Parametric inference and forecasting in continuously invertible volatility models

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Abstract: We introduce the notion of continuously invertible volatility models that relies on some Lyapunov condition and some regularity condition. We show that it is almost equivalent to the volatilities forecasting efficiency of the parametric inference approach based on the Stochastic Recurrence Equation (SRE) given in Straumann (2005). Under very weak assumptions, we prove the strong consistency and the asymptotic normality of an estimator based on the SRE. From this parametric estimation, we deduce a natural forecast of the volatility that is strongly consistent. We successfully apply this approach to recover known results on univariate and multivariate GARCH type models where our estimator coincides with the QMLE. In the EGARCH(1,1) model, we apply this approach to find a strongly consistence forecast and to prove that our estimator is asymptotically normal when the limiting covariance matrix exists. Finally, we give some encouraging empirical results of our approach on simulations and real data.

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1. Introduction

Since the seminal paper of Engle (1982) and Bollerslev (1986), the General Autoregressive Conditional Heteroskedasticity (GARCH) type models have been successfully applied to volatility modeling. One of the drawback of GARCH type models is that they could not capture the leverage effect, the asymmetry and negative correlation between the movement in the rate of return on asset prices and volatility, documented by several researches (see Cont (2001), Bouchaud and Potters (2001)). Nelson (1991) is the first attempt to introduce the leverage effect to the ARCH models, see Remark 5 for details. EGARCH has inspired several authors to introduce other nonlinear GARCH models : APGARCH of Ding, Granger and Engle (1993), GJR-GARCH of Glosten, Jagannathan and Runkle (1993), TGARCH of Zakoïan (1994), etc. Since then, EGARCH types models have been used extensively in empirical researches (see Brandt and Jones (2006) among many others) and financial industry. Not surprisingly, theoretical investigations of EGARCH has attracted constant attention, see He, Teräsvirta and Malmsten (2002), Harvey (2010) and Rodriguez and Ruiz (2009). However, apart from some very special cases studied in Straumann and Mikosch (2006), the asymptotic theory of the estimator for EGARCH remains underdeveloped. Our study provides a first attempt to fully understand the asymptotic of the estimator based on the SRE given in Straumann (2005) for general volatility models including EGARCH. We provide sufficient conditions under which the estimator is asymptotically normal, a condition that most empirical studies do not verify, see Remark 6. Our study suggests that, for empirical purpose, a parsimonious variant of EGARCH (12) should be used instead.

Let us work in a general volatility mode of the form $X_t = \Sigma_t^{1/2} \cdot Z_t$ where Σ_t is the covariance matrix and where the innovations Z_t are normalized, centered independent identical distributed (iid) random vectors. The natural filtration $\{\mathcal{F}_t\}_{t\in\mathbb{Z}}$ is generated by this innovation process $\{Z_t\}_{t\in\mathbb{Z}}$. Moreover, it is assumed that the transformed covariances satisfy some (possibly non-linear) SRE, i.e. there exist a function h and some $\mathcal{F}_{t-1}$ measurable random function ψ_t such that the following relation $(h(\Sigma_k))_{k\le t} = \psi_t((h(\Sigma_k))_{k\le t-1}, \theta_0)$ holds. All the randomness of the process (X_t) comes from the innovations Z_t. It is assume that the expression of ψ_t with respect to $(Z_k)_{k\le t-1}$ is known, but the parameter of interest θ_0 is unknown. In this parametric framework, which includes all the classical models of GARCH and EGARCH types, forecasting the volatility completely relies on the ability of inferring θ_0. The present paper gives a general procedure based on the asymptotic properties of the SREs to computes an approximation $\hat{\theta}_n$ as a measurable function of the observations $(X_t)_{1\le t\le n}$ which converge to θ_0.

A SRE is said to be convergent if it has a unique stationary non anticipative solution. The functional process (ψ_t) is generated by the filtration of the innovations $\sigma((Z_k)_{k\le t-1})$. It is then a well-defined stationary ergodic process. The convergence of a SRE such as the one associated to the ψ_t leads to the existence of the stationary process (X_t). If the functions ψ_t are Lipschitz, sufficient Lyapunov conditions (also necessary in the linear case) are stated in Bougerol and Picard (1992) and Bougerol (1993). However, the volatility process does not satisfy a unique SRE, see Remark 1. Optimal stationary conditions might apply to another SRE (think of the GARCH(p,q) model case). In the sequel, we assume that the optimal SRE generated by the innovations is both known and convergent. Then the process (X_t) is well defined, stationary and ergodic. Our first result in Proposition 1 asserts that, as any solutions of a convergent SRE, (X_t) has finite log-moments.

Assuming that the model to be continuously invertible **(CI)**, we study the estimator $\hat{\theta}_n$ given by the procedure based on the SRE of Straumann (2005). Using only log-moments properties ensured by Proposition 1 and the locally uniform convergence of regular functions, we prove that $\hat{\theta}_n$ is strongly consistent as soon as the model is identifiable. We then give a natural strongly consistent forecaster of the volatility. The estimator $\hat{\theta}_n$ is moreover asymptotically normal as soon as the limiting variance exists. Modulo these weak and technical assumptions, we have the "equivalences"

Convergence of the SRE generated by the innovations $(Z_t, Z_{t-1}, \ldots)$	$\Longleftrightarrow$ (A)	stationarity, ergodicity and log-moments
Convergence of the SRE generated by the observations $(X_t, X_{t-1}, \ldots)$	$\Longleftrightarrow$ (B)	invertibility, forecasting and statistical inference

Let us detail the consequences of this table in the GARCH and EGARCH type models:
The equivalence (A) has been extensively studied in GARCH type models since the seminal papers Nelson (1990) and Bougerol and Picard (1992) in the univariate case. For multivariate GARCH models the equivalence (A) has recently been studied by Francq and Zakoïan (2011). That the SRE generated by the observations converges for GARCH type models is straightforward, see Straumann and Mikosch (2006) for discussions in the AGARCH case. Thus the equivalence (B) directly gives asymptotic properties of the statistical inference based on the SRE for these models. As the inference based on the SRE coincides with the Quasi Maximum Likelihood Estimator (QMLE), we recover existing results in Subsection 3.5: for GARCH(p,q) models we recover the same results of Francq and Zakoïan (2004) that refine Berkes, Horvath and Kokoszka (2003), for AGARCH(p,q) and CCC-GARCH(p,q) models we refine the results of Straumann and Mikosch (2006) and Francq and Zakoïan (2011).
In the EGARCH type models, the SRE generated by the innovations converges as it coincides with the SRE of an ARMA process. Thus it admits a stationary ergodic solution with finite moments, see Nelson (1991). The right implication (B) proved in Section 3 is new for that type of process as the inference based on the SRE differs from the QMLE (see Subsection 4.5 for details). It enables us to prove in Section 4 the asymptotic properties of the inference and the forecasting of the volatility for invertible EGARCH(1,1) models. For proving the asymptotic normality, we show a necessary and sufficient condition for the existence of the asymptotic variance. Then we provide some encouraging empirical results of our approach in Section 5 on simulations and on real data. A first step on the reverse part of the equivalence (B) has been done recently by Sorokin (2011). He proves that forecast of the volatility based on the SRE is inconsistent for some non invertible models. Finally, notice that the statistical inference of θ_0 is possible without assuming the invertibility: it has been successfully done by Zaffaroni (2009) using the approach of Whittle.

An outline of the paper can be given as follows. In Section 2, we discuss the standard notions of invertibility and introduce the continuous invertibility. In Section 3 our main results on the statistical inference based on the SRE are stated. We apply this results to the EGARCH(1,1) model in Section 4. Finally, in Section 4, we report and discuss simulations results and empirical findings. The Appendices contain technical proofs and calculations.

2. Continuously invertible volatility models

2.1. *The general volatility model*

In this paper, (Z_t) is a stationary ergodic sequence of real vectors called the innovations. Let us denote $\mathcal{F}_t$ the filtration generated by $(Z_t, Z_{t-1}, \ldots)$ and let us consider the general volatility model

$$X_t = \Sigma_t^{1/2} \cdot Z_t, \qquad \text{where} \qquad (h(\Sigma_k))_{k \le t} = \psi_t((h(\Sigma_k))_{k \le t-1}, \theta_0) \tag{1}$$

with an injective function h from the space of real matrices of size $k \times k$ to an auxiliary separable metric space F and $\psi_t(\cdot, \theta_0)$ is a $\mathcal{F}_{t-1}$ adapted random function from the space of the sequences of elements in the image of h to itself. Let us denote ℓ the inverse of h (from the image of h to the space of real matrices of size $k \times k$) and call it the link function.

2.2. *Convergent SRE and stationarity*

A first question regarding this very general model is wether or not a stationary solution exists. As the sequence of the transformed volatilities $(h(\Sigma_k))_{k \le t}$ is a solution of a fixed point problem, we recall the following result due to Bougerol (1993). Let (E, d) be a complete separable metric space. A map $f : E \to E$ is a Lipschitz map if $\Lambda(f) = \sup_{(x,y) \in E^2} d(f(x), f(y))/d(x, y)$ is finite. For any sequence of random element in (E, d), (X_t) is said to be exponential almost sure convergence to 0 $X_t \xrightarrow{\text{e.a.s.}} 0$ as $t \to \infty$ if for $X_t = o(e^{-Ct})$ a.s. for some $C > 0$.

Theorem 1. *Let (Ψ_t) be a stationary ergodic sequence of Lipschitz maps from E to E. Suppose that $\mathbb{E}[\log^+(d(\Psi_0(x), x))] < \infty$ for some $x \in E$, that $\mathbb{E}[\log^+ \Lambda(\Psi_0)] < \infty$ and that for some integer $r \ge 1$,*

$$\mathbb{E}[\log \Lambda(\Psi_0^{(r)})] = \mathbb{E}[\log \Lambda(\Psi_0 \circ \cdots \circ \Psi_{-r+1})] < 0.$$

Then the SRE $X_t = \Psi_t(X_{t-1})$ for all $t \in \mathbb{Z}$ is convergent: it admits a unique stationary solution $(Y_t)_{t \in \mathbb{Z}}$ which is ergodic and for any $y \in E$

$$Y_t = \lim_{m \to \infty} \Psi_t \circ \cdots \circ \Psi_{t-m}(y), \quad t \in \mathbb{Z}.$$

The Y_t are measurable with respect to the $\sigma(\Psi_{t-k}, k \ge 0)$ and

$$d(\tilde{Y}_t, Y_t) \xrightarrow{e.a.s.} 0, \quad t \to \infty$$

such that $\tilde{Y}_t = \Psi_t(\tilde{Y}_{t-1})$ for all $t > 0$.

Remark that the sufficient Lyapunov assumptions $\mathbb{E}[\log \Lambda(\Psi_0^{(r)})] < 0$ is necessary in the linear case, see Bougerol and Picard (1992). The logarithmic moments of the solution of a convergent SRE is proved in the following result that seems to be new:

Proposition 1. *Under the assumptions of Theorem* (1) *the unique stationary solution also satisfies* $\mathbb{E}[\log^+(d(Y_0, x))] < \infty$.

Proof. See Appendix 1. □

In order to apply Theorem 1 in our case, let us denote by E the separable metric space of the sequences of elements in the image of h. Equipped with the metric $\sum_{j\geq 1} 2^{-j} d(x_j, y_j)/(1 + d(x_j, y_j))$, the space E is complete. A sufficient condition for stationarity of (X_t) is that the SRE driven by (ψ_t) converges in E. It simply expresses as the Lyapunov condition $\mathbb{E}[\log \Lambda(\psi_0^{(r)})] < 0$ for some integer $r \geq 1$ and some logarithmic moments. This assumption of stationarity is sufficient but not optimal in many cases:

Remark 1. *The state space of the SRE* (1)*, denoted E, in its most general form, is a space of infinite sequences. However in all classical models we can find a lag p such that $(h(\Sigma_k))_{t-p+1\leq k\leq t} = \psi_t((h(\Sigma_k))_{t-p\leq k\leq t-1}, \theta_0)$. The state space E is now the finite product of p spaces. It can be equipped by unbounded metrics such that $p^{-1}\sum_{j=1}^{p} d(x_j, y_j)$ or $\sqrt{\sum_{j=1}^{p} d^2(x_j, y_j)}$. The product metric has to be carefully chosen as it changes the value of the Lipschitz coefficients of the ϕ_t. Yet, even if the products spaces are embedded, the smallest possible lag p in the SRE yields the sharpest Lyapunov condition. Finally, if E has a finite dimension and if the condition of convergence of the SRE expresses in term of the top Lyapunov coefficient, one can choose any metric induced by any norm, see Bougerol (1993) for details.*

We prefer to work under the less explicit assumption

(ST) The process (X_t) satisfying (1) exists. It is a stationary, non anticipative and ergodic process with finite logarithmic moments.

In view of Proposition 1, it is reasonable to require that solution have finite logarithmic moments.

2.3. The invertibility and the observable invertibility

Now that under **(ST)** the process (X_t) is stationary and ergodic, we investigate the question of invertibility. We want to emphasis that the invertibility depends on the convergence of the SRE, governed by a Lyapunov condition. Following Tong (1993), we say that a volatility model is invertible if the volatility can be expressed as a function of the past observed values:

Definition 1. *The model is invertible if the sequence of the volatilities (Σ_t) is adapted to the filtration $(\mathcal{G}_{t-1})$ generated by $(X_{t-1}, X_{t-2}, \cdots)$.*

It is natural to assume invertibility to be able to forecast the volatility. This notion of invertibility is very weak and consists in restricting the underlying filtration $(\mathcal{F}_t)$ of the SRE to $(\mathcal{G}_{t-1})$. Indeed, under **(ST)** then $\mathcal{G}_t \subseteq \mathcal{F}_t$ is well defined and by using $Z_t = \Sigma_t^{-1} \cdot X_t$ in ψ_t we can transform the general model as

$$(h(\Sigma_k))_{k\le t} = \phi_t((h(\Sigma_k))_{k\le t-1}, \theta_0) \tag{2}$$

for some ergodic and stationary sequence (ϕ_t) adapted to $(\mathcal{G}_{t-1})$. Thus the invertibility is implied by Theorem 1 if the ϕ_t are Lipschitz maps such that for some $r > 0$,

$$\mathbb{E}[\log \Lambda(\phi_0(\cdot, \theta_0)^{(r)})] < 0. \tag{3}$$

The Remark 1 also holds for the SRE driven by (ϕ_t): the state space E of the SRE and the product metric must be chosen carefully. The condition (3) (with the optimal choice of the state space E and its metric) is called the condition of invertibility.

Proposition 2. *Under* **(ST)** *and* (3), *the general model* (2) *is invertible.*

Another notion of invertibility is the one introduced in Straumann and Mikosch (2006). We call it observable invertibility. Let us assume that there exists some approximations $\hat\phi_t$ of ϕ_t such that $\hat\phi_t$ is a measurable function of the observations $(X_{t-1}, X_{t-2}, \dots, X_1)$.

Definition 2. *The model is observably invertible if and only if the solution of the approximative SRE*

$$(h(\hat\Sigma_k))_{k\le t} = \hat\phi_t((h(\hat\Sigma_k))_{k\le t-1}, \theta_0) \tag{4}$$

is convergent, i.e. $\|\hat\Sigma_t - \Sigma_t\| \to 0$ *in probability as* $t \to \infty$.

Remark that in general the approximative SRE does not fit the conditions of Theorem 1 and in particular $(\hat\phi_t)$ is not necessarily stationary and ergodic. However, the useful Property below ensures that an invertible model is also observably invertible. It is a straightforward Corollary of Theorem 2.10 of Straumann and Mikosch (2006) and our Proposition 1

Proposition 3. *If* **(ST)** *and* (3) *hold, the link function ℓ is continuous and it exists $x \in E$ such that $d(\hat\phi_t(x), \phi_t(x)) \xrightarrow{e.a.s.} 0$ and $\Lambda(\hat\phi_t(\cdot, \theta_0) - \phi_t(\cdot, \theta_0)) \xrightarrow{e.a.s.} 0$ as $t \to \infty$, then the model is observably invertible.*

Remark 2. *Classical models satisfy an SRE for finite p lags of volatilities $(h(\Sigma_k))_{t-p+1\le k\le t} = \phi_t((h(\Sigma_k))_{t-p\le k\le t-1}, \theta_0)$ and for some ϕ_t generated by only a finite of past observation $(X_{t-1}, \dots, X_{t-q})$. In this context, the approximative SRE coincides with the initial ones, i.e. one can choose $\hat\phi_t = \phi_t$ for $t > q$. Therefore, conditions of Proposition 3 hold systematically; invertibility and observable invertibility are equivalent, i.e. they are induced by the same Lyapunov condition. As the initial values (for $0 \le t \le q$) in the SRE can be chosen arbitrarily from Theorem 1, with some abuse of notation we will work in the sequel with $\hat\phi_t = \phi_t$ for $t \ge 1$.*

2.4. *The continuous invertibility*

We have seen that the existing invertibility notions can be expressed in term of Lyapunov conditions. We introduce the notion of continuous invertibility in term of a Lyapunov condition and some regularity on the model. Let us consider models with parametric functions having continuous Lipschitz coefficients:

(CL) For any metric spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, a function $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z}$ satisfies (**CL**) if there exists a continuous function $\Lambda_f : \mathcal{Y} \mapsto \mathbb{R}^+$ such that $\Lambda(f(\cdot, y)) \le \Lambda_f(y)$ for all $y \in \mathcal{Y}$.

From Remark 2, let us consider the functional SRE of the form

$$(\hat{g}_k(\theta))_{t-p+1\le k\le t} = \phi_t((\hat{g}_k(\theta))_{t-p\le k\le t-1}, \theta), \qquad \forall \theta \in \Theta, \qquad \forall t \ge 1, \tag{5}$$

with any fixed initial value $(\hat{g}_k(\theta))_{1-p\le k\le 0}$. We introduce the condition of continuous invertibility:

(CI) Assume that the SRE (5) holds with ϕ_t satisfying **(CL)** for stationary (Λ_{ϕ_t}) under Lyapunov condition $\sup_\Theta \mathbb{E}[\log \Lambda_{\phi_0}^{(r)}(\theta)] < 0$ and $\mathbb{E}[\sup_\Theta \log \Lambda_{\phi_0}^{(r)}(\theta)] < \infty$. Assume there exists an $y \in E$ such that $\mathbb{E}[\sup_\Theta \log^+(d(\phi_0(y,\theta), y))] < \infty$.

The condition **(CI)** implies the standard invertible conditions given in Subsection 2.3 for all $\theta \in \Theta$ and in particular for the unknown θ_0. It also implies the local uniform regularity of the solution $g_t(\cdot)$ of the functional SRE $(g_k(\cdot))_{t-p+1\le k\le t} = \phi_t((g_k(\cdot))_{t-p\le k\le t-1}, \cdot)$ for all $t \in \mathbb{Z}$.

Theorem 2. *Assume that (Σ_t^2) is a stationary and ergodic process such that* **(CI)** *holds. Then the functions $g_t(\cdot)$ are continuous for all $\theta \in \Theta$ and all $t \in \mathbb{Z}$. Moreover, for any $\theta \in \Theta$ there exists an $\epsilon > 0$ such that $\hat{g}_t(\theta)$ satisfying (5) satisfies*

$$\lim \sup_{\theta' \in \overline{B}(\theta,\epsilon)\cap\Theta} d(\hat{g}_t(\theta'), g_t(\theta')) \xrightarrow{e.a.s.} 0. \tag{6}$$

Proof. See Appendix 1. □

3. Statistical inference under continuous invertibility

3.1. *Statistical inference based on the SRE*

Here we describe the approach in Straumann (2005). Assume that (5) holds with θ_0 unknown and θ_0 belonging in a known compact set Θ. Consider

$$\hat{\theta}_n = \mathrm{argmin}_{\theta\in\Theta} \hat{S}_n(\theta)$$

the M-estimator associated with the SRE criteria function

$$n\hat{S}_n(\theta) = \sum_{t=1}^{n} \hat{s}_t(\theta) = \sum_{t=1}^{n} 2^{-1} \left(X_t^T \ell(\hat{g}_t(\theta))^{-1} X_t + \log(\det(\ell(\hat{g}_t(\theta)))\right). \qquad (7)$$

where $(\hat{g}_t)$ is obtained from the approximative SRE (5).

Remark 3. *This statistical procedure does not coincides with the Quasi Maximum Likelihood approach that is not always observable, see Subsection 4.5 for the EGARCH(1,1) example.*

3.2. Strong consistency of the parametric inference

From now on, we assume that the innovations process (Z_t) is iid:

(IN) The Z_t are iid variables such that $\mathbb{E}[Z_1^T Z_1] =$ is the identity matrix.

Under **(ST)** and **(IN)**, $\mathbb{E}[X_t^T \cdot X_t \mid \mathcal{F}_0] = \Sigma_t$ in agreement with the definition of the volatility. The following assumption implies that the volatilities are invertible matrices:

(IV) The functions ℓ^{-1} and $\log(\det(\ell))$ are Lipschitz satisfying $\det(\ell(g_0(\theta))) \geq C(\theta)$ for some continuous function $C : \Theta \mapsto (0, \infty)$.

Remark 4. *The SRE criteria converges to the a possibly degenerate limit*

$$S(\theta) = \mathbb{E}[s_0(\theta)] = 2^{-1}\mathbb{E}\left[X_0^T \ell(g_0(\theta))^{-1} X_0 + \log(\det[\ell(g_0(\theta))])\right]$$

Remark that $S(\theta_0) = 2^{-1}\mathbb{E}[Z_0^T Z_0 + \log(\det(\Sigma_0))]$ is finite under **(ST)** *and* **(IN)** *because the volatilities have finite log moments as solutions of an SRE, see Proposition 1. But $S(\theta)$ may be infinite for $\theta \neq \theta_0$.*

If the model is identifiable, the estimator $\hat{\theta}_n$ is strongly consistent:

Theorem 3. *Assume that (***ST***) and* **(CI)** *are satisfied on the compact set Θ. If* **(IN)** *and* **(IV)** *are satisfied and the model is identifiable, i.e. $g_0(\theta) = h(\Sigma_0)$ iff $\theta = \theta_0$, then $\hat{\theta}_n \to \theta_0$ a.s. for any $\theta_0 \in \Theta$.*

Proof. First, remark that with no loss of generality we can always restrict in (5) to $\det(\ell(\hat{\phi}_t(\cdot, \theta))) \geq C(\theta)$. We adapt the proof of Jeantheau (1993), keeping his notation, $s_{*t}(\theta, \rho) = \inf\{s_t(\theta'), \theta' \in \overline{B}(\theta, \rho)\}$ and $\hat{s}_{*t}(\theta, \rho) = \inf\{\hat{s}_t(\theta'), \theta' \in \overline{B}(\theta, \rho)\}$. Let us recall Theorem 5.1 in Jeantheau (1993) : The M-estimator associated with the loss (7) is strongly consistent under the hypothesis H1-H6:

H1 Θ is compact.
H2 $\hat{S}_n(\theta) \to S(\theta)$ a.s. under the stationary law P_{θ_0}.
H3 $S(\theta)$ admits a unique minimum for $\theta = \theta_0$ in Θ. Moreover for any $\theta_1 \neq \theta_0$ we have:

$$\lim \inf_{\theta \to \theta_1} S(\theta) > S(\theta_1).$$

H4 $\forall\theta\in\Theta$ and sufficiently small $\rho>0$ the process $(\hat s_{*t}(\theta,\rho))_t$ is ergodic.
H5 $\forall\theta\in\Theta$, $\mathbb{E}_{\theta_0}[s_{*1}(\theta,\rho)]>-\infty$.
H5 $\lim_{\rho\to 0}\mathbb{E}_{\theta_0}[s_{*1}(\theta,\rho)]=\mathbb{E}[s_{*1}(\theta)]$.

Let us check **H1-H6** in our case. **H1** is satisfied by assumption. **H2** is verified in two steps. First, by the e.a.s. convergence given by Theorem 2, arguments of Straumann (2005) and the Lipschitz properties of ℓ^{-1} and $\log(\det(\ell))$ we obtain

$$\frac{1}{n}\sum_{t=1}^{n}\sup_{\overline{B}(\theta,\epsilon)}|\hat s_t(\theta')-s_t(\theta')|\to 0\qquad P_{\theta_0}-a.s.$$

Second we use that (s_t) is an ergodic sequence. Using Proposition 1.1 of Jeantheau (1993), $n^{-1}\sum_{t=1}^n s_t(\theta)$ converges P_{θ_0}-a.s. to $S(\theta)$ (taking values in $\mathbb{R}\cup\{+\infty\}$) as the s_t are bounded from below:

$$\frac{1}{n}\sum_{t=1}^{n}|s_t(\theta)-S(\theta)|\to 0\qquad P_{\theta_0}-a.s.$$

Combining this two steps leads to **H2**. The first part of **H3** is checked similarly than in (ii) p.2474 of Straumann and Mikosch (2006) and with the help of the Remark 4. Remark that S has a unique minimum iff $\mathbb{E}[\mathrm{Tr}(\Sigma_0\cdot\ell(g_0(\theta))^{-1})-\log(\det(\Sigma_0\cdot\ell(g_0(\theta))^{-1}))]$ has a unique minimum. As this criteria is the integrand of a sum of the $\lambda_i-\log(\lambda_i)$ where the λ_i are positive eigenvalues, we conclude under the identifiability condition from the property $x-\log(x)\geq 1$ for all $x>0$ with equality iff $x=1$. The second part is checked using the fact that

$$\liminf_{\theta\to\theta_1}S(\theta)\geq\mathbb{E}[\liminf_{\theta\to\theta_1}s_0(\theta)]=\mathbb{E}[s_0(\theta_1)]=S(\theta_1)$$

where the first inequality was already used for proving Theorem 2 and the first equality comes from the local continuity of g_0 and ℓ. **H4** is satisfied from the ergodicity of $(\hat s_t)$. **H5** and **H6** follows from Theorem 2 that ensures the continuity of the function s_{*1} and by the lower bounded assumption on $\det(\ell)$, see Proposition 1.3 of Jeantheau (1993). □

3.3. *Volatility forecasting*

From the inference of θ_0, we deduce a natural forecast of the volatility $\hat\Sigma_t=\ell(\hat g_t(\hat\theta_t))$. It is strongly consistent:

Proposition 4. *Under the conditions of Theorem 3 then* $\|\hat\Sigma_t-\Sigma_t\|\to 0$ *a.s. as* $t\to\infty$.

Proof. It is a direct consequence of Theorems 2 and 3 that assert the a.s. convergence of $\hat\theta_t$ toward θ_0 and the local uniform convergence of $\hat g_t$ toward g_t. Remark that for t sufficiently large such that $\hat\theta_t\in\overline{B}(\theta,\epsilon)$, a ball where the uniform Lyapunov condition $\mathbb{E}[\log\Lambda_\infty(\phi_t(\cdot))]<0$ is satisfied. Thus $\hat g_t(\hat\theta_t)-g_t(\theta_t)\to 0$

a.s. and by continuity of ℓ and g_t and from the identification $\Sigma_t = \ell(g_t(\theta_0))$ the result follows if $g_t(\hat{\theta}_t)$ converges to $g_t(\theta_0)$. For proving it, we use

$$d(g_t(\hat{\theta}_t), g_t(\theta_0)) \le \Lambda_\infty(\phi_t(\cdot))d(g_{t-1}(\hat{\theta}_t), g_{t-1}(\theta_0)) + w_t(\hat{\theta}_t)$$

where $w_t(\hat{\theta}_t) = d(\phi_t(g_{t-1}(\theta_0), \hat{\theta}_t), \phi_t(g_{t-1}(\theta_0), \theta_0))$. The RHS term satisfies an SRE of linear stationary maps satisfying the Lyapunov condition. We apply Theorem 1 as for any $\hat{\theta}_t$, by assumption $\mathbb{E}\log^+(w_0(\hat{\theta}_t))$ is uniformly bounded by $\mathbb{E}[\sup_\Theta \log^+(2d(\phi_t(y,\theta), y))] < \infty$. We get

$$d(g_t(\hat{\theta}_t), g_t(\theta_0)) \le \sum_{i=0}^{\infty} \Lambda_\infty(\phi_t(\cdot)) \cdots \Lambda_\infty(\phi_{t-i+1}(\cdot)) w_{t-i}(\hat{\theta}_t).$$

Conditioning on $(\hat{\theta}_t)$, the upper bound is a stationary normally convergent series of functions and

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=0}^{\infty} \Lambda_\infty(\phi_t(\cdot)) \cdots \Lambda_\infty(\phi_{t-i+1}(\cdot)) w_{t-i}(\hat{\theta}_t) \to 0\Big)\\
&= \mathbb{E}\Big[\mathbb{P}\Big(\sum_{i=0}^{\infty} \Lambda_\infty(\phi_t(\cdot)) \cdots \Lambda_\infty(\phi_{t-i+1}(\cdot)) w_{t-i}(\hat{\theta}_t) \to 0 \mid (\hat{\theta}_t)\Big)\Big]\\
&= \mathbb{E}\Big[\mathbb{P}\Big(\sum_{i=0}^{\infty} \Lambda_\infty(\phi_0(\cdot)) \cdots \Lambda_\infty(\phi_{-i+1}(\cdot)) w_{-i}(\hat{\theta}_t) \to 0 \mid (\hat{\theta}_t)\Big)\Big]\\
&= \mathbb{E}[1] = 1,
\end{aligned}$$

the last inequalities following from the continuity of normally convergent series of functions, $\hat{\theta}_t \to \theta_0$ and $w_i(\hat{\theta}_t) \to w_i(\theta_0) = 0$ for all i a.s. as $t \to \infty$. □

3.4. Asymptotic normality of the parametric inference

Classical computations show easily that if the M-estimator $\hat{\theta}_n$ is asymptotically normal then the asymptotic variance is given by the expression

$$\mathbf{V} = \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}$$

with $\mathbf{P} = \mathbb{E}[\mathbb{H}s_0(\theta_0)]$ and $\mathbf{Q} = \mathbb{E}[\nabla s_0(\theta_0)\nabla s_0(\theta_0)^T]$, where $\mathbb{H}s_0(\theta_0)$ is the Hessian matrix of $s_0(\theta_0)$. Let $\mathcal{V} = \overline{B}(\theta_0, \epsilon) \subset \Theta$ with $\theta_0 \in \overset{\circ}{\Theta}$ with $\epsilon > 0$ chosen in accordance with Theorem 2, i.e. such that $\mathbb{E}[\log(\sup_{\mathcal{V}} \Lambda_{\phi_0})] < 0$.

(AV) Assume that $\mathbb{E}(\|Z_0 Z_0^T\|^2) < \infty$ and that the functions ℓ and ϕ_t are 2-times continuously differentiable on the compact set Θ that coincides with the closure of its interior.

The next assumption is used to ensure that g_0 is 2-times differentiable.

(DL) The partial derivatives $\Phi_t = D_x(\phi_t), = D_\theta(\phi_t), = D^2_{x^2}(\phi_0)$, $D^2_{\theta,x}(\phi_0)$ or $D^2_{\theta^2}(\phi_0)$ satisfy **(CL)** for stationary (Λ_{Φ_t}) with $\mathbb{E}[\sup_{\mathcal{V}} \log(\Lambda_{\Phi_0})] < \infty$.

The following moments assumptions ensure the existence of $\mathbf{Q}$ and $\mathbf{P}$:

(MM) Assume that $\mathbb{E}[\|\nabla s_0(\theta_0)\|^2] < \infty$ and $\mathbb{E}[\|\mathbb{H}s_0(\theta_0)\|] < \infty$.

These moments assumptions holds only for $\theta = \theta_0$; they are simpler to verify than for the moment conditions for $\theta \neq \theta_0$ due to the specific form of the derivatives of the SRE criteria , see Remark 4 and computations in Bardet and Wintenberger (2009). The next assumption is classical and ensures to the existence of $\mathbf{P}^{-1}$:

(LI) The components of the vector $\nabla g_0(\theta_0)$ are linearly independent.

Finally, the last assumption is specific to the SRE approach. It ensures that $\nabla \hat{s}_t(\theta)$ is a good approximation of $\nabla s_t(\theta)$ uniformly on $\mathcal{V}$:

(LM) Assume that $y \to \nabla \ell^{-1}(y)$ and $y \to \nabla \log(\det(\ell(y)))$ are Lipschitz functions.

Theorem 4. *Under the assumptions of Theorem 3,* **(AV)**, **(DL)**, **(MM)**, **(LI)** *and* **(LM)** *then the asymptotic variance* $\mathbf{V}$ *is well defined and the statistical inference is asymptotically normal, i.e.*

$$\sqrt{n}(\hat{\theta}_n - \hat{\theta}_0) \to \mathcal{N}(0, \mathbf{V})$$

in distribution for any $\theta_0 \in \overset{\circ}{\Theta}$ *with* $\mathbf{V}$ *that is invertible.*

Proof. First, remark that from **(DL)** and Proposition 1 applied to $\sup_{\mathcal{V}} g_0(\theta)$ then $\mathbb{E}[\sup_{\mathcal{V}} \log^+(\|\Phi_0(\theta)\|)] < \infty$ for $\Phi_0(\theta) = D_\theta(\phi_0)(g_0(\theta), \theta)$ or $D^2_{x^2}(\phi_0)(g_0(\theta), \theta)$ or $D^2_{\theta,x}(\phi_0)(g_0(\theta), \theta)$ or $D^2_{\theta^2}(\phi_0)(g_0(\theta), \theta)$. Using the existence of these logarithmic moments and the relation $\mathbb{E}[\log(\sup_{\mathcal{V}} \Lambda_{\phi_0})] < 0$, we apply recursively the Theorem 1 and prove the existence of continuous first and second derivatives of $(g_t(\theta))$ on $\mathcal{V}$ as solutions of functional SRE. The asymptotic normality follows from a Taylor development on the first partial derivatives of S_n (see Section 5 of Bardet and Wintenberger (2009) for more details):

$$\nabla_i S_n(\hat{\theta}_n) - \nabla_i S_n(\theta_0) = \mathbb{H}S_n(\tilde{\theta}_{n,i})(\hat{\theta}_n - \theta_0)$$

Then the asymptotic normality follows from the following sufficient conditions:

1. $n^{-1/2}\nabla S_n(\theta_0) \to \mathcal{N}(0, \mathbf{Q})$,
2. $\|n^{-1}\mathbb{H}S_n(\tilde{\theta}_n) - \mathbf{P}\|$ converges a.s. to 0 for any sequence $(\tilde{\theta}_n)$ converging a.s. to θ_0 and $\mathbf{P}$ is invertible,
3. $n^{-1/2}\|\nabla \hat{S}_n(\hat{\theta}_n) - \nabla S_n(\hat{\theta}_n)\|$ converges a.s. to 0.

Due to its specific expression and that (Z_t) is a normalized difference of martingales sequence with finite moments of order 4, $(\nabla S_n(\theta_0))$ is a martingale, see Bardet and Wintenberger (2009) for detailed computations. Under **(MM)**, the CLT for differences of martingale applied to $(\nabla S_n(\theta_0))$ leads to the first condition. The first part of the second condition are derived from similar arguments than in the proof of Proposition 4 and an application of the Cesaro mean theorem ensuring that $n^{-1}\|\mathbb{H}S_n(\tilde\theta_n) - \sum_{t=1}^n \mathbb{H}s_t(\theta_0)\| \to 0$ a.s. The ergodic Theorem on $(\mathbb{H}s_t(\theta_0))$ with **(MM)** leads to $\|n^{-1}\mathbb{H}S_n(\tilde\theta_n) - \mathbf{P}\| \to 0$ a.s. The fact that $\mathbf{P}$ is invertible follows from **(LI)**, see Bardet and Wintenberger (2009) for detailed computations. Finally the third condition is obtained by applying Theorem 2.10 of Straumann and Mikosch (2006) on the SRE satisfied by (∇g_t) and its approximative SRE satisfied by $(\nabla \hat g_t)$ uniformly on $\mathcal{V}$. Remark that the condition $\mathbb{E}[\log(\sup_{\mathcal{V}}\|\nabla g_0\|)] < \infty$ of Theorem 2.10 of Straumann and Mikosch (2006) is automatically satisfied using the Proposition 1 on the functional SRE of ∇g_t and the local uniform norm on $\mathcal{V}$. Thus $\sup_{\mathcal{V}}\|\nabla\hat g_t - \nabla g_t\| \xrightarrow{\text{e.a.s.}} 0$ as $t\to\infty$ and Lipschitz conditions on $\nabla\ell^{-1}$ and $\nabla\log(\det(\ell))$ in **(LM)** and arguments similar than in Straumann (2005) leads to the desired result. □

3.5. A first application to GARCH type models

In the GARCH type models, the stationarity assumption **(ST)** is crucial, whereas the continuous invertibility condition **(CI)** is automatically satisfied due to the form of the model. Sufficient conditions for **(ST)** have been extensively studied in the literature and they can be necessary in linear cases. The asymptotic properties of the QMLE (that coincides with the SRE based inference in these cases) follow from Theorems 3 and 4. Thus, we recover and slightly refine existing results in the AGARCH and CCC-GARCH models (we refer the reader to Straumann (2005) and Francq and Zakoïan (2011) respectively for details in these both cases).

First, let us consider the univariate APGARCH(p,q) model introduced in Ding, Granger and Engle (1993), Zakoïan (1994) and studied in Straumann (2005):

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p}\alpha_i(|X_{t-i}| - \gamma X_{t-i})^2 + \sum_{j=1}^{q}\beta_j\sigma_{t-j}^2, \qquad t\in\mathbb{Z},$$

where $\alpha_0>0$, $\alpha_i,\beta_j\ge 0$ and $|\gamma|\le 1$ (it coincides with the GARCH(p,q) model if $\gamma=0$. Then we derive the strong consistency and the asymptotic normality directly from our Theorems 3 and 4. The conditions we obtained coincides with these of Theorem 5.5 and Theorem 8.1 of Straumann and Mikosch (2006) except that the technical (and superfluous) condition (8.1) of Straumann and Mikosch (2006) is not needed in our approach.

Second, let us consider the multivariate CCC-GARCH(p,q) model introduced by Bollerslev (1990), first studied in Jeantheau (1998) and refined in Francq and Zakoïan (2011)

$$Diag(\Sigma_t^2) = A_0 + \sum_{i=1}^{q} A_i Diag(X_{t-i}X_{t-1}^T) + \sum_{i=1}^{p} B_i Diag(\Sigma_{t-i}^2)$$

and $(\Sigma_t^2)_{i,j} = \rho_{i,j}\sqrt{(\Sigma_t^2)_{i,i}(\Sigma_t^2)_{j,j})}$ for all (i,j), where $Diag(M)$ is the vector of the diagonal elements of M. A necessary and sufficient conditions for **(ST)** is given in term of top Lyapunov condition in Francq and Zakoïan (2011). We recover the strong consistency and the asymptotic normality of Francq and Zakoïan (2011) directly from our Theorems 3 and 4 except that we do not need the superfluous condition $\forall\theta \in \Theta, \quad |B_\theta(z)| = 0 \implies |z| > 1$ assumed in **A2** of Francq and Zakoïan (2011). It is due to our local uniform approach of the SRE that improves the classical uniform approach used in Francq and Zakoïan (2011). The inequality (4.10) of Francq and Zakoïan (2011) does not always holds in our context.

4. Application to the EGARCH(1, 1) model

4.1. Definition of the model

Let (Z_t) be an iid sequence of random variables not concentrated on two points such that $\mathbb{E}(Z_0^2) = 1$. The EGARCH(1, 1) model introduced by Nelson (1991) is an AR(1) model for $\log\sigma_t^2$,

$$X_t = \sigma_t Z_t \quad \text{with} \quad \log\sigma_t^2 = \alpha_0 + \beta_0 \log\sigma_{t-1}^2 + W_{t-1}(\theta_0)$$

where $W_t(\theta_0) = \gamma_0 Z_t + \delta_0 |Z_t|$ are the innovations of this AR(1) model. Let $\theta_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0)$ be the unknown parameter. Assume that $\theta_0 \in \Theta$ where Θ is a compact subset of $\mathbb{R}\times]-1,1[\times\{(\delta,\gamma) \in \mathbb{R}^2;\ \delta \geq |\gamma|\}$ (see the next subsection for a discussion on Θ). The restriction on the parameter β_0 ($|\beta_0| < 1$) is sufficient for the existence of a stationary solution (assumption **(ST)** holds). Then we have a MA(∞) representation for the logarithm of the squared volatility:

$$\log\sigma_t^2 = \alpha_0(1-\beta_0)^{-1} + \sum_{k=1}^{\infty} \beta_0^{k-1} W_{t-k}(\theta_0). \tag{8}$$

The moments assumptions on Z_t ensures that the process $(\log\sigma_t^2)$ is ergodic, strongly and weakly stationary. Then the volatilities process (σ_t^2) is also ergodic and strongly stationary. However, it does not necessarily have finite variance.

Remark 5. *The EGARCH(1, 1) model takes into account some stylized facts such as the asymmetry in the squared volatility: if $Z_t > 0$, then $\log\sigma_t^2 = \alpha_0 + \beta_0\log\sigma_{t-1}^2 + (\gamma_0 + \delta_0)Z_{t-1}$ and $\log\sigma_t^2 = \alpha_0 + \beta_0\log\sigma_{t-1}^2 + (\gamma_0 - \delta_0)Z_{t-1}$ otherwise. So conditioning on the sign of the innovation, Z_t, the change of the log-volatility $\log(\sigma_{t+1}^2/\sigma_t^2)$ is asymmetric.*

4.2. Invertibility

The invertibility of the stationary solution of the EGARCH(1, 1) model does not hold in general. A sufficient condition for invertibility is given in Straumann and Mikosch (2006): as $(\log \sigma_t^2)$ satisfies the SRE

$$\log \sigma_t^2 = \alpha_0 + \beta_0 \log \sigma_{t-1}^2 + W_{t-1}(\theta_0) \exp\Big(- \frac{\log \sigma_{t-1}^2}{2} \Big), \quad \text{for all } t \in \mathbb{Z},$$

if the above SRE admits a unique stationary solution, then the model is invertible. Keeping the notation of our Section 2, the function h is here the logarithmic function and the SRE (4) holds with (ϕ_t) defined by

$$\phi_t(\cdot;\theta) : s \mapsto \alpha + \beta s + (\gamma X_{t-1} + \delta\, |X_{t-1}|) \exp(-s/2) \tag{9}$$

We check that the ϕ_t are random functions generated by $\mathcal{G}_{t-1}$. As in Straumann and Mikosch (2006) we restrict $\phi_t(\cdot;\theta)$ on the complete separable metric space $[\alpha/(1-\beta), \infty)$ equipped with $d(x,y) = |x-y|$. Then for any $\theta_0 \in \Theta$, as **(ST)** is satisfied, $(\phi_t(\cdot;\theta_0))$ is a stationary ergodic sequence of Lipschitz maps from $[\alpha_0/(1-\beta_0), \infty)$ to $[\alpha_0/(1-\beta_0), \infty)$ with the Lipschitz coefficient

$$\Lambda(\phi_t(\cdot,\theta_0)) \le \max\{|\beta_0|, |2^{-1}(\gamma_0 X_{t-1} + \delta_0 |X_{t-1}|) \exp(-2^{-1}\alpha_0(1-\beta_0)) - \beta_0|\}.$$

Thus a sufficient condition for the invertibility condition (3) (with $r = 1$) is

$$\mathbb{E}[\log(\max\{|\beta_0|, |2^{-1}(\gamma_0 X_{t-1} + \delta_0 |X_{t-1}|) \exp(-2^{-1}\alpha_0/(1-\beta_0)) - \beta_0|\})] < 0. \tag{10}$$

Remark 6. *The condition $\delta \ge |\gamma|$ is fundamental. If it is not satisfied and Z is unbounded, the innovations $W_t(\theta)$ may take any negative values. The logarithms of the squared volatility in (8) may also take any negative values and the ϕ_t are no longer globally Lipschitz functions. Empirical studies as Brandt and Jones (2006) do not work under this constraint and the model is not invertible (the volatilities forecasts based on the SRE are unstable). We suggest to use a parsimonious variant of EGARCH (12) to avoid this phenomena.*

4.3. Condition on the compact set Θ

Let us detail in the sequel the compact sets Θ such that the relation (10) is satisfied for any $\theta_0 \in \Theta$. From the MA(∞) representation (8) of $\log \sigma_t^2$ we rewrite the condition (10) as

$$\mathbb{E}\Big[\log\Big(\max\Big\{ |\beta|, \Big| 2^{-1} \exp\Big(2^{-1} \sum_{k=0}^{\infty} \beta^k (\gamma Z_{-k-1} + \delta\, |Z_{-k-1}|) \Big) \times (\gamma Z_0 + \delta\, |Z_0|) - \beta \Big| \Big\} \Big) \Big] < 0 \tag{11}$$

which does not depend on α and is easier to check. Using the Monte Carlo algorithm, assuming the Z_t to be normally distributed, we report in Figure 1 the largest values of β that satisfies the condition (11) on a grid of values of γ and δ.

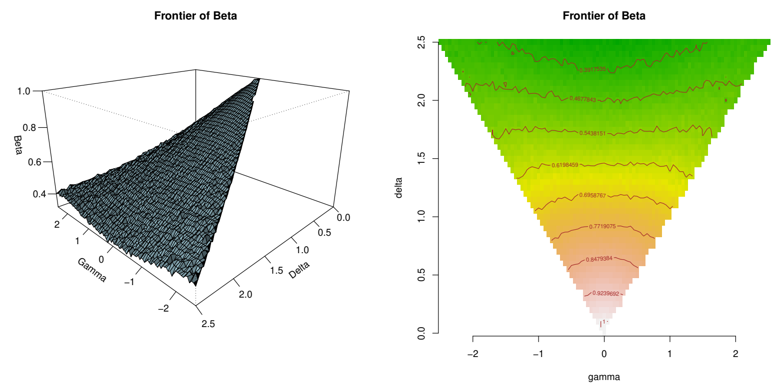


FIG 1. *Perspective and contour plots of the admissible region for continuous invertibility.*

Graphically, it seems that the most restrictive condition on β is when (γ, δ) is far away from $(0,0)$. However, notice that $|\beta|$ is never constrained to 0 as for $\beta = \gamma = 0$ the condition (11) becomes $\mathbb{E}[\log(\delta|Z_0|/2)] < \mathbb{E}[\delta|Z_0|/2]$ which is always satisfied as $\log(x) \leq x - 1$ for all $x > 0$. We then conjecture that the admissible set for θ is unbounded. Finally, remark that when $\beta = 0$, the EGARCH(1, 1) model degenerates to a sub-model $Y_t = \exp(\alpha + \gamma Z_{t-1} + \delta\,|Z_{t-1}|)Z_t$ for which the admissible parameters set is well known, see Straumann (2005) for details.

4.4. *Asymptotic properties of the parametric inference and forecast*

With the reasonable choice of Θ made in the last subsection, we know that **(ST)** is satisfied. Moreover, **(CL)** is automatically satisfied as

$$(\Lambda_{\phi_t}(\theta)) = (\max(|\beta|, |2^{-1}(\gamma X_{t-1} + \delta|X_{t-1}|)\exp(-2^{-1}\alpha/(1-\beta)) - \beta|))$$

are stationary continuous functions of θ. Thus, as a corollary of Theorem 3 and Proposition 4, we get

Corollary 1. *If Θ satisfying the condition (11) and $\theta_0 \in \Theta$, then $\hat{\theta}_n \to \theta_0$ and $\hat{\sigma}_n^2 - \sigma_n^2 \to 0$ a.s. as $n \to \infty$ with $\hat{\sigma}_t^2 = \exp(\hat{g}_t(\hat{\theta}_n))$.*

Proof. The condition **(CI)** follows from $\mathbb{E}[\log \Lambda(\phi_t(,\theta))] < 0$ by assumption of Θ and $\mathbb{E}[\sup_\Theta \log \Lambda(\phi_t(,\theta))] < \infty$ since $\mathbb{E}\log|X_{t-1}| = E(\log\sigma + \log|Z_{t-1}|) < \infty$ as $\log \sigma_t^2$ has a MA(∞) representation (8) and Z is integrable. Moreover as $\log^+(d(\phi_0(0,\theta), 0)) = \log^+|\alpha + (\gamma X_{-1} + \delta|X_{-1}|)|$ then $\mathbb{E}[\sup_\Theta \log^+(d(\phi_0(y,\theta), y)] < \infty$ for $y = 0$.
In the EGARCH(1, 1) model the link function is the exponential function $\ell(x) = \exp(x)$ and since we have $\log\sigma_t^2 \geq \alpha/(1-\beta)$, $1/\ell(x) = \exp(-x)$ is a Lipschitz function ($\log(\det(\ell)) = id$ is also a Lipschitz function). Moreover the volatility

process (σ_t^2) is bounded from below by $C(\theta) = \exp(\alpha/(1-\beta))$. Finally, the identifiability condition $g_0(\theta) = h(\theta_0)$ iff $\theta = \theta_0$ is checked in Section 5.1 of Straumann and Mikosch (2006). □

As a corollary of Theorem 4 we get the asymptotic normality of the inference in the EGARCH(1,1) model. It holds under the following necessary and sufficient condition of the existence of the asymptotic variance $\mathbf{V}$ which is detailled in Appendix 2 and 3 :

(MM') The innovation satisfy $\mathbb{E}[Z_0^4] < \infty$ and $\beta^2 - \delta\mathbb{E}|Z_0| + (\delta^2+\gamma^2)/4on < 1$.

Corollary 2. *Assume that Θ is well chosen as in Corollary 1 and that* **(MM')** *holds then $\sqrt{n}(\hat{\theta}_n - \theta_0) \to \mathcal{N}(0,\mathbf{V})$ in law with an invertible matrix $\mathbf{V}$.*

Proof. By definition, (ϕ_t) is 2-times continuously differentiable and simple computations give $D_x(\phi_t)(x,\theta) = \beta - 2^{-1}(\gamma X_{t-1} + \delta|X_{t-1}|)\exp(-x/2)$, $D_\theta(\phi_t)(x,\theta) = (1, x, X_{t-1}\exp(-x/2), |X_{t-1}|\exp(-x/2))^T$, $D^2_{x^2}(\phi_t)(x,\theta) = 4^{-1}(\gamma X_{t-1} + \delta|X_{t-1}|)$ $\exp(-x/2)$, $D^2_{x,\theta}(\phi_t)(x,\theta) = (0, 1, 2^{-1}X_{t-1}\exp(-x/2), 2^{-1}|X_{t-1}|\exp(-x/2))^T$ and $D^2_{\theta^2}(\phi_t)(x,\theta) = 0$. Moreover, as the link function is $\ell(x) = \exp(x)$ is also 2-times continuously differentiable, the last assertion of the condition **(AV)** of Theorem 4 holds. The fact that **(MM)** holds under the conditions $\mathbb{E}[Z_0^4] < \infty$ and $\beta^2 - \delta\mathbb{E}|Z_0| + (\delta^2+\gamma^2)/4 < 1$ is technical and postponed to the Appendix 2. The fact that **(LI)** holds if Z_0 is not concentrated on two points is classical and proved in the Appendix 3 for the sake of completeness. Assumption **(DL)** is satisfied from the expressions of the derivatives (that are Lipschitz functions) and as all the logarithmic moments are finite due to $\mathbb{E}[\log(X_{t-1}^2)] < \infty$. Finally **(LM)** is automatically satisfied due to the specific expression of the link function. Thus Theorem 4 applies. □

4.5. *The asymptotic variance* $\mathbf{V}$

Computations give $\mathbf{V} = (\mathbb{E}Z_0^4 - 1)\mathbf{B}^{-1}$ with $\mathbf{B} = \mathbb{E}[\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T]$. Lemma 1 in Appendix 2 states that the conditions $\beta^2 - \delta\mathbb{E}|Z_0| + (\delta^2+\gamma^2)/4 < 1$ and $\mathbb{E}[Z_0^2] = 1$ are necessary and sufficient for the existence of $\mathbf{B}$. Assuming the Z_t to be normally distributed, we report in Figure 2 the largest values of β that satisfies the conditions (11) and $\beta^2 - \delta\mathbb{E}|Z_0| + (\delta^2+\gamma^2)/4 < 1$ on a grid of values of γ and δ such that $\gamma^2+\delta^2 \le 4$. The additional condition does not affect much the constraint of β on this region of (γ,δ). However, for $\gamma^2+\delta^2 > 4$, the condition of existence of $\mathbf{V}$ imposes to exclude also small values of β starting from $\beta = 0$. Then the resulting admissible region is now bounded and seems to be convex.

The explicit computation of $\mathbf{B}$ is technical and is given in the Appendix 4. Remark that the quasi likelihood (computable if the observations Z_t were observable) is equal to $2^{-1}\sum_{t=1}^n (X_t^2/\hat{h}_t(\theta) + \log(\hat{h}_t(\theta)))$ where

$$\hat{h}_t(\theta) = \alpha + \beta\hat{h}_{t-1}(\theta) + (\gamma Z_{t-1} + \delta|Z_{t-1}|),\ \ t > 1,\ \text{ and }\ \hat{h}_1(\theta) = \zeta(\theta) \text{ fixed.}$$

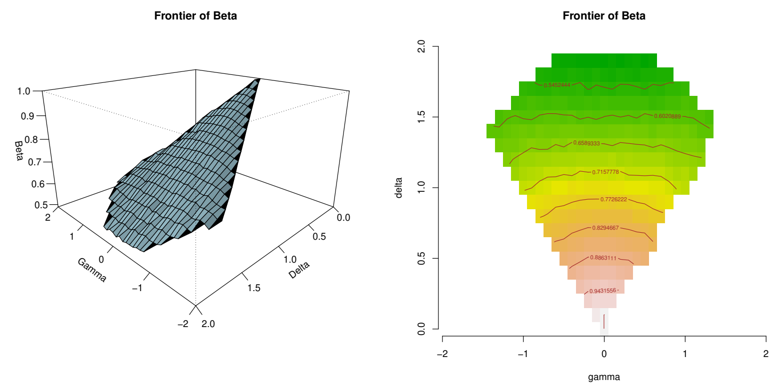


FIG 2. *Perspective and contour plots of the admissible region for the asymptotic normality.*

The likelihood $2^{-1}\sum_{t=1}^{n}(X_t^2/h_t(\theta)+\log(h_t(\theta)))$ coincides with SRE criteria $S_n(\theta)$ at the point θ_0 as $h_t(\theta_0) = g_t(\theta_0) = \sigma_t^2$. However, the two criteria differ for $\theta \neq \theta_0$. To see it, from the MA(∞) representation (8) we can easily compute the partial derivative of h_0 with respect to β:

$$\frac{\partial h_0}{\partial \beta}(\theta) = \frac{\alpha\beta}{(1-\beta)^2} + \sum_{k\geq 1} k\beta^{k-1}(\gamma Z_{-k-1} + \delta|Z_{-k-1}|).$$

The value of this partial derivative is different from $\partial h_0/\partial\beta(\theta)$, even for $\theta = \theta_0$. Thus the estimator given by our SRE approach and the QMLE are different. Moreover, the asymptotic variances of the two estimators differ and even for gaussian innovations (Z_t) our estimator $\hat{\theta}_n$ is not asymptotically efficient in the EGARCH(1, 1) case.

5. Numerical parametric inferences and forecasting

5.1. Monte Carlo analysis of the estimation risk

Let us study, by Monte Carlo simulation of 1000 replications of the sample path of different sizes $T = 512, 1034$ or 2048, the risk for estimating a parameter $\theta_0 \in \Theta$. We assume the innovations that drive Egarch model are iid standard normal. See Table 1. The columns "rmse" give the empirical Root Mean Square Error (RMSE) computed over the 1000 replications. The columns "napp" give the normal approximation of the RMSE. From the exact computation of $\mathbf{V}$ in Appendix 4, we compute the value of the asymptotic variance at the point θ_0 and divide the corresponding standard deviation by $\sqrt{T}$. From Table 1, the normal approximation seems to hold for T larger than 1024. Also, in accordance with our theoretical results, the mean values of the estimators $\hat{\theta}_n$ over monte carlo replications are more concentrated as the sample size is larger.

TABLE 1
Statistical inference and normal approximation

	T=	512			1024			2048		
θ	θ_0	mean	rmse	napp	mean	rmse	napp	mean	rmse	napp
α	-0.399	-.381	.127	.059	-.393	.041	.042	-.396	.030	.030
β	.9	.874	.170	.023	.897	.017	.016	.899	.012	.011
γ	-.3	-.300	.057	.045	-.301	.033	.032	-.299	.023	.023
δ	.5	.488	.097	.075	.492	.052	.053	.496	.038	.038

5.2. *Estimation of the asymptotic covariance matrix*

Assuming that $\mathbb{E}[Z_0^4] = 3$ in the EGARCH(1,1) model, we have two ways of estimating the asymptotic covariance matrix of our estimator $\hat{\theta}_n$. The first one relies on the plug-in $\mathbf{V}^{TH}(\hat{\theta}_n)$ in the explicit formula $\theta \to \mathbf{V}^{TH}(\theta)$ given in the Appendix 4. The second one relies on the SRE satisfied by $\hat{g}_t(\theta)$:

$$\begin{aligned}\nabla \hat{g}_t(\theta) = (1, \hat{g}_{t-1}(\theta), X_{t-1}\exp(-\hat{g}_{t-1}(\theta)/2), |X_{t-1}|\exp(-\hat{g}_{t-1}(\theta)/2))^T \\ + (\beta - (\gamma X_{t-1} + \delta|X_{t-1}|)\exp(-\hat{g}_{t-1}(\theta)/2)/2)\nabla \hat{g}_{t-1}(\theta).\end{aligned}$$

Running the SRE over $1 \le t \le n$ we obtain n values $\nabla \hat{g}_t(\theta)$, then we approximate $\mathbf{B}$ by $n^{-1}\sum_{t=1}^n \nabla \hat{g}_t(\theta)\nabla \hat{g}_t(\theta)^T$ and finally an approximation of $\mathbf{V} = 2\mathbf{B}^{-1}$ by inverting numerically the approximation of $\mathbf{B}$.

We compare the risk of the two estimation procedure associated with the Riemannian distance for symmetric positives definitive matrix defined by

$$d(A,B) = \sqrt{\sum_{k=1}^{4} \log^2 \nu_k(AB^{-1})}$$

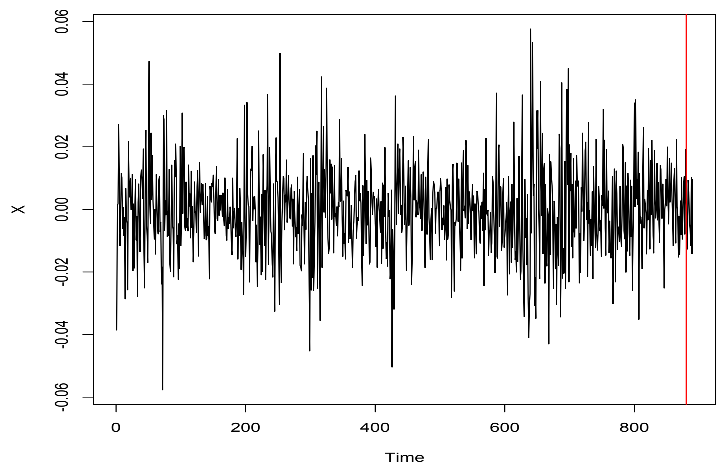
where $\nu_1(AB^{-1}), \ldots, \nu_4(AB^{-1})$ are the eigenvalues of the matrix AB^{-1}. To estimate the risk we sample randomly 100 parameters $\theta_1, \ldots, \theta_{100}$ in a compact set satisfying the conditions for the convergence of the SRE and the existence of $\mathbf{B}$. For each θ_k we simulate a path of $n = 512$, 1024 and 2048 observations and obtain the estimated values $\hat{\theta}_k$. Then we calculate the errors $d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\theta_k))$, $d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{TH}(\hat{\theta}_k))$,$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\hat{\theta}_k))$. We report the means of these distances in Table 2

TABLE 2

n	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\theta_k))$	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{TH}(\hat{\theta}_k))$	$d(\mathbf{V}^{TH}(\theta_k), \mathbf{V}^{SRE}(\hat{\theta}_k))$
512	.074	.788	.924
1024	.065	.767	.780
2048	.064	.426	.457

The table show that it is very safe to use $\mathbf{V}^{SRE}(\hat{\theta}_k)$. This might suggests that for general volatility models, we can safely use $\mathbf{V}^{SRE}(\hat{\theta}_k)$ if the explicit formula for $\mathbf{V}$ is not known.

FIG 3. *The daily log-return of the Standard & Poor's 500 data from Jan 4th, 2000 to Jul 22th, 2003.*



5.3. *Forecasting the volatility on real data*

We investigate the filtering and the forecasting problem of the daily log-return of the Standard & Poor's 500 data from Jan 4th, 2000 to Jul 22th, 2003 ($n = 890$).We also use the high frequency intra-day return(5 mins, 15mins, 65 mins) to calculate the daily realized volatilities as a proxies of the daily volatilities. More precisely, we consider the process (X_t) of the log difference of the daily close prices. The period has been chosen in order that the stationary condition might be reasonable (it is a period of high volatility). To test the stationarity, we perform a KPSS test and an augmented Dickey-Fuller on the logarithm of the X_t^2 (plotted in Figure 3) that give p-values 0.098 and less than 0.01 respectively. The unit root hypothesis can thus be reasonably rejected.

From now we assume that the general volatility model (1) is satisfied by the real data, i.e. $X_t = \sigma_t Z_t$ with a known parametric form ψ_t. Below we investigate some of the more classical models: the GARCH(1,1) model,the GARCH(1,1) model with Student innovations, the APGARCH(1,1) model and the EGARCH(1,1) model. Let us consider the classical in sample and out of sample procedures with two steps: for the in sample procedure, first we estimate $\hat{\theta}_n$ on the 890 first observations and then we investigate the performance of the natural forecast $\hat{\sigma}_t = \ell(\hat{g}_t(\hat{\theta}_n))$ of the volatility σ_t for $1 \leq t \leq 890$. For the out of sample procedure, first we estimate $\hat{\theta}_n$ on the 880 first observations and then we investigate the performance of the natural forecast $\hat{\sigma}_t = \ell(\hat{g}_t(\hat{\theta}_n))$ of the volatility σ_t for $881 \leq t \leq 890$ (the red line in figure 3 splits the training and validation data sets). One difficulty of evaluating the forecasting performance is that we could not observe the true volatility process. However, with the 5 mins high frequency prices data in hand, we bypass the problem: the Realized Volatil-

TABLE 3
In sample performances of the different forecasts

QLIK	X_t^2	5min RV	15min RV	65min RV
GARCH(1,1)	-7.438	-7.517	-7.592	-7.607
GARCH(1,1) Student	-7.439	-7.516	-7.589	-7.604
APGARCH(1,1)	**-7.489**	-7.528	-7.613	-7.618
Rolling Volatility	-7.444	-7.473	-7.542	-7.570
Riskmetrics	-7.429	-7.510	-7.583	-7.597
EGARCH(1,1)	-7.487	**-7.537**	**-7.619**	**-7.626**

ity (RV) is used as a consistent proxy of the conditional volatility. To measure the forecasting performance we use the quasi likelihood (QLIK) criteria. As noticed by Patton (2011), it is robust with respect to unbiased proxy. Moreover, we claim that it is more relevant than the Mean Square Error criteria as it does not involve moments of order larger than one of the volatilities. For the sake of completeness, we also give the QLIK criteria for forecasting the proxies obtained by 2 out of sample procedures, the Riskmetrics one and the rolling volatility one.

To sum up, we list the parametric forms of the volatilities in the different models that are investigated here:

- GARCH(1,1): $\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2$,
- APGARCH(1,1): $\sigma_t^\delta = \omega + \alpha(|X_{t-1}| - \gamma X_{t-1})^\delta + \beta\sigma_{t-1}^\delta$,
- Rolling Volatility (60days): the moving average $1/60\sum_{i=1}^{60} X_{t-i}^2$,
- Riskmetrics (Exponentially weighted moving average model)

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)X_{t-1}^2 \qquad \text{where} \qquad \lambda = .94,$$

- EGARCH(1,1): $\log\sigma_t^2 = \alpha + \beta\log\sigma_{t-1}^2 + (\gamma X_{t-1} + \delta|X_{t-1}|)/\sigma_t$.

Denoting $(r_{1,m,t},\dots,r_{m,m,t})$ the m intra day returns, we define the daily RV with m-frequency as

$$\mathrm{RV}_t^{(m)} = \sum_{i=1}^{i=m} r_{i,m,t}^2.$$

Finally, the QLIK criteria of the forecasting of the proxies RV by $\hat\sigma^2$ is

$$\mathrm{QLIK}(\hat\sigma^2) = \sum_{t=1}^{n} \log(\hat\sigma_t^2) + \frac{RV_t}{\hat\sigma_t^2}.$$

We report the in sample performances in Table 3, the out of sample ones in Table 4. The best forecast for each proxy is bolded. The performance of the EGARCH(1,1) is close to the best one in each case.

Applying our approach in the EGARCH(1,1) model, we find the estimations $\hat\alpha_n = -0.312$, $\hat\beta_n = 0.976$, $\hat\gamma_n = -0.122$ and $\hat\delta_n = 0.122$ in the in sample procedure, $\hat\alpha_{n'} = -0.324$, $\hat\beta_{n'} = 0.974$, $\hat\gamma_n = -0.123$ and $\hat\delta_n = 0.123$. These values are

TABLE 4
Out of sample performances of the different forecasts

QLIK	X_t^2	5min RV	15min RV	65min RV
GARCH(1,1)	**-8.285**	-8.153	**-8.192**	**-8.170**
GARCH(1,1) Student	-8.226	-8.131	-8.153	-8.111
APGARCH(1,1)	-8.226	-8.130	-8.152	-8.111
Rolling Volatility	-8.195	-8.096	-8.128	-8.111
Riskmetrics	-8.053	-7.978	-7.998	-7.977
EGARCH(1,1)	-8.272	**-8.155**	-8.184	-8.135

TABLE 5
Confident intervals given by the normal approximation in the EGARCH type model $\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \eta X_{t-1}^- / \sigma_t$

In-S.	Value	Confident interval	Out of S.	value	Confident interval
α	-.312	[-.450, -.175]	α	-.324	[-.464, -.185]
β	.976	[.962, .989]	β	.974	[.961,.988]
η	.243	[.171, .315]	η	.246	[.172,.320]

at the frontier of the linear constraint $\delta \geq |\gamma|$ and satisfy the Lyapunov condition and the condition of existence of $\mathbf{V}$. The EGARCH(1,1) model degenerates to the following model:

$$\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \eta X_{t-1}^- / \sigma_t. \tag{12}$$

where $\eta = \delta - \gamma$. It is a parsimonious model with only 3 parameters and the constraint $\eta \geq 0$ ensures the positivity of the innovations $\eta X_{t-1}^- / \sigma_t$ of the AR(1) model. As our estimations are not on the frontier $\eta = 0$, the asymptotic normality holds. Plugging our estimations in the explicit formula of the asymptotic variance given in Appendix 4 provides the 95% confident intervals for (α, γ, η) that are reported in Table 5. Remark that the behavior of the model $\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \eta X_{t-1}^- / \sigma_t$ differs completely wether the observations are positive or negative. It is in accordance with the plot in Figure 3: isolated extremes have negative values. It explains why the non symmetric AGARCH and EGARCH models have better in sample performance, see Table 3. For the out of sample performance, the EGARCH and the GARCH models have the best forecasting performances, see Table 4.

We draw the 15 min realized volatilities and the in sample forecasts for $540 \leq t \leq 890$ in Figure 4 (we only consider the last observations where the recurrent formula given by Riskmetrics should be the best) and the out of sample forecasts for $881 \leq t \leq 890$ in Figure 5. Graphically, the forecasts of the EGARCH model are satisfactory because it follows the global fluctuations of the realized volatilities. It is very close to the forecasts of the APGARCH model: their fluctuations are similar and their values are close (the EGARCH forecasts are slightly smaller). But as the APGARCH model relies on 5 unknown coefficients, we prefer to work with the more parsimonious EGARCH model (that degenerates to a form with only 3 unknown parameters).

FIG 4.

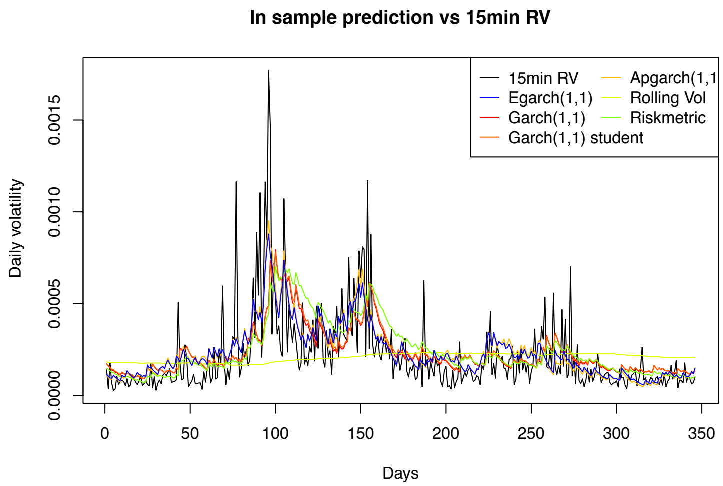
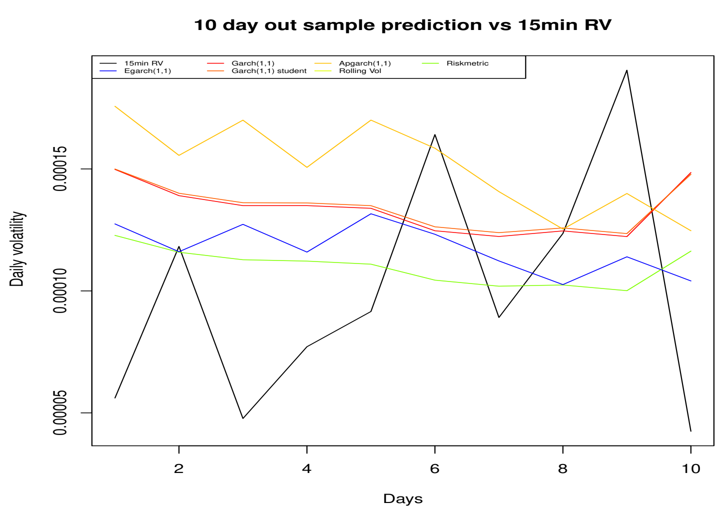

In sample prediction vs 15min RV
15min RV
Egarch(1,1)
Garch(1,1)
Garch(1,1) student
Apgarch(1,1
Rolling Vol
Riskmetric
Daily volatility
Days

FIG 5.


10 day out sample prediction vs 15min RV
15min RV
Egarch(1,1)
Garch(1,1)
Garch(1,1) student
Apgarch(1,1)
Rolling Vol
Riskmetric
Daily volatility
Days

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Appendix 1: Proofs of Section 1

Proof. (Proposition 1) First remark that $\log(1+y+z) \le \log(1+y)+\log(1+z)$ for any $y,z \ge 0$ and that $\mathbb{E}[\log(1+\Lambda(\Psi_0))] < \infty$ and $\mathbb{E}[\log(1+d(\Psi_0(x),x))] < \infty$ by assumption. Second, we remark that $\Lambda(f\circ g) \le \Lambda(f)\Lambda(g)$ for any Lipschitz functions f and g entails that

$$\begin{aligned}\log(1+d(x,\Psi_0\circ\cdots\circ\Psi_{-m}(x))) &\le \log(1+d(x,\Psi_0\circ\cdots\circ\Psi_{1-m}(x))) \\ &\quad +\log(1+\Lambda(\Psi_0\circ\cdots\circ\Psi_{1-m})d(\Psi_{-m}(x),x)). \qquad (13)\end{aligned}$$

Third, we assert that the ergodic theorem in $\mathbb{L}^1$ leads to the existence of $\rho < 1$ such that

$$\frac{1}{n}\sum_{j=1}^{n}\Lambda(\Psi_{(j-1)r}\circ\cdots\circ\Psi_{1-jr}) \to \log(\rho) \qquad \text{in} \qquad \mathbb{L}^1.$$

Using again that $\Lambda(f \circ g) \le \Lambda(f)\Lambda(g)$ for any Lipschitz functions f and g we infer $\overline{\lim}_{m\to\infty} m^{-1}\log(\Lambda(\Psi_0 \circ \cdots \circ \Psi_{-m})) \le \log(\rho)$ in $\mathbb{L}^1$. Thus it exists a r. v. $M > 0$ such that $\mathbb{E}[|\log(M)|] < \infty$ and $m^{-1}\log(\Lambda(\Psi_0 \circ \cdots \circ \Psi_{-m})) - \log(\rho) \le \log(M)$, i.e. $\Lambda(\Psi_0 \circ \cdots \circ \Psi_{-m}) \le (M\rho)^m$ a.s. for all $m \ge 0$. Using this bound in (13) we get, writing $v_m = \log(1 + d(x, \Psi_0 \circ \cdots \circ \Psi_{-m}(x)))$:

$$v_m \le v_{m-1} + \log(1 + (M\rho)^{m-1} d(\Psi_{-m}(x), x)).$$

As $\log(1 + yz) \le y^\omega \log(1 + z)$ for any $0 < y < 1$, $z \ge 0$ and $0 < \omega < 1$ we get by a straightforward recurrence

$$v_m \le v_0 + \sum_{j=0}^{m-1} \rho^{j\omega} \log(1 + M^j d(\Psi_0(x), x)).$$

As $\log(1 + yz) \le \log(1 + y^2 + z^2) \le 2(\log(1 + y) + \log(1 + z))$ for any $y, z \ge 0$ we assert that $\mathbb{E}[\log(1 + M^j d(\Psi_0(x), x))] \le Cj$ for some $C > 0$ depending on the (finite) logarithmic moments of M and $d(\Psi_0(x), x))$. Finally, $\lim_{m\to\infty} v_m = \log(1 + d(Y_o, x))$ is dominated by an integrable r.v. and the result follows dominated integration. □

Proof. (Theorem 2) For any $\rho > 0$, let us write $\Lambda_*^{(r)}(\theta, \rho) = \sup\{\Lambda_{\phi_0}^{(r)}(\theta'), \theta' \in \overline{B}(\theta, \rho) \cap \Theta\}$, where $\overline{B}(\theta, \rho)$ stands for the closed ball centered at θ with radius ρ. As $\mathbb{E}[\sup_\Theta \log \Lambda_{\phi_0}^{(r)}(\theta)] < \infty$, by the dominated convergence Theorem we obtain $\lim_{\rho\to 0} \mathbb{E}(\Lambda_*^{(r)}(\theta, \rho)) = \mathbb{E}(\lim_{\rho\to 0} \Lambda_*^{(r)}(\theta, \rho))$. But $\lim_{\rho\to 0} \Lambda_*^{(r)}(\theta, \rho) = \Lambda_{\phi_0}^{(r)}(\theta)$ by continuity of $\Lambda_{\phi_0}^{(r)}(\theta)$, we finally obtain

$$\lim_{\rho\to 0} \mathbb{E}(\Lambda_*^{(r)}(\theta, \rho)) = \mathbb{E}(\limsup_{\theta'\to\theta} \Lambda_{\phi_0}^{(r)}(\theta))) = \mathbb{E}(\Lambda_{\phi_0}^{(r)}(\theta)) < 0.$$

Thus, there exists an $\epsilon > 0$ such that $\mathbb{E}(\Lambda_*^{(r)}(\theta, \epsilon)) < 0$.

Let us now work on $C(\overline{B}(\theta, \epsilon) \cap \Theta)$, the complete metric space of continuous functions from $\overline{B}(\theta, \epsilon) \cap \Theta$ to $\mathbb{R}$ equipped with the supremum norm $d_\infty = \sup_{\overline{B}(\theta,\epsilon)\cap\Theta} d$. In this setting $(\hat{g}_t)$ satisfy a functional SRE $(\hat{g}_k)_{k\le t} = \phi_t((\hat{g}_k)_{k\le t-1})$

with Lipschitz constants satisfying

$$\begin{aligned}
\Lambda_\infty(\phi_t^{(r)}(\cdot)) &\le \sup_{s_1,s_2\in C(\overline{B}(\theta,\epsilon)\cap\Theta)} \frac{d_\infty(\phi_t^{(r)}(s_1),\phi_t^{(r)}(s_2))}{d_\infty(s_1,s_2)} \\
&\le \sup_{s_1,s_2\in C(\overline{B}(\theta,\epsilon)\cap\Theta)} \frac{\sup_{\overline{B}(\theta,\epsilon)\cap\Theta} d(\phi_t^{(r)}(s_1(\theta'),\theta'),\phi_t^{(r)}(s_2(\theta'),\theta')}{d_\infty(s_1,s_2)} \\
&\le \sup_{s_1,s_2\in C(\overline{B}(\theta,\epsilon)\cap\Theta)} \frac{\sup_{\overline{B}(\theta,\epsilon)\cap\Theta} \Lambda(\phi_t^{(r)}(\cdot,\theta'))d(s_1(\theta')s_2(\theta'))}{d_\infty(s_1,s_2)} \\
&\le \sup_{s_1,s_2\in C(\overline{B}(\theta,\epsilon)\cap\Theta)} \frac{\sup_{\overline{B}(\theta,\epsilon)\cap\Theta} \Lambda(\phi_t^{(r)}(\cdot,\theta'))d_\infty(s_1,s_2)}{d_\infty(s_1,s_2)} \\
&\le \sup_{\overline{B}(\theta,\epsilon)\cap\Theta} \Lambda(\phi_t^{(r)}(\cdot,\theta')) \le \sup_{\overline{B}(\theta,\epsilon)\cap\Theta} \Lambda_{\phi_t}^{(r)}(\theta') \le \Lambda_*^{(r)}(\theta,\epsilon)).
\end{aligned}$$

As $\mathbb{E}[\sup_{\overline{B}(\theta,\epsilon)\cap\Theta}\log^+(d(\phi_t(y,\theta'),y))] \le \mathbb{E}[\sup_\Theta \log^+(d(\phi_t(y,\theta),y))]$ is finite we can apply Theorem 1. By recurrence $\phi_t\circ\cdots\circ\phi_{t-m}(\zeta_0)\in C(\overline{B}(\theta,\epsilon)\cap\Theta)$ is continuous in θ and so is g_t as the convergence holds uniformly on $\overline{B}(\theta,\epsilon)\cap\Theta$. It is true for any $\theta\in\Theta$ and the result follows. □

Appendix 2: checking the assumption (MM)

Similar computations have been done in Demos and Kyriakopoulou (2009). Remember that $\mathbf{V}=\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}$ with $\mathbf{P}=\mathbb{E}[\mathbb{H}s_0(\theta_0)]$ and $\mathbf{Q}=\mathbb{E}[\nabla s_0(\theta_0)\nabla s_0(\theta_0)^T]$. Let us first prove the three identities $\mathbf{P}=2^{-1}\mathbf{B}$, $\mathbf{Q}=4^{-1}(\mathbb{E}Z_0^4-1)\mathbf{B}$ and thus $\mathbf{V}=(\mathbb{E}Z_0^4-1)\mathbf{B}^{-1}$ with $\mathbf{B}=\mathbb{E}\left[\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T\right]$. For the first identity, we compute

$$\begin{aligned}
\mathbf{P} &= 2^{-1}\mathbb{E}\left[(\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T Z_0^2 + \mathbb{H}g_t(\theta_0)(1-Z_0^2)\right] \\
&= 2^{-1}\mathbb{E}[\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T] = 2^{-1}\mathbf{B}.
\end{aligned}$$

For the second identity, we compute

$$\begin{aligned}
\mathbf{Q} &= \mathbb{E}\left[\frac{1}{4}\mathbb{E}\left[\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T(1-\mathbb{Z}_t^2)^2\right]|\mathcal{F}_{t-1}\right] \\
&= 4^{-1}\mathbb{E}[(1-\mathbb{Z}_0^2)^2]\mathbb{E}[\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T] = 4^{-1}(\mathbb{E}Z_0^4-1)\mathbf{B}
\end{aligned}$$

and the third identity follows the first ones. Thus, for checking the assumption **(MM)**, it is enough to check that diagonal coefficients $\mathbf{B}_{ii}$ are well defined when $\mathbb{E}(Z_0^4)<\infty$. Let us denote $W_t=\gamma Z_t+\delta|Z_t|$, $U_t=(1,\log\sigma_t^2,Z_t,|Z_t|)$ and $V_t=\beta-\frac{1}{2}(\gamma Z_t+\delta|Z_t|)$. Then $(\nabla g_t(\theta_0))$ is the solution of the linear SRE

$$\nabla g_t(\theta_0)=U_{t-1}+V_{t-1}\nabla g_{t-1}(\theta_0)=\sum_{l=1}^{\infty}\left(U_{t-l}\prod_{k=1}^{l-1}V_{t-k}\right).$$

Using the convention $\prod_{k=1}^{0} V_{t-k} = 1$, we obtain the expression

$$\nabla g_t(\theta_0) = \sum_{l=1}^{\infty} \left(U_{t-l} \prod_{k=1}^{l-1} V_{t-k} \right).$$

More precisely, we have the expressions: then

$$\mathbf{B}_{11} = \mathbb{E}\left(\frac{\partial g_t(\theta_0)}{\partial \theta_1}\right)^2 = \mathbb{E}\left[\sum_{l=1}^{\infty} \prod_{k=1}^{l-1} V_{t-k}\right]^2,$$

$$\mathbf{B}_{22} = \mathbb{E}\left(\frac{\partial g_t(\theta_0)}{\partial \theta_2}\right)^2 = \mathbb{E}\left[\sum_{l=1}^{\infty} \log \sigma_{t-l}^2 \prod_{k=1}^{l-1} V_{t-k}\right]^2,$$

$$\mathbf{B}_{33} = \mathbb{E}\left(\frac{\partial g_t(\theta_0)}{\partial \theta_3}\right)^2 = \mathbb{E}\left[\sum_{l=1}^{\infty} Z_{t-l} \prod_{k=1}^{l-1} V_{t-k}\right]^2,$$

$$\mathbf{B}_{44} = \mathbb{E}\left(\frac{\partial g_t(\theta_0)}{\partial \theta_i}\right)^2 = \mathbb{E}\left[\sum_{l=1}^{\infty} |Z_{t-l}| \prod_{k=1}^{l-1} V_{t-k}\right]^2.$$

To prove that condition **(MM)** is satisfied, i.e. that $\sum_{i=1}^{4} \mathbf{B}_{ii} < \infty$, we use the following Lemma

Lemma 1. $\sum_{i=1}^{4} \mathbf{B}_{ii} < \infty$ *iff* $\mathbb{E}V_0^2 < 1$ *iff* $\beta^2 - \delta\mathbb{E}|Z_0| + (\delta^2 + \gamma^2)/4 < 1$.

Proof. That the first coefficient B_{11} is finite comes easily:

$$\begin{aligned}
\mathbf{B}_{11} = \mathbb{E}(\sum_{l=1}^{\infty} \prod_{k=1}^{l-1} V_{t-k})^2 &= \mathbb{E}(\sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \prod_{k=1}^{l-1} V_{t-k} \prod_{k'=1}^{l'-1} V_{t-k'}) \\
&= \mathbb{E}(2 \sum_{l \geq 1}^{\infty} (\prod_{k=1}^{l-1} V_{t-k})^2 \sum_{l' > l}^{\infty} \prod_{k'=l}^{l'-1} V_{t-k'} + \mathbb{E} \sum_{l=1}^{\infty} (\prod_{k=1}^{l-1} V_{t-k}{}^2) \\
&= 2 \sum_{l \geq 1}^{\infty} (\mathbb{E}V_0^2)^{l-1} \frac{\mathbb{E}V_0}{1 - \mathbb{E}V_0} + \frac{1}{1 - \mathbb{E}V_0^2} \\
&= 2 \frac{1}{1 - \mathbb{E}V_0^2} \times \frac{\mathbb{E}V_0}{1 - \mathbb{E}V_0} + \frac{1}{1 - \mathbb{E}V_0^2}.
\end{aligned}$$

For the second coefficient $\mathbf{B}_{22}$, it is more complicated. We need some preliminary work. We know that $W_t = \gamma Z_t + \delta |Z_t| = 2(\beta - V_t)$, and

$$\log \sigma_t^2 = \frac{\alpha}{1-\beta} + \sum_{k=1}^{\infty} \beta^{k-1} W_{t-k} = \frac{\alpha + 2\beta}{1-\beta} - 2 \sum_{k=1}^{\infty} \beta^{k-1} V_{t-k}$$

so, we decompose $\mathbf{B}_{22}$ into three parts,

$$\begin{aligned}
\mathbf{B}_{22} &= \mathbb{E}\left[\sum_{l=1}^{\infty}\log\sigma_{t-l}^2\prod_{k=1}^{l-1}V_{t-k}\right]^2 \\
&= \mathbb{E}\left[\sum_{l=1}^{\infty}\left(\frac{\alpha+2\beta}{1-\beta}-2\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\right)\prod_{k'=1}^{l-1}V_{t-k'}\right]^2 \\
&= (\frac{\alpha+2\beta}{1-\beta})^2\mathbb{E}\left[\sum_{l=1}^{\infty}\prod_{k'=1}^{l-1}V_{t-k'}\right]^2+4\mathbb{E}\left[\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\prod_{k'=1}^{l-1}V_{t-k'}\right]^2 \\
&\quad -4\times\frac{\alpha+2\beta}{1-\beta}\mathbb{E}\left[\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\prod_{k'=1}^{l-1}V_{t-k'}\right].
\end{aligned}$$

That the first term of the sum is finite is already known. For the last term, it is straightforward from $\mathbb{E}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\prod_{k'=1}^{l-1}V_{t-k'}=(1-\beta)^{-1}\mathbb{E}V_0/(1-\mathbb{E}V_0)$. For the second term of the sum, we need an expansion

$$\begin{aligned}
&\left[\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\prod_{k'=1}^{l-1}V_{t-k'}\right]^2 \\
&= 2\times\sum_{1\le l<l'<\infty 1}^{\infty}\sum_{p,q=1}^{\infty}\beta^{p+q-2}V_{t-l-p}V_{t-l'-q}\prod_{p'=1}^{l-1}V_{t-p'}^2\prod_{q'=l}^{l'-1}V_{t-q'} \\
&\quad +\sum_{l=1}^{\infty}\sum_{p,q=1}^{\infty}\beta^{p+q-2}V_{t-l-p}V_{t-l-q}\prod_{p'=1}^{l-1}V_{t-p'}^2 \\
&= 4\times\sum_{1\le l<l'<\infty}\sum_{1\le p<q<\infty}\beta^{p+q-2}V_{t-l-p}V_{t-l'-q}\prod_{p'=1}^{l-1}V_{t-p'}^2\prod_{q'=l}^{l'-1}V_{t-q'} \\
&\quad +2\times\sum_{1\le l<l'<\infty}\sum_{p=1}^{\infty}\beta^{2p-2}V_{t-l-p}V_{t-l'-p}\prod_{p'=1}^{l-1}V_{t-p'}^2\prod_{q'=l}^{l'-1}V_{t-q'} \\
&\quad +2\sum_{l=1}^{\infty}\sum_{1\le p<q<\infty}\beta^{p+q-2}V_{t-l-p}V_{t-l-q}\prod_{p'=1}^{l-1}V_{t-p'}^2+\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\beta^{2p-2}V_{t-l-p}^2\prod_{p'=1}^{l-1}V_{t-p'}^2
\end{aligned}$$

and in expectation we obtain a bounded term if $\mathbb{E}V_0^2<1$:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\beta^{k-1}V_{t-l-k}\prod_{k'=1}^{l-1}V_{t-k'}\right]^2\\
&= 4\times\frac{\mathbb{E}V_0^2}{1-\mathbb{E}V_0^2}\left[\frac{\beta}{(1-\beta)(1-\beta^2)}\frac{\mathbb{E}V_0}{1-\mathbb{E}V_0}-\frac{1}{(1-\beta)(1-\beta^2)}\frac{\mathbb{E}V_0\beta}{1-\beta^2\mathbb{E}V_0}\right]\\
&\quad +4\times\frac{\beta(\mathbb{E}V_0)^3}{(1-\beta)(1-\beta^2)(1-\beta^2\mathbb{E}V_0)}\frac{1}{1-\mathbb{E}V_0^2}\\
&\quad +2\times\frac{1}{1-\beta^2}\frac{\mathbb{E}V_0^2}{1-\mathbb{E}V_0^2}\left[\frac{\mathbb{E}V_0}{1-\mathbb{E}V_0}-\frac{\mathbb{E}V_0}{1-\beta^2\mathbb{E}V_0}\right]\\
&\quad +2\frac{1}{1-\mathbb{E}V_0^2}\frac{(\mathbb{E}V_0)^3}{(1-\beta^2)(1-\beta^2\mathbb{E}V_0)}\\
&\quad +2\frac{1}{1-\mathbb{E}V_0^2}(\mathbb{E}V_0)^2\frac{\beta}{(1-\beta)(1-\beta^2)}+\frac{\mathbb{E}V_0^2}{1-\mathbb{E}V_0^2}\frac{1}{1-\beta^2}.
\end{aligned}
$$

That $\mathbf{B}_{33}$ is finite under $\mathbb{E}V_0^2<1$ comes from

$$
\begin{aligned}
\mathbf{B}_{33} &= \mathbb{E}\left[\sum_{l=1}^{\infty}Z_{t-l}\prod_{k=1}^{l-1}V_{t-k}\right]^2\\
&= 2\mathbb{E}\sum_{l=1}^{\infty}\sum_{l'>l}^{\infty}Z_{t-l}Z_{t-l'}\prod_{k=1}^{l-1}V_{t-k}\prod_{k'=1}^{l'-1}V_{t-k'}+\mathbb{E}\sum_{l=1}^{\infty}Z_{t-l}^2(\prod_{k=1}^{l-1}V_{t-k})^2\\
&= \mathbb{E}Z_0^2\sum_{l=1}^{\infty}(\mathbb{E}V_0^2)^{l-1}=\frac{\mathbb{E}Z_0^2}{1-\mathbb{E}V_0^2}.
\end{aligned}
$$

That the last coefficient is also finite comes form the computation

$$
\begin{aligned}
&\mathbf{B}_{44} = \mathbb{E}\left[\sum_{l=1}^{\infty}|Z_{t-l}|\prod_{k=1}^{l-1}V_{t-k}\right]^2 =\\
&2\mathbb{E}\sum_{l=1}^{\infty}\sum_{l'>l}^{\infty}|Z_{t-l}|\left|Z_{t-l'}\right|\prod_{k=1}^{l-1}V_{t-k}\prod_{k'=1}^{l'-1}V_{t-k'}+\mathbb{E}\sum_{l=1}^{\infty}Z_{t-l}^2(\prod_{k=1}^{l-1}V_{t-k})^2\\
&= 2\sum_{l=1}^{\infty}\sum_{l'>l}^{\infty}\mathbb{E}\left|Z_{t-l'}\right|\mathbb{E}\left(\prod_{k=1}^{l-1}V_{t-k}^2\right)\mathbb{E}\left(|Z_{t-l}|\prod_{k'=l}^{l'-1}V_{t-k'}\right)+\frac{\mathbb{E}Z_0^2}{1-\mathbb{E}V_0^2}\\
&= 2\sum_{l=1}^{\infty}\sum_{l'>l}^{\infty}(\mathbb{E}\,|Z_0|)(\mathbb{E}V_0^2)^{l-1}(\mathbb{E}\,|Z_0|\,V_0)\mathbb{E}V_0^{l'-l-1}+\frac{\mathbb{E}Z_0^2}{1-\mathbb{E}V_0^2}\\
&= \frac{2\mathbb{E}\,|Z_0|\,(\mathbb{E}\,|Z_0|\,V_0)}{(1-\mathbb{E}V_0)(1-\mathbb{E}V_0^2)}+\frac{\mathbb{E}Z_0^2}{1-\mathbb{E}V_0^2}.
\end{aligned}
$$

□

Appendix 3: checking the assumption (LI)

Let $x_0 \in \mathbb{R}^4$ be a vector such that $\nabla g_0(\theta_0)^T x_0 = 0$ a.s. Since $(\nabla g_t(\theta_0))_{t\in\mathbb{Z}}$ is stationary, then we also have $\nabla g_1(\theta_0)^T x_0 = 0$ a.s. We know that $\nabla g_1(\theta_0) = U_0(\theta_0) + V_0(\theta_0)\nabla g_0(\theta_0)$, then we deduce

$$U_0(\theta_0)^T x_0 = \begin{pmatrix} 1 \\ \log \sigma_0^2 \\ Z_0 \\ |Z_0| \end{pmatrix}^T x_0 = 0 \text{ a.s.}$$

which is impossible for $x_0 \neq 0$ if Z_0 is not concentrated on two points, see Lemma 8.2 of Straumann and Mikosch (2006) for more details.

Appendix 4: exact computation of V

Remember that $\mathbf{V} = (\mathbb{E}Z_0^4 - 1)\mathbf{B}^{-1}$ with $\mathbf{B} = \mathbb{E}\nabla g_t(\theta_0)(\nabla g_t(\theta_0))^T$. According the notation of Appendix 2, denote $W_t = \gamma Z_t + \delta|Z_t|$, $U_t = (1, \log\sigma_t^2, Z_t, |Z_t|)$, $V_t = \beta - \frac{1}{2}(\gamma Z_t + \delta\,|Z_t|)$ and $\mathbb{E}\nabla g_t(\theta_0) = \mathbf{G}$. Remark that $\mathbb{E}W_t = \delta\mathbb{E}\,|Z_0|$, $\mathbb{E}W_t^2 = (\gamma^2+\delta^2)\mathbb{E}Z_0^2$, $\mathbb{E}V_t = \beta - \frac{1}{2}\delta\mathbb{E}\,|Z_t|$, $\mathbb{E}V_t Z_t = -\frac{1}{2}\gamma\mathbb{E}Z_0^2$, $\mathbb{E}V_t\,|Z_t| = \beta\mathbb{E}\,|Z_t| - \frac{1}{2}\delta\mathbb{E}Z_0^2$, $\mathbb{E}V_t^2 = \beta^2 - \beta\delta\mathbb{E}\,|Z_t| + \frac{1}{4}(\gamma^2+\delta^2)\mathbb{E}Z_0^2$ and $\mathbb{E}\log\sigma_t^2 = \frac{\alpha+\delta\mathbb{E}|Z_0|}{1-\beta}$. We also have

$$\begin{aligned}
\mathbb{E}(\log\sigma_t^2)^2 &= \mathrm{Var}\,(\log\sigma_t^2) + (\mathbb{E}\log\sigma_t^2)^2 \\
&= \sum_{k=1}^{\infty}\beta^{2(k-1)}\mathrm{Var}\,(W_{t-k}) + \frac{(\alpha+\delta\mathbb{E}|Z_0|)^2}{(1-\beta)^2} \\
&= \frac{\gamma^2+\delta^2-(\delta\mathbb{E}|Z_0|)^2}{1-\beta^2} + \frac{(\alpha+\delta\mathbb{E}|Z_0|)^2}{(1-\beta)^2}
\end{aligned}$$

and $\mathbb{E}V_tU_t = (\mathbb{E}V_0, \mathbb{E}V_0\mathbb{E}\log\sigma_0^2, -\frac{1}{2}\gamma, \beta\mathbb{E}\,|Z_t| - \frac{1}{2})^T$. Since $\mathbb{E}\,\|\nabla g_t(\theta_0)\|_2^2 < \infty$ from Appendix 2, taking expectation on both side of the equation

$$\mathbf{G} = \mathbb{E}\nabla g_t(\theta_0) = \mathbb{E}U_{t-1} + \mathbb{E}V_{t-1}\mathbb{E}\nabla g_{t-1}(\theta_0) = \mathbb{E}U_{t-1} + \mathbb{E}V_{t-1}\mathbf{G}$$

so $\mathbf{G} = \mathbb{E}U_0/(1-\mathbb{E}V_0) = (1-\beta+\frac{1}{2}\delta\mathbb{E}|Z_0|)^{-1}(1, \mathbb{E}\log\sigma_t^2, 0, \mathbb{E}|Z_0|)^T$. Using again the SRE, we have

$$\begin{aligned}
\nabla g_t(\theta_0)\,(\nabla g_t(\theta_0))^T &= [U_{t-1} + V_{t-1}\nabla g_{t-1}(\theta_0)]\,[U_{t-1} + V_{t-1}\nabla g_t(\theta_0)]^T \\
&= U_{t-1}U_{t-1}^T + V_{t-1}\left[U_{t-1}(\nabla g_{t-1}(\theta_0))^T + \nabla g_{t-1}(\theta_0)U_{t-1}^T\right] \\
&\quad + V_{t-1}^2\left[\nabla g_{t-1}(\theta_0)\,(\nabla g_{t-1}(\theta_0))^T\right]
\end{aligned}$$

so

$$\mathbf{B} = \mathbb{E}U_{t-1}U_{t-1}^T + \mathbb{E}\left(V_{t-1}\left[U_{t-1}(\nabla g_{t-1}(\theta_0))^T + \nabla g_{t-1}(\theta_0)U_{t-1}^T\right]\right) + \mathbb{E}V_{t-1}^2\mathbf{B}$$

and $\mathbf{B} = (1 - \mathbb{E}V_0^2)^{-1}(\mathbb{E}U_0U_0^T + \mathbf{F})$ where $\mathbf{F} = \mathbb{E}\left[V_0\left[U_0(\nabla g_0(\theta_0))^T + \nabla g_0(\theta_0)U_0^T\right]\right]$. As we have

$$\mathbb{E}V_tU_t(\nabla g_t(\theta_0))^T = \mathbb{E}\begin{pmatrix} V_t \\ V_t\log\sigma_t^2 \\ V_tZ_t \\ V_t\,|Z_t| \end{pmatrix}(\nabla g_t(\theta_0))^T = \begin{pmatrix} \mathbb{E}V_t\mathbf{G}^T \\ \mathbb{E}V\mathbb{E}\log\sigma_t^2(\nabla g_{t-1}(\theta_0))^T \\ \mathbb{E}V_tZ_t\mathbf{G}^T \\ \mathbb{E}V_t\,|Z_t|\,\mathbf{G}^T \end{pmatrix}$$

it remains to calculate

$$\begin{aligned}\mathbb{E}\left[\nabla g_t(\theta_0)\log\sigma_t^2\right] &= \mathbb{E}\nabla g_t(\theta_0)\left(\frac{\alpha+2\beta}{1-\beta} - 2\sum_{k=1}^{\infty}\beta^{k-1}V_{t-k}\right) \\ &= \frac{\alpha+2\beta}{1-\beta}\mathbf{G} - 2\mathbb{E}\sum_{l=1}^{\infty}\left(U_{t-l}\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\sum_{k_2=1}^{\infty}\beta^{k_2-1}V_{t-k_2}\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}\sum_{l=1}^{\infty}\left(U_{t-l}\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\sum_{k_2=1}^{\infty}\beta^{k_2-1}V_{t-k_2} \\ &= \mathbb{E}U_{t-1}V_{t-1} + \mathbb{E}\sum_{l=2}^{\infty}\left(U_{t-l}\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\sum_{k_2=1}^{l-1}\beta^{k_2-1}V_{t-k_2} \\ &\quad + \mathbb{E}\sum_{l=2}^{\infty}\left(U_{t-l}\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\beta^{l-1}V_{t-l} + \mathbf{A} \\ &= \sum_{l=2}^{\infty}\mathbb{E}U_0(\mathbb{E}V_0)^{l-2}\mathbb{E}V_0^2\frac{1-\beta^{l-1}}{1-\beta} + \sum_{l=1}^{\infty}\mathbb{E}V_0U_0(\mathbb{E}V_0)^{l-1}\beta^{l-1} + \mathbf{A} \\ &= \frac{\mathbb{E}U_0\mathbb{E}V_0^2}{1-\beta}\left(\frac{1}{1-\mathbb{E}V_0} - \frac{\beta}{1-\beta\mathbb{E}V_0}\right) + \frac{\mathbb{E}V_0U_0}{1-\beta\mathbb{E}V_0} + \mathbf{A}.\end{aligned}$$

Now we treat the last term. Remark that U_t and V_{t-1} are independent except for their second coordinates, then for $j \neq 2$:

$$\begin{aligned}\mathbf{A}_j &= \mathbb{E}\sum_{l=1}^{\infty}\left(U_{t-l,j}\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\sum_{k_2=l+1}^{\infty}\beta^{k_2-1}V_{t-k_2} \\ &= \sum_{l=1}^{\infty}\left(\mathbb{E}U_{0,j}(\mathbb{E}V_0)^{\ell-1}\right)\sum_{k_2=l+1}^{\infty}\beta^{k_2-1}\mathbb{E}V_0 = \frac{\beta\mathbb{E}U_{0,j}\mathbb{E}V_0}{(1-\beta)(1-\beta\mathbb{E}V_0)}.\end{aligned}$$

For $j = 2$, we get

$$\begin{aligned}\mathbf{A}_2 &= \mathbb{E}\sum_{l=1}^{\infty}\left(\log\sigma_{t-l}^2\prod_{k_1=1}^{l-1}V_{t-k_1}\right)\sum_{k_2=l+1}^{\infty}\beta^{k_2-1}V_{t-k_2} \\ &= \sum_{l=1}^{\infty}(\mathbb{E}V_0)^{\ell-1}\mathbb{E}\left(\log\sigma_{t-l}^2\sum_{k_2=l+1}^{\infty}\beta^{k_2-1}V_{t-k_2}\right).\end{aligned}$$

Remembering that

$$\log \sigma_t^2 = \frac{\alpha + 2\beta}{1-\beta} - 2\sum_{k=1}^{\infty} \beta^{k-1} V_{t-k} \Leftrightarrow \sum_{k_2=l+1}^{\infty} \beta^{k_2-1} V_{t-k_2} = \frac{\beta^l}{2}\left(\frac{\alpha+2\beta}{1-\beta} - \log \sigma_{t-l}^2\right)$$

we finally obtain

$$\mathbf{A}_2 = \frac{\beta}{2(1-\beta E V_0)}\left(\frac{\alpha+2\beta}{1-\beta}\mathbb{E}\log \sigma_0^2 - \mathbb{E}(\log \sigma_0^2)^2\right).$$