Prevention of stock accumulation by restricting polluters' resources

Halkos, George

University of Thessaly, Department of Economics

April 2011

Prevention of stock accumulation by restricting polluters' resources

by

George E. Halkos and George J. Papageorgiou
University of Thessaly, Department of Economics
Korai 43, 3833, Volos Greece

Abstract
In this paper we claim that the disadvantage in the pollution control is not primarily the accumulated stock of pollutants, which is an accomplished fact, but the use of the available inputs in production in conjunction with the available equipment are the sources of pollutants accumulation. In most cases pollution is an irreversible fact and consequently, the main concern of a social planer should be the discovery of effective ways to reduce the sources (inputs and equipment) that generate pollutants. Using both optimal control and differential game approaches, we study the intertemporal strategic interactions between polluters and the social planer. We find that the establishment of cyclical strategies in a polluter's optimal control problem requires that the polluter's discount rate must be greater than the marginal resources' growth. For the saddle point stability, the marginal resources growth has to be equal or less than zero. Assuming constant elasticity for the polluters' resources reduction function and linearity for the rest of the functions, we find that the pollution game yields constant optimal Nash strategies. Finally, we provide analytical expressions of these strategies as well as the steady state value of the resources' stock.

Keywords: Pollution control; optimal control; differential games; Nash equilibrium.

JEL Classifications: C61, C62, D43, H21.

1. Introduction

Analyzing pollution control issues for developed and developing countries has become an important multi–disciplinary topic. Since the design of efficient action against pollution has to take into consideration the response of victims, game theory can be used as an appropriate tool. In this paper we claim that the disadvantage in the pollution control is not the accumulated stock of pollutants, which is an accomplished fact, but the use of the available inputs in production is the causality of the pollutants accumulation. In most cases, pollution is an irreversible fact and consequently, the main concern of the social planer should be the discovery of effective ways to reduce the sources (inputs and equipment) that generate pollutants. We use both optimal control and differential game approaches to study the intertemporal strategic interactions between the polluters and the social planer.

The pollutants accumulation is a major problem in our world and finding a way to effectively reduce, while maintaining the standards of the production process, is a great challenge facing capitalistic societies. The clean environment is obviously a public good. Conversely, all the "dirty" production process that creates pollutants accumulation, e.g. emissions caused by uncontrolled production, constitutes a public bad. But which of the factors of production process generates pollutants? Clearly uncontrolled, with respect to the environment, production involves antiquated equipment that emits more than permissible and therefore constitutes a polluters' "bad weapon". It is a usual phenomenon the old production equipment - which used to be the main production equipment for the Western developed countries - to change hands moving to the Southern or Eastern developing countries at a low acquisition cost. Similarly, all the extracted depletable resources which are used as inputs in the production are sources of pollution. The power of such a "dirty" production process rests upon the accumulation of a stock of resources, consequently depending on the financial capital for these resources that emits more and therefore accumulates pollutants.

On the other hand, in early days of applications of dynamic systems to economic problems, it was recognized that the optimal solution of infinite time problems may be characterized by multiple equilibrium points. Finding multiple equilibrium points in economic models is not an attractive solution for the policy makers. But the recognition of multiple optimal stable equilibria may be crucial in order to locate the thresholds separating the basins of attraction surrounding these different equilibria. Starting at a threshold, a rational economic agent is indifferent between moving toward one or the other equilibrium, but a small movement away from the threshold can "destroy" this indifference, leading in a unique optimal course of action.

Since the introductory one sector, with a convex – concave production function, optimal growth model of Skiba (Skiba, 1978), there has been a lot of progress towards the cyclical solution strategies generated in intertemporal dynamic economic models. Wirl (1995) exploring the optimality of cyclical exploitation of renewable resources stocks, reconsidering a model of Clark *et al* (1979), concludes that equilibrium that falls below the maximum sustainable yield but that exceeds the intertemporal harvest rule due to the positive spillovers allows for optimal, long run, cyclical harvest strategies.

Limit cycles, according to Poincare – Bendixson condition (Hartman, 1982) which also restricted in planar systems, has the intuitive explanation which says that if a trajectory of a continuous dynamical system stays in a bounded region forever, it has to approach "something". This "something" is either a point or a cycle. So if it is not a point, then it must be a cycle. This gives rise to cyclical policies in economic models, e.g. if a policy trajectory, say an abatement pollution policy, is restricted in a bounded planar space then this policy sooner or later will retrace its previous steps.

The Poincare – Andronov – Hopf theorem (Kuznetsov, 2004), which applies in a higher than the two dimensional systems, gives sufficient conditions for the existence of limit cycles of nonlinear dynamical systems. Informally, one can think of this theorem as requiring that equilibrium must suddenly change from a sink to a source with variation of a parameter. Arithmetically this requires that a pair of purely imaginary eigenvalues exists for a particular value of the bifurcation parameter and that the real part of this pair of eigenvalues changes smoothly its sign as the parameter is altered from below its actual value to above.

Hence, analogously to equilibrium, the stability of limit cycles is of great importance for the long run behavior of a dynamical system. But since the existence and therefore stability of a limit cycle is highly dependent on an arbitrarily chosen bifurcation parameter we have to deal with the qualitative analysis of such a problem. Economic mechanisms that may be a source of limit cycles, as mentioned by Dockner and Feichtinger (1995) are: (i) complementarity over time, (ii) dominated cross effects with respect to capital stocks, and (iii) positive growth of equilibrium.

The main contribution of our paper is that it considers the pollution control problem not in its irreversible aspect, as a stock of accumulated pollutants, but also as a stock of resources that potentially may damage the environmental quality. Consequently, from this perspective, one can prevent the accumulated stock by weakening the polluters' resources. As such resources, one may take the example of non-renewable: as far as equipment is concerned, we take into consideration that the antiquated equipment once used by the Western countries end-up at the less developed once, at a low acquisition cost; and these resources are considered as the polluters' bad resources.

The problem is modeled first as an optimal control problem and then as a differential game for which we explore the Nash equilibrium and try to investigate the existence of limit cycles and consequently the existence of cyclical strategies of the instrument variables. The environmental pollution control game takes place between the government, acting as the social planer, and polluters for which the resources used in production accumulate pollutants. Such pollutants accumulation and regulation control models can be fount, among others, in Forster (1980) concerning optimal energy use model; in Xepapadeas (1992) regarding environmental policy design and non-point source pollution and so on.

The remainder of the paper is organized as follows. Section 2 introduces the polluter's optimal control model and gives a necessary condition for cyclical strategies. Section 3 investigates the differential game between the government and the polluter and calculates the Nash equilibrium strategies and the players' value functions. The last section concludes the paper.

2. The Polluter's Optimal Control Model

Let us denote by $x(t)$ the instantaneous resources available to the representative polluter at time t. Without any counter pollution action undertaken, and also without any actions on behalf of the polluters, the stock of resources grows according to the function $e(x)$, which is considered as growth function, obviously dependent on the available resources.

Every polluter wishes to maximize the present value of utility, which is derived from two sources: first, the stock of its available resources, which is ready to send out emissions; second, from the emissions realization today, which causes damages to the available resources and, therefore, reduces the availability of the polluter's resources. The second part of utility, i.e. emission realizations, is certainly a kind of utility enjoyed by the polluter, measured in terms of damages made in the resources, e.g. big damages stems from the resource's intensive usage which causes higher emissions and, in turn, the polluter enjoys a higher amount of utility which is derived from the high level of emissions. For the sake of simplicity, the utility function is separable into these different kinds of benefits, and is, moreover, linear in the state variables. Thus, the objective of the representative polluter is

$$\max \int_0^\infty e^{-\rho t} \left(ax + \beta D - \gamma \tfrac{1}{2} u^2 \right) dt$$

Maximization process that takes place is subject to the available resources equation of motion. The change in this stock is equal to the difference between the resources' growth, mentioned above, less the damages in the resources caused by emissions realizations. Moreover, the resources' growth function, $e(x)$, follows the logistic law $e(x) = x(1-x)$. This specification is chosen because of its wide use in the literature, its plausibility and its convenience, but is not crucial for the model. Logistic growth, first proposed by Verhulst (1845), arises from the more general equation $\dot{x} = rx\left|1 - x/K\right|^a \operatorname{sign}(1 - x/K)$, where r the intrinsic growth, K the carrying capacity and a a positive constant playing the role of the penalty in a population model. Gatto *et al.* (1988) prove the optimality of the logistic growth function in both linear ($a = 1$) and nonlinear ($a \neq 1$) cases, and draw the optimal trajectories in both cases.

In the optimal control model we introduce a second state variable, D, which describes losses made in the resources from the emissions realizations, i.e. from the intensive use both of equipment and from the 'dirty' resources that generate pollutants. Clearly, the polluter losses, D, is a function dependent on the intensity of the inputs and equipment usage $\nu(t)$. Thus, we accept for simplicity the following form of the losses function: $D(\nu(t)) = \nu(t)$. Moreover, the second state variable changes sluggishly due to the necessary installation of equipment, to hiring and firing workers, etc. To simplify as far as possible, the original control ν now becomes the sum of historical adjustments. Current adjustments, the new control u, may be costly. For the sake of simplicity the changes in available resources reductions entails a quadratic cost of the form $\frac{1}{2}\gamma u^2$, which is separated from the original objective to differentiate from other mechanisms for limit cycles. After the above simplified assumptions the two state variables optimal control problem denoted as follows:

$$\max \int_0^\infty e^{-\rho t}\left(ax+\beta\nu-\gamma\tfrac{1}{2}u^2\right)dt \tag{1}$$

$$\dot{x}=x(1-x)-\nu, \quad x(0)=x_0 \tag{2}$$

$$\dot{\nu}=u \tag{3}$$

The following equations summarize the sufficient optimality conditions for the optimal control problem (1) – (3).

$$H=ax+\beta\nu-\tfrac{1}{2}\gamma u^2+\lambda_1\left[x(1-x)-\nu\right]+\lambda_2 u \tag{4}$$

$$H_u=-\gamma u+\lambda_2=0 \tag{5}$$

$$\dot{\lambda}_1=\left(\rho-(1-2x)\right)\lambda_1-a, \qquad \lim e^{-\rho t}\lambda_1(t)x(t)=0 \tag{6}$$

$$\dot{\lambda}_2=\rho\lambda_2-\beta+\lambda_1, \qquad \lim e^{-\rho t}\lambda_2(t)\nu(t)=0 \tag{7}$$

where H is the Hamiltonian function, λ_1, λ_2 are the costate variables associated with the states x, ν respectively. Substituting equation (5), $u=\lambda_2/\gamma$, into the second state equation we derive the canonical system without the control, as follows

$$\dot{x}=x(1-x)-\nu, \tag{8.1}$$

$$\dot{\nu}=\lambda_2/\gamma \tag{8.2}$$

$$\dot{\lambda}_1=\left(\rho-(1-2x)\right)\lambda_1-a, \tag{8.3}$$

$$\dot{\lambda}_2=\rho\lambda_2-\beta+\lambda_1 \tag{8.4}$$

Differential equations (8.1) – (8.4) determine the optimal intertemporal evolution of the system.

2.1. Stability Properties of Equilibrium

We continue exploring the existence of stable limit cycles, by considering system (8.1) – (8.4) on a bifurcation parameter. For the existence of stable limit cycles, therefore for the possibility of cyclical strategies, the following three conditions must hold:

i. The existence of a pair of purely imaginary roots of the characteristic equation of the Jacobian of the dynamic system for a particular parameter value

ii. The crossing of the imaginary axis at non – zero velocity when the bifurcation parameter is changed

iii. A negative coefficient of the so called normal form, which guarantees the stability of cycles.

Given these three conditions, stable limit cycles generically exist for a one – sided neighborhood around the bifurcation point.

The well known Dockner's formula is useful to check the first two conditions of the Hopf bifurcations (Dockner, 1985). With this formula we are able to calculate the four eigenvalues $r_i,\ i=1,\ldots 4$ of the Jacobian of the canonical equations in $(x,\nu,\lambda_1,\lambda_2)$. The eigenvalues are given by the following expression

$$r_{1,2,3,4}=\frac{\rho}{2}\pm\sqrt{\left(\frac{\rho}{2}\right)^2-\frac{K}{2}\pm\frac{1}{2}\left(\sqrt{K^2-4\|J\|}\right)} \tag{9}$$

where
$$K=\begin{Vmatrix}\dfrac{\partial\dot{x}}{\partial x} & \dfrac{\partial\dot{x}}{\partial\lambda_1}\\ \dfrac{\partial\dot{\lambda}_1}{\partial x} & \dfrac{\partial\dot{\lambda}_1}{\partial\lambda_1}\end{Vmatrix}+\begin{Vmatrix}\dfrac{\partial\dot{\nu}}{\partial\nu} & \dfrac{\partial\dot{\nu}}{\partial\lambda_2}\\ \dfrac{\partial\dot{\lambda}_2}{\partial\nu x} & \dfrac{\partial\dot{\lambda}_2}{\partial\lambda_2}\end{Vmatrix}+2\begin{Vmatrix}\dfrac{\partial\dot{x}}{\partial\nu} & \dfrac{\partial\dot{x}}{\partial\lambda_2}\\ \dfrac{\partial\dot{\lambda}_1}{\partial\nu} & \dfrac{\partial\dot{\lambda}_1}{\partial\lambda_2}\end{Vmatrix} \tag{10}$$

Condition (i) requires a parameter constellation such that two purely imaginary eigenvalues, denoted $r_{1,2}=\pm wi$, result from (9). This in turn implies the following necessary conditions, as Lemma 2, in Dockner and Feichtinger (1991). More precisely, the same lemma considers the positive quadrant of the $(K,\ \|J\|)$ plane as the set of potential candidates for limit cycles.

$$K>0 \text{ and } \|J\|>0 \text{ so that } \|J\|=\left(\frac{K}{2}\right)^2+\rho^2\frac{K}{2} \tag{11}$$

Equation (11) determines the bifurcation point. Condition (ii) of the Hopf's theorem states that the derivative $\mathrm{d}\,\mathrm{Re}\left(r_i(\upsilon)\right)/\mathrm{d}\upsilon$, with $\boldsymbol{\upsilon}$ the chosen bifurcation parameter, must not vanish at the bifurcation point. Moreover conditions (i) and (ii) are the necessary conditions for the existence of the limit cycle, while condition (iii) is the sufficient one. The Jacobian J of the canonical equations (8.1) – (8.4), evaluated at an equilibrium, is given below:

$$J = \begin{bmatrix} 1-2x & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/\gamma \\ 2b & 0 & \rho-1+2x & 0 \\ 0 & 0 & 1 & \rho \end{bmatrix} \tag{12}$$

The determinant of the Jacobian is

$$\|J\| = 2b/\gamma \tag{13}$$

while

$$K = \begin{Vmatrix} 1-2x & 0 \\ 2\lambda_1 & \rho-1+2x \end{Vmatrix} + \begin{Vmatrix} 0 & 1/\nu \\ 0 & \rho \end{Vmatrix} + 2\begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix} = e'(\rho - e') \tag{14}$$

where e' is the first derivative of the resource's logistic growth function $e = x(1-x)$.

The analysis is straightforward. The solution of the system (8.1) – (8.4) in the steady states, eliminating the controls and making the appropriate substitutions, is the following

$$x^* = \frac{1}{2}\left[\frac{a}{\beta} - (\rho-1)\right] \tag{15}$$

$$\nu^* = \frac{1}{2}\left[\frac{a}{\beta} - (\rho-1)\right] - \frac{1}{4}\left[\frac{a}{\beta} - (\rho-1)\right]^2 \tag{16}$$

$$\lambda_1^* = \beta \tag{17}$$

$$\lambda_2^* = 0 \tag{18}$$

Hence, the properties $K = e'(\rho - e') > 0,\ \|J\| = \beta/2\gamma > 0$ are necessary for the Jacobian to posses a pair of purely imaginary eigenvalues, while the latter is the necessity for a Hopf bifurcation. The first property $e' = 1-2x > 0$ is satisfied only if $x^* < \frac{1}{2}$. It remains to show that $\rho > e'$ is valid at the steady states. The positive externality of the stock implies $e' > 0$. From a simple inspection of (8.3) and (17), at the steady state, we see that $\lambda_1^* = \beta > 0$ and $\lambda_1^* = 1/(\rho - e')$. Consequently, it remains the denominator of the last expression of the shadow price λ_1^* to be positive, therefore, we always have $\rho > e'$. The second condition, $\|J\| > 0$, is always valid. From now onwards it remains to choose the bifurcation parameter, i.e. parameter γ, and find an explicit and analytical expression of the critical value, $\bar{\gamma}$, for the bifurcation point. Figure 1 outlines that no concern for resources, $a = 0$, and a sufficiently conservational attitude, which means a/β quite large, restrict the domain where complex patterns may be optimal. In economic terms, moderate concern for the polluter's resource stock, a small relative to β, may give raise to cyclical strategies, while strong concern or respectively no concern, $a=0$, are stabilizing.

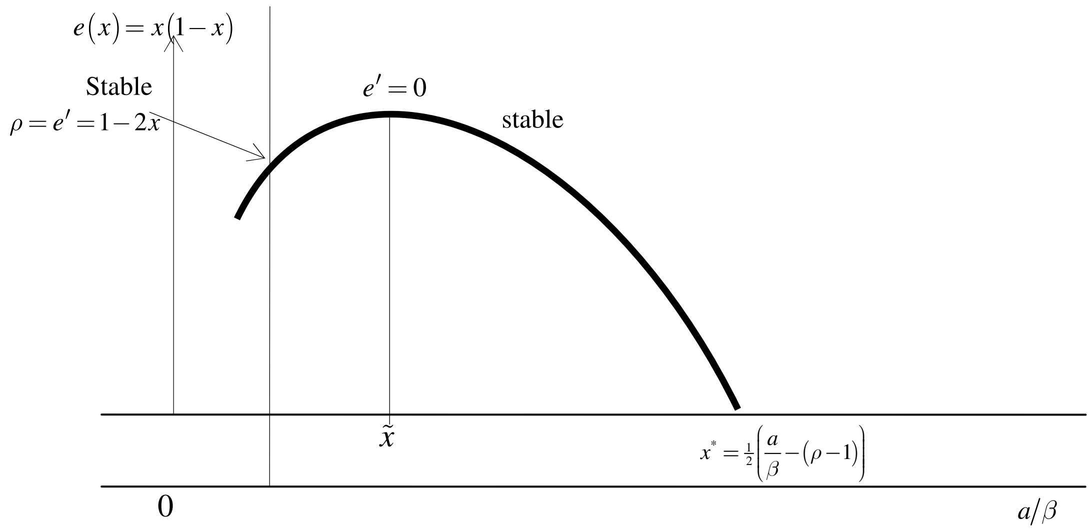

Figure 1: Stability properties with respect to a/β and associated stationary polluter's resource stock

The following proposition summarizes previous discussion about the existence of cyclical strategies for the representative polluter's optimal control problem and also gives the necessary condition for the saddle point stability of the same system.

Proposition 1:

The necessary condition to establish cyclical strategies for the representative polluter's optimal control problem (1) – (3) requires that the polluter's discount rate must be greater than the marginal resources' growth, i.e. $\rho > e'$. *For the same problem and for the saddle point stability, it suffices the marginal resources growth to be equal or less than zero. That is* $e'(x) \leq 0$.

3. The differential game

Let us, as in previous section, denote by $x(t)$ the instantaneous resources available to the polluters at time t. Without any counter pollution action undertaken and also without any actions of the polluters the stock of resources grows according to the function $e(x)$, which is considered as growth function, obviously dependent on the available resources, satisfying the conditions $e(0)=0,\ e(x)>0$ for all $x\in(0,K)$, $e(x)<0$ for all $x\in(K,\infty)$, $e''(x)\leq 0$. Carrying out emissions is costly for the polluters, e.g. compliance costs and damages in the available equipment, also reducing their capital available to the production process. This clearly affects negatively the resources of the polluters. The reduction of the growth of the resource stock, however, does not only depend on the intensity of emissions $\nu(t)$, but is also influenced by the counter pollution measures $u(t)$ undertaken by the government or by any groups of agents e.g. volunteers they fight against pollution. We set as instrument variables for both sides the intensity of emissions $\nu(t)$ and antipollution actions $u(t)$, which are assumed non-negatives $\nu(t)\geq 0,\quad u(t)\geq 0$.

Analogously to the models of optimal harvesting natural resources one can thought as "harvesting" the resources of polluters and this harvesting is denoted by $h(u,\nu)$. Combining the growth $e(x)$ with the harvesting $h(u,\nu)$ the state dynamics can be written as

$$\dot{x}=e(x)-h(u,\nu),\qquad x(0)=x_0>0 \tag{19}$$

Along a trajectory the non negativity constraint is imposed, that is

$$x(t)\geq 0\quad \forall t\geq 0 \tag{20}$$

With the assumption of emission's compliance costs and the damages incurred in equipment due to the intensive usage, a higher intensity of emissions and also the counter pollution measures leads to stronger reduction of the polluters' resources and therefore we assume the partial derivatives of the harvesting function that reduces polluters' resources $h(u,\nu)$ to be positive, i.e. $h_u>0,\ h_\nu>0$. Moreover the law of diminishing returns is applied only for the antipollution actions undertaken, that is $h_{uu}<0$ and for simplicity we assume $h_{\nu\nu}=0$. Additionally, we assume that the

Inada conditions, which guarantee that the optimal strategies are nonnegative, holds true, i.e.

$$\begin{aligned}&\lim_{u\to 0} h_u(u,\nu)=\infty, &&\lim_{u\to\infty} h_u(u,\nu)=0\\ &\lim_{\nu\to 0} h_\nu(u,\nu)=0, &&\lim_{\nu\to\infty} h_\nu(u,\nu)=\infty\end{aligned} \tag{20a}$$

The utility functions the two players want to maximize defined as follows: Player 1, say the government or any group of pollution fighters, derive instantaneous utility, on one hand from the emission reductions, on the other hand from their antipollution effort $u(t)$ which gives rise to increasing and convex costs $a(u)$. Additionally a high stock of resources that generates pollutants and a high level of emissions cause disutility, which are described by the increasing functions $d(x)$ and $\psi(\nu)$, respectively. After all the present value of utility is described by the following functional

$$J_1=\int_0^\infty e^{-\rho_1 t}\left[h(u,\nu)-d(x)-\psi(\nu)-a(u)\right]dt \tag{21}$$

Player 2, the polluters, enjoy utility $\upsilon(x)$ from the available resources $x(t)$, but also from their emissions at intensity ν, which is described by the function $\beta(\nu)$. For the utilities $\upsilon(x)$ and $\beta(\nu)$ we assume that are monotonically increasing functions with decreasing marginal returns, that is $\upsilon'(x)>0,\ \beta'(\nu)>0$ and $\upsilon''(x)<0,\ \beta''(\nu)<0$. So, player's 2 utility function is defined, in its additively separable form, as:

$$J_2=\int_0^\infty e^{-\rho_2 t}\left[\upsilon(x)+\beta(\nu)\right]dt \tag{22}$$

3.1 Nash Equilibrium

In this section we calculate the Nash equilibrium of the pollution differential game. The concept of open loop Nash equilibrium is based on the fact that every player's strategy is the best reply to the opponent's exogenously given strategy. Obviously, equilibrium holds if both strategies are simultaneously best replies.

Following Dockner *et al* (2000), we formulate the current value Hamiltonians for both players, as follows

$$H_1=h(u,\nu)-d(x)-\psi(\nu)-a(u)+\lambda\big(e(x)-h(u,\nu)\big)$$

$$H_2=\upsilon(x)+\beta(\nu)+\mu\big(e(x)-h(u,\nu)\big)$$

The first order conditions, for the maximization problem, are the following system of differential equations for both players:

First, the maximized Hamiltonians are

$$\frac{\partial H_1}{\partial u} = (1-\lambda)h_u(u,\nu) - a'(u) = 0 \tag{23}$$

$$\frac{\partial H_2}{\partial \nu} = \beta'(\nu) - \mu h_\nu(u,\nu) = 0 \tag{24}$$

and second the costate variables are defined by the equations

$$\dot{\lambda} = \rho_1\lambda - \frac{\partial H_1}{\partial x} = \lambda[\rho_1 - e'(x)] + d'(x) \tag{25}$$

$$\dot{\mu} = \rho_2\mu - \frac{\partial H_2}{\partial x} = \mu[\rho_2 - e'(x)] + \upsilon'(x) \tag{26}$$

The Hamiltonian of the player 1, H_1, is concave in the control u as far as long $\lambda < 1$ and is guaranteed by the assumptions on the signs of the derivatives, i.e. $h_{uu} < 0,\ h_{\nu\nu} = 0$ and from the decreasing marginal returns on the polluters' utilities, i.e. $\upsilon''(x) < 0,\ \beta''(\nu) < 0$. Moreover, optimality condition (23) implies that the adjoint variable λ is positive only if the regulator's marginal utility h_u exceeds the marginal costs, since $\lambda = \left(h_u(u,\nu) - a'(u)\right) / h_u(u,\nu)$.

We also assume linearity of the model. To be more precise we specify the following functions of the game to be in linear form:

i. the polluters resources growth function in the form $e(x) = \omega \cdot x$, where ω is the growth rate,

ii. the disutility function, $d(x)$, which stems from the high stock of the polluters resources, in the form $d(x) = d \cdot x$

iii. the disutility $\psi(\nu)$, derived from the level of emissions realizations, in the form $\psi(\nu) = \psi \cdot \nu$, and finally

iv. the abatement cost is in the form $u(t) = a \cdot u$

and all the constants involved are positive numbers, that is $\omega,\ d,\ \psi,\ a > 0$. From the polluters' side, the functions that maximized are specified linear, i.e. the utilities arisen from the resources stock and emissions realizations are written as $\upsilon(x) = \upsilon \cdot x(t)$ and $\beta(\nu) = \beta \cdot \nu(t)$ respectively.

After the above simplified specifications the canonical system of equations (23) - (26) can be rewritten as follows

$$\frac{\partial H_1}{\partial u} = (1-\lambda) h_u(u,\nu) - a = 0 \tag{27}$$

$$\frac{\partial H_2}{\partial \nu} = \beta - \mu h_\nu(u,\nu) = 0 \tag{28}$$

$$\dot{\lambda} = \rho_1 \lambda - \frac{\partial H_1}{\partial x} = \lambda[\rho_1 - \omega] + d \tag{29}$$

$$\dot{\mu} = \rho_2 \mu - \frac{\partial H_2}{\partial x} = \mu[\rho_2 - \omega] - \upsilon \tag{30}$$

and the limiting transversality conditions has to hold

$$\lim_{t\to\infty} e^{-\rho_1 t} x(t)\lambda(t) = 0, \quad \lim_{t\to\infty} e^{-\rho_2 t} x(t)\mu(t) = 0 \tag{31}$$

The analytical expressions of the adjoint variables (λ, μ), solving equations (29)-(30), are respectively:

$$\lambda(t) = \frac{d}{-\rho_1 + \omega} + e^{(\rho_1 - \omega)t} C_1 \tag{32}$$

$$\mu(t) = -\frac{\upsilon}{-\rho_2 + \omega} + e^{(\rho_2 - \omega)t} C_2 \tag{33}$$

In order the transversality conditions to satisfied it is convenient to choose the constant steady state values, and therefore the adjoint variables collapses to the following constants

$$\lambda = \frac{-d}{\rho_1 - \omega}, \quad \mu = \frac{\upsilon}{\rho_2 - \omega} \tag{34}$$

To ensure certain signs for the adjoints (34) we impose another condition on the discount rates, which claim that discount rates are greater than the resource's growth, i.e. we impose the condition

$$\rho_i > \omega, \quad i = 1, 2$$

thus, the constant adjoint variables has the negative and positive signs respectively. The above condition seems to be restrictive but can be justified as otherwise optimal solutions do not exist. Indeed, choosing $\rho_2 < \omega$, the polluters' discount rate to be lower than the resource's growth rate, their objective functional becomes unbounded in the case they choose to send out no emissions. Similarly, choosing the government's discount rate lower than the growth rate the associated adjoint variable

λ becomes a positive quantity in the long run. As a shadow price is implausible to be positive for optimal solutions, the above reasoning is sufficient for the assumption $\rho_i > \omega,\ i = 1,2$.

Once the concavity of the Hamiltonians, with respect to the strategies, for both players is satisfied the first order conditions guarantee its maximization. Now, we choose the function's $h(u,\nu)$ specification, i.e. the specification of the function that reduces the polluters' resources. This function is depending on the intensity of emissions and also depending on the abatement actions undertaken by the regulator. We choose a similar to Cobb – Douglas production function specification, which characterized by constant elasticities, and is in the following form

$$h(u,\nu) = u^{\sigma}\nu^{\zeta} \qquad 0 < \sigma < 1 < \zeta$$

The rest of the paper is devoted to the calculations of the explicit formulas at the Nash equilibrium.

3.2. Optimal Nash Strategies

Applying first order conditions (9), (10) for the chosen specification function

$$h_u(u,\nu) = \frac{a}{1-\lambda} \quad \Leftrightarrow \quad \sigma u^{\sigma-1}\nu^{\zeta} = \frac{a}{1-\lambda} \tag{35}$$

$$h_{\nu}(u,\nu) = \frac{\beta}{\mu} \quad \Leftrightarrow \quad \zeta u^{\sigma}\nu^{\zeta-1} = \frac{\beta}{\mu} \tag{36}$$

The combination of (35) and(36), using the Cobb–Douglas type of specification, reveals an existing interrelationship between the strategies, that is

$$h(u^*,\nu^*) = (u^*)^{\sigma}(\nu^*)^{\zeta} \quad \Leftrightarrow \quad \frac{au^*}{\sigma(1-\lambda)} = \frac{\beta\nu^*}{\zeta\mu} \quad \Leftrightarrow \quad \nu^* = u^*\frac{a\zeta\mu}{\sigma(1-\lambda)\beta} \tag{37}$$

Expression (37) now predicts the interrelationship between the player's Nash strategies, for which the result of comparison between them is dependent on the constant parameters and on the constant adjoint variables, as well.

Substituting back (37) into (36) we are able to find the analytical expressions of the strategies, after the following algebraic calculations. Expression (36) now becomes:

$$\left(u^*\right)^{\sigma+\zeta-1} = \left[\frac{a}{\sigma(1-\lambda)}\right]^{1-\zeta}\left(\frac{\zeta\mu}{\beta}\right)^{1-\zeta}\left(\frac{\mu\zeta}{\beta}\right)^{-1} = \left[\frac{a}{\sigma(1-\lambda)}\right]^{1-\zeta}\left(\frac{\mu\zeta}{\beta}\right)^{-\zeta}$$

and from the latter the analytical expressions for the equilibrium strategies is derived in a more comparable form now:

$$u^* = \left[\frac{a}{\sigma(1-\lambda)}\right]^{\frac{1-\zeta}{\sigma+\zeta-1}} \left(\frac{\mu\zeta}{\beta}\right)^{\frac{-\zeta}{\sigma+\zeta-1}} \tag{38}$$

$$\nu^* = \left[\frac{a}{\sigma(1-\lambda)}\right]^{\frac{\sigma}{\sigma+\zeta-1}} \left(\frac{\zeta\mu}{\beta}\right)^{\frac{\sigma-1}{\sigma+\zeta-1}} \tag{39}$$

Further substitutions in the equation of the resources accumulation, $\dot{x} = \omega x - u^{\sigma}\nu^{\zeta}$, yield the following steady state value of the stock

$$x^{ss} = \frac{1}{\omega}\left[\frac{a}{(1-\lambda)\sigma}\right]^{\frac{\sigma}{\sigma+\zeta-1}} \left(\frac{\zeta\mu}{\beta}\right)^{\frac{-\zeta}{\sigma+\zeta-1}} \tag{40}$$

We summarize the above discussion in a proposition.

Proposition 2:

Assuming the function which reduces the polluters' resources to exhibit constant elasticity and all the other functions to be linear, then the pollution game yields constant optimal Nash strategies. The analytical expressions of the strategies are given by (38) and (39) for the government and the polluters respectively. The steady state value of the resources' stock is given by the expression (40).

3.3. The Value Functions

In this section we compute the analytical expressions for the values of objective functions of the players. For this purpose we make use the constancy of the strategies *(38), (39)* computed above. We denote the pair of the constant strategies as $(\bar{u},\bar{\nu})$. Note that constant strategies, leads to a constant function $\bar{h} = h(\bar{u},\bar{\nu})$ which is the aforementioned function that reduces the polluters resources. The equation of the resources' accumulation, now can be solved explicitly with the following analytical solution

$$x(t) = \left(x_0 - \frac{\bar{h}}{\omega}\right)e^{\omega t} + \frac{\bar{h}}{\omega} \tag{41}$$

x_0 is the initial stock of the polluters resources. Note that expression (41) leads us to assume a sufficiently high initial stock of resources, specifically $x_0 \geq \bar{h}/\omega$, in order to satisfy the non-negativity condition $x(t) > 0$.

The earlier computed constant strategies and the linearity assumption of the value functionals for both government and polluters, gives us the advantage to calculate a linear integral. Thus, for the value function of player 1, we have:

$$J_1 = \frac{1}{\rho_1}\left(\bar{h} - a \cdot \bar{u} - \psi \cdot \bar{\nu}\right) - d\int_0^\infty e^{-\rho_1 t} x(t)\,dt \tag{42}$$

The value of the integral in (42) can be computed, giving

$$\int_0^\infty e^{-\rho_1 t} x(t)\,dt = \frac{\rho_1 x_0 - \bar{h}}{\rho_1(\rho_1 - \omega)}$$

The government's value function (42) now takes the following form:

$$J_1 = \frac{\bar{h}}{\rho_1}\left(1 + \frac{d}{\rho_1 - \omega}\right) - \frac{a\bar{u}}{\rho_1} - \frac{\psi\bar{\nu}}{\rho_1} - \frac{dx_0}{\rho_1 - \omega} \tag{43}$$

which is again a constant.

Similarly, thanks to the model's linearity, the polluters' value function can be calculated analytically yielding the following constant expression:

$$J_2 = \frac{1}{\rho_2}\left(\beta\bar{\nu} + \frac{\upsilon\left(\rho_2 x_0 - \bar{h}\right)}{\rho_2 - \omega}\right) = -\frac{\upsilon\bar{h}}{\rho_2(\rho_2 - \omega)} + \frac{\beta\bar{\nu}}{\rho_2} + \frac{\upsilon x_0}{\rho_2 - \omega} \tag{44}$$

4. Conclusions

This paper investigates the dynamics of pollution together with the actions undertaken for counter pollution. For this purpose a model of environmental pollution is setup, for which the crucial assumption made is not the traditional one for the pollutants accumulation. Instead, we claim that the disadvantage in the pollution control is not the accumulated stock of pollutants, an irreversible fact, but rather the use of the available "bad" inputs together with the antiquated equipment, used in the production process. We called the production inputs and the available equipment used "the polluter's resources", as the causality of the pollutants accumulation.

We model the polluter's resources as an accumulated stock and the damages made in the resources, due to the intensive use, as the sum of the historical adjustments. Thus, the simple model became an optimal control model with two state variables. In the solution, we explore the possibility of cyclical strategies and we found a necessary condition to establish that cyclical strategies. We extend the one person optimal control model in a simultaneous (Nash) differential game, for which the government undertakes the usual role of regulation.

Specifically, we find that the necessary condition for establishing cyclical strategies in a polluter's optimal control problem requires that the polluter's discount rate must be greater than the marginal resources' growth. Similarly and for the saddle point stability, the marginal resources growth has to be equal or less than zero. Assuming constant elasticity for the polluters' resources reduction function and linearity for the rest of the functions, we show that the pollution game yields constant optimal Nash strategies. The analytical expressions of the strategies for the government and the polluters as well as the steady state value of the resources' stock are provided.

The government's instrument variable was the counter pollution actions undertaken, while the polluter's instrument variable was the intensity of emissions realizations. In the Nash equilibrium of the game we calculate the optimal strategies for both players, and we found under linearity assumptions and for a specific resources' growth function that the strategies are constants. Finally, the analytical expressions of the value functions for both players are computed.

References

Clark, C., W., Clarke, F., H., and Munro G., R., (1979). The optimal exploitation or renewable resources stocks: Problems of irreversible investment, *Econometrica*, **47**, 25 – 47.

Dockner, E. (1985). Local stability analysis in optimal control problems with two state variables. In G. Feichtinger, Ed. *Optimal Control Theory and Economic Analysis*. Amsterdam: North Holland

Dockner, J., Feichtinger, G., (1991). On the Optimality of Limit Cycles in Dynamic Economic Systems, *Journal of Economics*, **53**, 31 – 50.

Gatto, M., Muratori, S., and Rinaldi, S., (1988), On the Optimality of the Logistic Growth, *Journal of Optimization Theory and Applications*, **57(3)**, 513 – 517.

Forster, B., (1980), Optimal Energy Use in a Polluted Environment, *Journal of Environmental Economics and Management*, **7**, 321 – 333.

Grass, D., Gaulkins, J., Feichtinger, G., Tragler, G., Behrens, D., (2008), *Optimal Control of Nonlinear Processes. With Applications in Drugs, Corruption and Terror*, Springer, Berlin.

Hartman, P., (1982). *Ordinary differential equations,* (2nd ed.), Basel: Birkhauser

Kuznetsov, Y., (2004). *Elements of Applied Bifurcation Theory*, (3rd ed.), Springer.

Skiba, A., K., (1978), Optimal Growth with a Convex – Concave Production Function, *Econometrica*, **46**, 527 – 539.

Wirl, F., (1995), The Cyclical Exploitation of Renewable Resources Stocks May Be Optimal, *Journal of Environmental Economics and Management*, **29**, 252 – 261.

Xepapadeas, A., (1992), Environmental Policy Design and Dynamic Non – Point Source Pollution, *Journal of Environmental Economics and Management*, **23**, 22 – 39.