Fibrations of financial events

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FIBRATIONS OF FINANCIAL EVENTS

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Abstract

In this paper we shall prove that the plane of financial events, introduced and applied to financial problems by the author himself (see [2], [3] and [4]) can be considered as a fibration in two different ways. The first one, *the natural one*, reveals itself to be isomorph to the tangent-bundle of the real line, when the last one is considered as a differentiable manifold in the natural way; the second one is a fibration induced by the *status of compound interest capitalization* at a given rate $i \in]-1, \to [$. Moreover, in the paper we define on the first fibration an affine connection, also in this case induced by the status of compound interest at a given rate i. The final goal of this paper is the awareness that all the effects determined by the status of compound interest are *nothing but* the consequences of the fact that the space of financial events is a fibration endowed with a particular affine connection, so they are consequences of *purely geometric properties*, at last, depending upon the curvature determined by the connection upon the fibration. A natural preorder upon the set of fibers of the second fibration is considered. Some remarks about the applicability to economics and finance of the theories presented in the paper and about the possible developements are made in the directions followed in papers [1], [5], [6], [7], [8] of the author.

1   Preliminaries

For the general theory of fibrations we follow [9]. A fibration or fiber space is a pair $\mathcal{F} = (X, \pi)$, where

   i) X is a non-empty set, said the underlying set of the fiber space;

   ii) π is a surjection of X onto a non-empty set B, called the base of the fiber space;

   iii) for any point b in B there is a subset U of B containing b, a set F_b and a bijection

$$h : U \times F_b \to \pi^{-1}(U)$$

such that

$$\pi(h(y,t)) = y,$$

for each y in U and t in F_b. In other terms,

$$\pi \circ h = \mathrm{pr}_1^{U \times F_b},$$

where $\mathrm{pr}_1^{U \times F_b}$ is the first projection of the cartesian product $U \times F_b$.

Let k be a natural number (an integer greater or equal to 0) a C^k-fibration or fiber space of class C^k is a pair $\mathcal{F} = (X, \pi)$, where

   i) X is a C^k-manifold, said the underlying set of the fiber space;

   ii) π is a surjection of X onto a C^k-differentiable manifolds $(B, \mathcal{A})$;

iii) for any point b in B there is an open neighborhood U of b in B, a differentiable manifold $(F, \mathcal{A}_F)$ and a C^k-diffeomorphism

$$h : U \times F \to \pi^{-1}(U)$$

such that

$$\pi(h(y, t)) = y,$$

for each y in U and t in F.

2 Fibrations on financial events plane

In this section we introduce the basic concepts of the paper.

Theorem 1. *The space of financial events $\mathbb{R}^2$ is a smooth fiber space in the following two ways:*
1) the trivial one $(\mathbb{R}^2, \mathrm{pr}_1)$;
2) $\mathcal{F}_i = (\mathbb{R}^2, \pi_i)$ with $i > -1$ and π_i the below surjection

$$\pi_i : \mathbb{R}^2 \to \mathbb{R} : (t, c) \mapsto (1 + i)^{-t} c.$$

Proof. Straightforward by definition of fiber space. ∎

Definition 1. *We call the fibration $(\mathbb{R}^2, \mathrm{pr}_1)$ **natural fibration of the financial events plane**. We call the fibration $\mathcal{F}_i = (\mathbb{R}^2, \pi_i)$ **fibration induced on the financial events plane by the compound capitalization at rate** $i > -1$.*

Let us examine the fibration $(\mathbb{R}^2, \mathrm{pr}_1)$:

- the base of the fibration is the time-line $\mathbb{R}$;

- for each time t, the fiber $\left(\mathbb{R}^2\right)_t$ is the straight-line $\mathrm{pr}_1^-(t) = \{t\} \times \mathbb{R}$, that is the equivalence class generated by the null event $(t, 0)$ by means of the equivalence relation "to have the same time";

- this fibration is a fibred space of fiber-type $\mathbb{R}$, since each fiber is diffeomorphic to the standard manifold $\mathbb{R}$.

Let us examine the second fibration $\mathcal{F}_i = (\mathbb{R}^2, \pi_i)$:

- the base of the fibration is the capital-line;

- for each element c of the capital-line, the fiber $(\mathbb{R}^2)_c$ is the set-curve $\pi_i^-(c) = \mathrm{gr}(M_c)$, graph of the function

$$M_c : \mathbb{R} \to \mathbb{R} : t \mapsto (1 + i)^t c,$$

the so called capital-evolution of the event $(0, c)$. The fiber $\pi_i^-(c)$ is nothing but the class of equivalence generated by the event $(0, c)$ by means of the equivalence relation $\sim_i$ induced by the compound capitalization at rate i, that is the binary relation defined by

$$e_0 \sim_i e \ \ \text{iff} \ \ \pi_i(e_0) = \pi_i(e),$$

the equivalence class generated by an event e shall be denoted also by $[e]_i$;

- this fibration is a fibred space of fiber-type $\mathbb{R}$, since each fiber is diffeomorphic to the standard manifold $\mathbb{R}$ (since each fiber is a set-curve).

Remark (the fibration induced by a capitalization factor at 0). If $f : [0, +\infty] \to \mathbb{R}$ is a capitalization factor of class C^k, that is a positive function from the time semi-line $[0, +\infty]$ into the capital line $\mathbb{R}$ of class C^k such that $f(0) = 1$, we can build up a C^k-fibration $([0, +\infty] \times \mathbb{R}, \pi_f)$, defined by

$$\pi_f : [0, +\infty] \times \mathbb{R} \to \mathbb{R} : (t, c) \mapsto f(t)^{-1}c.$$

Even more generally, we can define a C^0-fibration $(\mathbb{R}^2, \pi_f)$ by

$$\pi_f : \mathbb{R}^2 \to \mathbb{R} : (t, c) \mapsto \begin{cases} f(t)^{-1}c & if \quad t \geq 0 \\ f(-t)c & if \quad t < 0 \end{cases},$$

and this fibration is at least of class C^1 if $k > 0$. Indeed, setting $g_>(t) = f(t)^{-1}$ and $g_<(t) = f(-t)$, we have $g'_<(t) = -f'(-t)$ and $g'_>(t) = -f'(t)f(t)^{-2}$, from which

$$g'_>(0) = g'_<(0) = -f'(0).$$

3 Properties of the fibration induced by the compound interest

Theorem 2. *Let, for any real $i > -1$, $\mathcal{F}_i = (\mathbb{R}^2, \pi_i)$ be the fibration induced by the compound capitalization at rate i. Then, for any two rates i and i' the two fibrations $\mathcal{F}_i$ and $\mathcal{F}_{i'}$ are isomorph, being the bijection $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$g(t, c) = (t, (u')^t u^{-t}) = (t, (u'/u)^t c),$$

for any financial event (t, c), where $u = 1 + i$ and $u' = 1 + i'$, an $\mathbb{R}$-isomorphism.

Proof. An isomorphism of a C^k-fibration $\mathcal{F} = (X, \pi)$ onto another C^k-fibration $\mathcal{F}' = (X', \pi')$ with the same base B is a pair of C^k-functions (id_B, g), with $g : X \to X'$, such that

$$\pi' \circ g = \pi.$$

Put $u = 1 + i$, $u' = 1 + i'$ and consider the bijection $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(t, c) = (t, (u')^t u^{-t} c) = (t, (u'/u)^t c),$$

for any financial event (t, c), then the pair $(\mathrm{id}_\mathbb{R}, g)$ is an isomorphism of $\mathcal{F}_i$ onto $\mathcal{F}_{i'}$. Indeed we have

$$\begin{aligned} \pi_{i'}(g(t, c)) &= \pi_{i'}((t, (u')^t u^{-t} c)) = \\ &= (u')^{-t}(u')^t u^{-t} c = \\ &= \pi_i(t, c), \end{aligned}$$

for each financial event (t, c). ∎

Remark. Another way to prove that the two above induced fibrations are isomorph is to prove that, for every c_0 belonging to the common base $\mathbb{R}$ there is an isomorphism $g_{c_0} : X_{c_0} \to X'_{c_0}$. Indeed, define

$$g_{c_0}(t, c) = (t, (u'/u)^t c),$$

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for every event (t, c) in the fiber $X_{c_0} = [(0, c_0)]_i$. We note that if (t, c) is a financial event of the fiber generated by the event $(0, c_0)$, it has the form $(t, c_0 u^t)$, applying the function g_{c_0} we obtain

$$
\begin{aligned}
g_{c_0}(t, c) &= (t, (u'/u)^t c) = \\
&= (t, (u'/u)^t c_0 u^t) = \\
&= (t, (u')^t c_0),
\end{aligned}
$$

that is an event of the fiber $X'_{c_0} = [(0, c_0)]_{i'}$: in other words, we pull back the event e along the fiber X_{c_0} to the event $e_0 = (0, c_0)$ and then we push forward the event e_0 to $e' = (t, (u')^t c_0)$ along the fiber X'_{c_0}.

Corollary 1. *Let, for any real $i > -1$, $\mathcal{F}_i = (\mathbb{R}^2, \pi_i)$ be the fibration induced by the compound capitalization at rate i. Then, $\mathcal{F}_i$ is trivializable for every rate i.*

Proof. It derives from the circumstance that the fibration $\mathcal{F}_0$ (corresponding to the rate 0%) is trivializable, in fact the projection π_0 acts as follows

$$
\pi_0 : \mathbb{R}^2 \to \mathbb{R} : (t, c) \mapsto (1 + 0)^{-t} c = c,
$$

and then the projection π_0 is nothing but pr_2 on the cartesian product of the time-line T times the capital line C; now it is clear that this fibration is isomorph to the fibration $(C \times T, \mathrm{pr}_1)$. The conclusion follows from the fact that each fibration $\mathcal{F}_i$ is isomorph to the fibration $\mathcal{F}_0$. ∎

4 Sections of the fibration induced by the compound interest

Theorem 3. (the sections of the fibrations $\mathcal{F}_i$). *Let C be the real line of capitals and let E be the plane of financial events. Then, a curve $s : C \to E$ defined by $s(c) = (s_1(c), s_2(c))$, for every capital c, is a section of the fibration $\mathcal{F}_i$ if and only if*

$$
s_2(c) = c(1 + i)^{s_1(c)},
$$

for every capital c.

Proof. The curve s is a section of the fibration $\mathcal{F}_i$, by definition, if and only if

$$
\pi_i(s(c)) = c,
$$

for every capital c. This last relation means that

$$
\pi_i(s_1(c), s_2(c)) = (1 + i)^{-s_1(c)} s_2(c) = c,
$$

for any capital c, that is

$$
s_2(c) = c(1 + i)^{s_1(c)},
$$

for any capital c. ∎

Remark. In other words, the above theorem states that are sections of the fibration induced by the compound capitalization at rate i only those curves $s : C \to E$ of the form

$$
s(c) = \left(s_1(c), c(1 + i)^{s_1(c)} \right),
$$

for every c in C and for any function $s_1 : C \to E$.

We can restate the above theorem as follows.

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Theorem 4. *(the sections of the fibrations $\mathcal{F}_i$).* *Let C be the capital line, T the time line and let E be the plane of financial events. Then, a curve $s : C \to E$ is a section of the fibration $\mathcal{F}_i = (E, \pi_i)$ if and only if there if a function $f : C \to T$ such that*

$$s(c) = (f(c), u^{f(c)}c),$$

for every capital c.

Remark. The fibration $\mathcal{F}_0$ is the pair (E, pr_2), thus every its section $s : C \to E$ has the form

$$s(c) = (f(c), c),$$

for every c in C, where $f : C \to T$ is any function of the capital line into the time line. Since any fibration $\mathcal{F}_i$ is isomorphic to the fibration $\mathcal{F}_0$, the sections of $\mathcal{F}_i$ can be obtained by the section of $\mathcal{F}_0$ applying the canonical isomorphism of $\mathcal{F}_0$ into $\mathcal{F}_i$, that is the bijection $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(t, c) = (t, u^t c) = (t, u^t c),$$

for any financial event (t, c), where $u = 1 + i$; applying the isomorphism g to the section s, we obtain the curve $g \circ s : C \to E$, that is the curve defined by

$$g \circ s(c) = g(f(c), c) = (f(c), u^{f(c)}c),$$

for any capital c: so we obtained newly the above theorem.

In a perfectly analogous way we can extend the above theorem as follows.

Theorem 5. *(the sections of the fibrations $\mathcal{F}_i$).* *Let C' be a part of the capital line, T' be a part of the time line and let E be the plane of financial events. Then, a curve $s : C' \to E$ is a section of the fibration $\mathcal{F}_i = (E, \pi_i)$ upon the part C' if and only if there if a function $f : C' \to T'$ such that*

$$s(c) = (f(c), u^{f(c)}c),$$

for every capital c in C'.

5   Capital evolutions as sections in the compound interest

We devote this paragraph to solve this problem important in the applications:

- Let $M : T \to C$ be a function from the time line into the capital line, called a capital evolution. There are sufficient conditions to assure that the graph of the function M, the subset $\mathrm{gr}(M)$ of the financial events plane E, is the trace of a section $s : C \to E$?

At this purpose we have the following complete result.

Theorem 6. *Let $M : T \to C$ be a function from the time line into the capital line. Then, the graph of the function M, the subset $\mathrm{gr}(M)$ of E, is the trace of a section $s : C \to E$ if and only if there exists a bijection $f : C \to T$ such that*

$$M(t) = u^t f^-(t),$$

for each time t.

Proof. **Sufficiency.** Let us suppose there exist a bijection $f : C \to T$ such that

$$M(t) = u^t f^-(t),$$

for each time t. Then, let $s : C \to E$ be the curve defined by

$$c \mapsto (f(c), u^{f(c)}c),$$

for every capital c. For each t in T (by surjectivity of the function f) there is a capital c in C such that $f(c) = t$, hence we have

$$(t, M(t)) = (f(c), u^t f^-(t)) = (f(c), u^{f(c)}c) = s(c),$$

so any point $(t, M(t))$ of the graph of M is a point of the curve s, that is

$$\text{gr}(M) \subseteq s(C);$$

we have now to prove that $s(C) \subseteq \text{gr}(M)$, indeed, let c be a capital, then by surjectivity of the reciprocal function f^-, there is a time t such that $f^-(t) = c$, now

$$s(c) = (f(c), u^{f(c)}c) = (f(c), u^t f^-(t)) = (t, M(t)),$$

as we desire. **Necessity.** Suppose now that the graph of M is the trace of a section s, this is equivalent to say (by the above characterization of sections) that there is a function $f : C \to T$ (not necessarily a bijection) such that, for each time t in T, we have

$$(t, M(t)) = (f(c), u^{f(c)}c),$$

for some c in C. First of all, we have to prove that the function f is bijective. In fact, let c and c' be two capitals such that $f(c) = f(c')$, since $f(c)$ is in T, we have

$$\begin{aligned}
(f(c), M(f(c)) &= s(c) = (f(c), u^{f(c)}c), \\
(f(c'), M(f(c')) &= s(c') = (f(c'), u^{f(c')}c'),
\end{aligned}$$

from which

$$\begin{aligned}
u^{f(c)}c &= M(f(c)) = \\
&= M(f(c')) = \\
&= u^{f(c')}c' = \\
&= u^{f(c)}c',
\end{aligned}$$

and we conclude $c = c'$. The function f is then injective, it is surjective since for every t there is a c such that $t = f(c)$. Concluding the relation $M(t) = u^t f^-(t)$, is an obvious consequence of the relations $t = f(c)$ and $M(t) = u^{f(c)}c$ by means of bijectivity. The theorem is proved. ∎

We conclude the section with a little (sometimes useful) result.

Proposition 1. *Let $M : T \to C$ be a capital evolution. Then, the graph of M is the trace of a section $s : C \to E$ of the fibration $\mathcal{F}_i$ if and only if the mapping*

$$h : \text{gr}(M) \to C : (t, c) \mapsto cu^{-t}$$

is a bijection.

Proof. **Necessity.** Let the graph of M be a section, then there is a bijection $f : C \to T$ such that
$$M(t) = u^t f^-(t),$$
for each time t. We have
$$
\begin{aligned}
h(t,c) &= h(t, M(t)) = \\
&= h(t, u^t f^-(t)) = \\
&= u^t f^-(t) u^{-t} = \\
&= f^-(t),
\end{aligned}
$$
and this prove that h is a bijection. **Sufficiency.** Let the mapping h be a bijection, we put
$$v(t) = h(t, M(t)),$$
it is clear that v is a bijection, moreover
$$v(t) = h(t, M(t)) = M(t) u^{-t},$$
from which, setting $f = v^-$, we deduce, for each t in T,
$$M(t) = v(t) u^t = f^-(t) u^t,$$
as we desired. ∎

Analogous result we have for the evolutions defined on a part of the time line.

Theorem 7. *Let T' be a part of the time line, C' be a part of the capital line and $M : T' \to C'$ be a capital evolution. Then, the graph of the function M, that is a subset $\mathrm{gr}(M)$ of the rectangle $T' \times C'$, is the trace of a section $s : C' \to E$ upon the part if and only if there exists a bijection $f : C' \to T'$ such that*
$$M(t) = u^t f^-(t),$$
for each time t in T'.

Example. Let i be a positive real, T be the time line, let $C_>$ be the semi-line of strictly positive capital and let $M : T \to C_>$ be a surjective C^1-capital evolution such that M' is strictly negative. Then, the graph of M is the trace of a section of the fibration $\mathcal{F}_i$ upon $C_>$. Indeed, put
$$v(t) = M(t) u^{-t},$$
we have
$$v'(t) = M'(t) u^{-t} - M(t) u^{-t} \ln u < 0,$$
for any time t, so the function v is strictly decreasing (hence injective) and surjective since M is so, and the claim is proved taking for f the inverse of v.

6 Connections on the financial fibration and capitalization laws

Consider the trivial financial fibration (E, pr_1), where E is the rectangle $U \times \mathbb{R}$ product of an open subset of the time-line times the capital axis $\mathbb{R}$.

Definition 2. (of local discount factor). *Let t be a time in U and let $F : V \to \mathbb{R}$ be an application of a neighborhood V of the time-vector 0 into the discount factor line $\mathbb{R}$. The mapping F is said a (local) discount law in U if verifies the following properties:*
  i) the translation $t + V$ is contained in the open subset U;
  ii) the discount factor $F(h)$ is positive, for every time-vector h;
  iii) the F-discount factor at time 0 is 1, $F(0) = 1$;
  i) F is of class C^1 in V.

Definition 3. (of financial translation induced by a discount factor). *We call, for every time-vector h in $\mathbb{R}$, such that $t + h$ lies in U, **financial translation from the fiber E_t to the fiber E_{t+h} induced by the discount law** F the mapping*

$$\tau_h : \{t\} \times \mathbb{R} \to \{t + h\} \times \mathbb{R}$$

defined by

$$\tau_h : (t, c) \mapsto (t + h, F(h)^{-1}c),$$

for every financial event $e = (t, c)$ of the fiber E_t.

Theorem 8. *The financial translation τ_h induced by a discount law F is a linear isomorphism of the fiber $E_t = \{t\} \times \mathbb{R}$ onto the fiber $E_{t+h} = \{t + h\} \times \mathbb{R}$ and the application τ of $V \times \mathbb{R}$ into $U \times \mathbb{R}$ defined by*

$$\tau : (h, c) \mapsto (t + h, F(h)^{-1}c)$$

is of class C^1. The derivative $\tau'(0, c)$ of the application τ at the point $(0, c)$ is the linear mapping of $\mathbb{R} \times \mathbb{R}$ into itself

$$(k, v) \mapsto (k, v - F'(0)kc).$$

Theorem 9. *Let F be a discount law. Then, the mapping $(k, c) \mapsto F'(0)kc$ is a bilinear application of $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, we denote it by Γ_t (and we call it the Cristoffel bilinear form)*

$$(k, c) \mapsto \Gamma_t(k, c) = F'(0)kc.$$

Conversely, if we have a bilinear application $(k, c) \mapsto \Gamma_t(k, c)$ and if we put

$$F(h) = 1 + \Gamma_t(h, 1),$$

the function F is a discount factor such that

$$F'(0)kc = \Gamma_t(k, c).$$

Since $E = U \times \mathbb{R}$ and since the event $e = (t, c)$ is a point of a fiber E_t, the tangent space $T_{(t,c)}(E)$ can be identified with the product $T_t(U) \times T_c(\mathbb{R})$, and this product can be itself identified with the product $(\{t\} \times \mathbb{R}) \times (\{c\} \times \mathbb{R})$.

Definition 4. (of local connection induced by a discount factor). *Let $\mathcal{T}$ be the time line endowed with its natural structure of C^∞ manifold. We call the application C_t of the product*

$T_t(\mathcal{T}) \times E_t$ into the tangent bundle $T(E)$ of the fibration (E, pr_1), union of the (disjoint) tangent spaces $T_e(E) = \{e\} \times \mathbb{R}^2$, with e varing in E, defined by

$$C_t : T_t(\mathcal{T}) \times E_t \to T(E) : ((t,k), e) \mapsto (e, (k, -F'(0)kc)),$$

local connection at time t induced by the discount law F. The local connection C_t associate with a couple of (applied) vectors $(t, k) \in T_t(\mathcal{T})$, $(t, c) \in E_t$ the applied vector at the event $e = (t, c)$ given by

$$C_t((t,k), (t,c)) = ((t,c), (k, -\Gamma_t(k,c))).$$

Definition 5. *(of global connection induced by a discount factor).* *Let F be a global discount law. The connection induced by the capitalization factor F is the mapping*

$$C : T(\mathcal{T}) \oplus E \to T(E) : ((t,k), e) \mapsto C_t((t,k), e),$$

where $T(\mathcal{T}) \oplus E$ is the union of the (disjoint) rectangles $T_t(\mathcal{T}) \times E_t$, i.e., the rectangles $(\{t\} \times \mathbb{R}) \times (\{t\} \times \mathbb{R})$.

Application. Consider an event $e = (t, c)$ and a capitalization law $u : \mathbb{R} \to \mathbb{R}$, that is a mapping verifing the following properties:

- the capitalization factor $u(h)$ is positive, for every time-vector h;

- the u-capitalization factor at time 0 is 1, i.e. $u(0) = 1$;

- u is of class C^1

The capital-evolution of the event e determined by the capitalization factor u is by definition the mapping

$$M : T \to C : t \mapsto u(t)c.$$

We note that the moltiplicative inverse $v = u^{-1}$ is a discount law. Let us consider the connection induced by the discount factor v:

$$C_t : T_t(\mathcal{T}) \times E_t \to T(E) : ((t,k), e) \mapsto (e, (k, -F'(0)kc)).$$

Suppose that each event $e = (t, c)$ has a capitalization-time t, that is we suppose that e is the state at t of the event $e_0 = (0, cu(t)^{-1})$, the financial translation induced by the capitalization law u is defined by

$$\tau_h : (t, c) \mapsto (t + h, u(t+h)u(t)^{-1}c),$$

so, concerning the discount law we have

$$v(h)^{-1} = u(t+h)u(t)^{-1},$$

deriving we obtain

$$-v(h)^{-2}v'(h) = u'(t+h)u(t)^{-1},$$

and considering the Cristoffel bilinear form, we have

$$
\begin{aligned}
-\Gamma_t(k,c) &= -v'(0)kc = \\
&= u'(t)u(t)^{-1}kc = \\
&= \delta(t)kc,
\end{aligned}
$$

where $\delta(t) := u'(t)u(t)^{-1}$ is the instant force of interest (by definition) at time t of the capitalization law u.

References

[1] David Carfì, *Optimal boundaries for decisions*, Atti dell'Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali Vol. LXXXVI, (2008)

[2] David Carfì, *Structures on the space of financial events*, Atti dell'Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali, Vol. LXXXVI, (2008)

[3] David Carfì, Giovanni Caristi, *Financial dynamical systems*, Differential Geometry - Dynamical Systems, Vol.10, (2008)

[4] David Carfì, *The family of transformations associated with a financial law*, Atti dell'Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali, Vol. LXXXI-LXXXII, (2004)

[5] David Carfì, *S-Linear Algebra in Economics and Physics*, APPLIED SCIENCES Vol. 9, 2007

[6] David Carfì, *Prigogine approach to irreversibility for Financial and Physical applications*, Supplemento Atti dell'Accademia Peloritana dei Pericolanti di Messina, Proceedings Thermocon'05. (2008)

[7] David Carfì,*Dyson formulas for Financial and Physical evolutions in $\mathcal{S}'$*, Communications to SIMAI congress, vol. 2 (2007)

[8] D. Carfì, *Feynmann's transition amplitudes in the space $\mathcal{S}'$*, Atti della Accademia Peloritana dei Pericolanti, classe di scienze Fisiche Matematiche e Naturali. Volume on line LXXXIII (2005)

[9] J. A. Dieudonnè, *Element d'Analyse, vol. 3*, Edition J. Gabay.