Agreeing to disagree with generalised decision functions

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Abstract We develop a framework that allows us to emulate standard results from the "agreeing to disagree" literature with generalised decision functions (e.g. Bacharach (1985)) in a manner the avoids known incoherences pointed out by Moses and Nachum (1990). Avoiding the incoherences requires making some sacrifices: For example, we must require the decision functions to be independent of interactive information, and, the language in which the states are described must be "rich" - in some well-defined sense. Using weak additional assumptions, we also extend all previous results to allow agents to base their decisions on possibly false information. Finally, we provide agreement theorems in which the decision functions are not required to satisfy the Sure-Thing Principle (a central assumption in the standard results).

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JEL classification D80, D83, D89.

1 Introduction

The agreement theorem of Aumann (1976) states that if agents have a common prior on some event, then if their posteriors are common knowledge, these posteriors must be equal, *even if the agents' updates are based on different information.* This was proved for posterior probabilities in the context of a *partitional* information structure.

Briefly, Ω is a finite set of states and any of its subsets E is an *event.* For each agent $i \in N$ there is an *information function* $I_i : \Omega \to 2^{\Omega}$; the *information cell* $I_i(\omega)$ is

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the set of states that i conceives as possible at state ω, and for each $i \in N$, it is assumed that (i) $\omega \in I_i(\omega)$, and (ii) $I_i(\omega)$ and $I_i(\omega')$ are either identical or disjoint, so the set $\mathcal{I}_i = \{I_i(\omega)|\omega \in \Omega\}$ partitions the state space. Furthermore, agent i is said to "know" event E at state ω if $\omega \in I_i(\omega) \subseteq E$; and an operator $\mathbf{K}_i(.)$ is defined, where "i knows event E" is the event $\mathbf{K}_i(E) = \{\omega \in \Omega|I_i(\omega) \subseteq E\}$. Informally, E is *common knowledge* for a group of agents $G \subseteq N$ if everyone knows that E, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it, and so on ad infinitum. Note that in this framework, the knowledge operator inherits the following properties: (i) $\mathbf{K}_i(E \cap F) = \mathbf{K}_i(E) \cap \mathbf{K}_i(F)$, (ii) $\mathbf{K}_i(E) \subseteq E$, (iii) $\mathbf{K}_i(E) \subseteq \mathbf{K}_i(\mathbf{K}_i(E))$ and (iv) $\Omega \backslash \mathbf{K}_i(E) \subseteq \mathbf{K}_i(\Omega \backslash \mathbf{K}_i(E))$.
The robustness of the agreement result was tested through various generalisations. Still operating in a partitional structure, Cave (1983) and Bacharach (1985) independently extended the probabilisitic result to general decision functions, $D_i : \mathcal{F} \rightarrow \mathcal{A}$, that map from a field $\mathcal{F}$ of subsets of Ω into an arbitrary set $\mathcal{A}$ of *actions*. To derive the result, it is assumed that agents have the same decision function (termed "like-mindedness"), and that the decision functions satisfy what we call the Disjoint Sure-Thing Principle ($DSTP$): $\forall E \in \mathcal{E}$, if $D_i(E) = x$ then $D_i(\cup_{E \in \mathcal{E}} E) = x$, where $\mathcal{E}$ is a set of disjoint events.[1] The following states their result.[2]

> *If agents i and j are "like-minded", decision functions satisfy $DSTP$, information is partitional, and it is common knowledge at some state ω that i takes action x and j takes action y, then $x = y$.*

Moses and Nachum (1990) criticise the result above on the grounds that defining decisions over unions of information cells, as required by the $DSTP$ is meaningless in the context of generalised decision functions. Bacharach's decision functions map from subsets of Ω to capture the idea that actions must be contingent upon the agent's *information* - in a similar manner to the way in which posterior probabilities are contingent upon the information function at a given state. And, $DSTP$ is intended to capture the intuition that if one chooses to do x in every case where one is "better informed" (e.g. $D_i(I_i(\omega)) = x$ and $D_i(I_i(\omega')) = x$), then one must also choose to do x when one is more "ignorant". However, one's decision when one is being more ignorant in this case is taken to be $D_i(I_i(\omega) \cup I_i(\omega')) = x$. This is problematic because $I_i(\omega) \cup I_i(\omega')$ has no defined informational content: The primitives defining that determine the informational content of sets of states are the information cells, but this union is merely a collection of states, so it is not obvious what the agent knows in this case, since it is not an information cell.

[1]The $DSTP$ is trivially satisfied when the decision functions are posterior probabilities.

[2]Note that Aumann (1976) can be derived as a corollary by defining a common prior probability distribution over the states, and by setting, for an event E, $D_i^E(I_i(\omega)) = \Pr(E|I_i(\omega))$.

There is a deeper problem however, which we can illustrate with the following example. Suppose the state space is $\Omega = \{\omega, \omega'\}$, and two agents i and j such that $I_i(\omega) = \{\omega\}$ and $I_i(\omega') = \omega'$, and $I_j(\omega) = \{\omega, \omega'\}$. Suppose for example, that ω is the state in which a coin is facing heads up, whereas ω' is the state in which the coin is facing tails up. The set of states in which i knows which side is up is $\{\omega, \omega'\}$; and since $I_j(\omega) \subseteq \{\omega, \omega'\}$, we can interpret the event $E = \{\omega, \omega'\}$ as "Agent j knows that i knows which side is up". Note that at each state, i knows E. But now, suppose we take the union $I_i(\omega) \cup I_i(\omega')$. Now we may ask, what is the informational content of this set? Well, on the one hand, since $I_i(\omega) \cup I_i(\omega') \subseteq E$, it would appear that i knows E. That is, i knows that j knows that i knows which side is up. On the other hand, it is not possible that i knows E because now, it is no longer the case that i knows which side is up! This example suggests that although cell union may be appropriate to capture "more ignorance" in a single-agent setting, an incoherence arises where there is interactive information - events of the type: i knows that j knows that E.
Moses and Nachum (1990) propose their our solution to the generalised agreement theorem by defining a projection from states to an arbitrary set, intended to capture the information at each state that is relevant to the decision, and the decision functions map relevant information into actions. Now, relevant information is defined over a variety of sets of states, so the above criticism is resolved. However they require a *stronger* version of the Sure-Thing Principle, which does not require the "disjointness" of the relevant information, which we term the Non-Disjoint Sure-Thing Principle, $NDSTP$.
More recently, Aumann and Hart (2006) use the framework developed in Aumann (1999) to reproduce the results of Bacharach and of Moses & Nachum in a coherent approach. Our approach is largely similar to theirs.

In an altogether different strand of the literature, Samet (1990) and Collins (1997) prove agreement theorems, restricting themselves to decision functions as posterior probabilities, but in a *non-partitional* information structure. This is an important line of investigation since partitional structures imply that agents can only know what is the case; in other words, agents cannot base their decisions on false information. But surely, it is perfectly plausible for rational agents to do so. The culprit is the assumption that for all $\omega \in \Omega$, $\omega \in I_i(\omega)$ since the "actual" state is always included in the set of states that the agent considers possible. Instead, Collins (1997) imposes (i) $I_i(\omega) \neq \emptyset$ and (ii), if $\omega' \in I_i(\omega)$ then $I_i(\omega') = I_i(\omega)$. Now, it is possible that $\omega \notin I_i(\omega)$ - in which case ω is called a *blindspot* for i since at that state the agent considers it impossible - and the operator $\mathbf{K}$ is now interpreted as a "belief" operator (since it is possible to believe what is false, but not to know it; in particular, it is now no longer necessary that $\mathbf{K}_i(E) \subseteq E$).

The result requires what we term the Zero-Priors assumption:[3] The prior probability distribution must assign zero probability to every state that is a blindspot for *every* agent. It is justified on the grounds that the states that an agent does not consider possible should not affect the agent's decision. However, this assumption is forcefully criticised by Collins (1997): Although it seems reasonable to say that i's prior must assign zero probability to the states that i considers impossible, it is not reasonable to also require i's prior to also assign zero probability to the states that j considers impossible (although i might consider them possible).
Finally, in a similar vein, Bonanno and Nehring (1998) prove an agreement theorem in a non-partitional information structure. They do this by assuming "quasi-coherence" (defined later), and over functions that satisfy a "properness" condition. If the function is "quantitative", properness implies the Disjoint Sure-Thing Principle (in a manner that does not avoid the Moses and Nachum (1990) criticism); and when it is "qualitative", properness is equivalent to the Non-Disjoint Sure-Thing Principle. Of course, this implies that the interpretation of properness depends on the type of function that is used.

In this paper, we use standard concepts from epistemic logic to derive agreement theorems with generalised decision functions in both partitional and non-partitional models, that are analogous to the results mentioned above, but that do not suffer from the incoherences pointed out by Moses and Nachum (1990).[4] Avoiding the incoherences requires making some sacrifices: For example, we must require the decision functions to be independent of interactive information, and, the language in which the states are described must be "rich" - in some well-defined sense. Finally, we are also able to prove agreement theorems in which less restrictions are imposed on the decision functions. Namely, we no longer require them to satisfy the Sure-Thing Principle (whether disjoint or not).

In section 2, we introduce the basic concepts that we use from epistemic logic. In section 3, we expand the standard epistemic logic framework to encompass decision functions, and we state our main assumptions. We derive our main results in partitional models in section 4, and in non-partitional models in section 5. All proofs are in the appendix.

[3] Termed "consistency" in Samet (1990).

[4] Although still working with a partitional structure, Samet (2010) takes an altogether different approach, deriving a generalised agreement theorem by assuming an "interpersonal" Sure-Thing Principle ($ISTP$), which is a condition imposed on decision functions *across* different agents. The generalisation of his result in our framework to non-partitional structures is the subject of a companion paper (Tarbush (2011)).

2 Epistemic Logic

This section introduces concepts from epistemic logic. All the definitions and results in this section are standard (e.g. see Chellas (1980) and van Benthem (2010) for general reference).

We must develop the *language* that our results will be stated in. This will consist in defining a *syntax* - which determines which symbols or chains of symbols are part of the language (e.g. "dog" is permissible in the English language, but "a@b6tt" is not) -, and in defining *semantics* - which assigns a meaning to the symbols and thus determines a grammar (e.g. "The dog is sick" is semantically permissible in the English language, but "Dog towards rain table" is not) -.
A *proposition* is a sentence, usually represented by a lower case letter. For example, "The dog is sick", and, "It is raining" can represented by p and q respectively. Propositions can be combined in various ways using the standard *Boolean operators*: not, and, or, if...then, if and only if, which are represented by the following symbols respectively $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. An example of a combination of propositions is "The dog is sick *and* it is raining" (formally $p \wedge q$).
Finally, we will also allow for *modal* operators in our language. These are operators that qualify an entire proposition. For example, "I know that the dog is sick" is made up of the proposition "The dog is sick" and the modal operator "I know that". We will have two basic symbols for modal operators in our language, namely $\Box_i$ and C_G, although their exact interpretation will be developed later. Essentially, depending on semantics which we will specify later, $\Box_i p$ will either stand for "Agent i knows that p", or "Agent i believes that p", whereas $C_G p$ will either stand for "It is common knowledge among the subset of agents G that p", or "It is common belief among the subset of agent G that p".
Propositions are *atomic* if they do not contain any operators (whether Boolean or modal), and are thus reduced to the most basic building block. For example, "The dog is sick" contains no operators, so cannot be made more basic, whereas "The dog is sick and it is raining" can be reduced to two propositions, so is not atomic.
A *formula* is any chain of symbols that is acceptable in the language. Formally, we construct the syntax, or the set of formulas in our language, as follows:

Definition 1 (Basic syntax)**.** Define a finite set of atomic *propositions*, $\mathcal{P}$, which consists of all propositions that cannot be further reduced. Let N denote the set of all agents. We then inductively create all the *formulas* in our language, $\mathcal{L}$, as follows:
(i) Every $p \in \mathcal{P}$ is a formula.
(ii) If ψ is a formula, so is $\neg\psi$.
(iii) If ψ and ϕ are formulas, then so is $\psi \circ \phi$, where $\circ$ is one of the following

$$\begin{tikzcd} \omega \arrow[loop left, "{i,j}"] \arrow[r, leftrightarrow, "j"] & \omega' \arrow[loop right, "{i,j}"] \end{tikzcd}$$

Figure 1: Example of a Kripke structure

Boolean operators: $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.
(iv) If ψ is a formula, then so $\bullet\psi$, where $\bullet$ is one of the modal operators $\Box_{i\in N}$ or $C_{G\subseteq N}$.
(v) Nothing else is a formula.

So far, we have pure uninterpreted syntax. Indeed, "Agent i knows that it is raining and knows that it is not raining" is a formula of our language (represented as $\Box_i q \wedge \Box_i \neg q$), but surely it cannot be true. We therefore introduce the semantics of our language to determine the truth or falsity of formulas. To do this we use standard Kripke semantics.

Definition 2 (Kripke semantics)**.** A *frame* is a pair $\langle \Omega, R_{i\in N}\rangle$, where Ω is a finite, non-empty set of *states* (or "possible worlds"), and $R_i \subseteq \Omega \times \Omega$ is a binary relation for each agent i, also called the *accessibility relation* for agent i. A *model* on a frame $\langle \Omega, R_{i\in N}\rangle$, is a triple $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$, where $\mathcal{V} : \mathcal{P} \times \Omega \to \{0, 1\}$ is a *valuation map.*

Definition 3 (Truth)**.** We say that a proposition $p \in \mathcal{P}$ is *true* at state ω in model $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$, denoted $\mathcal{M}, \omega \models p$, if and only if $\mathcal{V}(p, \omega) = 1$. Truth is then extended inductively to all other formulas ψ as follows:
(i) $\mathcal{M}, \omega \models \neg\psi$ if and only if it is not the case that $\mathcal{M}, \omega \models \psi$.
(ii) $\mathcal{M}, \omega \models (\psi \wedge \phi)$ if and only if $\mathcal{M}, \omega \models \psi$ and $\mathcal{M}, \omega \models \phi$.[5]
(iii) $\mathcal{M}, \omega \models \Box_i\psi$ if and only if $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\mathcal{M}, \omega' \models \psi$.
(iv) $\mathcal{M}, \omega \models C_G\psi$ if and only if $\forall \omega' \in \Omega$ accessible from ω in a finite sequence of R_i ($i \in G \subseteq N$) steps, $\mathcal{M}, \omega' \models \psi$.

The above definitions can be illustrated by the model $\mathcal{M} = \langle \Omega, R_{i\in N}, \mathcal{V}\rangle$ represented in Figure 1. The state space is $\Omega = \{\omega, \omega'\}$. The accessibility relations for agents i and j are as represented in the figure. Namely, $R_i = \{(\omega, \omega), (\omega', \omega')\}$ and $R_j = \{(\omega, \omega), (\omega', \omega'), (\omega, \omega'), (\omega', \omega)\}$. Finally, we can let $\mathcal{P} = \{h, t\}$, $\mathcal{V}(h, \omega) = 1$, and $\mathcal{V}(t, \omega') = 1$. From this alone, we can generate several new formulas. For example, note that every state ω'' that is accessible from ω for agent i is such that $\omega'' \models h$ (indeed, the only state that i can "access" from ω is ω itself, and h is true at ω). Therefore, by the definition of truth, we have that $\omega \models \Box_i h$. Similarly, we have $\omega' \models \Box_i t$. On the other hand, we have $\omega \models \neg\Box_j h$. This is because from ω, j can "access" the state ω' in which h is not true, but rather t is true.

[5] The truth of formulas involving the other Boolean operators are similarly defined.

We can even go further. One can verify that $\omega \models \Box_i h \vee \Box_i t$ and $\omega' \models \Box_i h \vee \Box_i t$; and therefore, $\omega \models \Box_j(\Box_i h \vee \Box_i t)$ and $\omega' \models \Box_j(\Box_i h \vee \Box_i t)$. In fact, since for any state accessible from ω in a finite sequence of R_k ($k \in \{i, j\}$) steps, it is the case that $\Box_i h \vee \Box_i t$, we can also conclude that $\omega \models C_{\{i,j\}}(\Box_i h \vee \Box_i t)$.

We can note that some formulas, such as $\Box_i h$ are only true at some state of the model, whereas others, like $\Box_i h \vee \Box_i t$ are true at *every* state in the model. The latter are said to be *valid in the model*. But there are even more general levels of validity. For example, suppose we keep the same states and accessibility relations as the model in Figure 1, but modify the valuation map. Then, we obtain a set of new models, all with the same frame. The formulas that remain true at every state of each of these models are said to be *valid in the frame*. Even more generally, we can allow the frame itself to vary, but within a class of frames. For example, we could consider all the frames in which for every $\omega \in \Omega$, $(\omega, \omega) \in R_i$ for each $i \in N$. The formulas that remain true at every state of every model in every frame within this class are said to be *valid (within this class of frames)*. Formally, we have the following definition.

Definition 4 (Validity)**.** Formula ψ is valid in a model $\mathcal{M}$, denoted $\mathcal{M} \models \psi$ iff $\forall \omega \in \Omega$ in $\mathcal{M}$, $\omega \models \psi$. Formula ψ is valid in a frame $\langle \Omega, R_{i \in N} \rangle$, denoted $\langle \Omega, R_{i \in N} \rangle \models \psi$, iff $\forall \mathcal{M}$ over $\langle \Omega, R_{i \in N} \rangle$, $\mathcal{M} \models \psi$. Formula ψ is $\mathcal{T}$-valid (or valid), denoted $\models \psi$, iff $\forall \langle \Omega, R_{i \in N} \rangle \in \mathcal{T}$ ($\mathcal{T}$, a class of frames), $\langle \Omega,\ R_{i \in N} \rangle \models \psi$.

The frame classes can be determined by the conditions that are imposed on the accessibility relations. The following gives a selection of conditions that are often used to classify frames.

Definition 5 (Conditions on frames)**.** We say that a frame $\langle \Omega, R_{i \in N} \rangle$ is,
(i) Reflexive if $\forall i \in N$, $\forall \omega \in \Omega, \omega R_i \omega$.
(ii) Symmetric if $\forall i \in N$, $\forall \omega, \omega' \in \Omega$, if $\omega R_i \omega'$ then $\omega' R_i \omega$.
(iii) Transitive if $\forall i \in N$, $\forall \omega, \omega', \omega'' \in \Omega$, if $\omega R_i \omega'$ and $\omega' R_i \omega''$ then $\omega R_i \omega''$.
(iv) Euclidean if $\forall i \in N$, $\forall \omega, \omega', \omega'' \in \Omega$, if $\omega R_i \omega'$ and $\omega R_i \omega''$ then $\omega' R_i \omega''$.
(v) Serial if $\forall i \in N$, $\forall \omega \in \Omega, \exists \omega' \in \Omega, \omega R_i \omega'$.

We will be interested in two particular classes of frames. One of them is the $S5$ class, which consists of all frames that are reflexive, symmetric and transitive. The other class, known as $KD45$, is the class of all frames that are transitive, Euclidean and serial.

We have so far, in our example in Figure 1, been careful not to interpret the symbol $\Box$ as a knowledge operator. Indeed, to allow such an interpretation, we must guarantee that the operator possesses the properties that one might expect of knowledge. For example, one distinguishing characteristic of knowledge is that one cannot know what is false. So, we must at least impose the restriction that the formula $\Box_i\psi \rightarrow \psi$ for any agent i and any formula ψ, be *valid*.

It turns out that the following formulas are valid in $S5$ frames:
(i) Distribution: $\Box_i(\psi \rightarrow \phi) \rightarrow (\Box_i\psi \rightarrow \Box_i\phi)$.
(ii) Veracity: $\Box_i\psi \rightarrow \psi$.
(iii) Positive introspection: $\Box_i\psi \rightarrow \Box_i\Box_i\psi$.
(iv) Negative introspection: $\neg\Box_i\psi \rightarrow \Box_i\neg\Box_i\psi$.
In fact, the converse also holds: Namely, if we require (i) - (iv) to be validities, then the frame must be $S5$.

The formulas (i) - (iv) happen to be precisely the properties that are considered to be the defining characteristics of knowledge (Early formal philosophical underpinnings can be found in Hintikka (1962)). For example, veracity states that if i knows that ψ, then ψ must be true. In other words, one cannot know what is false. Positive introspection states that if i knows that ψ, then i knows that i knows that ψ. That is, if one knows something, then one knows that one knows it. Finally, negative introspection states that if i does not know that ψ then i knows that i does not know that ψ. That is, if one does not know something, then one knows that one does not know it. Admittedly these are properties of a very strong notion of what knowledge means. However, they are taken as standard, and we will not discuss their justification. In fact, they are completely analogous to the properties of the knowledge operator **K** mentioned in the introduction.

Given the above, we can return to the model given in Figure 1. One can verify that the model has an $S5$ frame, and the modal operators can thus be interpreted as knowledge and common knowledge.
In fact, the model can be seen as a representation of the following scenario: Suppose agents i and j are in a room with a box. Inside the box is a coin, which can either have the heads side facing up or the tails side facing up. Let h be the proposition "The coin is heads side up", and t be "The coin is tails side up". Suppose that i can look directly into the box, but j cannot; however, j can see that i can look into the box.
It was shown previously that $\omega \models \Box_i h$, which means that in the state in which the coin is indeed heads side up, agent i knows this (since he can see it). Also, $\omega' \models \Box_i t$ means that in the state in which the coin is tails side up, i also knows this. Furthermore, we had that $\omega \models \neg\Box_j h$, so in the state in which the coin is heads side up, j does not know that the coin is heads side up.
Note that the modal formulas in the above paragraph have a single modal operator,

so are said to have a *modal depth* of 1. However, a formula such as $\Box_i \Box_j \psi$ has nested modal operators, and has a modal depth of 2. In our example, the formula $\Box_j(\Box_i h \vee \Box_i t)$, interpreted as "$j$ knows that i knows which side of the coin is facing up", has a modal depth of 2.[6] Clearly, *interactive* knowledge - of the form "I know that you know..." - requires a modal depth of at least 2. A formal definition of this notion is given below.

Definition 6 (Modal depth)**.** The modal depth $md(\psi)$ of a formula ψ is the maximal length of a nested sequence of modal operators. This can be defined by the following recursion on our syntax rules: (i) $md(p) = 0$ for any $p \in \mathcal{P}$, (ii) $md(\neg\psi) = md(\psi)$, (iii) $md(\psi \wedge \phi) = md(\psi \vee \phi) = md(\psi \to \phi) = md(\psi \leftrightarrow \phi) = \max(md(\psi), md(\phi))$, (iv) $md(\Box_i \psi) = 1 + md(\psi)$, (v) $md(C_G \psi) = 1 + md(\psi)$.

Finally, returning to our example one last time, we showed that $C_{\{i,j\}}(\Box_i h \vee \Box_i t)$ is valid in the model. This is interpreted as it being *common knowledge* among i and j that i knows which side is facing up, in the sense that they both know this, both know that they know it, both know that they know that they know it, and so on ad infinitum. This is the interpretation of the C_G operator because, completely generally, if $\mathcal{M}, \omega \models C_G \psi$, then one can generate any formula of finite modal depth of the form $\Box_i \Box_j ... \Box_r \psi$ with $i, j ... r \in G$, and this formula will be true at ω in model $\mathcal{M}$.[7]

3 Models with information and decisions

All the definitions in this section are completely general, so hold for arbitrary frame classes. It will firstly be useful to define the following, frequently used concept.

Definition 7 (Component)**.** For any $\omega \in \Omega$, we will denote the set of all states that are accessible from ω in a finite sequence of R_i $(i \in G)$ steps, by $\Omega_G(\omega)$. We will call this set the *component* of ω.

Now, let P be a finite set of atomic propositions. Since P is finite, its closure under the standard Boolean operators, denoted P^*, is tautologically finite.[8] So

[6] Obviously, we could have created a more complicated model representing a situation where j does not see that i can see into the box. That is, a situation in which j does not know that i knows which side is facing up.

[7] Note that the definition of the operator C_G is drawn from van Benthem (2010), where it is also mentioned that an alternative definition can be given: One can define a new accessibility relation R^*_G for the whole group G as the reflexive transitive closure of the union of all separate relations R_i $(i \in G)$, and then simply let $\mathcal{M}, \omega \models C_G \psi$ if and only if $\forall \omega' \in \Omega$, if $\omega R^*_G \omega'$ then $\mathcal{M}, \omega' \models \psi$.

[8] In the sense that there is only a finite number of inequivalent formulas (so p and $p \wedge p$ count as one).

P^* is just the set of all possible inequivalent formulas that can be created out of the propositions in P and the Boolean operators. Let Ψ_0^r be the set of all possible modal formulas that can be generated from P^* with modal depth 0 up to r for an arbitrary $r \in \mathbb{N}_0$. Again, since P^* is finite, so is Ψ_0^r, so $|\Psi_0^r| = m$, for some $m \in \mathbb{N}$; and note that $\Psi_0^0 = P^*$.[9]

Definition 8 (New operators)**.** For each agent $i \in N$ create a set of modal operators, $O_i = \{\Box_i, \hat{\Box}_i, \dot{\Box}_i\}$, where for every formula ψ, $\hat{\Box}_i\psi := \Box_i\neg\psi$ and $\dot{\Box}_i\psi := \neg(\Box_i\psi \vee \hat{\Box}_i\psi)$.
In $S5$, $\hat{\Box}_i\psi$ stands for "Agent i knows that it is not the case that ψ", and $\dot{\Box}_i\psi$ stands for "Agent i does not know whether it is the case that ψ".

Definition 9 (Kens)**.** Order the set Ψ_0^r into a vector of length m: $(\psi_1, \psi_2, ..., \psi_m)$, and for each agent $i \in N$, create the sets

$$U_i = \{(\nu_i^1\psi_1 \wedge \nu_i^2\psi_2 \wedge ... \wedge \nu_i^m\psi_m) | \forall n \in \{1, ..., m\}, \nu_i^n \in O_i\}$$

$$V_i = \{\nu_i \in U_i | \models \neg(\nu_i \leftrightarrow (p \wedge \neg p))\}$$

A *ken* ($\nu_i \in V_i$) for agent i, describes i's information concerning every formula in Ψ_0^r. So, calling $\nu_i^n\psi_n$ the n^{th} *entry* of i's ken, $\nu_i^n\psi_n$ states whether i knows that the formula ψ_n is the case, or knows that it is not the case, or does not know whether it is the case.
Note that V_i is a restriction of U_i to the set of kens that are not logically equivalent to a contradiction; so only the logically consistent kens are considered.

The following lemma shows that at each state, there exists a ken for each agent which holds at that state, and moreover, that any two different kens must be contradictory at any given state.

Lemma 1. *(i)* $\forall\omega \in \Omega, \exists\nu_i \in V_i, \omega \models \nu_i$, *(ii)* $\forall\omega \in \Omega, \forall\nu_i, \mu_i \in V_i$, *if* $\nu_i \neq \mu_i$ *then* $\omega \models \neg(\nu_i \wedge \mu_i)$.

By the above lemma, there is a unique ken in V_i that holds at a given state. So for any $\nu_i \in V_i$, if $\omega \models \nu_i$, we can index the ken by the state, denoting it, $\nu(\omega)_i$.

Definition 10 (Informativeness)**.** Create an order $\succsim \subseteq V_i \times V_j$ for all $i, j \in N$. We say that the ken ν_i is *more informative* than the ken μ_j, denoted $\nu_i \succsim \mu_j$, if and only if whenever i knows that ψ then j either also knows that ψ or does not know whether ψ, and whenever i does not know whether ψ, then so does j.[10]

[9] If $P = \{p, q\}$, then one can generate 20 inequivalent formulas: 2 from p alone, 2 from q alone and 16 out of p and q together, so $|P^*| = 20$.

[10] Formally, (i) if $\nu_i^n\psi_n = \Box_i\psi_n$ then $(\mu_j^n\psi_n = \Box_j\psi_n$ or $\mu_j^n\psi_n = \dot{\Box}_j\psi_n)$, (ii) if $\nu_i^n\psi_n = \hat{\Box}_i\psi_n$ then $(\mu_j^n\psi_n = \hat{\Box}_j\psi_n$ or $\mu_j^n\psi_n = \dot{\Box}_j\psi_n)$, and (iii) if $\nu_i^n\psi_n = \dot{\Box}_i\psi_n$ then $(\mu_j^n\psi_n = \dot{\Box}_j\psi_n)$.

Note that $\succsim$ is not a complete order on kens. For example, consider any two kens ν_i and μ_i for agent i, in which the n^{th} entry is $\nu_j^n \psi_n = \Box_i \psi_n$ and $\mu_j^n \psi_n = \hat{\Box}_i \psi_n$. These two kens would not be comparable with $\succsim$.
Finally, note that $\nu_i \sim \mu_j$ denotes $\nu_i \succsim \mu_j$ and $\mu_j \succsim \nu_i$; which is interpreted as ν_i and μ_j carrying the *same* information, but seen from the perspectives of agents i and j respectively.

The infimum of ν_i and μ_i, denoted $\inf\{\nu_i, \mu_i\}$, is the most informative ken that is less informative than ν_i and μ_i. Incidentally, this is a main point where our analysis differs from Bacharach (1985): An incoherence exists in his framework because the union of two information cells is not itself an information cell. However, we do not encounter this conceptual difficulty because the infimum of two kens (our analogue to cell union) is itself a ken, as shown below.

Lemma 2. *For any $\nu_i, \mu_i \in V_i$, $\inf\{\nu_i, \mu_i\}$ exists in V_i and is characterised by:*

$$\inf\{\nu_i, \mu_i\}^n \psi_n = \Box_i \psi_n \textit{ iff } (\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n)$$
$$\inf\{\nu_i, \mu_i\}^n \psi_n = \hat{\Box}_i \psi_n \textit{ iff } (\nu_i^n \psi_n = \mu_i^n \psi_n = \hat{\Box}_i \psi_n)$$
$$\inf\{\nu_i, \mu_i\}^n \psi_n = \dot{\Box}_i \psi_n \textit{ iff } (\nu_i^n \psi_n \neq \mu_i^n \psi_n \textit{ or } \nu_i^n \psi_n = \mu_i^n \psi_n = \dot{\Box}_i \psi_n)$$

Definition 11 (Decision function)**.** For each $i \in N$, $D_i : V_i \to \mathcal{A}$, is the *decision function* of agent i, where $\mathcal{A}$ is a set of *actions*.

Definition 12 (Action function)**.** For all $\nu_i \in V_i$, $\models \nu_i \to d_i^{D_i(\nu_i)}$

The *action function* d_i selects the action that is actually chosen at each state.[11] "$D_i(\nu_i) = x$" is read as "if i's ken is ν_i, then i's decision is to do x", whereas "d_i^x" is read as "i performs action x". So although the decision function, D_i, determines what the agent *would do* over *all* possible kens, $d_i^{D_i(\nu_i)}$ is the *formula* - added to the syntax - describing the agent performing the action that her decision function requires her to take given the ken she has at each particular state.[12]

Definition 13 (Richness)**.** (i) The language in a component $\Omega_G(\omega)$ is *rich* if and only if for all $i \in G$, and any pair $\nu(\omega')_i, \mu(\omega'')_i$ such that $\nu(\omega')_i \neq \mu(\omega'')_i$ and $\omega', \omega'' \in \Omega_G(\omega)$, there is $n \in \{1, ..., m\}$ such that $\nu_i^n = \Box_i$ and $\mu_i^n = \hat{\Box}_i$.
(ii) The language in a component $\Omega_G(\omega)$ is *very rich* if and only if for all $i \in G$, and any $\nu(\omega')_i$ such that $\omega' \in \Omega_G(\omega)$, there is no $n \in \{1, ..., m\}$ such that $\nu_i^n = \dot{\Box}_i$.

[11] Lemma 1 guarantees that the action function is well-defined.

[12] Technically, we let all propositions of the form "$D_i(\nu_i) = x$" live in a set $\mathcal{D}$, and all propositions of the form "d_i^x" live in a set $\mathcal{Q}$. Then the set of a propositions is $\mathcal{P} = P \cup \mathcal{D} \cup \mathcal{Q}$, so the valuation function is $\mathcal{V} : \mathcal{P} \times \Omega \to \{0, 1\}$.

Essentially, the language in a component is rich if any two distinct kens in the component for agent i are incomparable via $\succsim$. In other words, any two distinct kens must be contradictory about *some* “fact” - i.e. formula - (in the sense that in one ken, the agent knows that it is true, whereas in the other ken, the agent knows that it is false). Richness is how we capture the idea of “disjointness” in our framework.[13] Furthermore, when the language is very rich in a component, there is no ken in which the agent is “unsure” about some fact, in the sense that he/she does not know whether it is true or false. That is, of every fact at every state of the component, the agent either knows that it is true, or knows that it is false. Below, we show how richness can be given an interpretation in terms of “signals”, which is closer to the discussion given in Aumann et al. (2005).

3.1 Main assumptions

We will assume two distinct versions of the Sure-Thing Principle, and prove an agreement theorem with each respectively. The first version is the analogue of the “non-disjoint” version used by Moses and Nachum (1990), which we state as a formula, $NDSTP$, that we assume to be valid:

Assumption 1 (Non-Disjoint Sure-Thing Principle).
At every state $\omega \in \Omega$, it is true that for every agent $i \in N$ and all $\nu_i, \mu_i \in V_i$, if $D_i(\nu_i) = D_i(\mu_i)$ then $D_i(\inf\{\nu_i, \mu_i\}) = D_i(\nu_i)$.

This states that whenever an agent would take the same decision given the information ν_i and μ_i, then the agent would take the same decision over the infimum of those kens - i.e. in the situation in which the agents is “just” less informed. The second version of the Sure-Thing Principle, which we call $DTSP$, is closer to the original one used by Bacharach (1985), because it requires disjointness. In our framework, $DSTP$ is simply $NDSTP$ but is only required to hold over kens that are expressed in a “rich” language.

Assumption 1’ (Disjoint Sure-Thing Principle).
Let $T_i = \{(\nu_i, \mu_i) \in V_i \times V_i | \exists n$ such that $\nu_i^n = \Box_i$ and $\mu_i^n = \hat{\Box}_i\}$. At every state $\omega \in \Omega$, it is true that for every agent $i \in N$ and all $\nu_i, \mu_i \in T_i$, if $D_i(\nu_i) = D_i(\mu_i)$ then $D_i(\inf\{\nu_i, \mu_i\}) = D_i(\nu_i)$.

The above versions of the Sure-Thing Principle can be illustrated by means

[13] Note that richness is analogous to what we understand as the requirement that all knowledge be “elementary” in Aumann and Hart (2006); and is intended to capture the idea that the information be “disjoint” (in line with the Sure-Thing Principle of Bacharach (1985)).

of the following example.[14] Suppose i sends out an invitation for a dinner party to Alice, Bob and Charlie, and define ν_i to be the ken in which i knows that Alice is coming to the dinner, but does not know whether Bob is coming to the dinner and does not know whether Charlie is coming to the dinner ($\nu_i = \Box_i a \wedge \dot{\Box}_i b \wedge \dot{\Box}_i c$). Furthermore, let μ_i be the ken in which i knows that Bob is coming to the dinner, but does not know whether Alice is coming to the dinner and does not know whether Charlie is coming to the dinner ($\mu_i = \dot{\Box}_i a \wedge \Box_i b \wedge \dot{\Box}_i c$). Suppose furthermore, that $D_i(\nu_i) = D_i(\mu_i)$. The $NDSTP$ would then require that $D_i(\inf\{\nu_i, \mu_i\}) = D_i(\dot{\Box}_i a \wedge \dot{\Box}_i b \wedge \dot{\Box}_i c) = D_i(\nu_i)$. That is, i must take the same decision when i does not know anything about whether any guests are coming to the dinner.

The above example illustrates how strong an assumption the $NDSTP$ is: The agent is required to make the same decision, jumping directly from ν_i and μ_i to a situation in which essentially, nothing is known. But many other kens could have been cycled through as well, and the same decision would have been required! For example $\dot{\Box}_i a \wedge \dot{\Box}_i b \wedge \Box_i c$.

To remedy this, suppose we reformulated the situation as "The agent knows that Alice sent an RSVP and knows that Bob and Charlie did not". Letting a' stand for "Alice sent an RSVP", we have $\nu'_i = \Box_i a' \wedge \hat{\Box}_i b' \wedge \hat{\Box}_i c'$. Similarly, we have $\mu'_i = \hat{\Box}_i a' \wedge \Box_i b' \wedge \hat{\Box}_i c'$. Now, the pair of kens $\nu''_i = \nu_i \wedge \nu'_i$ and $\mu''_i = \mu_i \wedge \mu'_i$ is "rich" in the sense that there is a proposition, namely a' for which $\Box_i a'$ in one ken, and $\hat{\Box}_i a'$ in the other. In fact, ν''_i and μ''_i include all the information, including all the information about how the information was acquired, i.e. the "signals" (in the form of propositions regarding whether or not the guests sent an RSVP). Aumann et al. (2005) argue that in this case, the $DSTP$ is a reasonable assumption, so if one takes the same decision in the case of ν''_i and μ''_i, then the same decision must be taken over $\inf\{\nu''_i, \mu''_i\}$.

Note that the pair of kens ν'_i and μ'_i that only consider information regarding signals and discard the rest, are "very rich" in the sense that everything is solely expressed in terms of "knowing that" or "knowing that not".

Assumption 2 (Like-mindedness)**.** *At every state $\omega \in \Omega$, it is true that for any $\nu_i \in V_i$ and $\nu_j \in V_j$, if $\nu_i \sim \nu_j$ then $D_i(\nu_i) = D_j(\nu_j)$.*

The assumption of like-mindedness captures the idea that the agents would take the same decision if they had the same information.

[14] Note that the assumptions can be stated completely formally as valid formulas:
$\models NDSTP := \bigwedge_{i\in N} \bigwedge_{\nu_i,\mu_i \in V_i} [D_i(\nu_i) = D_i(\mu_i) \to D_i(\inf\{\nu_i,\mu_i\}) = D_i(\nu_i)]$
$\models DSTP := \bigwedge_{i\in N} \bigwedge_{(\nu_i,\mu_i) \in T_i} [D_i(\nu_i) = D_i(\mu_i) \to D_i(\inf\{\nu_i,\mu_i\}) = D_i(\nu_i)]$
And, they are guaranteed to be well-defined by Lemma 2.

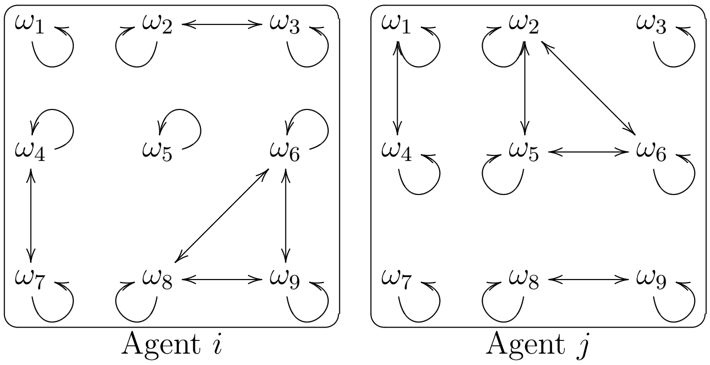

Figure 2: State space in $S5$

Assumption 3 (State-independent decision functions)**.** *The decision function is invariant across all states for each agent. That is, for any $\omega', \omega'' \in \Omega$, if $\omega' \models D'_i(.)$ and $\omega'' \models D''_i(.)$, then $\omega' \models D'_i(.) = D''_i(.)$ and $\omega'' \models D'_i(.) = D''_i(.)$.*[15]

In a sense, this allows us to imagine our models being constructed as follows: Formulas of the form d_i^x are made true or false at given states by the valuation map. But, there exists some "global" decision function D_i for each agent such that $D_i(\nu(\omega)_i) = x$ whenever $\omega \models d_i^x$.

4 Generalised results in $S5$

In $S5$, the accessibility relation R_i of each agent i is reflexive, symmetric and transitive. So it is an equivalence relation for each $i \in N$. Let $I_i(\omega) = \{\omega' \in \Omega | \omega R_i \omega'\}$ be the *information cell* of i at ω. One can verify that the set $\mathcal{I}_i = \{I_i(\omega) | \omega \in \Omega\}$ is a partition of the state space Ω - we thus have a *partitional* model.
We provide a schematic representation of an $S5$ model in Figure 2. The state space is $\Omega = \{\omega_1, ..., \omega_9\}$. The partition for agent i is given by the set $\mathcal{I}_i = \{\{\omega_1\}, \{\omega_5\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_7\}, \{\omega_6, \omega_8, \omega_9\}\}$. Agent j's partition is found similarly. Furthermore, $\Omega_{\{i,j\}}(\omega_1) = \{\omega_1, \omega_4, \omega_7\}$, and $\Omega_{\{i,j\}}(\omega_2) = \Omega \backslash \Omega_{\{i,j\}}(\omega_1)$.

The following lemma states that in $S5$, the information cells of every agent exhaust any component.

Lemma 3. *$\forall i \in G$, $\bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \Omega_G(\omega)$.*

The lemma below states that kens are identical across all the states that are in the same information cell.

[15] This could equivalently be stated as $\nu(\omega)_i \sim \nu(\omega')_i \rightarrow D_i(\nu(\omega)_i) = D_i(\nu(\omega')_i)$.

Lemma 4. *If for some $\omega' \in I_i(\omega)$, $\omega' \models \nu_i$, then for all $\omega'' \in I_i(\omega)$, $\omega'' \models \nu_i$.*

It will now be useful to introduce a new definition which will eventually allow us to provide a semantic characterisation of $\inf\{\nu_i, \mu_i\}$ for any kens $\nu_i, \mu_i \in V_i$.

Definition 14 (Cell merge)**.** Consider a model in $S5$, $\mathcal{M} = \langle \Omega, R_{i\in N}, V\rangle$. Let $I_i(\omega) = \{\omega'' \in \Omega | \omega R_i \omega''\}$ and $I_i(\omega') = \{\omega'' \in \Omega | \omega' R_i \omega''\}$. Create a new model $\mathcal{M}(I_i(\omega), I_i(\omega')) = \langle \Omega', R'_{i\in N}, V'\rangle$ where,

$$
\begin{aligned}
&\Omega' = \Omega \\
&R'_i = R''_i \cup R_i|_{\Omega\setminus I_i(\omega)\cup I_i(\omega')} \\
&\text{ where } R''_i = \{(\omega'', \omega''') \in \Omega\times\Omega | \omega'', \omega''' \in I_i(\omega)\cup I_i(\omega')\} \\
&\text{ and } R_i|_{\Omega\setminus I_i(\omega)\cup I_i(\omega')} = \{(\omega'', \omega''') \in R_i | \omega'', \omega''' \in \Omega\setminus I_i(\omega)\cup I_i(\omega')\} \\
&R'_j = R_j \text{ for all } j \neq i \\
&V' = V
\end{aligned}
$$

One can verify that the model $\mathcal{M}(I_i(\omega), I_i(\omega'))$ is itself a model in $S5$, but where the cells $I_i(\omega)$ and $I_i(\omega')$ are *merged* to form a single information cell (with all the accessibility relations appropriately "rewired"), yet leaving the rest of the original model, $\mathcal{M}$, unchanged.

For sake of illustration, let us return to the example given in Figure 2. Let the model represented be $\mathcal{M}$. We can, for example, create the "merged" model, $\mathcal{M}(I_i(\omega_4), I_i(\omega_5))$, in which j's information partition is unchanged, but i's partition is now $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_7\}, \{\omega_6, \omega_8, \omega_9\}\}$.

The following lemmas provides a semantic characterisation of $\inf\{\nu_i, \mu_i\}$ in $S5$, which turns out to be the ken that holds in a model in which the information cells, at which ν_i and μ_i hold, are merged.

Lemma 5. *Consider Ψ_0^r with $r = 0$.*
If $\mathcal{M}, \omega \models \nu_i$ and $\mathcal{M}, \omega' \models \mu_i$, then for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega''' \models \inf\{\nu_i, \mu_i\}$.

Lemma 6. *Consider Ψ_0^r with $r = 0$ and let $G = \{i, j\}$.*
For any $\Omega_G(\omega)$, $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$.

Finally, we are in a position to state our agreement results in $S5$:

Theorem 1. *Consider Ψ_0^r with $r = 0$, suppose $NDSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, and the system is $S5$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

Theorem 2. *Consider Ψ_0^r with $r = 0$, suppose assumptions $DSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, the language is rich in every component, and the system is $S5$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \rightarrow (x = y)$.*

4.1 Discussion

The intuition behind the results is that each agent has some ken at the actual state ω, and based on this ken, say $\nu(\omega)_i$, the agent actually takes the action d_i^x. However, the Sure-Thing Principle allows us to discover that i's decision *would* also be x *if i's information were* $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}$. This is *not* the ken that i has ω, so i's action is not taken based on this ken. However, responding to Moses and Nachum (1990), it has a clear interpretation: It is the most informative ken that is less informative than any ken that i has at any state in the common knowledge component; and if this *were* i's ken, then i's decision would be x. However, over a similar ken, we find that j's decision would be y. But since this is the same uninformative ken, and agents are like-minded, we conclude that $x = y$.
Note the role of the infimum of kens in the theorems: Effectively, it only preserves those propositions that *both* agents know. Any proposition p where i knows that p while j knows that $\neg p$, or where i knows that p and j does not know whether p is discarded. That is, implicitly, the only information that becomes relevant for the decisions of the agents is the information on which they already agree.

Theorem 1 in particular, highlights an awkwardness of the agreement results: If we require the weaker version of the Sure-Thing Principle to hold ($DSTP$), then whether or not the agreement theorem holds depends on the richness of the language. In other words, it depends on the way in which the environment is described! (That is, if the language were not rich enough in every component, then agreement would not necessarily follow).

We should note that in $S5$, if we assume the language to be *very rich* in some component, this has the remarkable implication that both agents must have the same information at every state of the component.

Proposition 1. *Suppose the language is very rich in some component $\Omega_G(\omega)$. Then, for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu_i \sim \mu_j$.*

A direct corollary of this is that agents cannot agree to disagree if they are like-minded, even if the decision functions do not satisfy any version of the Sure-Thing Principle. Agreement becomes trivial since they always have the same information.

Theorem 3. *Suppose the agents are like-minded and the language is very rich in every component, and the system is $S5$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

For a discussion of this theorem, see Theorem 7 below.

Regarding both theorems, it should be noted that they are stated with *global* assumptions, but *local* assumptions would have sufficed: We could have required that the assumed validities hold true at each state of the component rather than at every state of the state space. This highlights to point that even if all the conditions required for the agreement results to hold are not satisfied everywhere in a model, there may exist "pockets" in the state space in which agents cannot agree to disagree - because the conditions do hold in those components - and others where they can agree to disagree.

Furthermore, both rely on the restriction that only Ψ_0^r with $r = 0$ be considered (that is, $\Psi_0^0 = P^*$). This means that decisions cannot be based on formulas involving nested modal operators; that is, on *interactive* information.[16] This is analogous to the assumption made in Aumann and Hart (2006) that decisions be *substantive*: "Only knowledge of elementary facts matters, not knowledge about knowledge (i.e. interactive knowledge)".[17] To see how restrictive this assumption is, let us return to our example of the coin in the box given in Figure 1. Suppose that i and j are required to write what side of the coin is facing up on a piece of paper. If they get it right, they earn a prize. Now, if j's decision can depend on the fact that she know that i knows what side is facing up, j can write: "The side that is facing up is the one that i says is facing up". However, if this interactive information must be ignored, this strategy is, as far as j is concerned, just as good as simply guessing, since she might as well not know that i knows which side is facing up.

The reason for the restriction is that Lemma 5 does not hold for Ψ_0^r if $r > 0$. This is because the truth of formulas of a modal depth one or greater is fully determined by the accessibility relations of all agents. The trouble is that by moving from the model $\mathcal{M}$ to a merged model $\mathcal{M}(I_i(\omega), I_i(\omega'))$, we are modifying the accessibility relations, and there is no guarantee that truth of higher depth formulas will remain unchanged, so kens in the merged model may be incomparable (via $\succsim$) with the kens in the original model.

Figure 3 provides a counter-example to Lemma 5 when $r > 0$: Suppose that in both

[16]Note: Tarbush (2011) finds that a distinguishing feature of the agreement result in Samet (2010) is that it holds for all $r \geq 0$.

[17]This avoids the criticism of Moses and Nachum (1990) concerning the like-mindedness assumption.

$$
\begin{tikzcd}
\omega \arrow[loop left, "{i,j}"] \arrow[r, leftrightarrow, "j"] & \omega' \arrow[loop right, "{i,j}"]
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\omega \arrow[loop left, "{i,j}"] \arrow[r, leftrightarrow, bend left, "i"] \arrow[r, leftrightarrow, bend right, "j"'] & \omega' \arrow[loop right, "{i,j}"]
\end{tikzcd}
$$

$$\mathcal{M} \qquad\qquad \mathcal{M}(I_i(\omega), I_i(\omega'))$$

Figure 3: Merge with $r > 0$

models, $\omega \models p$ and $\omega' \models \neg p$. One can verify that for all $\omega \in \Omega$, $\mathcal{M}, \omega \models \Box_i \hat{\Box}_j \dot{\Box}_i p$, and $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega \models \hat{\Box}_i \hat{\Box}_j \dot{\Box}_i p$. Therefore, whatever ken i might have in the merged model, it is incomparable (via $\succsim$) with her kens in the original model.
Of course, the upshot of this is that, given our other assumptions, agents *can* agree to disagree if their decision functions are allowed to depend on interactive information.

Finally in $S5$, we can show that our framework can be mapped directly into that of Bacharach (1985), and is essentially identical to that of Aumann and Hart (2006) (see Appendix B). However, the framework developed here has some advantages: (i) The use of epistemic logic allows for a very transparent account of the conditions on the modal depth of formulas, (ii) the ordering $\succsim$ on kens gives a clear definition of informativeness, and hence of $\inf\{\nu_i, \mu_i\}$, (iii) explicitly modelling the accessibility relations between states allows us to easily consider extensions in a non-partitional state space, and finally (iv) our approach allows us to unify and compare the results of the literature in one methodological approach.

5 Generalised result in $KD45$

We have so far derive all our results within partitional models - that is, in $S5$ frames. However, the $\Box$ operator has very strong properties in such frames. In particular, one cannot "know" what is false. But there is nothing inherent to the notion of rationality that requires rational agents to base their decisions only on correct information. For this reason, we will consider $KD45$ frames, in which the accessibility relations are transitive, Euclidean and serial. The following formulas are valid in $KD45$ frames:
(i) Distribution: $\Box_i(\psi \to \phi) \to (\Box_i \psi \to \Box_i \phi)$.
(ii) Consistency: $\Box_i \psi \to \neg \Box_i \neg \psi$.
(iii) Positive introspection: $\Box_i \psi \to \Box_i \Box_i \psi$.
(iv) Negative introspection: $\neg \Box_i \psi \to \Box_i \neg \Box_i \psi$.
The converse also holds: Namely, if we require (i) - (iv) to be validities, then the frame must be $KD45$.

These validities describe the properties that we would require $\Box$ to satisfy in order to be interpreted as a *belief*, rather than a knowledge, operator (Again, see Hintikka (1962) for philosophical underpinnings). Essentially, the only difference is that unlike knowledge, belief is not *infallible*: By dropping reflexivity, it is possible to have $\Box_i p \wedge \neg p$ in a $KD45$ frame - that is, agents are allowed to believe what is false, and thus to base decision on *false* information. Note however, that agents are at least required to have consistent beliefs.
One can verify that all $S5$ frames are also $KD45$ frames, but the converse is not true. In fact, $S5 = KD45 +$ reflexivity.

We can provide a description of the links between states in a $KD45$ frame: Some sets of states within Ω are "completely connected", in the sense that the accessibility relation over states within such sets in an equivalence relation, so these sets have the same properties as information cells in $S5$; and, for each one of these completely connected sets there exists a (possibly empty) set of "associated" states that have arrows pointing from them to every state in the completely connected set, but with no arrow (by the same agent) pointing towards them. The set of all completely connected sets and their set of associated states exhaust the state space.
Formally, let $S_i(\omega) = \{\omega' \in \Omega | \omega E_i \omega'\}$, where E_i is an equivalence relation. We call this set of completely connected states the *information sink* of state ω for player i. The set S_i do not necessarily partition the state space, hence we have a *non-partitional* model. Note, that this way of defining the sink guarantees that if $S_i(\omega) \neq \emptyset$ then $\omega \in S_i(\omega)$. Furthermore, we define ω's *set of associated states* as $A_i(\omega) = \{\omega'' \in \Omega | \forall \omega''' \in S_i(\omega), \omega'' F_i \omega'''\}$, where F_i is now a simple arrow. So, note that now, for any agent i, we have that $R_i = E_i \cup F_i$. Finally, we can define $J_i(\omega) = S_i(\omega) \cup A_i(\omega)$, and note that $\mathcal{J}_i = \{J_i(\omega) | \omega \in \Omega\}$ exhausts the entire state space.

Proposition 2. *The above is a complete characterisation of the $KD45$ state space.*

We provide a schematic representation of a $KD45$ model in Figure 4. For example, i's information sink at state ω_4 is the set $S_i(\omega_4) = \{\omega_4, \omega_5\}$, and the set of associated states is $A_i(\omega_4) = \{\omega_1, \omega_2, \omega_3\}$. Furthermore, note for example that the component of state ω_1 is the set $\Omega_{\{i,j\}}(\omega_1) = \Omega \backslash \{\omega_1, \omega_7\}$, so it is now possible that $\omega \notin \Omega_G(\omega)$.
We can see how having false beliefs is possible in such frames: For example, let $\omega_1 \models \neg p$ and $\omega_4 \models p$. Then, $\omega_1 \models \Box_j p \wedge \neg p$. This also shows how it is only in the sets A_i that the agent could potentially hold false beliefs.

We will need to add the following assumptions to derive the main results:

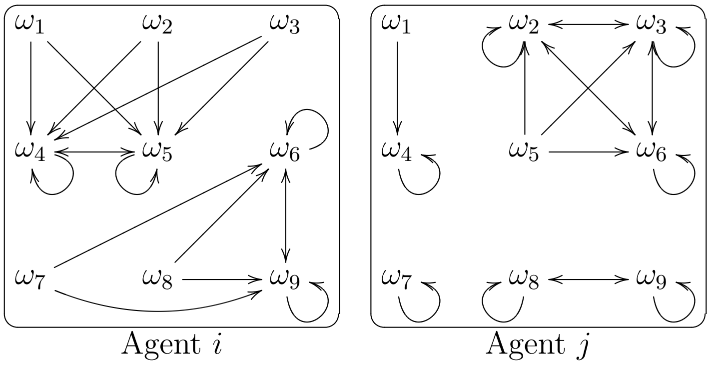

Figure 4: State space in $KD45$

Assumption 4 (Heterogeneity). *If for all $i \in G$, and for all $\omega' \in \Omega_G(\omega)$, $\nu(\omega')_i = \nu(\omega)_i$, then for all $\omega' \in \Omega_G(\omega)$, $\nu(\omega')_i \sim \nu(\omega')_j$.*[18]

This assumption is termed "heterogeneity" because it is equivalent to the statement: Either for all $\omega' \in \Omega_G(\omega)$, $\nu(\omega')_i \sim \nu(\omega')_j$; or, there exists an $i \in G$ such that $\nu(\omega')_i \neq \nu(\omega)_i$ for some $\omega' \in \Omega_G(\omega)$. That is, in any component, either the agents have the same information, or at least one of the agents has different information at a different state in the component.

The following lemmas are generalisations of the ones found for $S5$.

Lemma 7. *$\forall i \in G$, $\bigcup_{\omega' \in \Omega_G(\omega)} S_i(\omega') \subseteq \Omega_G(\omega) \subseteq \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$.*

Lemma 8. *If for some $\omega' \in J_i(\omega)$, $\omega' \models \nu_i$, then for all $\omega'' \in J_i(\omega)$, $\omega'' \models \nu_i$.*

Definition 15 (Sink merge)**.** Consider a model in $KD45$, $\mathcal{M} = \langle \Omega, R_{i \in N}, V \rangle$. Let $J_i(\omega) = S_i(\omega) \cup A_i(\omega)$ and $J_i(\omega') = S_i(\omega') \cup A_i(\omega')$. Create a new model

[18]Note that this is equivalently stated as: For all $\omega \in \Omega$, $\omega \models C_G(\nu(\omega)_i \wedge \mu(\omega)_j) \rightarrow \bigwedge_{\omega' \in \Omega_G(\omega)} (\nu(\omega')_i \sim \mu(\omega')_j)$. Obviously, this also implies that for all $\omega \in \Omega$, $\omega \models C_G(\nu(\omega)_i \wedge \mu(\omega)_j) \rightarrow (\nu(\omega)_i \sim \mu(\omega)_j)$.

$\mathcal{M}(J_i(\omega), J_i(\omega')) = \langle \Omega', R'_{i\in N}, V' \rangle$ where,

$\Omega' = \Omega$
$R'_i = E'_i \cup F'_i$
$E'_i = E''_i \cup E_i|_{\Omega \backslash S_i(\omega) \cup S_i(\omega')}$
where $E''_i = \{(\omega'', \omega''') \in \Omega \times \Omega | \omega'', \omega''' \in S_i(\omega) \cup S_i(\omega')\}$
and $E_i|_{\Omega \backslash S_i(\omega) \cup S_i(\omega')} = \{(\omega'', \omega''') \in E_i | \omega'', \omega''' \in \Omega \backslash S_i(\omega) \cup S_i(\omega')\}$
$F'_i = F''_i \cup F_i|_{\Omega \backslash A_i(\omega) \cup A_i(\omega')}$
where $F''_i = \{(\omega'', \omega''') \in \Omega \times \Omega | \omega'' \in A_i(\omega) \cup A_i(\omega'), \omega''' \in S_i(\omega) \cup S_i(\omega')\}$
and $F_i|_{\Omega \backslash A_i(\omega) \cup A_i(\omega')} = \{(\omega'', \omega''') \in F_i | \omega'', \omega''' \in \Omega \backslash A_i(\omega) \cup A_i(\omega')\}$
$R'_j = R_j$ for all $j \neq i$
$V' = V$

One can verify that the model $\mathcal{M}(J_i(\omega), J_i(\omega'))$ is itself a model in $KD45$, but where $J_i(\omega)$ and $J_i(\omega')$ are *merged* to form a new information sink with a set of associated states, yet leaving the rest of the original model, $\mathcal{M}$, unchanged.

For sake of illustration, let us return to the example given in Figure 4. Let the model represented be $\mathcal{M}$. We can, for example, create the "merged" model, $\mathcal{M}(J_j(\omega_1), J_j(\omega_8))$, in which i's accessibility relation is unchanged, but j now has a sink $S_i(\omega_8) = \{\omega_4, \omega_8, \omega_9\}$ and a set of associated states $A_j(\omega_8) = \{\omega_1\}$. That is, there is an equivalence relation over the states in $S_i(\omega_8)$, and there are arrows from ω_1 pointing to each of the states in $S_i(\omega_8)$; and, the relations between the rest of the states remain as they were in the original model for j.

Lemma 9. *Consider Ψ_0^r with $r = 0$.*
If $\mathcal{M}, \omega \models \nu_i$ and $\mathcal{M}, \omega' \models \mu_i$, then for all $\omega'' \in J_i(\omega) \cup J_i(\omega')$, $\mathcal{M}(J_i(\omega), J_i(\omega')), \omega'' \models \inf\{\nu_i, \mu_i\}$.

Lemma 10. *Consider Ψ_0^r with $r = 0$, and suppose heterogeneity holds.*
Let $G = \{i, j\}$. For any $\Omega_G(\omega)$, $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$.

We can now state our generalised agreement results in $KD45$.

Theorem 4. *Consider Ψ_0^r with $r = 0$, suppose $NDSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, heterogeneity holds, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

Theorem 5. *Consider Ψ_0^r with $r = 0$, suppose $DSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, heterogeneity holds, the language is rich in every component, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then, $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.*

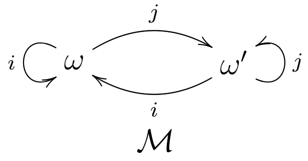

Figure 5: Example of disagreement when heterogeneity does not hold

5.1 Discussion

Heterogeneity requires that at least some agent has some variation in her information in the set $\Omega^S(\omega)$ (or that the agents' information be the same). Note that the assumption is *always* satisfied in an $S5$ model.

Proposition 3. *Let the system be $S5$ and $G = \{i, j\}$. If for all $i \in G$, and for all $\omega' \in \Omega_G(\omega)$, $\nu(\omega')_i = \nu(\omega)_i$, then for all $\omega' \in \Omega_G(\omega)$, $\nu(\omega')_i \sim \nu(\omega')_j$.*

We construct a model in which heterogeneity fails (where both agents have no variation in their information), and show that the agents can agree to disagree. Consider the model represented in Figure 5 and suppose that $\omega \models p$, and $\omega' \models \neg p$. In this model, at *every* state, i believes that p is the case, whereas j believes that $\neg p$ is the case. So we can let i's decision at every state be x while letting j's be y.
An interpretation of this example is that the agents are *systematically biased in the way they acquire new information.* For example, suppose Alice and Bob have a decision function whereby they leave the country if they believe that taxes will rise after the election, and stay if they believe that taxes stay the same. Now, suppose that in state ω, Alice consults one expert, and in ω', she consults another, but both experts tell her that taxes will rise; so Alice would always come to believe that taxes will rise, so she decides to leave the country. On the other hand, in state ω, Bob consults one expert, and another in ω', but in both cases, he is told that taxes will not rise, so he always comes to believe that they will not rise, and thus decides to stay.
Now, even though it is the case that Bob knows that Alice will leave the country, and he knows that she has the same decision function as he does, he cannot "update" his decision when he is given the information about her decision, because there is simply no other information that he deems it is possible to acquire.

5.1.1 A taxonomy of conditions

In this section, we contrast and compare various conditions that have been used in the literature in relation to agreement theorems. This will allow us to place heterogeneity in relation to more familiar conditions, and also to provide a discussion

of the richness assumption in $KD45$.
Each condition will be given semantically (a), and essentially syntactically (b).[19]

Definition 16 (Condition 1)**.** Condition (1.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') = S_j(\omega')$. Condition (1.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\nu(\omega')_i \sim \nu(\omega')_j$.

Definition 17 (Condition 2)**.** Condition (2.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \subseteq S_j(\omega')$. Condition (2.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\nu(\omega')_i \succsim \nu(\omega')_j$, and there exists an $\omega'' \in \Omega_G(\omega)$ such that $\nu(\omega'')_j \succsim \nu(\omega'')_i$.

Condition 1 states that in any component, there must exist a state in which both agents have the same ken. Syntactically: The agents must jointly consider it possible that they have the same information. Condition 2 states that in any component, a state must exist in which i's ken is more informative than j's, and a state must exist in which j's ken is more informative than i's.

Definition 18 (Condition 3)**.** Condition (3.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\cup_{\omega'' \in \Omega_G(\omega')} S_i(\omega'') = \cup_{\omega'' \in \Omega_G(\omega')} S_j(\omega'')$. Condition (3.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that for all $\omega'' \in \Omega_G(\omega')$, $\omega'' \models (\bigwedge_{n \in \{1,\dots,m\}} \bigwedge_{i \in G} \Box_i \psi_n \rightarrow \psi_n)$.

Definition 19 (Condition 4)**.** Condition (4.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \subseteq \cup_{\omega'' \in \Omega_G(\omega)} S_j(\omega'')$. Condition (4.b): Heterogeneity.

Condition 3 and 4 are clearly weaker counterparts of conditions 1 and 2 respectively. Their direct interpretation is not obvious. However, their syntactic implications are interpretable: Condition (3.b) is what Bonanno and Nehring (1998) term *quasi-coherence*: "agents consider it jointly possible that they commonly believe that what they believe is true". They show that it is equivalent to the impossibility of unbounded gains from betting (with moderately risk averse agents), which gives it normative appeal. Condition (4.b) is simply heterogeneity: "if agents' beliefs (kens) are commonly believed, then their beliefs (kens) must be the same".

Definition 20 (Condition 5)**.** Condition (5.a): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $S_i(\omega') \cap S_j(\omega') \neq \emptyset$. Condition (5.b): For all $\omega \in \Omega$ and $i, j \in G$, there exists an $\omega' \in \Omega_G(\omega)$ such that $\omega' \models (\bigwedge_{n \in \{1,\dots,m\}} \neg(\Box_i \psi_n \wedge \hat{\Box}_j \psi_n))$.

[19]The (b) conditions are syntactic in the sense that they could be stated purely in our syntax, but they are complicated formulas so stating them explicitly might obscure their meaning. Heterogeneity is an example.

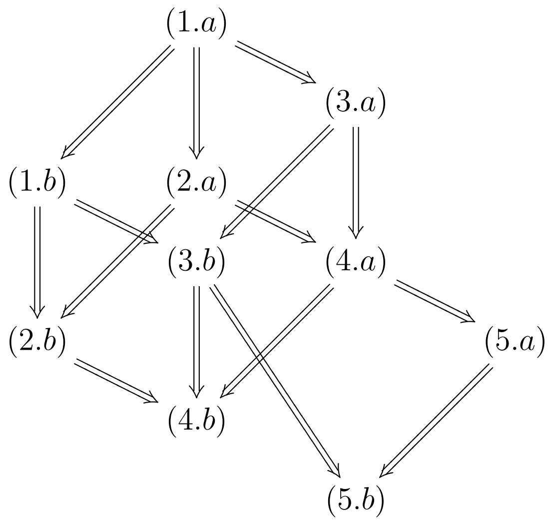

Figure 6: Taxonomy of conditions

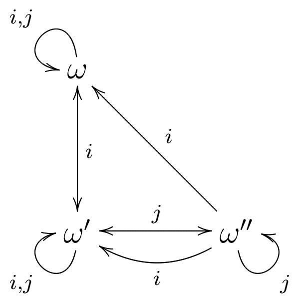

Figure 7: Quasi-coherence fails but heterogeneity holds

This condition states that it cannot be the case that all the information sinks are disjoint across agents. (Syntactically: The agents must jointly consider it possible that they are not completely at odds about *every* “fact” - i.e. one believes that it is the case while the other believes that it is not the case). Obviously, imposing such a condition would rule out the scenario represented in Figure 5.

Proposition 4. *The arrows (⇒) represent logical implication in Figure 6.*

Notably, it is shown that quasi-coherence implies heterogeneity. However, the converse does not hold, as shown in Figure 7. Suppose that $\omega \models p$, $\omega' \models p$ and $\omega'' \models \neg p$. Clearly, there is a state, namely ω'' in $\Omega_G(\omega)$ at which $(\omega'', \omega'') \notin R_i$ so quasi-coherence fails. However, at $\omega \models \Box_j p$ whereas $\omega'' \models \dot{\Box}_j p$ so heterogeneity holds.

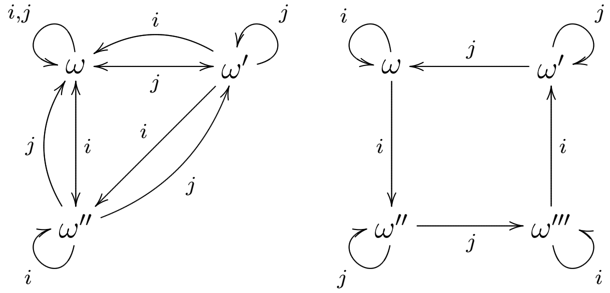

Figure 8: Condition (5.b) holds and heterogeneity fails (left); Heterogeneity holds and (5.b) fails (right)

On the other hand, there is no implication in either direction between heterogeneity and condition (5.b). In the model on the left in Figure 8, let $\omega \models p$, $\omega' \models \neg p$ and $\omega'' \models p$. It is easy to see that condition (5.b) holds since the sinks intersect at ω. However, $\Box_i p$ holds at every state while $\dot{\Box}_j p$ holds at every state, so heterogeneity fails. However, in the model on the right, let $\omega \models p \wedge q$, $\omega' \models \neg q \wedge \neg p$, $\omega'' \models q \wedge \neg p$ and $\omega''' \models p \wedge \neg q$. One can verify that at every state, there exists a proposition ψ such that $\Box_i \psi \wedge \hat{\Box}_j \psi$, so (5.b) fails. However, there is variation in the agents' kens across states, so heterogeneity holds.

It is interesting to note that the difficulty in fully characterising heterogeneity only in terms of sets of states and accessibility relations can be seen as offering vindication to the approach adopted in this paper: The syntactic approach allows us to avoid incoherences, and may allow us to consider models that a purely semantic approach may not.
The upshot of our analysis is that *if* one is prepared to accept the assumptions for the results in $S5$, then it is only a small step to also accept the results in $KD45$ - without having to resort to anything as strong as the Zero-Priors assumption.

In contrast with $S5$, assuming that the language is *very rich* does not imply that the agents have the same information at every state of every component. However, it does nevertheless yield striking results.

Proposition 5. *Suppose the language is very rich in some component $\Omega_G(\omega)$. Then, condition (5.b) implies (1.b).*

The implication of this proposition is that (5.b) together with a very rich language imply heterogeneity. Therefore, we obtain the following theorem:

Theorem 6. *Consider Ψ_0^r with $r = 0$, suppose $DSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, condition (5.b) holds,* *the language is very rich, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then,* $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.

5.2 Agreement without the Sure-Thing Principle

Here, we present here a theorem, similar to Theorem 3, which does not restrict the decision functions to satisfy the Sure-Thing Principle. Firstly, reasonable decision functions that violate the principle may arguably exist, and secondly, such a theorem determines the conditions under which there is agreement even when the principle is violated behaviourally (which is common, as surveyed in Shafir (1994)).

Theorem 7. *Suppose agents are like-minded and condition (1.b) hold, and the system is $KD45$. Let $G = \{i, j\} \subseteq N$. Then,* $\models C_G(d_i^x \wedge d_j^y) \to (x = y)$.

This result has several striking features: Firstly, it does not assume anything about the decision functions, other than the requirement of like-mindedness. Therefore, this theorem applies to all decision functions, including the ones that do not satisfy the Sure-Thing Principle. Secondly, it makes no requirement on the richness of the language. Thirdly, it does not require any restriction on r, the modal depth of formulas. This means that decisions can be based on interactive information. That is, formulas of the form: i believes that j believes that p. Finally, it does not require decision functions to be independent of states, which implies that the theorem holds even if the decision functions themselves are not commonly believed.

Of course, the main driver here is condition (1.b), which states that it must be commonly possible for the agents to have the same information. In $S5$, this condition has the same effect as assuming a *very* rich language in $S5$; namely, it implies that the agents have the same information at the same states in every component. However, this implication does not hold in $KD45$, so unlike Theorem 3, the agreement in Theorem 7 is not trivial.

Appendix A

Proof of Lemma 1 (i) Consider an arbitrary $i \in N$ and $\omega \in \Omega$, and suppose that $\omega \models \psi$, for some formula $\psi \in \Psi_0^r$. It must be the case that either (i.a) $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\omega' \models \psi$, or (i.b) $\forall \omega' \in \Omega$, if $\omega R_i \omega'$ then $\omega' \models \neg\psi$, or (i.c) $\exists \omega', \omega'' \in \Omega$, such that $\omega R_i \omega'$ and $\omega R_i \omega''$, and $\omega' \models \psi$ and $\omega'' \models \neg\psi$ (i.e. neither (i.a) nor (i.b)). If (i.a) is the case, then $\omega \models \Box_i \psi$. If (i.b) is the case, then $\omega \models \hat{\Box}_i \psi$, and finally, if (i.c) is the case, then $\omega \models \dot{\Box}_i \psi$. Therefore, in all cases, the operator over ψ belongs to the set O_i, and since this holds for any $\psi \in \Psi_0^r$, it holds for each entry of a ken. Furthermore, $\models$ can only generate consistent lists of formulas, so kens cannot be inconsistent. This implies that a ken must exist that belongs to V_i.
(ii) Consider an arbitrary $i \in N$ and $\omega \in \Omega$. Let $\nu_i, \mu_i \in V_i$, and consider the n^{th} entry of each ken such that $\nu_i^n \psi_n \neq \mu_i^n \psi_n$. Case (ii.a): Suppose $\omega \models \nu_i^n \psi_n = \Box_i \psi_n$. So, $\forall \omega' \in \Omega$, if $\omega R_i \omega'$, then $\omega' \models \psi_n$. By definition, this rules out the possibility that also, $\omega \models \hat{\Box}_i \psi_n$, or $\omega \models \dot{\Box}_i \psi_n$. For cases (ii.b), $\omega \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$, and (ii.c), $\omega \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$, proceed analogously to (ii.a).

Proof of Lemma 2 For ease of notation, let $\inf\{\nu_i, \mu_i\} = \eta_i$.
(a) Suppose $\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n$. Then, if $\nu_i \succsim \eta_i$ and $\mu_i \succsim \eta_i$, it must be the case that $\eta_i^n \psi_n = \Box_i \psi_n$ or $\eta_i^n \psi_n = \dot{\Box}_i \psi_n$. However, if the latter, then η_i would not be maximal in the set $\{\eta_i \in V_i | \nu_i \succsim \eta_i \text{ and } \mu_i \succsim \eta_i\}$. Therefore, $\eta_i^n \psi_n = \Box_i \psi_n$. Conversely, suppose $\eta_i^n \psi_n = \Box_i \psi_n$. Furthermore, suppose, without loss of generality that $\mu_i^n \psi_n = \hat{\Box}_i \psi_n$ or $\mu_i^n \psi_n = \dot{\Box}_i \psi_n$. In the former case, η_i and μ_i would not be comparable, and in the latter case, η_i would be more informative than μ_i on that entry. Therefore, in either case, η_i would not belong to the set $\{\eta_i \in V_i | \nu_i \succsim \eta_i \text{ and } \mu_i \succsim \eta_i\}$. Therefore, $\nu_i^n \psi_n = \mu_i^n \psi_n = \Box_i \psi_n$. Proving cases (b), $\eta_i^n \psi_n = \hat{\Box}_i \psi_n$ iff ($\nu_i^n \psi_n = \mu_i^n \psi_n = \hat{\Box}_i \psi_n$) and (c), $\eta_i^n \psi_n = \dot{\Box}_i \psi_n$ iff ($\nu_i^n \psi_n \neq \mu_i^n \psi_n$ or $\nu_i^n \psi_n = \mu_i^n \psi_n = \dot{\Box}_i \psi_n$) can be done analogously to case (a).
Finally, suppose $\models \eta_i \leftrightarrow (p \wedge \neg p)$. Then, there exist n and n' such that $\eta_i^n \psi_n \leftrightarrow \neg \eta_i^{n'} \psi_{n'}$. But η_i^n is essentially generated by the conjunction of ν_i^n and μ_i^n. So, we have $(\nu_i^n \psi_n \wedge \mu_i^n \psi_n) \leftrightarrow \neg(\nu_i^{n'} \psi_{n'} \wedge \mu_i^{n'} \psi_{n'})$. But this implies that $\nu_i^n \psi_n \leftrightarrow \neg \nu_i^{n'} \psi_{n'}$ or $\mu_i^n \psi_n \leftrightarrow \neg \mu_i^{n'} \psi_{n'}$. That is, η_i is not in V_i if ν_i or μ_i are not in V_i. Therefore, $\eta_i \in V_i$.

Proof of Lemma 3 Suppose $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$. So, $\omega'' \in I_i(\omega')$ for some $\omega' \in \Omega_G(\omega)$. But, $\omega' R_i \omega''$, and there exists a sequence of R_i ($i \in G$) steps such that ω' is reachable from ω. Therefore, there exists a sequence, one step longer, such that ω'' is reachable from ω. So, $\omega'' \in \Omega_G(\omega)$. (And, note that $I_i(\omega'') \subseteq \Omega_G(\omega)$). Suppose $\omega'' \in \Omega_G(\omega)$. Reflexivity guarantees that $\omega'' \in I_i(\omega'')$. So, for some $\omega^* \in \Omega_G(\omega)$, $\omega'' \in I_i(\omega^*)$, so $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$.

Proof of Lemma 4 Suppose $\omega' \models \nu_i$ for some $\omega' \in I_i(\omega)$. Consider the n^{th}

entry of the ken, namely, $\nu_i^n\psi_n$.
(a) Suppose $\omega' \models \nu_i^n\psi_n = \Box_i\psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' R_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in I_i(\omega')$, $\omega'' \models \psi_n$. But since R_i is an equivalence relation, and $\omega' \in I_i(\omega)$, it follows that $I_i(\omega') = I_i(\omega)$. So, for all $\omega'' \in I_i(\omega)$, $\omega'' \models \psi_n$, from which it follows that for all $\omega'' \in I_i(\omega)$, $\omega'' \models \Box_i\psi_n$.
Case (b), $\omega' \models \nu_i^n\psi_n = \hat{\Box}_i\psi_n$ and (c), $\omega' \models \nu_i^n\psi_n = \dot{\Box}_i\psi_n$ are analogous to case (a).

Proof of Lemma 5 Suppose that for all $\omega' \in I_i(\omega)$, $\mathcal{M}, \omega' \models \nu_i$ and for all $\omega'' \in I_i(\omega')$, $\mathcal{M}, \omega'' \models \mu_i$. Consider the n^{th} entry of each of these kens, which are only defined for formulas in Ψ_0^0.
Case (a): Suppose that $\nu_i^n p_n = \mu_i^n p_n = \Box_i p_n$, then for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\omega''' \models p_n$, and therefore, for all $\omega''' \in I_i(\omega) \cup I_i(\omega')$, $\mathcal{M}(I_i(\omega), I_i(\omega')), \omega''' \models \inf\{\nu_i, \mu_i\}^n p_n = \Box_i p_n$.
Case (b), $\nu_i^n p_n = \mu_i^n \psi_n = \hat{\Box}_i p_n$, and (c) ($\nu_i^n p_n \neq \mu_i^n p_n$ or $\nu_i^n p_n = \mu_i^n p_n = \dot{\Box}_i p_n$) are treated analogously to case (a).

Proof of Lemma 6 By Lemma 1, for each $\omega' \in \Omega$, there is a ken that holds at ω'. That is, $\omega' \models \nu(\omega')_i$. By Lemma 4, we have that for all $\omega'' \in I_i(\omega')$, $\omega'' \models \nu(\omega')_i$. Now, consider the set of kens $\{\nu(\omega')_i | \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')\}$. By Lemma 5, it follows that for all $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')$, $\mathcal{M}(\{I_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu(\omega')_i | \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega')\}$. By Lemma 3, for all $\omega'' \in \Omega_G(\omega)$, $\mathcal{M}(\{I_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}$. So, in the model in which i's information cell is equal to $\Omega_G(\omega)$, leaving j's partition unchanged, $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. Reasoning similarly for agent j, in the model in which j's information cell is equal to $\Omega_G(\omega)$, leaving i's partition unchanged, $\inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. However, since $r = 0$, an agent i's ken only depends on i's accessibility relation (higher depth nested formulas are ignored). So, $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}$ and $\inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$ hold at every state $\omega' \in \Omega_G(\omega)$ of a model $\mathcal{M}^*$ in which all the set $I_i(\omega')$ are "merged" *and* all the sets $I_j(\omega')$ are merged. But $\cup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \cup_{\omega' \in \Omega_G(\omega)} I_j(\omega') = \Omega_G(\omega)$. That is, the agents have the same information cell in $\mathcal{M}^*$. Trivially, it follows that for all $\omega' \in \Omega_G(\omega)$, $\mathcal{M}^*, \omega' \models \inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$.

Proof of Theorem 1 Suppose $r = 0$, so $\Psi_0^0 = P^*$ and that $NDSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, and the system is $S5$. Arbitrarily choose $\omega \in \Omega$, and consider the set $\Omega_G(\omega)$. Consider the sets of kens $\nu(\omega)_i \in V_i$ and $\nu(\omega)_j \in V_i$, for all $\omega \in \Omega$. By Lemma 1 part (i), we have that $\omega \models \nu(\omega)_i \wedge \nu(\omega)_j$. Using the action function we have that $\omega \models d_i^{D_i(\nu(\omega)_i)} \wedge d_j^{D_j(\nu(\omega)_j)}$. Now, suppose that $\omega \models C_G(d_i^x \wedge d_j^y)$. By definition, $\forall \omega'' \in \Omega_G(\omega)$, $\omega'' \models d_i^x \wedge d_j^y$. Therefore, notably, since $\omega \in \Omega_G(\omega)$, we have that

$\omega \models D_i(\nu(\omega)_i) = x \wedge D_j(\nu(\omega)_j) = y$. It remains to show that $\omega \models (x = y)$.
Since the decision functions are the same across states, we can apply $NDSTP$ to obtain that for all $\omega'' \in \Omega_G(\omega)$, $\omega'' \models D_i(\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}) = x$.
Since $\omega \in \Omega_G(\omega)$, we have that $\omega \models D_i(\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\}) = x$. By a similar argument, one can show that $\omega \models D_j(\inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}) = y$.
By Lemma 6, $\inf\{\nu(\omega')_i | \omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j | \omega' \in \Omega_G(\omega)\}$. So, by like-mindedness, it follows that $\omega \models (x = y)$.

Proof of Theorem 2 Repeat the proof of Theorem 1, replacing Assumption $NDSTP$ with $DSTP$ and the assumption that the language is rich in every component.

Proof of Proposition 1 Consider an arbitrary $\omega' \in \Omega_G(\omega)$, and suppose $\omega' \models \Box_i \psi_n$. Then $\omega' \models \psi_n$ by reflexivity of R_i. But by reflexivity of R_j at ω', this implies that $\omega' \models \Box_j \psi_n \wedge \dot{\Box}_j \psi_n$. But if the language is very rich, the only possible case is $\omega' \models \Box_j \psi_n$.

Proof of Proposition 2 Let "i-arrow" refer to an arrow of i's accessibility relation. Firstly, we can show that $R_i = E_i \cup F_i$. An arbitrary $\omega \in \Omega$ either has an i-arrow pointing to it or it does not. If it does not, by seriality, it points to another state. If it does, then there exists a state ω' that points to ω which itself points to some state ω'' by seriality. Transitivity implies that ω' points to ω'' and Euclideaness implies that ω'' points to ω. From here it is easy to prove that ω, ω' and ω'' are in an equivalence class.
Secondly, we show that if $J_i(\omega') \neq J_i(\omega'')$ then $J_i(\omega') \cap J_i(\omega'') = \emptyset$. Suppose $\omega \in J_i(\omega') \cap J_i(\omega'')$. If $\omega \in S_i(\omega') \cap S_i(\omega'')$ then $S_i(\omega')$ and $S_i(\omega'')$ are indistinguishable, and one can verify that $J_i(\omega') = J_i(\omega'')$. If $\omega \in S_i(\omega') \cap A_i(\omega'')$ then ω both does have and does not have an i-arrow pointing to it. Finally, if $\omega \in A_i(\omega') \cap A_i(\omega'')$ then by Euclideaness, ω' and ω'' are indistinguishable, and $J_i(\omega') = J_i(\omega'')$.
Thirdly, we can show that $\cup_{\omega \in \Omega} J_i(\omega) = \Omega$. Suppose $\omega' \in \cup_{\omega \in \Omega} J_i(\omega)$, then by the definitions of S_i and A_i, $\omega' \in \Omega$. On the other hand, suppose $\omega \in \Omega$. Then if there is an i-arrow pointing to ω, $\omega \in S_i(\omega) \subseteq J_i(\omega)$. If there is no i-arrow pointing to it, then by seriality, there is an ω' that ω points to, so $\omega \in A_i(\omega') \subseteq J_i(\omega')$. So, $\omega \in \cup_{\omega \in \Omega} J_i(\omega)$.

Proof of Lemma 7 Suppose $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} S_i(\omega')$. So, $\omega'' \in S_i(\omega')$ for some $\omega' \in \Omega_G(\omega)$. But, $\omega' E_i \omega''$, and there exists a sequence of R_i ($i \in G$) steps such that ω' is reachable from ω. Therefore, there exists a sequence, one step longer, such that ω'' is reachable from ω. So, $\omega'' \in \Omega_G(\omega)$.

Suppose $\omega'' \in \Omega_G(\omega)$. Either ω'' has an i-arrow pointing towards it, in which case $\omega'' \in S_i(\omega'')$. So, $\omega'' \in S_i(\omega'') \cup A_i(\omega'') = J_i(\omega'')$, or, ω'' has no i-arrow pointing towards it, in which case, by seriality, there exists some ω''' such that $\omega'' \in A_i(\omega''')$. Note that ω''' must be in $\Omega_G(\omega)$ since it is reachable from ω''. So, $\omega'' \in S_i(\omega''') \cup A_i(\omega''') = J_i(\omega''')$. In either case, for some $\omega^* \in \Omega_G(\omega)$, $\omega'' \in J_i(\omega^*)$, so $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$.

Proof of Lemma 8 Suppose $\omega' \models \nu_i$ for some $\omega' \in J_i(\omega)$. Firstly, suppose $\omega' \in S_i(\omega)$, and consider the n^{th} entry of the ken, namely, $\nu_i^n \psi_n$.
(a) Suppose $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' E_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in S_i(\omega')$, $\omega'' \models \psi_n$. But since E_i is an equivalence relation, and $\omega' \in S_i(\omega)$, it follows that $S_i(\omega') = S_i(\omega)$. So, for all $\omega'' \in S_i(\omega)$, $\omega'' \models \psi_n$, from which it follows that for all $\omega'' \in S_i(\omega)$, $\omega'' \models \Box_i \psi_n$. Also, each $\omega''' \in A_i(\omega)$ has an arrow pointing to each state in $S_i(\omega)$, so for all $\omega^* \in S_i(\omega)$, if $\omega''' F_i \omega^*$, $\omega^* \models \psi_n$. So, for all $\omega''' \in A_i(\omega)$, $\omega''' \models \Box_i \psi_n$. It follows that for all $\omega'' \in J_i(\omega)$, $\omega'' \models \Box_i \psi_n$. Case (b), $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (c), $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ are analogous to case (a).
Now, suppose $\omega' \in A_i(\omega)$, and consider the n^{th} entry of the ken, namely, $\nu_i^n \psi_n$.
(d) Suppose $\omega' \models \nu_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in \Omega$, $\omega' F_i \omega''$ implies $\omega'' \models \psi_n$. So, for all $\omega'' \in S_i(\omega')$, $\omega'' \models \psi_n$. This implies that $\omega'' \models \Box_i \psi_n$ for all $\omega'' \in S_i(\omega)$, and $\omega''' \models \Box_i \psi_n$ for all other states $\omega''' \in A_i(\omega)$. It follows that for all $\omega'' \in J_i(\omega)$, $\omega'' \models \Box_i \psi_n$.
Case (e), $\omega' \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (f), $\omega' \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ are analogous to case (d).

Proof of Lemma 9 Suppose that for all $\omega \in J_i(\omega)$, $\mathcal{M}, \omega \models \nu_i$ and for all $\omega' \in J_i(\omega')$, $\mathcal{M}, \omega' \models \mu_i$. Consider the n^{th} entry of each of these kens, defined only for formulas in Ψ_0^0.
Case (a): Suppose that $\nu_i^n p_n = \mu_i^n p_n = \Box_i p_n$, then for all $\omega'' \in S_i(\omega) \cup S_i(\omega')$, $\omega'' \models p_n$, and therefore, following the proof of Lemma 8, for all $\omega'' \in J_i(\omega) \cup J_i(\omega')$, $\mathcal{M}(J_i(\omega), J_i(\omega')), \omega'' \models \inf\{\nu_i, \mu_i\}^n p_n = \Box_i p_n$.
Case (b), $\nu_i^n p_n = \mu_i^n \psi_n = \hat{\Box}_i p_n$, and (c) ($\nu_i^n p_n \neq \mu_i^n p_n$ or $\nu_i^n p_n = \mu_i^n p_n = \dot{\Box}_i p_n$) are treated analogously to case (a).

Proof of Lemma 10 By Lemma 1, for each $\omega' \in \Omega$, there is a ken that holds at ω'. That is, $\omega' \models \nu(\omega')_i$. By Lemma 8, we have that for all $\omega'' \in J_i(\omega')$, $\omega'' \models \nu(\omega')_i$. Now, consider the set of kens $\{\nu(\omega')_i | \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$. By Lemma 9, it follows that for all $\omega'' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$, $\mathcal{M}(\{J_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu(\omega')_i | \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$.
By Lemma 7, since $\Omega_G(\omega) \subseteq \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')$, it follows that for all $\omega'' \in \Omega_G(\omega)$, we have that $\mathcal{M}(\{J_i(\omega') | \omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu(\omega')_i | \omega' \in \bigcup_{\omega' \in \Omega_G(\omega)} J_i(\omega')\}$. Fur-

thermore, the kens that hold in states $(\bigcup_{\omega'\in\Omega_G(\omega)} J_i(\omega'))\backslash\Omega_G(\omega)$ must be identical to the ones that hold at the states in $\Omega_G(\omega)$, because all the states in the former set must be associated states, and thus the information that holds at them must be the same as the information that holds true in their respective information sinks, which are contained in $\Omega_G(\omega)$. Therefore, $\inf\{\nu(\omega')_i|\omega' \in \bigcup_{\omega'\in\Omega_G(\omega)} J_i(\omega')\} = \inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\}$. It follows therefore, that for all $\omega'' \in \Omega_G(\omega)$, we have that $\mathcal{M}(\{J_i(\omega')|\omega' \in \Omega_G(\omega)\}), \omega'' \models \inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\}$. So, in the model in which i's information sink plus associated states is equal to $\Omega_G(\omega)$, leaving j's accessibility relation unchanged, $\inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$. Reasoning similarly for agent j, in the model in which j's information sink plus associated states is equal to $\Omega_G(\omega)$, leaving i's accessibility relation unchanged, $\inf\{\nu(\omega')_j|\omega' \in \Omega_G(\omega)\}$ holds at every state in $\Omega_G(\omega)$.[20]
Now, since $r = 0$, an agent i's ken only depends on i's accessibility relation (higher depth nested formulas are ignored). So, $\inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\}$ and $\inf\{\nu(\omega')_j|\omega' \in \Omega_G(\omega)\}$ hold at every state $\omega' \in \Omega_G(\omega)$ of a model $\mathcal{M}^*$ in which all the set $S_i(\omega')$ are "merged" *and* all the sets $S_j(\omega')$ are merged. Now, by heterogeneity, it follows that for all $\omega' \in \Omega_G(\omega)$, $\mathcal{M}^*, \omega' \models \inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j|\omega' \in \Omega_G(\omega)\}$.[21]

Proof of Theorem 4 Suppose we restrict ourselves to Ψ_0^0 and that $NDSTP$ holds, the agents are like-minded, the decision functions are invariant across all states, heterogeneity holds, and the system is $KD45$. Arbitrarily choose $\omega \in \Omega$, and consider the set $\Omega_G(\omega)$. Consider the sets of kens $\nu(\omega)_i \in V_i$ and $\nu(\omega)_j \in V_i$, for all $\omega \in \Omega$. By Lemma 1 part (i), we have that $\omega \models \nu(\omega)_i \wedge \nu(\omega)_j$. And, by the action function, we have that $\omega \models d_i^{D_i(\nu(\omega)_i)} \wedge d_j^{D_j(\nu(\omega)_j)}$. Now, suppose that $\omega \models C_G(d_i^x \wedge d_j^y)$. By definition, $\forall\omega'' \in \Omega_G(\omega)$, $\omega'' \models d_i^x \wedge d_j^y$. Therefore, $\forall\omega'' \in \Omega_G(\omega)$, $\omega'' \models D_i(\nu(\omega'')_i) = x \wedge D_j(\nu(\omega'')_j) = y$. By Lemma 8, the kens are uniform across the sets J, and furthermore, the decision functions are invariant across state, so even if $\omega \notin \Omega_G(\omega)$, we must have $\omega \models D_i(\nu(\omega)_i) = x \wedge D_j(\nu(\omega)_j) = y$. It remains to show that $\omega \models (x = y)$.
Since the decision functions are the same across states, we can apply Assumption $NDSTP$ to obtain that for all $\omega'' \in \Omega_G(\omega)$, $\omega'' \models D_i(\inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\}) = x$. By a similar argument, one can show that $\omega \models D_j(\inf\{\nu(\omega')_j|\omega' \in \Omega_G(\omega)\}) = y$. By Lemma 10, $\inf\{\nu(\omega')_i|\omega' \in \Omega_G(\omega)\} \sim \inf\{\nu(\omega')_j|\omega' \in \Omega_G(\omega)\}$. So, by like-mindedness, it follows that $\omega \models (x = y)$.

[20] Note that the set $\Omega_G(\omega)$ does not change as a result of the sink merge operation: No state in $\Omega_G(\omega)$ becomes connected to a state outside the set, and states within the set can only gain connections, never lose any.

[21] We require heterogeneity since there is no guarantee that $\cup_{\omega'\in\Omega_G(\omega)} S_i(\omega') = \cup_{\omega'\in\Omega_G(\omega)} S_j(\omega')$, and an agent i's ken essentially depends only on the sets S_i.

Proof of Theorem 5 Repeat proof of Theorem 4, replacing $NDSTP$ with $DSTP$ and the assumption that the language is rich in every component.

Proof of Proposition 3 Suppose that for every $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_i$. Consider the n^{th} entry of the kens.
Case (a): Suppose that $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_i^n \psi_n = \Box_i \psi_n$. Then, for all $\omega'' \in I_i(\omega')$, $\omega'' \models \psi_n$. But, by Lemma 3, since $\bigcup_{\omega' \in \Omega_G(\omega)} I_i(\omega') = \Omega_G(\omega)$, it follows that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \psi_n$.
Furthermore, by Lemma 3, $\bigcup_{\omega' \in \Omega_G(\omega)} I_j(\omega') = \Omega_G(\omega)$. Therefore, no matter what information cell j might be in, ψ_n will be true at each state in that information cell. Therefore $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \Box_j \psi_n$. That is, the n^{th} entry of the kens carry the same information.
Case (b): $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_i^n \psi_n = \hat{\Box}_i \psi_n$ is treated analogously to case (a).
Case (c): Suppose that $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_i^n \psi_n = \dot{\Box}_i \psi_n$. Then, there exists ω'' and ω''' with $\omega' R_i \omega''$ and $\omega' R_i \omega'''$, such that $\omega'' \models \psi_n$ and $\omega''' \models \neg \psi_n$. It follows that there exists $\omega'', \omega''' \in \Omega_G(\omega)$ such that $\omega'' \models \psi_n$ and $\omega''' \models \neg \psi_n$. Now, suppose that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_j^n \psi_n = \Box_j \psi_n$ or that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_j^n \psi_n = \hat{\Box}_j \psi_n$. If for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \Box_j \psi_n$, then (as above) for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \psi_n$, which contradicts the fact that $\omega''' \models \neg \psi_n$. Similarly, if for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \hat{\Box}_j \psi_n$, then (as above) for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \neg \psi_n$, which contradicts the fact that $\omega'' \models \psi_n$. Therefore, $\forall \omega' \in \Omega_G(\omega)$, $\omega' \models \dot{\Box}_j \psi_n$.
Since the above cases exhaust every possibility of an entry in a ken, and since the entry was chosen arbitrarily, it follows that for all $\omega' \in \Omega_G(\omega)$, $\omega' \models \nu(\omega)_i \sim \nu(\omega)_j$.

Proof of Proposition 4 The implications among the conditions expressed semantically are simple.
Now, we can show that for any $\omega \in \Omega$ such that $S_i(\omega) \subseteq S_j(\omega)$, if $\omega \models \nu_i \wedge \nu_j$ then $\omega \models \nu_i \succsim \nu_j$; which would establish the semantic to syntactic implications for conditions 1 and 2. Consider some arbitrary state $\omega \in \Omega$. Suppose $S_i(\omega) \subseteq S_j(\omega)$ and $\omega \models \nu_i \wedge \nu_j$. Consider the nth entry of these kens. (a) Suppose $\omega \models \nu_i^n \psi_n = \Box_i \psi_n$, and suppose that $\omega \models \nu_j^n \psi_n = \hat{\Box}_j \psi_n$. Then, $\forall \omega' \in S_j(\omega)$, $\omega' \models \neg \psi_n$. But if $S_i(\omega) \subseteq S_j(\omega)$, then $\forall \omega' \in S_i(\omega)$, $\omega' \models \neg \psi_n$, which contradicts the statement that $\omega \models \Box_i \psi_n$. Therefore, $\omega \models (\nu_j^n \psi_n = \Box_j \psi_n \vee \nu_j^n \psi_n = \dot{\Box}_j \psi_n)$. Cases (b), $\omega \models \nu_i^n \psi_n = \hat{\Box}_i \psi_n$ and (c) $\omega \models \nu_i^n \psi_n = \dot{\Box}_i \psi_n$ can dealt with analogously to case (a).
Now we can show that (3.a) implies (3.b): By the definition of (3.a), there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. So, at each one of those states, $\Box_i \psi_n \to \psi_n$ for all formulas and all agents.
We can show that (4.a) implies (4.b): Suppose that (4.a) holds, but not (4.b). So let $\omega' \models \nu(\omega)_i \wedge \mu(\omega)_j$ for all $\omega' \in \Omega_G(\omega)$, and yet, $\nu(\omega)_i$ is different from

$\mu(\omega)_j$. Case (a): At $\omega' \in S_i(\omega')$, for some ψ_n, we have $\omega' \models \Box_i\psi_n \wedge \hat{\Box}_j\psi_n$. But if $S_i(\omega') \subseteq S_j(\omega'')$ for some ω'', then $\omega'' \models \psi_n$, in which case we cannot have $\hat{\Box}_j\psi_n$ at every state in the component. Case (b): At $\omega' \in S_j(\omega')$, for some ψ_n, we have $\omega' \models \Box_i\psi_n \wedge \dot{\Box}_j\psi_n$. But if $S_j(\omega') \subseteq S_i(\omega'')$ for some ω'', then $\omega''' \models \neg\psi_n$ for some $\omega''' \in S_i(\omega'')$, in which case we cannot have $\Box_i\psi_n$ at every state in the component. All other cases are trivial, or resemble one of the above.
We can show that (5.a) implies (5.b): Suppose $\omega' \in S_i(\omega') \cap S_j(\omega')$. Suppose that for some ψ_n, $\omega \models \Box_i\psi_n$. By reflexivity of R_i, $\omega \models \psi_n$. Now, suppose $\omega \models \hat{\Box}_j\psi_n$. By reflexivity of R_j at ω', $\omega' \models \neg\psi_n$, a contradiction.
Finally, that (1.b) implies (2.b) implies (4.b) is trivial. Also, that (1.b) implies (3.b) is trivial.
We can show that (3.b) implies (4.b): Suppose (3.b) holds and that (4.b) does not hold. (3.b) implies that there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. Suppose $\omega' \models \nu(\omega)_i \wedge \mu(\omega)_j$ for all $\omega' \in \Omega_G(\omega)$, and yet, $\nu(\omega)_i$ is different from $\mu(\omega)_j$. Let ω'' be reachable from ω'. Case (a): suppose that at ω'', for some ψ_n, we have $\omega'' \models \Box_i\psi_n \wedge \hat{\Box}_j\psi_n$. By reflexivity of both R_i and R_j, $\omega'' \models \psi_n \wedge \neg\psi_n$, a contradiction. Case (b): $\omega'' \models \Box_i\psi_n \wedge \dot{\Box}_j\psi_n$. Then, for some reachable ω''', $\omega''' \models \neg\psi_n$. Since R_i is reflexive at ω''', it cannot be the case that $\omega''' \models \Box_i\psi_n$, thus contradicting the assumption that i's ken is the same across each state in the component.
Finally, we can show that (3.b) implies (5.b): (3.b), implies that there is a state $\omega' \in \Omega_G(\omega)$ such that every state reachable from ω' is reflexive in both R_i and R_j. Let ω'' be reachable from ω'. Suppose that at ω'', for some ψ_n, we have $\omega'' \models \Box_i\psi_n \wedge \hat{\Box}_j\psi_n$. By reflexivity of both R_i and R_j, $\omega'' \models \psi_n \wedge \neg\psi_n$, a contradiction.

Proof of Proposition 5 Suppose that there exists $\omega' \in \Omega_G(\omega)$ such that for all $n \in \{1, ..., m\}$, $\omega' \models \neg(\Box_i\psi_n \wedge \hat{\Box}_j\psi_n)$. So, suppose $\omega \models \Box_i\psi_n$. This implies that $\omega' \models \dot{\Box}_j\psi_n \vee \Box_j\psi_n$. But since the language is very rich, we are only left with $\omega' \models \Box_j\psi_n$. This is true for all propositions. Therefore, if (5.b) holds, then there is a state in which the agents have the same ken. That is, condition (1.b) holds.

Proof of Theorem 7 Suppose that there is some $\omega \in \Omega$ such that $\omega \models C_G(d_i^x \wedge d_j^y) \wedge (x \neq y)$. Then, for all $\omega' \in \Omega_G(\omega)$, $\omega' \models D_i(\nu(\omega')_i) = x \neq y = D_j(\mu(\omega')_j)$. By like-mindedness, it follows that $\omega' \models \neg(\nu(\omega')_i \sim \mu(\omega')_j)$. But this contradicts (1.b). Namely, that there exists a state $\omega'' \in \Omega_G(\omega)$ such that $\omega'' \models \nu(\omega'')_i \sim \mu(\omega'')_j$.

Appendix B

Map to Bacharach (1985)

Note that by Lemma 4, if $D_i(\nu(\omega)_i)$ for some $\omega \in \Omega$, then for all $\omega' \in I_i(\omega)$, $D_i(\nu(\omega')_i) = D_i(\nu(\omega)_i)$. So, for each agent $i \in N$, we can define a function $H_i : 2^\Omega \to \mathcal{A}$, where $H_i(I_i(\omega)) = D_i(\nu(\omega)_i)$. Furthermore, we can define another function $h_i : \Omega \to \mathcal{A}$ such that for all $\omega' \in I_i(\omega)$, $h_i(\omega') = H_i(I_i(\omega))$. This thus defines a map from our decision functions into Bacharach's framework.
Finally, we can also define Bacharach's Sure-Thing Principle: If $H_i(I_i(\omega)) = H_i(I_i(\omega'))$ and clearly $I_i(\omega) \cap I_i(\omega') = \emptyset$, then $H_i(m(I_i(\omega), I_i(\omega'))) = H_i(I_i(\omega))$. Now, $m(I_i(\omega), I_i(\omega'))$, in Bacharach (1985) would simply be equal to $I_i(\omega) \cup I_i(\omega')$.

Map to Aumann & Hart (2006)

Aumann and Hart (2006) derive their agreement theorem using the framework developed in Aumann (1999) for the analysis of interactive knowledge in a partitional state space. Essentially, restricting ourselves to Ψ_0^0, we can define a mapping $\nu_i \mapsto e \in P^*$, where,

$$e := \bigwedge_{x \in \{p_n | \nu_i^n p_n = \Box_i p_n\}} x \wedge \bigwedge_{y \in \{\neg p_n | \nu_i^n p_n = \hat{\Box}_i p_n\}} y$$

That is, e is the conjunction of all the propositions (or their negation) that i knows, and all the propositions p_n for which $\nu_i^n p_n = \dot{\Box}_i p_n$ are ignored.
Given this, if we have $\nu(\omega)_i$, then e is the "minimal" formula that i knows at ω, in the sense that if $\Box_i e'$ then $e \to e'$. Note furthermore, that given our "richness" assumption on Ψ_0^0, we have that if $e \neq e'$ then $\neg(e \wedge e')$, and if $\nu_i \mapsto e'$ and $\mu_i \mapsto e''$, then $\inf\{\nu_i, \mu_i\} \mapsto (e' \vee e'')$. Given this map, our decision functions $D_i : V_i \to \mathcal{A}$ become $H_i : P^* \to \mathcal{A}$.
The Disjoint Sure-Thing Principle now becomes,

$$\models \bigwedge_{i \in N} \bigwedge_{e, e' \in P^*} [H_i(e) = H_i(e') \wedge \neg(e \wedge e') \to H_i(e \vee e') = H_i(e)]$$

which is the formulation given in Aumann and Hart (2006).

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