $S \subseteq N$, they denote $S^c = N \setminus S$ and $\mathbf{0}_S$ the zero vector in $\mathbb{R}^S$. Given a solution ψ, a game $(N, \mathbf{m}, v) \in MC$, and $S \subseteq N$, the **reduced game** $\Big(N, (\mathbf{m}_S, \mathbf{0}_{S^c}), v^{\psi}_{S,m}\Big)$ with respect to ψ, S and $\mathbf{m}$ is defined by

$$v^{\psi}_{S,\mathbf{m}}(\mathbf{x}, \mathbf{0}_{S^c}) = v(\mathbf{x}, \mathbf{m}_{S^c}) - \sum_{i\in S^c} \psi_{i,m_i}\Big(N, (\mathbf{x}, \mathbf{m}_{S^c}), v\Big) \quad \text{for all } \mathbf{x} \in M^S.$$

Furthermore, they defined the consistency property only for the $H\&R$ Shapley value with symmetric form as follows.

Consistency : A solution ψ on MC satisfies **consistency** if for all $(N, \mathbf{m}, v) \in MC$ and all $S \subseteq N$,

$$\psi_{i,j}\Big(N, (\mathbf{m}_S, \mathbf{0}_{S^c}), v^{\psi}_{S,\mathbf{m}}\Big) = \psi_{i,j}(N, \mathbf{m}, v) \quad \text{for all } i \in S \text{ and } j \in M_i^+.$$

Clearly, the reduced game defined by Hwang and Liao' papers is a special case of w-reduced game. Reducing on a set of players is a special case of reducing on an action vector. It is easy to see the following.

Given $(N, \mathbf{m}, v) \in MC$, a solution ψ on MC and $S \subseteq N$. Let $\mathbf{z} = (\mathbf{m}_S, \mathbf{0}_{S^c})$, by definitions of $v^{\psi}_{\mathbf{z}}$ and $v^{\psi}_{S,\mathbf{m}}$, we have that $v^{\psi}_{\mathbf{z}}(\mathbf{y}) = v^{\psi}_{S,\mathbf{m}}(\mathbf{y})$ for all $\mathbf{y} \leq \mathbf{z} = (\mathbf{m}_S, \mathbf{0}_{S^c})$. Hence, if a solution satisfies w-Consistency, then it satisfies Consistency.

Comment 8 None of the above definitions are well-defined, hence Hsiao didn't send the following Theorem for publication until 2005. In 2005, we found that Hwang and Liao was using the results in [7], especially the explicit formula (A3) of potential function to re-produce many papers, and did not tell the readers the originality of (A3). After that finding, Hsiao started to send [11] for publication, the purpose of that is just to tell the referees and the editors that the explicit formula of the potential function is originally from [7].

The following Theorem is for general case of γ^w, Hwang and Liao published special cases.

Theorem O-2([7], [10], [17]) The solution γ^w is w-consistent.

5 Characterization

In [17], under our supervision, Liao had the following definition.

Definition O-4 A solution function ϕ^w is standard for two-person games if

$$\begin{aligned}\psi^w_{i,k}(N, \mathbf{x}, v) = & \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ z_j = x_j}} \left[\frac{w(z_i)}{w(z_i) + w(z_j)}\right] \cdot \left[v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}^{\{i\}})\right] \\ & + \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ \mathbf{z} \leq \mathbf{x} \\ z_j \neq x_j}} \left[\frac{w(z_i)}{w(z_i) + w(z_j)}\right] \cdot \left[v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}^{\{i\}})\right] \\ & - \sum_{t=1}^{k} \sum_{\substack{z_i = t \\ \mathbf{z} \leq \mathbf{x} z_j \neq x_j}} \left[\frac{w(z_i)}{w(z_j) + w(z_j + 1)}\right] \cdot \left[v(\mathbf{z}) - v(\mathbf{z} - \mathbf{e}^{\{i\}}\right]\end{aligned}$$

where $\mathbf{x} = (0, .., x_i, 0, .., x_j, 0, ..0)$

In [17], under our supervision, Liao had the following axiomatization which is the parallel of Hart and Mas-Colell's (1989) axiomatization of the Shapley value by applying

consistency. However, since the definitions of reduced game and related consistency in [17] are not well-defined, the following theorem is incomplete.
Theorem O-3 ([17]) Given a weight function w. A solution ψ^w satisfies ST and w-CON if and only if $\psi^w = \gamma^w$.

Liao re-define **Standard for two-person game** as follows.

Standard for two-person game: For all $(N, \mathbf{m}, v) \in MC$ with $|S(\mathbf{m})| \leq 2$, $\psi^w = \gamma^w$.

With the above new definition Hwang and Liao re-produced similar Theorem in many publications for P&Z Shapley value Symmetric form of H& R Shapley value and D&P Shapley. However, since the original definition of reduced game in [3](Hart 1989) is not well-defined, the publications need revision.

Conclusion After, Hsiao and Raghavan(1992, 1993) extended the traditional cooperative games to the multi-choice cooperative games, researchers may try to extend any result of a traditional game to a multi-choice game. However, if we do not have a reasonable real-world example to justify the value of extending the result to a multi-choice game, then what we have done might be just rewriting the result in terms of the notations of multi-choice games.

A player in the traditional games has only two choices while a player in a multi-choice game has more than two choices with well-ordered action levels. Therefore, in a multi-choice game, a player has a finite well-ordered action set $\{\sigma_0, \sigma_1, ..., \sigma_m\}$ and may raise his action level from σ_0-doing nothing to action σ_k with $k > 1$ all in once. Hence, in a multi-choice game, we may consider a players **whole** reward for raising his action from σ_0 to σ_k. Therefore, in a multi-choice game if we consider a players reward separately for raising action levels one by one from σ_j to σ_{j+1}, then there is very little difference between studying a multi-choice game and studying a traditional game. Essentially, many researchers are writing the traditional games in terms of multi-choice games. Since Chapter 2 in [18], Liao considers a players reward separately for raising action levels one by one from σ_j to σ_{j+1}, we are not interested in Chapter 2.

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