Non-negative demand in newsvendor models:The case of singly truncated normal samples

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ABSTRACT

This paper considers the classical newsvendor model when demand is normally distributed but with a large coefficient of variation. This leads to observe with a non-negligible probability negative values that do not make sense. To avoid the occurrence of such negative values, first, we derive generalized forms for the optimal order quantity and the maximum expected profit using properties of singly truncated normal distributions. Since truncating at zero produces non-symmetric distributions for the positive values, three alternative models are used to develop confidence intervals for the true optimal order quantity and the true maximum expected profit under truncation. The first model assumes traditional normality without truncation, while the other two models assume that demand follows (a) the log-normal distribution and (b) the exponential distribution. The validity of confidence intervals is tested through Monte-Carlo simulations, for low and high profit products under different sample sizes and alternative values for coefficient of variation. For each case, three statistical measures are computed: the coverage, namely the estimated actual confidence level, the relative average half length, and the relative standard deviation of half lengths. Only for very few cases the normal and the log-normal model produce confidence intervals with acceptable coverage but these intervals are characterized by low precision and stability.

Keywords: Inventory Management, Newsvendor model, Truncated normal, Demand estimation, confidence intervals, Monte-Carlo simulations

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1. INTRODUCTION

Newsvendor models are used to develop optimal order quantity decisions for products whose life-cycle of demand lasts a single relatively short period. In the classical form of the newsvendor model (Khouja, 1999), the optimal order quantity that maximizes expected profit is determined by equating the probability of demand not to exceed order quantity to a critical fractile whose value depends on selling price, salvage value, and purchase and shortage costs. When the critical fractile is greater than 0,5 (less than), the product is classified as high-profit product (low-profit) (Schweitzer and Cachon, 2000).

For such models, developing optimal inventory policies has been based on the assumption that parameters of demand distribution are known. But the extent of applicability of newsvendor models in inventory management to determine the level of customer service depends upon the estimation of demand parameters. And research on studying the effects of demand estimation on optimal inventory policies is limited (Conrad 1976; Nahmias, 1994; Agrawal and Smith, 1996; Hill, 1997; Bell, 2000). Besides, none of these works addressed the problem of how sampling variability of estimated values of demand parameters influences the quality of estimation concerning optimal inventory policies.

Assuming that demand follows the normal distribution, Kevork (2010) explored the variability of estimates for the optimal order quantity and the maximum expected profit. His analysis showed that the weak point of applying the classical newsvendor model to real life situations is the significant reductions in precision and stability of confidence intervals for the true maximum expected profit when high shortage costs occur. But coefficients of variation (CV) for the normal distributions that were used in Kevork's experimental framework never exceeded 0,2. The reason was that in the process of modeling demand by the normal distribution, the use of large CV results in probabilistic laws that generate negative values with a non-negligible probability (Lau, 1997; Strijbosch and Moors, 2006). The solution to avoid the occurrence of such negative values is to accept that demand follows a normal distribution singly truncated at zero.

Practically, truncated samples of normal distribution appear in cases where recorded measurements exist only for part of the variable, with Lee (1915) to provide the first solution for cases of normal demand in estimating population parameters from censored data. Fischer (1931) extracted maximum likelihood estimators for the mean and standard deviation. Hald (1952) presented among others the cumulative distribution and density of the truncated normal distribution. Halperin (1952a) examined large sample properties of truncated samples for a single parameter population and was the first to extract the maximum likelihood estimator in the case of truncated samples from exponential populations. Harpaz et al. (1982) used Bayesian methods to tackle the problem. Braden and Freimer (1991) examined the sufficiency issue in the case of truncated distribution and named the class of distributions with sufficient statistics as newsboy distributions. These distributions exclude normality.

There are a number of papers in the existing literature related to parameters' estimation of the original population, relying on data from the truncated distribution (Gupta 1952; Halperin 1952b). Cohen (1950, 1961, 1991) examined the maximum likelihood estimation for the doubly truncated normal distributions in an example of a non-steep exponential family. Cohen solved numerically the likelihood equation showing that there is one solution only if the coefficient of variation for the sample considered is less than 1. In the case of values greater than 1 the maximum likelihood estimator yields a distribution of the one-parameter exponential family. Davis (1952) showed the use of the singly truncated normal distribution in the notion of reliability. Castillo and Puig (1999) found that the likelihood ratio test for singly truncated normal against exponentiality can be found in terms of the coefficient of variation of the sample considered. Barr and Sherrill (1999) estimated the maximum likelihhod estimators for the mean and variance of a truncated normal distribution relying on the full sample from the original distribution.

Bebu and Mathew (2009) use normal or lognormal distribution to construct confidence intervals for the mean and variance of the limited or truncated random variables. They also report the coverage probability of the large sample confidence interval from the delta method where the coverage turns out to be below the nominal confidence level even in cases of samples sizes smaller than 80. Barndorff and Nielsen (1978) examined the maximum likelihood estimator for the doubly truncated bormal distributions as exponential family. Efron (1978) and Letac and Mora (1990) showed that the singly truncated normal distributions is an example of a non-steep exponential family. Expressions of the moments for doubly truncated distributions are presented for the Weibull and gamma distributions. Jawitz (2004) derives truncated moment expressions for normal, lognormal, gamma, exponential, Weibull and Gumbel distributions for the double truncated case. These distributions are presented in details in Johnson et al. (1994). Nauman and Buffham (1983) and Consortini and Conforti (1984) analyze the upper truncation on measured moments of exponential and lognormal distributions respectively.

Using, therefore, normal distributions with high CV, to ignore the possibility of a negative value to appear is a potential scenario. But in the environment of performing Monte-

Carlo simulations such negative values eventually will be present. Removing, however, such negative values from the data set, the distribution of the remaining positive observations will display an "artificial non-symmetric picture". This remark reveals a serious problem. When real life data follows skewed distributions, there is a certain degree of ambiguity whether skewness in the data is an outcome of truncation or it is due to some parent non-symmetric probabilistic laws generating the observed values.

The current paper addresses this problem by investigating the consequences of modeling positive demand data after truncation by (a) the log-normal distribution, (b) the exponential distribution, and (c) by the traditional normal distribution ignoring completely that truncations has already taken place. At a first stage, we derive generalized forms for the optimal order quantity and the maximum expected profit when demand is modeled as normal distribution singly truncated at zero. To do so, the expected profit is rewritten to a suitable form that enables the use of properties of truncated normal distributions. In a similar manner we derive the optimal order quantity and the maximum expected profit for the log-normal and the exponential using again properties of their corresponding truncated distribution. Under the three hypothetical distributions, appropriate estimators for the optimal order quantity and the maximum expected profit are considered and their asymptotic distributions are stated. Then the validity of the derived asymptotic confidence intervals using the three hypothetical distributions *"for the true optimal order quantity and the true maximum expected profit under truncation"* is explored for finite samples through Monte-Carlo simulations. The evaluation is based on three statistical criteria which are computed under different combinations of sample size, and values of coefficient of variation, for low and high profit products. For each combination, the criteria summarize the actual confidence level that the interval can succeed as well as the precision and stability that it can attain.

2. OPTIMAL ORDERING POLICIES WHEN DEMAND FOLLOWS THE TRUNCATED NORMAL

Under a normal demand singly truncated at zero with mean μ and variance σ^2, the profit function for the classical newsvendor problem given in Khouja (1999) is modified as:

$$\pi = \begin{cases} (p-c)Q - (p-v)(Q-D_t) & \text{if} \quad 0 < D_t \le Q \\ (p-c)Q + s(Q-D_t) & \text{if} \quad D_t > Q \end{cases}, \tag{1}$$

where, D_t the size for demand for period t, Q the order quantity, p the selling price per unit, c the purchase cost per unit, v the salvage value and s the shortage cost per unit. The expected value of (1) is derived in Appendix, and is given by

$$E(\pi) = (p-c)Q - (p-v)\left\{(Q-\mu) - \sigma\frac{\phi_\theta}{\Phi_\theta}\right\} + (p-v+s)\left\{(Q-\mu)\frac{1-\Phi_z}{\Phi_\theta} - \sigma\frac{\phi_z}{\Phi_\theta}\right\}, \tag{2}$$

with ϕ_z, ϕ_θ to be density functions of the standard normal evaluated respectively at $z = (Q-\mu)/\sigma$ and $\theta = 1/CV = \mu/\sigma$, CV the coefficient of variation, and Φ_z, Φ_θ the corresponding distribution functions.

The optimal order quantity, Q^*, maximizing (2) satisfies the equation

$$\Phi_z = \Pr(D_t \le Q) = \Pr\left(\frac{D_t-\mu}{\sigma} \le \frac{Q-\mu}{\sigma}\right) = \Pr(Z \le z_R) = 1 - \frac{c-v}{p-v+s}\Phi_\theta = R,$$

leading to

$$Q^* = \mu + z_R \cdot \sigma \tag{3}$$

where z_R is the inverse function of the standard normal evaluated at R. Replacing Q with Q^* into (2), the maximum expected profit is given by

$$E(\pi)^* = (p-c)\mu + (p-v)\sigma\frac{\phi_\theta}{\Phi_\theta} - (p-v+s)\sigma\frac{\phi_z}{\Phi_\theta}. \tag{4}$$

Truncating a normal distribution at point "zero" and taking only the positive values, an "*artificial non-symmetric picture*" is produced. This is evident from figures (1a) and (1b). Each graph refers to a single realization of 5000 observations from a normal distribution with mean 300 and standard deviations 300 and 450 respectively. It is obvious that raising CV skewness becomes more severe. Ignoring, therefore, truncation, we could model such a situation by assuming that a classical non-normal probabilistic law governs the generation of demand data. In the current work, we shall investigate the consequences of such an action by assuming that demand follows (a) the log-normal distribution, and (b) the exponential distribution.

Figure 1(a)

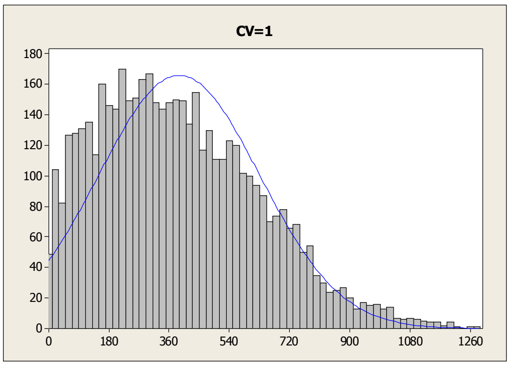
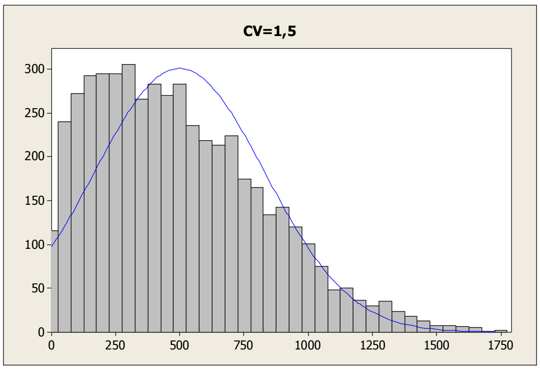


Figure 1(b)



Demand follows the log-normal distribution if $Y \sim N(\mu_{LN}, \sigma^2_{LN})$ and $D_t = e^Y$. In Appendix we show that the optimal order quantity, Q^*_{ln}, and the maximum expected profit, $E(\pi)^*_{ln}$, satisfy the equations

$$\ln Q^*_{ln} = \mu_{LN} + z_R \cdot \sigma_{LN} \tag{5}$$

and

$$\ln E(\pi)^*_{ln} = \mu_{LN} + \frac{\sigma^2_{LN}}{2} + \ln\left[(p - v + s)\Phi_{z_R - \sigma_{LN}} - s\right]. \tag{6}$$

On the other hand, when demand at period t follows the exponential distribution with mean λ, the expected value of (1) is derived in Appendix, and is given by

$$E(\pi)_{exp} = (p - c + s)Q - (p - v + s)Q + (p - v)\lambda - (p - v + s)\lambda e^{-Q/\lambda} \tag{7}$$

First order condition for maximizing (7) leads to $e^{-Q/\lambda} = 1 - (p - c + s)/(p - v + s)$, or

$$\Pr(D_t \le Q) = 1 - e^{-Q/\lambda} = \frac{p - c + s}{p - v + s} = R$$

Thus, the optimal order quantity is taken from $Q^* = -\lambda \ln(1 - R)$ and the maximum expected profit from $E(\pi)^*_{exp} = \lambda[(p - c) + (c - v)\ln(1 - R)]$.

3. ALTERNATIVE ESTIMATORS FOR Q* AND E(π)*

Let $D_1, D_2, \ldots, D_T$, be a sequence of random variables representing demand for a sample of T successive periods. In the current section, we shall evaluate confidence intervals for the true optimal order quantity given in (3) and the true maximum expected profit given in (4) using alternative estimators for each quantity based on the following three models:

Normal Model: In this case we shall ignore that truncation has taken place and we shall assume that demand follows the normal distribution with mean μ and variance σ^2. For this situation, kevork (2010) suggested the following 95% confidence intervals for Q^* and $E(\pi)^*$:

$$\hat{Q}^*_{NM} \pm 1{,}96 \frac{\hat{\sigma}}{\sqrt{T}} \sqrt{1 + \frac{z_R^2}{2}}, \tag{8}$$

$$\hat{E}(\pi)^*_{NM} \pm 1{,}96 \frac{\hat{\sigma}(p-c)}{\sqrt{T}} \sqrt{1 + \frac{1}{2}\left[\left(1 + \frac{s}{p-c}\right)\frac{\phi_{z_R}}{R}\right]^2}, \tag{9}$$

where $\hat{Q}^*_{NM} = \hat{\mu} + z_R \cdot \hat{\sigma}$, $\hat{E}(\pi)^*_{NM} = (p-c)\hat{\mu} - \left[(p-c+s)\phi_{z_R}\hat{\sigma}\right]/R$ are the corresponding estimators for period T+1, and $\hat{\mu}$, $\hat{\sigma}^2$ the maximum likelihood estimators for μ and σ^2.

Log-Normal Model: Denoting by $\hat{\mu}_{LN} = \sum_{t=1}^{T} \ln D_t \Big/ T$ and $\hat{\sigma}^2_{LN} = \sum_{t=1}^{T} \left(\ln D_t - \hat{\mu}_{LN}\right)^2 \Big/ T$, from (5) and (6) the following estimators for period T+1 are defined: $\ln \hat{Q}^*_{LN} = \hat{\mu}_{LN} + z_R \cdot \hat{\sigma}_{LN}$ and $\ln \hat{E}(\pi)^*_{LN} = \hat{\mu}_{LN} + \hat{\sigma}^2_{LN}/2 + \ln\left[(p-v+s)\Phi_{z_R - \hat{\sigma}} - s\right]$. Having been assumed that $\ln D_t \sim N\left(\mu_{LN}, \sigma^2_{LN}\right)$, and using form (12) of Kevork (2010), the suggested 95% confidence interval for Q^* will be

$$\exp\left(\ln \hat{Q}^*_{LN} \pm 1{,}96 \frac{\hat{\sigma}_{LN}}{T} \sqrt{1 + \frac{z_R^2}{2}}\right). \tag{10}$$

In Appendix, we also show that for T sufficiently large, an approximate 95% confidence interval for $E(\pi)^*$ will be

$$\exp\left[\ln \hat{E}(\pi)^*_{LN} \pm 1{,}96\frac{\hat{\sigma}_{LN}}{\sqrt{T}}\sqrt{\left\{1+\frac{1}{2}\left[\hat{\sigma}_{LN} - \frac{(p-v+s)\phi_{z_R-\hat{\sigma}_{LN}}}{(p-v+s)\Phi_{z_R-\hat{\sigma}_{LN}} - s}\right]^2\right\}}\right] \tag{11}$$

Exponential Model: Taking $\hat{\lambda} = \hat{\mu} = \frac{\sum_{t=1}^{T} D_t}{T}$, the following estimators are defined: $\hat{Q}^*_{EXP} = -\hat{\lambda}\ln(1-R)$ and $\hat{E}(\pi)^*_{EXP} = \hat{\lambda}[(p-c)+(c-v)\ln(1-R)]$. Using the central limit theorem for $\hat{\lambda}$, the suggested 95% confidence intervals for Q^* and $E(\pi)^*$ will be

$$\hat{Q}^*_{EXP} \pm 1{,}96\frac{\hat{\sigma}}{\sqrt{T}}\ln(1-R) \tag{12}$$

$$\hat{E}(\pi)^*_{EXP} \pm 1{,}96\frac{\hat{\sigma}}{\sqrt{T}}[(p-c)+(c-v)\ln(1-R)] \tag{13}$$

To study the performance of the three models, we generated 10000 replications of 1000 positive observations from the truncated normal with mean 300 and CV=1, and 1,5. More details for the random number generator which has been used can be found in kevork (2010). To obtain estimates for Q^* and $E(\pi)^*$ according to the three models, values for p, c, v, s have been chosen in order to satisfy the following principles:

(a) Increasing R larger profit margins were set due to the high/low profit product principle stated by Schweitzer and Cachon (2000).

(b) Salvage value was set less than the purchase cost

(c) With R being fixed, to avoid the problem of changing both s and v, we set $s = 0$, assuming that effective customer communication policies have been developed in order not to loose customers when they do not find the product which they are looking for.

Table 1 displays the values for p, c and v as well as the values for the true optimal order quantity and the maximum expected profit.

For each combination of R, CV and sample size T, three statistical measures were computed to summarize the performance of confidence intervals (8)-(13): (a) the Coverage (COV) computed as the percentage of confidence intervals containing the true value, (b) the Relative Average Half Length (RAHL) which is computed dividing the average half-length by Q^* or $E(\pi)^*$, and (c) the Relative Standard Deviation of Half Lengths (RSDHL) computed by dividing the standard deviation of half lengths by Q^* or $E(\pi)^*$.

Table 1: Input parameters for simulation experiments, μ=300, s=0

	R=0,4					R=0,8				
	p	c	v	Q*	E(π)*	p	c	V	Q*	E(π)*
CV=1	200	190	165,14	224,00	1205,41	200	160	147,53	552,49	11289,16
CV=1,5	200	190	149,32	185,99	959,25	200	160	145,39	678,73	13298,15

Table 2 displays the coverage attained by confidence intervals generated using the three models under consideration. For any R, coverage is reduced when sample size is getting larger. Besides, the three models fail to produce acceptable confidence intervals for low-profit products (R=0,4). The exponential model also fails to achieve acceptable coverage for both Q^* and $E(\pi)^*$ even in the case of high-profit products (R=0,8). For R=0,8, only the log-normal model achieves coverage of Q^* close or greater than 0,95 with sample sizes less than 50 observations. Regarding $E(\pi)^*$, acceptable confidence intervals can be produced either with the normal or the log-normal model, when CV takes values close to one for specific sample sizes. Particularly, with the normal model, acceptable coverage is attained with sample sizes around 20 observations, while with the log-normal model the sample size should be between 20 and 100 observations.

Table 2: Coverage of 95% confidence interval for Q^* and $E(\pi)^*$

		Q*				E(π)*			
		R=0,4		R=0,8		R=0,4		R=0,8	
Model		CV=1	CV=1,5	CV=1	CV=1,5	CV=1	CV=1,5	CV=1	CV=1,5
Normal	T								
	20	0,4993	0,077	0,8888	0,8145	0,9064	0,8404	0,9351	0,9229
	30	0,3234	0,0121	0,879	0,7513	0,8985	0,8006	0,933	0,9013
	40	0,1999	0,0017	0,8593	0,6881	0,8781	0,7503	0,9234	0,8846
	50	0,1146	0,0003	0,8472	0,6289	0,8682	0,7023	0,9206	0,8691
	100	0,0044	0	0,7617	0,381	0,786	0,452	0,8893	0,762
	500	0	0	0,2647	0,0037	0,1558	0,0004	0,5384	0,1363
	1000	0	0	0,0611	0	0,0068	0	0,2302	0,0068
Log-Normal	T								
	20	0,8321	0,4833	0,9912	0,982	0,4222	0,0717	0,9278	0,8978
	30	0,844	0,4098	0,9874	0,967	0,369	0,0305	0,9382	0,9049
	40	0,8397	0,3368	0,9783	0,9487	0,3195	0,0122	0,9425	0,9082
	50	0,8462	0,2779	0,968	0,9211	0,2814	0,0034	0,9492	0,9142
	100	0,8588	0,0978	0,8781	0,732	0,1298	0	0,9565	0,9166
	500	0,8371	0	0,124	0,0112	0,0002	0	0,8747	0,8291
	1000	0,8091	0	0,0034	0	0	0	0,7822	0,7166
Exponential	T								
	20	0,7824	0,5756	0,8747	0,8483	0	0	0,095	0,025
	30	0,7363	0,4084	0,8362	0,7924	0	0	0,0235	0,0031
	40	0,6835	0,2769	0,7896	0,7339	0	0	0,0054	0,0003
	50	0,6282	0,1802	0,7557	0,6777	0	0	0,0009	0
	100	0,407	0,0163	0,5646	0,4235	0	0	0	0
	500	0,0042	0	0,025	0,0025	0	0	0	0
	1000	0,0001	0	0	0	0	0	0	0

For the cases where acceptable coverage is attained table 3 displays RAHL and RSDHL. The following three important remarks are pointed out: (a) Precision and stability of confidence intervals is reduced as CV is getting larger, (b) the normal model produces confidence intervals for $E(\pi)^*$ with higher precision and stability compared with the log-normal model, and (c) wherever acceptable coverage is achieved, confidence intervals show low precision and stability. For example, with R=0,8, CV=1 and T=50:

(i) Confidence intervals for Q^* using the log-normal model will have an average total length of approximately 754 units

(ii)) Confidence intervals for $E(\pi)^*$ using the normal model will have an average total length of approximately 5353 monetary units

Table 3: Precision and stability of 95% confidence intervals for Q^* and $E(\pi)^*$ when R=0,8

	Q*				E(π)*			
	log-normal model				normal model		log-normal model	
	CV = 1		CV = 1,5		CV = 1		CV =1	
T	RAHL	RSDHL	RAHL	RSDHL	RAHL	RSDHL	RAHL	RSDHL
20	1,06523	0,54230	1,21044	0,60435	0,36642	0,05919	0,75795	0,19358
30	0,87585	0,34416	0,99711	0,38984	0,30344	0,03953	0,62675	0,12928
40	0,76031	0,25556	0,86511	0,28306	0,26411	0,02990	0,54596	0,09832
50	0,68221	0,20524	0,77624	0,22462	0,23707	0,02385	0,49004	0,07956
100							0,35011	0,04025

4. CONCLUSION

When real-life data (representing variables taking on only positive values) follows skewed distributions, we should not exclude the case that the observed skewness might be caused by a parent normal distribution with a large coefficient of variation for which truncation at point zero has been occurred. This case was investigated in the current work regarding optimal ordering policies for the classical newsvendor problem when shortage cost is zero. Particularly, we assumed that the true distribution generating demand data was the normal singly truncated at zero, and erroneously the remaining part of data after truncation had been modeled as (a) a log-normal, (b) an exponential, and (c) a traditional normal distribution without truncation.

Estimators for the optimal order quantity and the maximum expected profit were considered for each one of the three hypothetical distributions. Based on the asymptotic distribution of the estimators, alternative confidence intervals were suggested for the true optimal order quantity and the true maximum expected profit under truncation. The performance of the confidence intervals was evaluated through Monte-Carlo simulations in different sample sizes from the truncated normal with CV=1 and 1,5. The evaluation was based on the coverage, namely, the estimated actual confidence level the interval can attain, the average half length, and the standard deviation of half lengths. The latter two measures were divided by the true values of the optimal ordered quantity and the maximum expected profit.

For low profit products (R<0,5), confidence intervals derived using the three hypothetical distributions fail to attain coverage close to the nominal confidence level. For R high (e.g. R=0,8), confidence intervals of the normal and the log-normal distribution can succeed acceptable coverage but only for a limited range of small sample sizes. Unfortunately, for the three hypothetical distributions the coverage of their confidence intervals for both the true optimal order quantity and the maximum expected profit is reduced as the sample size is getting larger tending eventually to zero when the sample size becomes sufficiently large. But, even for those cases where acceptable confidence intervals in terms of coverage are produced, their precision and stability are too low offering little information at the stage of decision making.

APPENDIX

Proof of (2)

$$E(\pi) = (p-c)Q - (p-v)\{Q - E(D_t/0 < D_t \le Q)\}\cdot \Pr(0 < D_t \le Q) + \\ + s\{Q - E(D_t/D_t > Q)\}\cdot \Pr(D_t > Q). \tag{A1}$$

Using results on the truncated normal regarding its probability density function and mean (Maddala, 1983, p. 366), we obtain:

$$\Pr(0 < D_t \le Q) = \frac{1}{\Phi_\theta}\int_0^Q \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{Q-\mu}{\sigma}\right)^2} dx = \frac{1}{\Phi_\theta}\int_{-\frac{\mu}{\sigma}}^{\frac{Q-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = 1 - \frac{1-\Phi_z}{\Phi_\theta}, \tag{A2.1}$$

$$E(D_t/0 < D_t \le Q) = \mu - \sigma\frac{\phi_z - \phi_\theta}{\Phi_z - \Phi_\theta}, \tag{A2.2}$$

$$E(D_t/D_t > Q) = \mu + \sigma\frac{\phi_z}{1-\Phi_z}, \tag{A2.3}$$

The result follows after replacing (A2.1)-(A2.3) into (A1).

Proof of (5)

From Johnson et al. (1994, p. 241), and Maddala (1983, p. 369), we obtain:

$$\Pr(D_t \le Q) = \Pr(\ln D_t \le \ln Q) = \Pr\left(Z \le \frac{\ln Q - \mu_{LN}}{\sigma_{LN}}\right) = \Pr(Z \le z_{LN}) = \Phi_{z_{LN}} \tag{A3.1}$$

$$E(D_t / D_t \le Q) = e^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}} \frac{\Phi_{z_{LN} - \sigma_{LN}}}{\Phi_{z_{LN}}} \tag{A3.2}$$

$$E(D_t / D_t > Q) = e^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}} \frac{1 - \Phi_{z_{LN} - \sigma_{LN}}}{1 - \Phi_{z_{LN}}} \tag{A3.3}$$

Replacing (A3.1)-(A3.3) into (2),

$$E(\pi)_{LN} = (p - c + s)Q - (p - v + s)Q\Phi_{z_{LN}} - se^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}} + (p - v + s)e^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}} \Phi_{z_{LN} - \sigma_{LN}} \tag{A4}$$

The result follows from

$$\frac{dE(\pi)_{LN}}{dQ} = (p - c + s) - (p - v + s)\Phi_{z_{LN}} = 0$$

and

$$\Phi_{z_{LN}} = \Pr\left(Z \le \frac{\ln Q - \mu_{LN}}{\sigma_{LN}}\right) = \Pr(Z \le z_R) = \frac{p - c + s}{p - v + s} = R$$

having used the following derivatives:

$$\frac{d\Phi_{z_{LN}}}{dQ} = \frac{d\Phi_{z_{LN}}}{dz_{LN}} \cdot \frac{dz_{LN}}{d\ln Q} \cdot \frac{d\ln Q}{dQ} = \frac{\phi_{z_{LN}}}{\sigma Q}$$

$$\frac{d\Phi_{z_{LN} - \sigma_{LN}}}{dQ} = \frac{d\Phi_{z_{LN} - \sigma_{LN}}}{d(z_{LN} - \sigma_{LN})} \cdot \frac{d(z_{LN} - \sigma_{LN})}{d\ln Q} \cdot \frac{d\ln Q}{dQ} = \frac{1}{\sigma_{LN}} \frac{\phi_{z_{LN}}}{e^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}}}$$

$$\text{since } \phi_{z_{LN} - \sigma_{LN}} = Q \frac{\phi_{z_{LN}}}{e^{\mu_{LN} + \frac{\sigma_{LN}^2}{2}}}$$

Proof of (6)

The result follows after replacing $(p - c + s) = (p - v + s)\Phi_{z_{LN}}$, and

$$z_{LN} - \sigma_{LN} = \frac{\ln Q_{LN}^* - \mu_{LN}}{\sigma_{LN}} = z_R - \sigma_{LN},$$

into (A4).

Proof of (7)

The result follows after replacing the following relationships into (2):

$$\Pr(D_t \leq Q) = 1 - e^{-Q/\lambda}$$

$$E(D_t / D_t \leq Q) = -\frac{Qe^{-Q/\lambda} + \lambda e^{-Q/\lambda} - \lambda}{1 - e^{-Q/\lambda}}$$

$$E(D_t / D_t > Q) = \frac{Qe^{-Q/\lambda} + \lambda e^{-Q/\lambda}}{e^{-Q/\lambda}}$$

Proof of (11)

As $\ln D_t \sim N(\mu_{LN,} \sigma^2_{LN})$, then $\sqrt{T}\begin{bmatrix} \hat{\mu}_{LN} - \mu_{LN} \\ \hat{\sigma}^2_{LN} - \sigma^2_{LN} \end{bmatrix} \xrightarrow{d} N_2[\mathbf{0}, \mathbf{\Sigma}]$ where $\mathbf{\Sigma} = \begin{bmatrix} \sigma^2_{LN} & 0 \\ 0 & 2\sigma^4_{LN} \end{bmatrix}$.

Since $p\lim \ln \hat{E}(\pi)^*_{LN} = p\lim \hat{\mu}_{LN} + \frac{1}{2} p\lim \hat{\sigma}^2_{LN} + \ln\left[(p - v + s) \cdot \Phi_{z_R - (p\lim \hat{\sigma}^2_{LN})^{0,5}} - s\right] = \ln E(\pi)^*$,

by applying the multivariate delta method (knight, 1999),

$$\sqrt{T}\left[\ln \hat{E}(\pi)^*_{LN} - \ln E(\pi)^*\right] \xrightarrow{d} N(0, \mathbf{L}' \cdot \mathbf{\Sigma} \cdot \mathbf{L})$$

where

$$\mathbf{L}' = \left[\frac{\partial f(\hat{\mu}_{LN}, \hat{\sigma}^2_{LN})}{\partial \hat{\mu}_{LN}} \quad \frac{\partial f(\hat{\mu}_{LN}, \hat{\sigma}^2_{LN})}{\partial \hat{\sigma}^2_{LN}} \right]$$

and

$$f(\hat{\mu}_{LN}, \hat{\sigma}^2_{LN}) = \hat{\mu}_{LN} + \frac{1}{2}\hat{\sigma}^2_{LN} + \ln\left[(p - v + s)\Phi_{z_R - (\hat{\sigma}^2_{LN})^{0,5}} - s\right]$$

Evaluating the partial derivatives at $\hat{\mu}_{LN} = \mu_{LN}$ and $\hat{\sigma}^2_{LN} = \sigma^2_{LN}$, we take

$$\frac{\partial f(\hat{\mu}_{LN}, \hat{\sigma}^2_{LN})}{\partial \hat{\mu}_{LN}} = 1,$$

$$\frac{\partial f(\hat{\mu}_{LN}, \hat{\sigma}^2_{LN})}{\partial \hat{\sigma}^2_{LN}} = \frac{1}{2\sigma}\left[\sigma - \frac{(p - v + s)\phi_{z_R - \sigma}}{(p - v + s)\Phi_{z_R - \sigma} - s}\right],$$

and

$$\mathbf{L}' \cdot \mathbf{\Sigma} \cdot \mathbf{L} = \sigma^2 \left\{ 1 + \frac{1}{2}\left[\sigma - \frac{(p - v + s)\phi_{z_R - \sigma}}{(p - v + s)\Phi_{z_R - \sigma} - s}\right]^2 \right\}$$

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