The transfer of statistical equilibrium from physics to economics

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ABSTRACT

Two applications of the concept of statistical equilibrium, taken from statistical mechanics, are compared: a simple model of a pure exchange economy, constructed as an alternative to a walrasian exchange equilibrium, and a simple model of an industry, in which statistical equilibrium is used as a complement to the classical long period equilibrium. The postulate of equal probability of all possible microstates is critically re-examined. Equal probabilities are deduced as a steady state of linear and non-linear Markov chains.

Introduction

The concept of statistical equilibrium is a fundamental analytical tool in physics and particularly in statistical mechanics. After having borrowed the classical mechanics concept of equilibrium, economic theory has occasionally turned its attention to the other concept of probabilistic equilibrium. In fact, since the contributions which appeared in the 50s and the early 60s it is only recently that serious attempts have been made to revise and develop the notion of statistical equilibrium in economics. Past contributions include Champernown (1953), Simon-Bonini (1958), Newman-Wolf (1961) and Steindl (1962) and were mainly related to Gibrat's Law (1931) and to Pareto distribution. Recent works, explicitly linked to thermodynamics, are E. Farjoun - M. Machover (1983) and, in particular, Foley (1991, 1994).

In economics a statistical equilibrium is a most probable distribution of certain economic entities (say firms or individuals) which cannot all be distinguished one from another, rather than a particular configuration in which each entity is identified. In other words, this equilibrium is a macrostate with maximum number of realizations (microstates) and, as such, is a distinct concept from a state obtained by the simple inclusion of some random variable in the relations which determine a classical equilibrium.[1] In this work two applications of the concept of statistical equilibrium to economic theory will be formulated and compared using simple models. Furthermore it will be shown that, under sufficient conditions, a state of equal probability of microstates - a basic postulate in statistical mechanics - in the long period is consistent with unequal transition probabilities.

In section 1 the first application is a model of a pure exchange economy, constructed as a special case of Foley's (1994) model in which statistical equilibrium appears as an alternative to the Walrasian equilibrium. In section 2 the second application is a model of an industry in which statistical equilibrium is used as a complement to the classical long period equilibrium. It will be argued that only the latter application maintains the notion of statistical equilibrium adopted in the field of physics; whereas the former differs from it on an essential point and resolves itself into a concept of equilibrium similar to that of temporary equilibrium adopted in economics. In section 3 the postulate of equal probability is re-examined and a simple case of linear Markov chains is presented, in which equal probability is a steady state of a stochastic process. In section 4 this uniform probability outcome is generalized to non-linear Markov chains, applying a theorem proved by Fujimoto and Krause (1985).

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[1] See Parrinello (1990).

1. *Statistical equilibrium in a pure exchange economy*

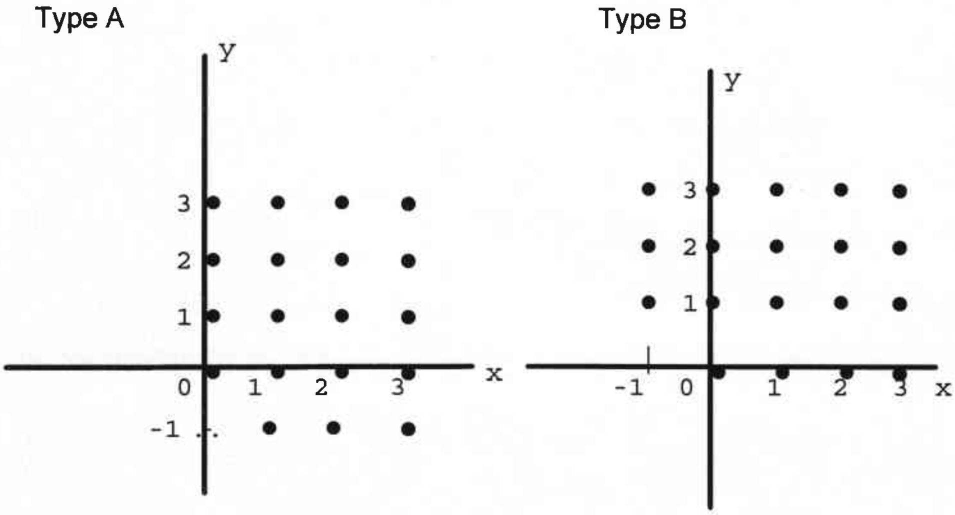
Let us make a simple example of statistical equilibrium for an exchange economy, as a special case of the statistical theory of markets developed by Foley (1991-1994). In this theory the elementary unit of analysis is the individual offer set :

"The market begins with agents defined by offer sets reflecting their information, technical possibilities, endowments and preferences" (p.324).
"In terms of standard production-exchange model,...., offer sets consist of technically feasible transactions leading to final consumption bundles that are preferred to initial endowments" (Foley p.324)

Suppose that there are only 4 agents a_1, a_2, b_1, b_2 and two goods X,Y the quantities of which are measured by integers. There is a total of 4 units of good X and 4 of good Y which are equally distributed at the beginning: each individual therefore has an endowment of one unit of each good. We will attribute to the agents very simple preferences: agents a_1, a_2 like good X, but are indifferent to good Y; whilst agents b_1, b_2 like good Y, but are indifferent to good X. An agent's transaction is a vector of quantities of the two goods with a plus sign to indicate a net acquisition, a minus sign for a net cession and zero if the initial endowment is maintained.

The offer set of an agent is the set of transactions which are weakly preferable to and feasible for him in relation to his initial endowment. In Foley's model the agents that have the same offer set are considered indistinguishable and represent a type of agent.

In the figures below the lattices represent parts of the offer sets of agents of type A and B as feasible transaction sets. The null transaction (0,0) is included among the possibilities.



In the example we can therefore find two types of agents: type A (to which a_1,a_2 belong) and type B (to which b_1,b_2 belong). These types can be identified by their offer sets which are distinct as far as their preferences are concerned, but not for their endowments.
Table I describes the feasible microstates of the exchange economy.

TABLE. I

(0 0)	(1 -1)	(-1 1)
a_1, a_2, b_1, b_2	--	
a_1, b_1	a_2	b_2
a_1, b_2	a_2	b_1
$a_2 b_1$	a_1	b_2
$a_2 b_2$	a_1	b_1
--	a_1, a_2	b_1, b_2

It is to be noted that no exchange takes place in the first microstate and each agent in the last one acquires one unit of his preferred good against one unit of his indifferent good. In the other four microstates two agents make one preferred transaction, whilst the other two remain in their initial position.

Let us consider the feasible statistical aggregates or <u>macrostates</u> of the exchange economy by treating agents of the same type as indistinguishable and grouping all microstates with the same distribution of types of agents. In the example we find three macrostates, two of which are made of only one microstate (the first and the last one represented in table I) and one made by four microstates (the others).

Let us assign equal probabilities to all feasible microstates.

A statistical equilibrium is a macrostate with maximum probability, that is with the maximum number of feasible equally probable microstates. In the example this macrostate is the one with four microstates, in which, for each type of agent, one of the two benefits from the exchange by acquiring a unit of his preferred good and by giving up a unit of the indifferent good, whilst the other agent remains at the status quo.

A market statistical equilibrium in the model developed by Foley possesses the following interesting features that contrast with those shared by a walrasian general equilibrium:

1.In general it is not Pareto-efficient;

2.It does not imply a uniform exchange ratio between each pair of commodities over all

transactions;

3.A uniform entropy price is associated to each good: this price is a shadow price determined by solving an entropy maximizing problem under the total endowment constraints.

Property 1 is straightforward in our exchange model, as the most probable macrostate is Pareto-inferior compared to that in which all the agents obtain a unit of the preferred good in exchange for the other good. Instead properties 2 and 3 are not evident in this simple model and we shall not be concerned with them for the sake of the following argument.

It should be emphasized that the statistical equilibrium of the exchange economy is determined by offer sets that depend on the initial endowments of each individual. In general the offer sets undergo endogenous change if the economy is conceived in real time. To make this point clear, it is sufficient to assume that the two goods are non perishable and that the economy is subject to two trials and two corresponding observations. Let us suppose that the following microstate is realized in the first trial:

(0 0)	(1 -1)	(-1 1)
a_1, b_1	a_2	b_2

Then at the second trial the individual endowment will differ from that at the beginning of the first trial. Therefore the types of agents and the number of each type will differ from the initial stage, even if we assume that the preferences do not change. Hence the macrostate which has been defined as statistical equilibrium at the first trial is no longer so at the second. At the second trial each agent will represent a distinct type:

a_1 with endowments (1,1) and offer set $\{(0\ 0)\ (1\ -1)\ (2\ -1)...\}$
a_2 with endowments (2,0) and offer set $\{(0\ 0)\}$
b_1 with endowments (1,1) and offer set $\{(0\ 0)\ (-1\ 1)\ (-1\ 2)\ ....\}$
b_2 with endowments (0,2) and offer set $\{(0\ 0)\}$;

At the second trial the agents a_2, b_2 prefer their respective initial endowments to the outcome of any feasible transaction; whilst the agents a_1, b_1 prefer any positive amount of the preferred good in exchange for the unit of the good they are indifferent towards. The feasible microstates after the second trial are described below

TABLE II

(0 0)	(1 -1)	(-1 1)
a_1, a_2, b_1, b_2	--	--
$a_2 b_2$	a_1	b_1

From the statistical point of view the sample space has changed. The two microstates in table II each have probability 1/2 at each trial. However, as the trials are repeated an indefinite number of times, the second microstate will be realized with probability 1 and when that happens the economy will have reached a Pareto-efficient configuration.[2] At that point the offer set of each agent will be represented by the null vector (0,0), that is by the absence of any further transaction.

One may well ask whether the statistical equilibrium of exchange, as defined above, preserves the concept of statistical equilibrium in physics. The answer is no. The latter has the relative persistence of its determinants in common with the classical equilibrium in economics; by contrast, the statistical exchange equilibrium, as illustrated in the example, does not possess this prerequisite and from this point of view it is similar to the concept of temporary equilibrium in economics. Furthermore, if the model of statistical equilibrium of the exchange economy is interpreted in real time, it becomes a model of statistical disequilibrium, with certain transition probabilities that imply an <u>absorbing</u> microstate. This state is a Pareto-efficient

[2]In a certain sense the agents a_1, a_2, b_1, b_2 could not be distinguished into type A and B right from the very first trial if "distinguishability" also requires "observability". In fact at the beginning all agents have the same initial endowments, whilst their preferences, the only feature which in this case identifies the types, are not observable characteristics.

equilibrium. The stochastic feature is inherent only in the adjustment (or relaxation) process but not in the final equilibrium of the exchange. In a more general model with many Pareto-efficient microstates, we would find a problem of indeterminacy similar to the one found in a Walrasian exchange model if we assume that transactions can occur at disequilibrium prices in the adjustment process towards equilibrium. In this case the convergence of the stochastic process towards a Walrasian equilibrium is possible, but in general this equilibrium state is not a walrasian equilibrium relatively to the initial endowments. As a consequence, the statistical exchange equilibrium is "statistical" only because it is reached by a succession of stochastic disequilibria when it is stable, but it is not statistical in so far as it coincides with a microstate which takes probability 1 at the limit.

2. *A model of statistical equilibrium of the industry*

Now we shift to a more positive application of statistical equilibrium. Let us suppose now that an industry is in a long period competitive equilibrium, under constant returns to scale at the firm level. Suppose that its product can take only integer numbers 1,2,3,... Let Q be the quantity produced and N the number of firms which can produce at the minimum cost per unit of output. With these hypotheses, if D is the demand for the product at the long period prices, the theory of classical equilibrium determines Q from the equation Q = D, but does not determine the size of each firm.

Let us consider now the feasible microstates of the industry which can be obtained by distributing in every possible way the N firms among the possible sizes measured by the quantities 0,1,2,3.... In order to illustrate this we will present an example similar to that used by others (A. F. Brown, 1967) to introduce the concept of statistical equilibrium with reference to the distribution of a given amount of energy among a given number of particles of a perfect gas. Let us assume $Q = 3$; $N = 4$ and call the four firms (a),(b),(c),(d). The 20 feasible microstates of the industry are described in the following table.

TABLE III

Firm size measured by amounts of output

	0	1	2	3
1	(a) (b) (c)			(d)
2	(a) (b) (d)			(c)
3	(a) (c) (d)			(b)
4	(b) (c) (d)			(a)
5	(a) (b)	(c)	(d)	
6	(a) (b)	(d)	(c)	
7	(a) (c)	(b)	(d)	
8	(a) (c)	(d)	(b)	
9	(a) (d)	(b)	(c)	
10	(a) (d)	(c)	(b)	
11	(b) (c)	(a)	(d)	
12	(b) (c)	(d)	(a)	
13	(b) (d)	(a)	(c)	
14	(b) (d)	(c)	(a)	
15	(c) (d)	(a)	(b)	
16	(c) (d)	(b)	(a)	
17	(a)	(b) (c) (d)		
18	(b)	(a) (c) (d)		
19	(c)	(a) (b) (d)		
20	(d)	(a) (b) (c)		

Let us adopt the term "macrostate" to indicate a statistical aggregate of microstates (a distribution), obtained by assuming that the firms are not distinguishable from each other. The firms are not distinguished either because we are not interested in their identification or because they cannot be distinguished. In our example three macrostates of the industry are feasible

- three inactive firms and a firm of size 3;

- two inactive firms, a size-1 firm and a size-2 firm;
- one inactive firm and three size-1 firms.

The first macrostate is generated by the first four microstates; the second by the next 12 and the third by the last four.

In the general case let us indicate

n = $(n_0, n_1, n_2, ..., n_Q)$ a feasible macrostate in which n_0 firms are size 0, n_1 firms are size 1,....,n_Q firms are size Q on the condition that

$$n_0 + n_1 + n_2 + ... n_Q = N \qquad (1)$$

Let w(**n**) be the number of feasible microstates with distribution **n** = $(n_0, n_1, n_2, ..., n_Q)$.

Combinatorial analysis gives

$$w(\mathbf{n}) = \frac{N!}{n_o! n_1! n_2! ... n_Q!} \qquad (2)$$

where by convention 0!=1.

A macrostate **n** = $(n_0, n_1, n_2, ..., n_Q)$ is feasible if it satisfies, besides the equality (1), the conservation condition of the total quantity Q

$$0n_0 + 1n_1 + 2n_2 + \cdots + Qn_Q = Q \qquad (3)$$

The total number of feasible microstates is

$$m = \sum_{\mathbf{n}} \mathrm{w}(\mathbf{n}),$$

with summation over all macrostates which satisfy (1) and(3).

We may have an idea of the order of change in w(**n**) in response to variations in **n,** as the number of firms is slightly larger than the number represented in the table, if we assume[3] N = 20 and Q = 20. In this case, we will have for the macrostate made up of 8 inactive firms, 6 size-1 firms, 4 size-2 firms and 2 size-3 firms:

$$w = \frac{20!}{8!6!4!2!} \cong 2x10^8.$$

By contrast for the macrostate in which all the firms are of a uniform size equal to 1 we find

$$w = \frac{20!}{20!} = 1.$$

It is clear how enormous the difference is between the multiplicity of microstates in the first case, which represents a decreasing distribution compared with the single

[3]This numerical example has been taken from Brown (1967. p. 123)

microstate with a uniform distribution. So far we have followed combinatorial analysis.

To move onto the concept of statistical equilibrium we have to assume a probability distribution. In statistical physics we find more than one assumption of probability on this point. In the so-called Maxwell-Boltzmann distribution equal probability is attributed to each microstate; different assumptions of probability can be found, however, at the basis of the Bose-Einstein and of the Fermi-Dirac distributions.[4] We will adopt the Maxwell-Boltzmann hypothesis of equiprobability initially as an a priori; then we will obtain this uniform probability from other assumptions.

In the example illustrated in table III, each microstate has probability 1/20; whilst the three macrostates have respectively probabilities 1/5, 3/5, 1/5. The statistical equilibrium of the industry is the second macrostate with probability 3/5. In this case the small number of microstates, used for the purpose of the exposition, does not yet enable us to attribute a useful theoretical role to this equilibrium macrostate. In fact statistical equilibrium needs a sufficiently large number N (a typical case is that of the particles of gas considered in statistical physics).

In general, in order to determine **n**, the following maximum problem has to be solved

$$\max_{n_0,n_1,\dots,n_Q} w(\mathbf{n}) = \frac{N!}{n_o!n_1!n_2!\dots n_Q!}$$

subject to $$n_0 + n_1 + n_2 + \dots . n_Q = N$$

$$0n_0 + 1n_1 + 2n_2 + \dots + Qn_Q = Q.$$

By adopting a similar demonstration to that given in statistical physics[5], the following solution, as shown in Appendix I, can be obtained by an approximation in the continuum and for N and Q large numbers.

$$\overline{n_s} = N \frac{e^{-\beta s}}{\sum_{s=0}^{Q} e^{-\beta s}}, \qquad s = 0,1, \dots, Q. \tag{4}$$

with $$\beta = \ln\left(1 + \frac{N}{Q}\right) > 0. \tag{5}$$

where ln is the natural logarithm.

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[4] For a comparison of Maxwell-Boltzmann's, Bose-Einstein's and Fermi-Dirac's so-called statistics, see W. Feller (1970).

[5] Similar demonstrations can be found in Fast (1970); Brown (1967), .Hollinger and Zenzen (1985).

Hence the most probable macrostate **n** *is a distribution of firms which decreases according to a geometric progression as the size increases; as*

$$\frac{n_0}{n_1} = \frac{n_1}{n_2} = \ldots\ldots\frac{n_{Q-1}}{n_Q} = e^{\beta}. \qquad (6)$$

From (5) we obtain

$$\frac{Q}{N} = \frac{1}{e^{\beta}-1}. \qquad (7)$$

Having reached this result, the initial assumption that Q, the quantity produced by the industry, is a quantity in non-statistical classical equilibrium, determined on the demand side, becomes important. Substituting Q = D in (7), we obtain:

$$\frac{D}{N} = \frac{1}{e^{\beta}-1} \qquad (8).$$

Equations (6) and (8) show that as the demand D increases, *ceteris paribus*, the coefficient β decreases and, therefore, the dispersion of firms among classes of ever increasing size grows. It is worth noting that the ratio D/N, demand per number of firms, plays a similar role to that played by temperature T in the corresponding physical problem determining the most probable distribution of particles or harmonic oscillators among a certain amount of energy.

Entropy

We can interpret the equilibrium of the industry in terms of entropy. Let p_i be the probability of microstate *i*; and let us measure the *improbability* of microstate *i* by the logarithm

$$\ln\frac{1}{p_i} = -\ln p_i.$$

We can then define entropy **S(n)** of the macrostate **n** the average improbability of the microstates of which it is composed, where the weights are the probabilities p_i:

$$\mathbf{S(n)} = -\sum_{i}^{w(\mathbf{n})} p_i \ln p_i$$

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If all microstates in the macrostate **n** have probability $p_i = \frac{1}{w(\mathbf{n})}$, the entropy of **n** is

$$S(\mathbf{n}) = \mathcal{L}n\ w(\mathbf{n})$$

and the entropy of a most probable macrostate $\overline{\mathbf{n}}$ is $S(\overline{\mathbf{n}}) = \mathcal{L}n\ w(\overline{\mathbf{n}})$. Therefore *a statistical equilibrium of the industry is a macrostate that has maximum entropy*; hence, under the assumption of equal probability, a macrostate with maximum number of possible realizations. As N increases, the ratio $\frac{w(\overline{\mathbf{n}})}{m}$ decreases, whilst $\frac{\mathcal{L}n\ w(\overline{\mathbf{n}})}{\mathcal{L}n\ m}$ tends to 1, where *m* is the total number of microstates. Then, for N large, the entropy of the industry in its most probable state can be written $S(w(\overline{\mathbf{n}})) \cong \mathcal{L}n m$. Since the improbability of a macrostate **n** is $\mathcal{L}n \frac{m}{w(\mathbf{n})} = \mathcal{L}n\ m - \mathcal{L}n\ w(\mathbf{n})$, we can also say that for a large N a statistical equilibrium belongs to a set of macrostates with almost zero improbability, in the sense that $w(\overline{\mathbf{n}})$ turns out incomparably greater than w(n) associated with any other macrostate **n** outside the equilibrium set.

This property means that a macroscopic regularity (equilibrium) exists in terms of firm distribution. Such regularity emerges in real time, if we suppose that the number of *potential* firms N and the quantity in demand D are stationary. A statistical equilibrium can therefore be considered like the image of a film, which is made up of the same *perceived* scene repeated on a large number of frames, interspersed every so often with pictures of other scenes: when the film is run at a sufficiently high speed, the viewer is hardly aware of these odd scenes at all, whilst he perceives the main scene. Leaving this metaphor to one side, it must be stressed that the notion of statistical equilibrium which has been formulated here, does not substitute the classical equilibrium of the industry, but it does presuppose it and stands as a complement to it. In fact the stationarity of Q is not a physical necessity (like energy conservation), but rather a property of classical equilibrium in which Q is determined by the effective demand at the long run competitive prices. It is to be noted that the stationarity of the most probable distribution of firms hides an incessant movement at a microeconomic level: if it were possible to observe the trajectory of each firm (a not so impossible task compared with the case of a trajectory of a particle in physics) over a sufficiently long period, a firm would be seen to move through the whole range of sizes and the industry would pass through all feasible microstates. This would be true in principle.

In economics, as in the physics of gas, the number of units involved has to be large for this concept to be of use for the analysis. Thus in the model of the industry the number of firms N has to be large *enough*. It must be noted, incidentally, that there are some difficulties in observing N, in so far as many potentially active firms are inactive in equilibrium. Also the quantity Q, which is measured by integers, had to be assumed to be large for the purpose of the solution given in appendix I. Clearly the problem of the numerosity of Q differs from the one concerning N, as it does not seem so harmful to assume a sufficient divisibility of the product.

3. *The choice of the sample space and the assumption of equal probability*

In all main formulations of the method of statistical equilibrium in physics (the Maxwell-Boltzmann distribution, the Bose-Einstein distribution and the Fermi-Dirac distribution), a set of feasible microstates (the sample space) is defined at a certain level of analysis and then equal probability is assigned to these microstates. This analytical level is chosen on the basis of the logic of the problem, of a separate theory or of an intuition, the usefulness of this choice being tested by its predictive capability. In applying the statistical equilibrium approach, two methodological pitfalls should be avoided. With respect to the phenomenon under investigation: a) the assumed sample space might lack persistency and b) the assumed microstates might not have equal probability. Let us examine now the applications of the statistical equilibrium approach to the exchange economy (section 1) and to the economy of the industry (section 2) at the light of the above criterion.

In the application to the exchange economy, the choice of the sample space and the hypothesis of equal probability must be assessed on the basis of some implicit assumption of "rational" individual behaviour.[6] In this case it is hard to explain why the probability of a Pareto-efficient microstate is and remains not greater than the

[6] The *principle of insufficient reason* has been called upon by Foley (1994) to justify the hypothesis of equal probabilities of the feasible microstates. Of course this principle is of little use for justifying the choice between feasible and unfeasible microstates.

probability of any inefficient microstate which were not Pareto-inferior to the initial state. For example, the microstate described in the first row of Table I does not represent any Pareto-improvement, but it has, nevertheless, been attributed the same probability as any of the other microstates (described in the other rows) that do in fact imply an improvement. We observe that the latter problem prevents the exchange statistical equilibrium from strengthening its theoretical role in the following case,' in which the difficulty arising from the non-persistence of the initial endowments does not arise. In the pure exchange economy let us assume the goods to be labour services, instead of durable goods, and the initial endowments to be made up only of persistent labour capacities of workers to provide those services. By this hypothesis, if we assign all the feasible microstates equal probability, it is possible to formulate a statistical equilibrium for the exchange of labour services in real time, instead of a temporary statistical equilibrium. In spite of this, there still remain the same objections to the hypothesis of equal probability: as if on each trial the agents described by the model would look for each other and accept with equal chance any transaction which does not entail an inferior position for them, compared to the absence of exchange. The rationality of these agents seem to be minimal. It would be more reasonable to attribute equal probability to those microstates which imply Pareto-efficient allocations of labour-services and lower probabilities to all the other microstates[7]. Unfortunately no general criterion seems to be available a priori for assigning non-uniform probabilities within exogeneously given offer sets.

Also in the application to the economy of the industry, illustrated in section 2, the appropriatness of the choice of the sample space and of the equal probability assumption can be questioned, albeit for different reasons. .

On the one hand, we observe that the choice of the sample space, made of all possible microstates of the industry, belongs to the general model of placing randomly a given number of balls (firms) in a given number of cells (firm sizes); then aggregation runs by treating the balls as indistinguishable, whereas the cells are kept as distinct entities. [8] This model might not be appropriate, if the distribution of many customers among many firms is an essential element in the enumeration of the microstates of a production system *with exchange.* Suppose for simplicity that in the model described by Table III there are three customers and each customer demands one unit of outptut, as if he would represent an economic "quantum". In this case,

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[7] Foley himself in his working paper (Foley, 1991) assumed as feasible only those microstates which imply Pareto-superior and efficient allocations.

[8] As Feller (1970) has warned us, meaningful statistical aggregates can be constructed as composed events by treating the cells, instead of the balls, as indistinguishable entities.

many microstates listed in table III must be re-interpreted as composed events: for instance row 1 would stiill describe a simple event with a single realization, in which firm (d) supplies one unit of output to each customer, whereas the other firms (a),(b),(c) are inactive; by contrast row 5 would describe a composed event with 3 realizations, as firm (c) can supply one unit of output to each of the three customers alternatively, whereas firm (d) supplies one unit to each of the two residual customers and firms (a), (b) remain inactive. From this perspective, many microstates in Table III should be conceived as macrostates that must be decomposed in further microstates by replacing the occupancy model of balls and cells with a model that counts all possible ways for assigning three quanta, initially distinguishable, to four particles, initially distinguishable as well. Only at this extended micro-level, the equal probability assumption should be applied and the definition of the macrostates should be chosen.

On the other hand, the equal probability assumption refers to absolute probabilities. It remains to be proved that a state of uniform absolute probability is a steady state outcome of a stochastic process and that this outcome is independent from the initial probability vector. In particular, in the industry model, gradual structural changes could be more probable than major alterations in the size of the firms during the same period of time. Thus, in the example described in Table III, it can be supposed that, if the initial microstate is the one described in line 1 (with firms a,b,c, inactive and firm d of size 3), it can be more probable that microstate 5 (with firms a and b still inactive, firm c at size 1 and firm d at size 2) will be realized in the following trial than microstate 2 (firms a,b,d inactive and firm c of size 3). However, under certain assumptions, these unequal conditional probabilities are compatible with equal absolute probability of all possible microstates. In particular it can be immediately proved[9], using the theory of Markov chains that, if the transition matrix is given and it is a doubly stochastic and primitive,[10] then all microstates take equal probabilities at the limit of a series of repeated trials and this uniform probability is independent of the initial microstate (or, more generally, of the initial probability vector). A special case of doubly stochastic transition matrix arises if we assume that reversibility exists in the probabilistic sense between each pair of microstates of the

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[9] See Feller (1970), chapter XV page 399; and Seneta (1973).

[10] Let p_{ij} be the transition probability from microstate i to microstate j in one trial and let $\mathbf{P} = \{p_{ij}\}$ the mxm transition matrix. $\mathbf{P}$ is called doubly stochastic if $\sum_i p_{ij} = 1$, $\sum_j p_{ij} = 1$, that is both the row sums and the column sums of $\mathbf{P}$ are unity. Primitivity of $\mathbf{P}$ implies that there exists some power matrix $\mathbf{P}^{(t)}$ of $\mathbf{P}$ whose elements are all strictly positive.

industry at each trial; that is the probability that the microstate i occurs, following the realization of the microstate j, is the same as the probability that j occurs, following the realization of *i*. This hypothesis is represented by a *mxm symmetrical* transition matrix.

In the next section it will be proved that equal probabilities can be deduced as a limit property under assumptions less restrictive than that of double stochasticity.

4. *Equal probability through non-linear Markov chains with lagged variables.*

Let us introduce time lags and write

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathbf{x}_{t-m}) \qquad \text{for } t = 0,1,2,....$$

The vector $\mathbf{x}_t \equiv (x_{t1}, x_{t2}, ..., x_{tn})'$ shows the absolute probability x_{ti} of microstate i in period t. A prime indicates transposition. With no fear of confusion, we also write $\mathbf{f} \equiv (f_1, f_2, ..., f_n)'$. Let $\mathbf{X}_t \equiv (\mathbf{x}_{t-m}, \mathbf{x}_{t-m+1}, ..., \mathbf{x}_t)'$. When the given function f is homogeneous of degree one in each vector variable and continuously differentiable, the above equation is now written as

$$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t, \qquad \text{(p)}$$

where $\mathbf{A}$ is *(nxm)* by *(nxm)* and

$$\mathbf{A} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ F^{(m)} & F^{(m-1)} & . & . & . & F^{(1)} \end{pmatrix}.$$

A typical element of $F_{ij}^{(k)}$ is $\dfrac{\delta f_i}{\partial x_{t-k,j}}$.

To apply the *Proposition* in Appendix II, the assumptions we now make are:

Ass 1. f_i is non decreasing in each variable.
Ass 2. f_i is homogeneous of degree one.
Ass 3. $F^{(m)}$ has at least one positive entry in each row as well as in each column.

$F^{(1)}$ has at least one positive diagonal entry.
Ass 4. $\mathbf{e} \equiv \mathbf{f}(\mathbf{e},\mathbf{e},...,\mathbf{e})$, where $\mathbf{e} \equiv (1/n,\ 1/n,.....,1/n)$.

Originally, $\mathbf{f}$ is defined on a subset of $\Re_+^{n\,\mathrm{x}\,m}$ because $\mathbf{x}_t$ is a vector of probability distribution. To satisfy Ass 1, $\mathbf{f}$ is first to be extended to the whole $\Re_+^{n\,\mathrm{x}\,m}$ in a natural way. Ass 3 is to assure the primitivity of the process, i.e. the matrix $\mathbf{A}$; while Ass 4 requires that if the equal probabilities have been observed in the consecutive m past periods, then that situation be continued as an equilibrium. It should be noted that this is more general than the assumption of double stochasticity in the linear case. More importantly, the equal distribution of the present period is not enough to ensure the equilibrium state to be repeated.

Now we can apply the *Proposition* in Appendix II, and can assert that starting from any $\mathbf{X}_0$ in $\Re_+^{n\,\mathrm{x}\,m}$, the process (p) , i.e. $\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t$, yields a series which converges to a <u>unique</u> X^*. By the special form of $\mathbf{A}$, we can deduce

$$X^* = (x^*, x^*, ..., x^*).$$

Finally by Ass 4, $\mathbf{x}^* = e$ if each $\mathbf{x}^*$ should be normalized so that it belongs to the unit simplex.

With time lags being introduced, a more natural interpretation of the model is now possible. That is, a society has a chain of memory, and accumulate the experience of shift from one microstate to another, and these piled up "experience" or "memory" affect the transition probabilities most plausibly in non-linear way. Besides, in the linear case, the speed of convergence is quick and at a geometric rate. Let us hope this nice property continues to hold also in the non-linear case, and establishes the equal probabilities in a blink as Nature may wish. Nature somehow seems to love "equality" or at least equal opportunities for all.

Strong ergodicity in the case of nonlinear positive mappings has been extended to the transformations on Banach spaces (see Fujimoto and Krause (1994)). The arguments above can hence be carried over to the spaces of an infinite dimension. It may serve also to give a lower-level foundation to the equal-share principle in thermodynamics and, in spirit, to that in quantum theory.

5. *Final considerations*

The two applications developed in sections 1 and 2 have enabled us to point out certain limitations on the transfer of the statistical equilibrium method from physics to economics. These limitations seem to hold beyond the specific cases examined above.

The basic difficulty for that transfer lies in the changeover from particles in physics to intelligent units with memory and learning skills. The method seems to be fully successful only in those theoretical areas of economics where the microstates cannot be ordered in terms of preferences or profitability and furthermore the determinants of statistical equilibrium are relatively persistent. These requirements have proved to be plausible in the application to the economics of the industry, but they appeared rather problematical in the application to a pure exchange economy. It should be noticed that, in the application to the economics of the industry, rationality is not absent, because it underlies the given demand D for the product, that can be interpreted as a classical equilibrium quantity determined by a wider model of the economy.

Although the arguments presented in these notes have shown some comparative advantage of the transfer of statistical equilibrium as a complement of classical long period equilibrium, against the transfer of the same concept as a substitute for a walrasian notion of equilibrium, some basic questions remain unanswered here for further applications of the notion of statistical equilibrium within the former approach.

First, are there really any important areas of indeterminacy still left by the classical equilibrium method apart from that of the constant returns industry examined here ? We believe this to be so, even under the assumption of free competition where there are no cases of indeterminacy due to strategic interactions among agents. One important case of indeterminacy can be found in classical theory of value and distribution if the labour force is supposed to be homogeneous as far as productive efficiency is concerned, but to have non-homogeneous tastes. Suppose an economic system with single product industries, constant returns to scale, free competition and a fixed interest rate. In this case the non substitution theorem holds: the choice of the cost-minimising technique and the long period prices of the commodities are uniquely determined, but the amount and the composition of employment in terms of individual tastes cannot be determined by the same economic criterion, even if some correspondences are supposed to exist among prices, incomes and effective demand. Hence the composition of social product remains indeterminate as well.

Secondly, in the case of the industry, it was assumed that the classical method of equilibrium first determines the total output at the long period prices and then the method of statistical equilibrium step in to fill the gap of indeterminacy left by the first stage of analysis. In the more general cases such a logical sequence in the application of two equilibrium methods might not work. This could happen in the industry example, if the demand D would be affected in turn by some characteristic value of the most probable structure of the industry itself, e.g. by the multiplier β which appears in the firm distribution function (4).

Thirdly, the concepts of Pareto-efficient and Pareto-superior states of the economy should be re-examined, if we adopt the method of statistical equilibrium. In particular the notion of expected utility seems more suitable compared to the deterministic one adopted in section 1 in order to characterize some properties of Foley's model.

It is left for future research programmes to explore the two routes through which the concept of statistical equilibrium can be exported from physics to economics. On the side of the classical approach, it is left to study whether that concept is capable of filling other gaps of indeterminacy and to ascertain to what extent the logical succession between the two stages of analysis mentioned above can be usefully maintained. On the side of the approach proposed by Foley, it is left a more ambitious task; in so far as that approach aims to replace other notions of equilibrium for theorizing the economic system as a whole.

Appendix I

Solve

$$\max_{n_0, n_1, \ldots, n_Q} w(\mathbf{n}) = \frac{N!}{n_o!n_1!n_2!\ldots n_Q!}$$

subject to $\quad n_0 + n_1 + n_2 + \ldots . n_Q = N \qquad$ (a)

$$0n_0 + 1n_1 + 2n_2 + \ldots + Qn_Q = Q \qquad \text{(b).}$$

For n large, the following approximation can be shown to hold using Stirling's formula:

$$\ln n! = n.\ln n - n$$

where $\ln$ is the natural logarithm.

We will treat $n_0, n_1, n_2, \ldots . n_Q$ as continuous variables and apply Lagrange multiplier method.

We obtain the solution:

$$\overline{n_s} = N \frac{e^{-\beta s}}{\sum_{s=0}^{Q} e^{-\beta s}}, \qquad s = 0, 1, \ldots, Q. \qquad \text{(c)}$$

where β is an undetermined coefficient.

Let us consider the geometric progression:

$$\sum_{s=0}^{Q} e^{-\beta s} = 1 + x + x^2 + \ldots . + x^Q, \text{ with } x = e^{-\beta}.$$

Hence, as $Q \to \infty$, $\sum_{s=0}^{Q} e^{-\beta s} \to \frac{1}{1-x}$. Substituting the limit in (c), we get:

$$\overline{n_s} = Ne^{-\beta s}(1 - e^{-\beta}) \text{ . s=0,1...Q} \qquad \text{(d)}$$

Substituting (d) in the constraint (b):

$$N(1 - e^{-\beta}).\sum_{s=0}^{Q} se^{-\beta s} = Q \qquad \text{(e)}$$

To solve (e) with respect to β, we use the equation which holds at the limit

$$\sum_{s=0}^{Q} e^{-\beta s} = \frac{1}{1 - x}.$$

Differentiating both sides of this equation, we get:

$$\sum_{s=0}^{Q} s e^{-\beta s} = \frac{e^{-\beta}}{(1-e^{-\beta})^2} \qquad \text{(f)}$$

By substitution of (f) in (e):

$$N \cdot \frac{e^{-\beta}}{1-e^{-\beta}} = Q \qquad \text{(g)}$$

From (g)

$$\beta = \ln(1 + \frac{N}{Q})$$

which make solution (c) determined.

Appendix II

Suppose there exist n microstates, and let $\boldsymbol{x}$ be an n-column vector whose i-th element represents the probability of microstate i. The symbol R^n denotes the Euclidean space of dimension n, R^n_+ the non negative ortant of R^n, and $S^n \equiv \{x \in R^n_+ | e' \cdot x = 1\}$, where $\boldsymbol{e}$ is an n-column vector whose elements are all unity. In R^n, an order $\geq$ is induced by the cone R^n_+. We write $\boldsymbol{x} > \boldsymbol{y}$ when $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. and also write $\boldsymbol{x} >> \boldsymbol{y}$ when $\boldsymbol{x} - \boldsymbol{y}$ is in the interior of R^n_+.

Now a given continuous transformation f maps R^n_+ into itself, and satisfies the following assumptions.

Assumption 1: f is *monotone, i.e.,* $f(x) \geq f(y)$ when $\boldsymbol{x} \geq \boldsymbol{y}$.

Assumption 2: f is *weakly homogeneous,* i.e., for any $x \in R^n_+$, $\lambda \in R_+$, we have $f(\lambda x) = h(\lambda) f(x)$, where *:* $h: R_+ \to R_+$ is such that $h(\lambda)/\lambda$ is non increasing and $h(0)=0$.

Assumption 3: f is *primitive, i.e.,* there exists a natural number m such that for $\boldsymbol{x}, \boldsymbol{y} \in R^n_+$, $\boldsymbol{x} > \boldsymbol{y}$ implies $f^m(\mathrm{x}) >> f^m(\mathrm{y})$.

Assumption 4: When $\boldsymbol{x} \in$ S, then $f(x) \in$ S.

Assumption 5: $f(e/n) = e/n$.

Using the theorem and corollary 1 in Fujimoto and Krause (1985), it is easy to show that Assumptions 1-4 are sufficient to have a unique strictly positive $\boldsymbol{x}^* \in$ S. Since $\boldsymbol{x}^*$ is unique, this must coincide with $\boldsymbol{e}/n$ because of Assumption 5. Summarized as

Proposition. Given Assumptions 1-5, starting from any $\boldsymbol{x} \in$ S, $f^t(\mathbf{x})$ converges to $\boldsymbol{e}/n$ as t goes to infinity.

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