Variable-population voting rules

Pivato, Marcus

Department of Mathematics, Trent University

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Marcus Pivato, Trent University, Canada

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Abstract

Let $\mathcal{X}$ be a set of social alternatives, and let $\mathcal{V}$ be a set of 'votes' or 'signals'. (We do not assume any structure on $\mathcal{X}$ or $\mathcal{V}$). A *variable population voting rule* F takes any number of anonymous votes drawn from $\mathcal{V}$ as input, and produces a nonempty subset of $\mathcal{X}$ as output. The rule F satisfies *reinforcement* if, whenever two disjoint sets of voters independently select some subset $\mathcal{Y} \subseteq \mathcal{X}$, the union of these two sets will also select $\mathcal{Y}$. We show that F satisfies reinforcement if and only if F is a *balance rule.* If F satisfies a form of neutrality, then F is satisfies reinforcement if and only if F is a scoring rule (with scores taking values in an abstract linearly ordered abelian group $\mathcal{R}$); this generalizes a result of Myerson (1995). We also discuss the sense in which the balance or scoring representation of F is unique. Finally, we provide a characterization of two scoring rules: *formally utilitarian* voting and *range voting.*

1 Introduction

Suppose a group of voters must collectively choose some policy from a set of alternatives, using a voting rule F. Suppose we split the voters into two subgroups, and each subgroup, using rule F, selects the alternative x. Then it seems desirable that the combined group, using F, should also select alternative x. We say the rule F satisfies *reinforcement*[1] if it has this property. Smith (1973) and Young (1974b, 1975) showed that 'scoring rules' are the only preference aggregation rules[2] which satisfy reinforcement and are anonymous and neutral (i.e. invariant under relabeling of the voters and/or alternatives). These results have led to characterizations of the Borda rule (Young, 1974a, 1975; Nitzan and Rubinstein, 1981), Kemeny rule (Young and Levenglick, 1978), and plurality rule (Richelson, 1978; Morkelyunas, 1982; Ching, 1996; Yeh, 2008) each as the only preference aggregation rule which satisfies reinforcement along with certain other axioms.

Myerson (1995) generalized the Smith-Young results from preference aggregators to abstract voting rules. In Myerson's framework, there is a finite set of social alternatives

[1]Smith (1973) calls this condition 'separability', while Young (1974b, 1975) calls it 'consistency'. Dhillon (1998) and Dhillon and Mertens (1999) refer to an analogous property as 'extended Pareto'.

[2]Smith (1973) and Young (1974b) consider rules which produce social preference relations as output, whereas Young (1975) considers rules which produce subsets of social alternatives as output; these two frameworks yield two slightly different definitions of 'reinforcement' and 'scoring rule'.

$\mathcal{X}$ and a finite set of 'signals' $\mathcal{V}$. A 'profile' assigns a signal to each voter, and an abstract voting rule selects some nonempty subset of $\mathcal{X}$ for each profile. A ***scoring rule*** is a voting rule where each element of $\mathcal{V}$ assigns a real-valued 'score' to each element of $\mathcal{X}$; the rule then selects the alternative(s) with the highest total score.[3] Myerson showed that if an abstract voting rule satisfied reinforcement, universal domain (i.e. it is defined for all profiles), a form of neutrality (i.e. all social alternatives are treated equally), and an Archimedean/continuity condition he called *overwhelming majority*, then it was a scoring rule. For example, approval voting (Brams and Fishburn, 1983) is a scoring rule —indeed, it is the only abstract voting rule which satisfies reinforcement along with certain other axioms (Fishburn, 1978; Morkelyunas, 1981; Alós-Ferrer, 2006).

In section 2, I extend Myerson's representation theorem, by considering infinite signal sets, and removing the hypotheses of universal domain and overwhelming majority. I do this by considering scoring rules where the scores can range over an abstract linearly ordered abelian group, instead of ranging over the real numbers.[4] I also characterize *balance rules* (another class introduced by Myerson (1995)), the *formally utilitarian* voting rule, and the *range voting* rule. Next, section 3 shows that the 'neutrality' (i.e. permutation-equivariance) properties of a voting rule can be reflected by corresponding neutrality properties of its balance representation or scoring representation. Section 4 considers the sense in which a real-valued balance representation or scoring representation of a voting rule is unique up to some set of 'rescalings'. Section 5 studies the relationship between balance and scoring rules in more detail. Section 6 concludes with some open problems. Appendix A provides some background on linearly ordered abelian groups. Appendix B contains the proofs of all results in the paper. Appendix C contains further uniqueness results.

2 Model and main results

Let $\mathcal{X}$ be a set of social alternatives, and let $\mathcal{V}$ be the set of possible signals which could be sent by each voter. (The sets $\mathcal{X}$ and $\mathcal{V}$ could be finite or infinite.) Let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ and $\mathbb{Z} := \{\pm n;\ n \in \mathbb{N}\}$. For any $\mathbf{n} \in \mathbb{Z}^{\mathcal{V}}$, let $\|\mathbf{n}\| := \sum_{v \in \mathcal{V}} |n_v|$. Define $\mathbb{N}^{\langle\mathcal{V}\rangle} := \{\mathbf{n} \in \mathbb{N}^{\mathcal{V}};\ \|\mathbf{n}\| < \infty\}$. If $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, then $\mathbf{n}$ represents an anonymous profile of voters: for each $v \in \mathcal{V}$, we interpret n_v as the number of voters sending the signal v, while $\|\mathbf{n}\|$ is the size of the whole population. Note that we do not fix $\|\mathbf{n}\|$ in advance. A ***domain*** is any collection of profiles $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $\mathbf{0} \in \mathcal{D}$. (The set $\mathbb{N}^{\langle\mathcal{V}\rangle}$ itself is the ***universal domain***.) A ***(variable population, anonymous) voting rule*** is a correspondence $F : \mathcal{D} \rightrightarrows \mathcal{X}$ such that $F(\mathbf{0}) = \mathcal{X}$. Thus, for all $\mathbf{d} \in \mathcal{D}$, the outcome $F(\mathbf{d}) \subseteq \mathcal{X}$ is a nonempty set (typically a singleton).

A ***linearly ordered abelian group*** is a triple $(\mathcal{R}, +, >)$, where $\mathcal{R}$ is a set, "+" is an abelian group operation, and ">" is a complete, antisymmetric, transitive binary relation such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$. (For example: the set $\mathbb{R}$ of real numbers is a linearly ordered abelian group, with the standard ordering and addition operator. So is any subgroup of $\mathbb{R}$. For any $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a linearly ordered abelian group

[3] Zwicker (2008) has shown that such scoring rules can also be interpreted as 'mean proximity rules': each element of $\mathcal{V}$ and $\mathcal{X}$ is represented as a vector in $\mathbb{R}^N$, and the social choice is the element of $\mathcal{X}$ which is closest to the vector average of the signals of the voters.

[4] Smith (1973) also considered linearly ordered abelian groups.

under vector addition and the lexicographic order.) Such groups are useful for representing infinite-horizon intertemporal preferences, non-probabilistic uncertainty, and preferences where some decision variables have lexicographical priority over others (Pivato, 2011).

For any $\mathbf{r} = (r_v)_{v\in\mathcal{V}} \in \mathcal{R}^{\mathcal{V}}$, we define a group homomorphism $\mathbf{r} : \mathbb{Z}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}$ by setting $\mathbf{r}(\mathbf{z}) := \sum_{v\in\mathcal{V}} z_v r_v$ for all $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$.[5] An *$\mathcal{R}$-valued score system* on $(\mathcal{X},\mathcal{V})$ is an $\mathcal{X}$-indexed collection $\mathsf{S} := \{\mathbf{s}^x\}_{x\in\mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$. For any domain $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$, the *scoring rule* determined by S is the voting rule $F_{\mathsf{S}} : \mathcal{D} \rightrightarrows \mathcal{X}$ defined as follows:

$$F_{\mathsf{S}}(\mathbf{d}) \quad := \quad \operatorname*{argmax}_{x\in\mathcal{X}} \ \mathbf{s}^x(\mathbf{d}), \text{ for all } \mathbf{d} \in \mathcal{D}.$$

Intuitively, $\mathbf{s}^x(\mathbf{d})$ is the 'score' which alternative x receives from the profile $\mathbf{d}$; each voter who sends the signal v contributes s^x_v 'points' to this score. The alternative with the highest score wins.

For example, *plurality vote* is a scoring rule with $\mathcal{V} = \mathcal{X}$, and $\mathcal{R} = \mathbb{Z}$, and $s^x_v = 1$ if $x = v$, while $s^x_v = 0$ if $x \neq v$. *Approval vote* is a scoring rule, where $\mathcal{V}$ is the set of all subsets of $\mathcal{X}$, and $\mathcal{R} = \mathbb{Z}$, and $s^x_v = 1$ if $x \in v$, while $s^x_v = 0$ if $x \notin v$. The *Borda rule* is a scoring rule where $\mathcal{V}$ is the set of all strict preference orders over $\mathcal{X}$, and $\mathcal{R} = \mathbb{Z}$, and $s^x_v = r$ if x is ranked rth place from the bottom in the preference order v. If $(\mathcal{Y}, d)$ is a metric space, and $\mathcal{X},\mathcal{V} \subseteq \mathcal{Y}$, then the *median rule* is the scoring rule where $\mathcal{R} = \mathbb{R}$, and $s^x_v = -d(x, v)$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. This rule picks the element(s) of $\mathcal{X}$ which minimize the average distance to the signals sent by the voters. (If $\mathcal{X} = \mathcal{V} \subseteq \mathbb{R}$ with the standard metric, then this is the usual notion of the median of a collection of real numbers.) For example, the *Kemeny rule* is a median rule where $\mathcal{X} = \mathcal{V}$ is the set of all strict preference orders over some set $\mathcal{A}$ of alternatives, and d is the *Kendall metric* on $\mathcal{X}$ (so $d(x, v)$ is the number of pairwise comparisons on which the orderings x and v disagree).

There are also several scoring rules where $\mathcal{R} = \mathbb{R}$ and $\mathcal{V}$ is some subset of $\mathbb{R}^{\mathcal{X}}$, and $s^{\mathbf{v}}_x := v_x$ for all $\mathbf{v} \in \mathcal{V}$ and $x \in \mathcal{X}$. *Formally utilitarian* voting is obtained by setting $\mathcal{V} := \mathbb{R}^{\mathcal{X}}$.[6] *Range voting* is obtained by setting $\mathcal{V} := [0,1]^{\mathcal{X}}$ (Smith, 2000; Gaertner and Xu, 2011). *Relative utilitarianism* is obtained by setting $\mathcal{V} := \{\mathbf{v} \in [0,1]^{\mathcal{X}};\ \min_{x\in\mathcal{X}} v_x = 0 \text{ and } \max_{x\in\mathcal{X}} v_x = 1\}$ (Dhillon, 1998; Dhillon and Mertens, 1999). Finally, *cumulative voting* is obtained by setting $\mathcal{V} := \{\mathbf{v} \in [0,1]^{\mathcal{X}};\ \sum_{x\in\mathcal{X}} v_x = 1\}$.

Now suppose we first apply one scoring rule F_a, and then use a second scoring rule F_b *only* to break any ties which arise in F_a. This can be modelled as an $\mathcal{R}$-valued scoring rule, where $\mathcal{R} = \mathbb{R}^2$ with the lexicographical order. For example, the procedure, "first apply approval vote; then break any ties using the Borda rule" can be modelled by defining $\mathcal{V} = \mathcal{V}_a \times \mathcal{V}_b$, where $\mathcal{V}_a := \{\text{all subsets of } \mathcal{X}\}$ and $\mathcal{V}_b := \{\text{all preference orders over } \mathcal{X}\}$. For any $(v_a, v_b) \in \mathcal{V}_a \times \mathcal{V}_b$, we set $s(v_a, v_b) := (s_a(v_a), s_b(v_b)) \in \mathbb{R}^2$, where s_a is the approval score system and s_b is the Borda score system (as described above).

[5]Thus, if $\mathcal{R} = \mathbb{R}$, then $\mathbf{r}(\mathbf{z}) = \mathbf{r} \bullet \mathbf{z}$, where "$\bullet$" is the inner product operation on $\mathbb{R}^{\mathcal{V}}$.

[6]'Formally' utilitarian voting corresponds to the *true* utilitarian social welfare order only if the scores assigned by each voter are given by her cardinal utility function. But these scores could also be some monotone transform of her cardinal utility function (e.g. the Nash SWO is obtained by adding the logarithms of voters' utilities). Or these scores could be completely unrelated to cardinal utility data. Hence the qualifier 'formally'.

Hahn's Embedding Theorem says that *any* linearly ordered abelian group is isomorphic to an ordered subgroup of a lexicographically ordered vector space $\mathbb{R}^{\mathcal{I}}$, where $\mathcal{I}$ is (possibly infinite) linearly ordered set.[7] Thus, *any* scoring rule can be interpreted as a (possibly infinite) chain of real-valued scoring rules, each acting as a tie-breaker for the prior ones.

Let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group. Let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain of profiles. An $\mathcal{R}$-*valued balance system* on $(\mathcal{X}, \mathcal{V}, \mathcal{D})$ is an $\mathcal{X}^2$-indexed collection $\mathsf{B} := \{\mathbf{b}^{x,y}\}_{x,y \in \mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$ such that $\mathbf{b}^{x,y} = -\mathbf{b}^{y,x}$ for all $x, y \in \mathcal{X}$ (in particular, $\mathbf{b}^{x,x} = 0$ for all $x \in \mathcal{X}$), and such that,

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{X}} \mathbf{b}^{x,y}(\mathbf{d}) \quad \geq \quad 0, \qquad \text{for all } \mathbf{d} \in \mathcal{D}. \tag{1}$$

We then define the *balance rule* $F_{\mathsf{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ as follows: for all $\mathbf{d} \in \mathcal{D}$ and $x \in \mathcal{X}$, we let $x \in F_{\mathsf{B}}(\mathbf{d})$ if and only if $\mathbf{b}^{x,y}(\mathbf{d}) \geq 0$ for all $y \in \mathcal{X}$. (The condition (1) is equivalent to stipulating that $F_{\mathsf{B}}(\mathbf{d}) \neq \emptyset$ for all $\mathbf{d} \in \mathcal{D}$.)

Example 2.1. (a) Let $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ be an $\mathcal{R}$-valued score system on $(\mathcal{X}, \mathcal{V})$. For all $x, y \in \mathcal{X}$, define $\nabla^{x,y}\mathsf{S} := \mathbf{s}^x - \mathbf{s}^y \in \mathcal{R}^{\mathcal{V}}$, to obtain a balance system $\nabla\mathsf{S} := \{\nabla^{x,y}\mathsf{S}\}_{x,y \in \mathcal{X}}$. Then $F_{\nabla\mathsf{S}}(\mathbf{n}) = F_{\mathsf{S}}(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$.

(b) Let $\mathcal{V}$ be the set of all nonstrict preference orders over $\mathcal{X}$. For all $x, y \in \mathcal{X}$ and $v \in \mathcal{V}$, define $b_v^{x,y} := 1$ if v prefers x to y, while $b_v^{x,y} := -1$ if v prefers y to x, and $b_v^{x,y} := 0$ if v is indifferent between x and y. Then F_{B} is the *Condorcet rule*: for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, we have $x \in F(\mathbf{n})$ if and only if x is a *Condorcet winner* in the profile $\mathbf{n}$ (i.e. for any other $y \in \mathcal{X}$, at least as many voters strictly prefer x over y as the number who strictly prefer y over x). Unfortunately, $F(\mathbf{n}) = \emptyset$ for some $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ (the 'Condorcet paradox'). Let $\mathcal{D} \subset \mathbb{N}^{\langle \mathcal{V} \rangle}$ be the set of all profiles having a Condorcet winner. Then $F_{\mathsf{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ is a balance rule. $\diamondsuit$

Let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$. A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies *reinforcement* if the following is true: for any $\mathbf{n}, \mathbf{m} \in \mathcal{D}$, if $F(\mathbf{n}) \cap F(\mathbf{m}) \neq \emptyset$, then $\mathbf{n} + \mathbf{m} \in \mathcal{D}$, and $F(\mathbf{n} + \mathbf{m}) = F(\mathbf{n}) \cap F(\mathbf{m})$.[8] Here, the profile $(\mathbf{n} + \mathbf{m})$ represents a union of two disjoint subgroups, represented by profiles $\mathbf{n}$ and $\mathbf{m}$. Reinforcement says: if $x \in \mathcal{X}$ and both $\mathbf{n}$ and $\mathbf{m}$ endorse x (i.e. $x \in F(\mathbf{n})$ and $x \in F(\mathbf{m})$), then we should have $x \in F(\mathbf{n} + \mathbf{m})$. Furthermore, in this case, $F(\mathbf{n} + \mathbf{m})$ should consist of *only* those $x \in \mathcal{X}$ which receive this joint endorsement. We now come to our first main result:

Theorem 2.2 *Let $\mathcal{X}$ and $\mathcal{V}$ be arbitrary sets, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be any domain, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a voting rule. Then F satisfies reinforcement if and only if F is a balance rule.*

Not every balance rule is a scoring rule, even if we require $\mathcal{D} = \mathbb{N}^{\langle \mathcal{V} \rangle}$ (see Example 5.2 below). Thus, we must add some other hypotheses to reinforcement to characterize scoring rules. Let $\Pi_{\mathcal{V}}$ be the group of all permutations of $\mathcal{V}$. For any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ and $\pi \in \Pi_{\mathcal{V}}$, we define $\pi(\mathbf{n}) := \mathbf{m}$, where $m_v := n_{\pi^{-1}(v)}$ for all $v \in \mathcal{V}$. Let $\Pi_{\mathcal{X}}$ be the group of all permutations of $\mathcal{X}$. A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is *neutral* if there exists a group homomorphism $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ (the *neutralizer*) such that, for all $\pi \in \Pi_{\mathcal{X}}$, if $\widetilde{\pi} := \nu(\pi)$,

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[7] See Hausner and Wendel (1952).

[8] Note that we do *not* require the domain $\mathcal{D}$ itself to be closed under addition. For example, the Condorcet rule in Example 2.1(b) satisfies reinforcement, but its domain is not additively closed.

then the domain $\mathcal{D}$ is $\widetilde{\pi}$-invariant, and $F(\widetilde{\pi}(\mathbf{d})) = \pi(F(\mathbf{d}))$ for all $\mathbf{d} \in \mathcal{D}$. Thus, every alternative in $\mathcal{X}$ is treated equally: for any $x, y \in \mathcal{X}$, and every profile $\mathbf{d} \in \mathcal{D}$ such that $x \in F(\mathbf{d})$, there exists some permutation $\mathbf{d}'$ of $\mathbf{d}$ such that $y \in F(\mathbf{d}')$.

For any $\pi \in \Pi_{\mathcal{V}}$ and any $\mathbf{r} \in \mathcal{R}^{\mathcal{V}}$, we define $\mathbf{r}\pi \in \mathcal{R}^{\mathcal{V}}$ by $(\mathbf{r}\pi)_v = r_{\pi(v)}$ for all $v \in \mathcal{V}$. Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a homomorphism. A score system $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ is ν-*neutral* if, for all $\pi \in \Pi_{\mathcal{X}}$ and $x, y \in \mathcal{X}$, if $\pi(y) = x$ and $\widetilde{\pi} := \nu(\pi)$, then $\mathbf{s}^x \widetilde{\pi} = \mathbf{s}^y$. Except for abstract median rules, all the scoring rules mentioned above (including the Kemeny rule) have neutral score systems, with the obvious neutralizers. A domain $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ is a *cone* if $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}$ whenever $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}$, and also, $\mathbf{d} \in \mathcal{D}$ whenever $n\,\mathbf{d} \in \mathcal{D}$ for some $n \in \mathbb{N}$. (For example, the universal domain $\mathbb{N}^{\langle \mathcal{V} \rangle}$ is a cone.) Here is our second main result:

Theorem 2.3 *Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be any set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a cone, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be any voting rule. Then F is neutral and satisfies reinforcement if and only if F is a scoring rule with a neutral score system.*

Thus, combining reinforcement with neutrality yields a scoring rule with a neutral score system. By combining reinforcement with weaker (but more technical) hypothesis, we can also obtain scoring rules with non-neutral score systems (see Proposition 5.3).

Theorem 2.3 considers a rule F defined on a domain $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$. Given a score system S, we have $F_{\mathsf{S}}(\mathbf{n}) \neq \emptyset$ for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$; thus, the scoring representation offers a way to 'extend' F from $\mathcal{D}$ to all of $\mathbb{N}^{\langle \mathcal{V} \rangle}$. However, we might still wish to restrict F_{S} to the smaller domain $\mathcal{D}$. For example, suppose we have adopted F on the basis of certain normative criteria which only make sense inside $\mathcal{D}$ (e.g. Condorcet consistency). If $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle} \setminus \mathcal{D}$, then we might regard the value of $F_{\mathsf{S}}(\mathbf{n})$ as a meaningless artifact; the normative criteria which justify F inside $\mathcal{D}$ do not apply at $\mathbf{n}$.[9]

A voting rule F satisfies *overwhelming majority*[10] if, for any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, there exists some $M \in \mathbb{N}$ such that, for all $m > M$, we have $F(m\mathbf{n} + \mathbf{n}') \subseteq F(\mathbf{n})$. This means: if one sub-population of voters (represented by $m\,\mathbf{n}$) is much larger than another sub-population (represented by $\mathbf{n}'$), then the choice of the combined population should be determined by the choice of the larger sub-population —except that the smaller sub-population may act as a 'tie-breaker' in some cases. Myerson (1995) showed that, if the voting rule in Theorem 2.3 satisfies overwhelming majority, then not only is it a scoring rule, but the score system is real-valued. Our next result makes an analogous statement for the balance rule in Theorem 2.2, without assuming neutrality. A voting rule $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ satisfies the *tie condition* (TC) if, for all distinct $x, y \in \mathcal{X}$:

(TC1) There exists some $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ with $F(\mathbf{n}) = \{x, y\}$.

(TC2) For any finite $\mathcal{W} \subseteq \mathcal{V}$, there exists some $\mathbf{m} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ such that $m_w > 0$ for all $w \in \mathcal{W}$, and $F(\mathbf{m}) \supseteq \{x, y\}$.

[9]A similar remark applies to a balance rule $F_{\mathsf{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$. There may exist other profiles $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle} \setminus \mathcal{D}$ which satisfy the nontriviality condition (1), so that $F_{\mathsf{B}}(\mathbf{n}) \neq \emptyset$. But the fact that F_{B} is well-defined outside $\mathcal{D}$ does not imply that society is normatively compelled to apply F_{B} outside $\mathcal{D}$.

[10]Sometimes this is called *continuity* or the *Archimedean property*.

For example: any nontrivial neutral balance rule satisfies TC (see Lemma B.9).

Proposition 2.4 *Let* $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ *be a balance rule satisfying* TC*. Then* F *satisfies overwhelming majority if and only if* $F = F_{\mathsf{B}}$ *for some real-valued balance system* B.

Theorem 2.3 makes the class of scoring rules quite attractive. What is the 'best' scoring rule? Our last two major results offers two possible answers to this question. Let $\mathcal{V}$ and $\mathcal{W}$ be two sets, and let $\alpha : \mathcal{W} \longrightarrow \mathcal{V}$. Define $\alpha_* : \mathbb{N}^{\langle \mathcal{W} \rangle} \longrightarrow \mathbb{N}^{\langle \mathcal{V} \rangle}$ as follows: for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{W} \rangle}$, and any $v \in \mathcal{V}$, $\alpha_*(\mathbf{n})_v := \sum\{n_w; \quad w \in \mathcal{W} \text{ and } \alpha(w) = v\}$. Given two voting rules $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ and $G : \mathbb{N}^{\langle \mathcal{W} \rangle} \rightrightarrows \mathcal{X}$, we say that F is *at least as expressive as* G if there is a some function $\alpha : \mathcal{W} \longrightarrow \mathcal{V}$ such that, for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{W} \rangle}$, $F(\alpha_*(\mathbf{n})) = G(\mathbf{n})$. Thus, for any $w \in \mathcal{W}$, voting for w in the rule G is effectively equivalent to voting for $\alpha(w)$ in F. Thus, the voters can express any profile of opinions via F which they could have expressed via G. The rule F is the *most expressive* member of some class of rules if it is at least as expressive as every other element of that class.[11]

Proposition 2.5 *Let* $\mathcal{X}$ *be a finite set. Formally utilitarian voting is the most expressive* $\mathcal{X}$*-valued voting rule which satisfies reinforcement, neutrality, and overwhelming majority.*

For any $v \in \mathcal{V}$, define $\mathbf{1}^v \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ by $(\mathbf{1}^v)_v := 1$, whereas $(\mathbf{1}^v)_w := 0$ for all $w \in \mathcal{V} \setminus \{v\}$. A voting rule $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ *admits minority overrides* if, for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, there is some $v \in \mathcal{V}$ such that $F(\mathbf{n} + \mathbf{1}^v) \neq F(\mathbf{n})$. Thus, regardless of the size of the populace and the weight of existing public opinion, a single voter can always cast a vote which changes the outcome. Such 'overrides' not only generate political instability; they are arguably undemocratic. It might be better if F did *not* admit minority overrides.[12] If $\mathcal{V}$ is finite, then any rule satisfying overwhelming majority will not admit minority overrides.[13] However, we will be interested in the case when $\mathcal{V}$ is infinite. For neutral voting rules, an absence of minority overrides is effectively equivalent to imposing upper and lower bounds on the scores which voters can assign to alternatives (see Lemma B.11). For example: formally utilitarian voting admits minority overrides, but all of the other aforementioned scoring rules do not.

Proposition 2.6 *Let* $\mathcal{X}$ *be a finite set. Range voting is the most expressive* $\mathcal{X}$*-valued voting rule which satisfies reinforcement, neutrality, overwhelming majority, and does not admit minority overrides.*

Despite Propositions 2.4, 2.5, and 2.6, overwhelming majority is not always normatively compelling. In some cases, a non-real-valued scoring system may be more appropriate.

[11] If we are interested in maximizing expressiveness, then it does not make sense to impose domain restrictions. That is why we have assumed the universal domain $\mathcal{D} = \mathbb{N}^{\langle \mathcal{V} \rangle}$ in this paragraph and the next two results.

[12] Of course, there will always be *some* profiles where a single voter can change the outcome; the point is that this should not be true for *all* profiles.

[13] *Proof.* Find $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ such that $|F(\mathbf{n})| = 1$. If $M \in \mathbb{N}$ is large enough, then overwhelming majority protects the profile $M\mathbf{n}$ from minority overrides.

Example 2.7. Let $\mathcal{X}$ be a two-dimensional policy space. That is: $\mathcal{X} \subset \mathcal{X}_1 \times \mathcal{X}_2$, where $\mathcal{X}_1$ is a space of alternatives in one 'policy dimension', while $\mathcal{X}_2$ is a space of alternatives along some other dimension. Note that $\mathcal{X}$ is a proper subset of $\mathcal{X}_1 \times \mathcal{X}_2$ —not all policy combinations are feasible. Suppose $\mathcal{X}_1$ is considered to be lexicographically prior to $\mathcal{X}_2$ (e.g. $\mathcal{X}_1$ represents basic human rights, while $\mathcal{X}_2$ represents GDP). For $j = 1, 2$, let $\mathcal{V}_j$ be a space of signals, and suppose we have decided to use the $\mathbb{R}$-valued score system ${}_j\mathsf{S} = \{{}_j\mathbf{s}^x\}_{x \in \mathcal{X}} \subset \mathcal{R}^{\mathcal{V}_j}$ on $(\mathcal{X}_j, \mathcal{V}_j)$. If we simply apply the scoring rules $F_{{}_1\mathsf{S}} : \mathbb{N}^{\langle \mathcal{V}_1 \rangle} \rightrightarrows \mathcal{X}_1$ and $F_{{}_2\mathsf{S}} : \mathbb{N}^{\langle \mathcal{V}_2 \rangle} \rightrightarrows \mathcal{X}_2$ separately, then we may end up selecting an element of $(\mathcal{X}_1 \times \mathcal{X}_2) \setminus \mathcal{X}$ —an infeasible policy. We could combine ${}_1\mathsf{S}$ and ${}_2\mathsf{S}$ into a single, $\mathbb{R}$-valued score system S on $(\mathcal{X}, \mathcal{V}_1 \times \mathcal{V}_2)$ by defining $s^{x_1;x_2}_{v_1,v_2} := {}_1s^{x_1}_{v_1} + {}_2s^{x_2}_{v_2}$ for all $(x_1, x_2) \in \mathcal{X}$ and $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$. But this would not respect the lexicographical priority of $\mathcal{X}_1$ over $\mathcal{X}_2$.

Instead, let $\mathcal{R} := \mathbb{R}^2$ with the vector addition operation '$+$' and the lexicographical ordering '$\succ$' (i.e. $(r_1, r_2) \succ (s_1, s_2)$ if and only if either $r_1 > s_1$, or $r_1 = s_1$ and $r_2 > s_2$). Then $(\mathcal{R}, +, \succ)$ is a linearly ordered abelian group. Define an $\mathcal{R}$-valued score system S on $(\mathcal{X}, \mathcal{V}_1 \times \mathcal{V}_2)$ by setting $s^{x_1;x_2}_{v_1,v_2} := ({}_1s^{x_1}_{v_1}, {}_2s^{x_2}_{v_2})$ for all $(x_1, x_2) \in \mathcal{X}$ and $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$. $\diamondsuit$

3 Generalized neutrality

The two results of this section show how the neutrality properties of a voting rule F can be 'encoded' in a balance system or scoring system for F. They are also key technical steps in the proof of Theorem 2.3.

Let $\Pi'_{\mathcal{X}} \subseteq \Pi_{\mathcal{X}}$ be any subgroup of permutations of $\mathcal{X}$, and let $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a group homomorphism. For any domain $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$, a voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is *ν-neutral* if, for all $\pi \in \Pi'_{\mathcal{X}}$, if $\widetilde{\pi} = \nu(\pi)$, then $\mathcal{D}$ is $\widetilde{\pi}$-invariant and $F(\widetilde{\pi}(\mathbf{d})) = \pi(F(\mathbf{d}))$ for all $\mathbf{d} \in \mathcal{D}$. (For example: let $\mathcal{X} = \mathcal{V}$ be a metric space, let $\Pi'_{\mathcal{X}}$ be the set of all self-isometries of $\mathcal{X}$, and let $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{X}} = \Pi_{\mathcal{V}}$ be the inclusion map. Then the median rule on $\mathcal{X}$ is ν-neutral.)

A balance system B is *ν-neutral* if, for all $x, y, x', y' \in \mathcal{X}$ and $\pi \in \Pi'_{\mathcal{X}}$, if $x' := \pi^{-1}(x)$ and $y' := \pi^{-1}(y)$ and $\widetilde{\pi} = \nu(\pi)$, then $\mathbf{b}^{x,y}\widetilde{\pi} = \mathbf{b}^{x',y'}$. A score system $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ is *ν-neutral* if, for all $\pi \in \Pi'_{\mathcal{X}}$ and $x, y \in \mathcal{X}$, if $\pi(y) = x$ and $\widetilde{\pi} := \nu(\pi)$, then $\mathbf{s}^x\,\widetilde{\pi} = \mathbf{s}^y$. A voting rule, balance system, or score system is *$\Pi'_{\mathcal{X}}$-neutral* if it is ν-neutral for some group homomorphism $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ (called the *neutralizer*).

Proposition 3.1 *Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be an arbitrary set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a scoring rule. Let $\Pi'_{\mathcal{X}}$ be any group of permutations of $\mathcal{X}$, and let $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a homomorphism. Then F is ν-neutral if and only if $F = F_{\mathsf{S}}$ for some ν-neutral score system S.*

The corresponding result for balance rules is a bit more complicated. Let $\Pi'_{\mathcal{X}}$ be a group of permutations of $\mathcal{X}$. We say that $\Pi'_{\mathcal{X}}$ is *doubly transitive* if, for all $x, y, x', y' \in \mathcal{X}$ with $x \neq y$ and $x' \neq y'$, there exists some $\pi \in \Pi'_{\mathcal{X}}$ such that $\pi(x) = x'$ and $\pi(y) = y'$.

A balance system B is *perfect* on the domain $\mathcal{D}$ if, for any $\mathbf{d} \in \mathcal{D}$, any $x \in F_{\mathsf{B}}(\mathbf{d})$ and any $y \in \mathcal{X} \setminus F_{\mathsf{B}}(\mathbf{d})$, we have $\mathbf{b}^{x,y}(\mathbf{d}) > 0$. (For instance: Example 2.1(a) is perfect, but Example 2.1(b) is not perfect.) Perfection simplifies the computation of $F(\mathbf{d})$: once we find one point $x \in F_{\mathsf{B}}(\mathbf{d})$, we have $F_{\mathsf{B}}(\mathbf{d}) = \{y \in \mathcal{X};\ \mathbf{b}^{x,y}(\mathbf{d}) = 0\}$.

Proposition 3.2 *Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be an arbitrary set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a balance rule. Let $\Pi'_{\mathcal{X}}$ be a doubly transitive group of permutations of $\mathcal{X}$, and let $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a homomorphism. Then F is ν-neutral if and only if $F = F_{\mathsf{B}}$ for some ν-neutral perfect balance system B.*

4 Uniqueness

Theorems 2.2 and 2.3 characterize when a voting rule admits a balance representation or scoring representation. Now we consider the uniqueness of these representations. Let $\mathsf{S} := \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ and $\widetilde{\mathsf{S}} := \{\widetilde{\mathbf{s}}^x\}_{x \in \mathcal{X}}$ be two real-valued score systems on $(\mathcal{X}, \mathcal{V})$. Say $\widetilde{\mathsf{S}}$ is an *affine transform* of S if there exists $r > 0$ and $\mathbf{t} \in \mathbb{R}^{\mathcal{V}}$ such that $\widetilde{\mathbf{s}}^x = r\,\mathbf{s}^x + \mathbf{t}$, for all $x \in \mathcal{X}$.

Proposition 4.1 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, and let $\Pi'_{\mathcal{X}}$ be a transitive group of permutations on $\mathcal{X}$. Let S and $\widetilde{\mathsf{S}}$ be two real-valued score systems on $(\mathcal{X}, \mathcal{V})$. Suppose the vectors $\{\mathbf{s}^x\}_{x \in \mathcal{X}}$ are linearly independent in $\mathbb{R}^{\mathcal{V}}$, and F_{S} is $\Pi'_{\mathcal{X}}$-neutral. Then:*

(a) *$F_{\mathsf{S}} = F_{\widetilde{\mathsf{S}}}$ on $\mathbb{N}^{\mathcal{V}}$ if and only if $\widetilde{\mathsf{S}}$ is an affine transform of S.*

(b) *If S and $\widetilde{\mathsf{S}}$ are $\Pi'_{\mathcal{X}}$-neutral, then $\mathbf{t}\,\widetilde{\pi} = \mathbf{t}$ for all $\pi \in \Pi'_{\mathcal{X}}$. In particular, if the group $\widetilde{\Pi}'_{\mathcal{X}} = \{\widetilde{\pi};\ \pi \in \Pi'_{\mathcal{X}}\}$ acts transitively on $\mathcal{V}$, then $\mathbf{t}$ is a constant vector.*

Example 4.2. Let $\mathcal{V} :=$ {all preference orders over $\mathcal{X}$}. For any $\pi \in \Pi_{\mathcal{X}}$, define $\widetilde{\pi} \in \Pi_{\mathcal{V}}$ such that $\widetilde{\pi}(v)$ prefers $\pi(x)$ to $\pi(y)$ if and only if v prefers x to y. Then $\widetilde{\Pi}_{\mathcal{X}}$ acts transitively on $\mathcal{V}$. Thus, if S and $\widetilde{\mathsf{S}}$ are neutral, real-valued score systems on $(\mathcal{X}, \mathcal{V})$, and $F_{\mathsf{S}} = F_{\widetilde{\mathsf{S}}}$, then there exists $r > 0$ and $t \in \mathbb{R}$ such that $\widetilde{s}^x_v = r\, s^x_v + t$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. $\diamondsuit$

Suppose $\mathcal{V}$ is finite and $\mathsf{B} = \{\mathbf{b}^{x,y}\}_{x,y \in \mathcal{X}}$ is a real-valued balance system on $\mathcal{V}$. Given a domain $\mathcal{D} \subseteq \mathbb{N}^{\mathcal{V}}$, let $\mathbb{R}_+[\mathcal{D}]$ be the set of all $\mathbb{R}_+$-linear combinations of elements of $\mathcal{D}$; this is a convex cone in $\mathbb{R}^{\mathcal{V}}_+$. The domain $\mathcal{D}$ is *thick* if $\mathcal{D}$ is closed under addition (i.e. $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}$ whenever $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}$) and $\mathbb{R}_+[\mathcal{D}]$ has nonempty interior in $\mathbb{R}^{\mathcal{V}}$. (For example, the universal domain $\mathbb{N}^{\mathcal{V}}$ is thick.) For any $x \in \mathcal{X}$, define

$$ {}_{\mathbb{R}}\mathcal{C}_x \quad := \quad \{\mathbf{r} \in \mathbb{R}_+[\mathcal{D}]\ ;\ \mathbf{b}^{x,y}(\mathbf{r}) \geq 0,\ \text{for all } y \in \mathcal{X}\}. \tag{2} $$

This is a convex cone which is relatively closed in $\mathbb{R}_+[\mathcal{D}]$. A subset $\mathcal{H} \subset \mathbb{R}^{\mathcal{V}}$ is a *hyperplane* if there is a nontrivial linear function $\phi : \mathbb{R}^{\mathcal{V}} \longrightarrow \mathbb{R}$ such that $\mathcal{H} = \ker(\phi)$. A subset $\mathcal{B} \subset \mathbb{R}^{\mathcal{V}}$ is *hyperplanar* if there exists a unique hyperplane $\mathcal{H}$ such that $\mathcal{B} \subseteq \mathcal{H}$. For any $x, y \in \mathcal{X}$, let $\mathcal{B}_{x,y} := {}_{\mathbb{R}}\mathcal{C}_x \cap {}_{\mathbb{R}}\mathcal{C}_y$; write "$x \underset{\mathsf{B}}{\sim} y$" if $\mathcal{B}_{x,y}$ is hyperplanar. The structure $(\mathcal{X}, \underset{\mathsf{B}}{\sim})$ is the *graph* of the rule F_{B}. (This graph can be nontrivial only if the domain $\mathcal{D}$ is thick.)

A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is *nondegenerate* if, for all $x \in \mathcal{X}$, there exists $\mathbf{d} \in \mathcal{D}$ with $F(\mathbf{d}) = \{x\}$. (For example: any nontrivial, neutral balance rule is nondegenerate; see Lemma B.8.) A balance system B is *nondegenerate on $\mathcal{D}$* if the rule $F_{\mathsf{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ is nondegenerate. We say B has *no zeros* if $\mathbf{b}^{x,y} \neq \mathbf{0}$ for any $x, y \in \mathcal{X}$ with $x \underset{\mathsf{B}}{\sim} y$. (For example, if S is a scoring system, and $\mathsf{B} = \nabla\mathsf{S}$ as in Example 2.1(a), then B has no zeros if $\mathbf{s}^x \neq \mathbf{s}^y$ for all distinct $x, y \in \mathcal{X}$, which holds if F_{S} is nondegenerate.) The next result says that, subject to these mild conditions, a $\mathbb{R}$-valued balance system B is uniquely determined by the structure of F_{B}, up to independent scalar multiplication of the balance vectors.

Proposition 4.3 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, and let $\mathcal{D} \subseteq \mathbb{N}^{\mathcal{V}}$ be a thick domain. Let B and $\widetilde{\mathsf{B}}$ be $\mathbb{R}$-valued balance systems on $\mathcal{X}$ with no zeros, with F_{B} nondegenerate. Then $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$ on $\mathcal{D}$ if and only if, for every $x \underset{\mathsf{B}}{\sim} y \in \mathcal{X}$, there is some $r_{x,y} > 0$ such that $\widetilde{\mathbf{b}}^{x,y} = r_{x,y} \cdot \mathbf{b}^{x,y}$.*

Let S be a real-valued score system on $(\mathcal{X}, \mathcal{V})$. For any $x, y \in \mathcal{X}$, write "$x \underset{\mathsf{S}}{\sim} y$" if $x \underset{\nabla S}{\sim} y$, where $\nabla\mathsf{S}$ is as in Example 2.1(a). For any $z \in \mathcal{X}$, if $x \underset{\mathsf{S}}{\sim} y$, $y \underset{\mathsf{S}}{\sim} z$, and $x \underset{\mathsf{S}}{\sim} z$, then write "$(x \underset{\mathsf{S}}{\sim} y) \equiv (y \underset{\mathsf{S}}{\sim} z)$". Thus, "$\equiv$" is a symmetric binary relation on the edges of the graph $(\mathcal{X}, \underset{\mathsf{S}}{\sim})$. Let "$\cong$" be the transitive closure of "$\equiv$"; then "$\cong$" is an equivalence relation on the edges. Say S is ***simple*** if every edge is equivalent to every other edge via this relation. (For example: if $x \underset{\mathsf{S}}{\sim} y$ for all $x, y \in \mathcal{X}$, then S is simple.)

Proposition 4.4 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, and let $\mathcal{D} \subseteq \mathbb{N}^{\mathcal{V}}$ be a thick domain. Let S be a simple, nondegenerate real-valued score system on $\mathcal{X}$, and let $\widetilde{\mathsf{S}}$ be another real-valued score system. Then $F_{\mathsf{S}} = F_{\widetilde{\mathsf{S}}}$ on $\mathcal{D}$ if and only if $\widetilde{\mathsf{S}}$ is an affine transform of S.*

Example 4.5. If S is not simple, then the conclusion of Proposition 4.4 does not hold. For example, suppose $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{X} = \{x, y, z\}$, and ${}_{\mathbb{R}}\mathcal{C}_x$, ${}_{\mathbb{R}}\mathcal{C}_y$, and ${}_{\mathbb{R}}\mathcal{C}_z$ are as shown in Figure 1(a). Then $x \underset{\mathsf{S}}{\sim} y$ and $y \underset{\mathsf{S}}{\sim} z$, but $x \underset{\mathsf{S}}{\not\sim} z$. (Thus, $(x \underset{\mathsf{S}}{\sim} y) \not\equiv (y \underset{\mathsf{S}}{\sim} z)$.) Define $\widetilde{\mathbf{s}}^x := \mathbf{s}^x + \mathbf{s}^y$, $\widetilde{\mathbf{s}}^y := 2\mathbf{s}^y$, and $\widetilde{\mathbf{s}}^z = 2\mathbf{s}^z$. Then: $\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^y = \mathbf{s}^x - \mathbf{s}^y$ and $\widetilde{\mathbf{s}}^y - \widetilde{\mathbf{s}}^z = 2(\mathbf{s}^y - \mathbf{s}^z)$, so Example 2.1(a) and Proposition 4.3 imply that $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$. But $\widetilde{\mathsf{S}}$ is not an affine transformation of S.

For another example, suppose $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{X} = \{w, x, y, z\}$, and ${}_{\mathbb{R}}\mathcal{C}_w$, ${}_{\mathbb{R}}\mathcal{C}_x$, ${}_{\mathbb{R}}\mathcal{C}_y$, and ${}_{\mathbb{R}}\mathcal{C}_z$ are as shown in Figure 1(b). Then $w \underset{\mathsf{S}}{\sim} x \underset{\mathsf{S}}{\sim} y \underset{\mathsf{S}}{\sim} z \underset{\mathsf{S}}{\sim} w$, but $w \underset{\mathsf{S}}{\not\sim} y$ and $x \underset{\mathsf{S}}{\not\sim} z$. Suppose further that $\mathbf{s}^w - \mathbf{s}^z = \mathbf{s}^x - \mathbf{s}^y$ (as indicated by the fact that the boundaries $\mathcal{B}_{w,z}$ and $\mathcal{B}_{x,y}$ are coplanar in $\mathbb{R}^3$). Define $\widetilde{\mathbf{s}}^w := \mathbf{s}^w + \mathbf{s}^x$, $\widetilde{\mathbf{s}}^x := 2\mathbf{s}^x$, $\widetilde{\mathbf{s}}^y := 2\mathbf{s}^y$, and $\widetilde{\mathbf{s}}^z = \mathbf{s}^z + \mathbf{s}^y$. Then clearly, $\widetilde{\mathbf{s}}^w - \widetilde{\mathbf{s}}^x = \mathbf{s}^w - \mathbf{s}^x$, $\widetilde{\mathbf{s}}^z - \widetilde{\mathbf{s}}^y = \mathbf{s}^z - \mathbf{s}^y$, and $\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^y = 2(\mathbf{s}^x - \mathbf{s}^y)$. Finally, $\widetilde{\mathbf{s}}^w - \widetilde{\mathbf{s}}^z = (\mathbf{s}^w + \mathbf{s}^x) - (\mathbf{s}^z + \mathbf{s}^y) = (\mathbf{s}^w - \mathbf{s}^z) + (\mathbf{s}^x - \mathbf{s}^y) = 2(\mathbf{s}^w - \mathbf{s}^z)$. Thus, Example 2.1(a) and Proposition 4.3 imply that $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$. But $\widetilde{\mathsf{S}}$ is not an affine transformation of S. $\diamondsuit$

Appendix C contains an analogue to Proposition 4.3 where B is not necessarily real-valued, and $\mathcal{X}$ and $\mathcal{V}$ can be infinite. We do not yet have analogues for Propositions 4.1 and 4.4.

5 From balance rules to scoring rules

Not every balance rule is a scoring rule.

Lemma 5.1 *Let $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a domain. A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is a scoring rule if and only if F is a balance rule with a balance system $\mathsf{B} = \{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}}$ satisfying:*

$$\mathbf{b}^{x,y}(\mathbf{d}) + \mathbf{b}^{y,z}(\mathbf{d}) \quad = \quad \mathbf{b}^{x,z}(\mathbf{d}), \qquad \textit{for all } x, y, z \in \mathcal{X} \textit{ and } \mathbf{d} \in \mathcal{D}. \tag{3}$$

Example 5.2. Let $\mathcal{X} := \{x, y, z\}$, let $\mathcal{V}$ be finite, and let $\mathcal{D} := \mathbb{N}^{\mathcal{V}}$. Let $\mathbb{L} \subset \mathbb{R}^{\mathcal{V}}$ be a proper linear subspace, and consider an $\mathbb{R}$-valued balance system B such that $\mathbf{b}^{x,z}, \mathbf{b}^{y,z} \in \mathbb{L}$, but $\mathbf{b}^{x,y} \notin \mathbb{L}$, and such that $x \underset{\mathsf{B}}{\sim} y \underset{\mathsf{B}}{\sim} z \underset{\mathsf{B}}{\sim} x$, as shown in Figure 1(c). If $\widetilde{\mathsf{B}}$ is any other $\mathbb{R}$-valued balance system such that $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$, then Proposition 4.3 implies that $\widetilde{\mathbf{b}}^{x,z}, \widetilde{\mathbf{b}}^{y,z} \in \mathbb{L}$, but $\widetilde{\mathbf{b}}^{x,y} \notin \mathbb{L}$. Thus, $\widetilde{\mathsf{B}}$ cannot satisfy condition (3); thus F_{B} is not a scoring rule.[14] $\diamondsuit$

[14]Myerson (1995, Example 2) gives another, more complicated example of a balance rule which is not a scoring rule.

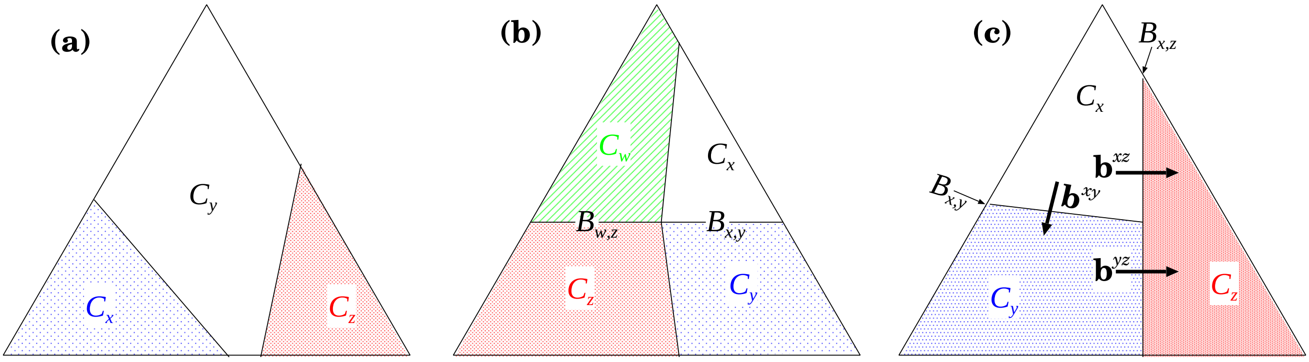

Figure 1: Let $\mathcal{V} = \{1, 2, 3\}$. These pictures are projections of $\mathbb{N}^{\mathcal{V}}$ onto the 'rational unit simplex' $\Delta_{\mathbb{Q}} := \{\mathbf{q} \in \mathbb{Q}^3_+;\ q_1 + q_2 + q_3 = 1\}$. (Thus, each d-dimensional feature in these pictures actually represents a $(d+1)$-dimensional cone in $\mathbb{R}^3_+$). If a rule $F : \mathbb{N}^{\mathcal{V}} \rightrightarrows \mathcal{X}$ satisfies reinforcement, then it is homogeneous: $F(M\mathbf{n}) = F(\mathbf{n})$ for all $M \in \mathbb{N}$. Thus, if we define $f(\mathbf{n}/\|\mathbf{n}\|) := F(\mathbf{n})$, then we obtain a correspondence $f : \Delta_{\mathbb{Q}} \rightrightarrows \mathcal{X}$, which completely encodes the structure of F. (a,b) Nonsimple scoring rules with non-unique scoring representations; see Example 4.5. (c) A balance rule which is not a scoring rule; see Example 5.2.

Let B be a real-valued balance system on $(\mathcal{X}, \mathcal{V})$. Recall the relation '$\underset{\mathsf{B}}{\sim}$' on $\mathcal{X}$ defined in section 4. Let $\mathcal{E} := \{(x, y) \in \mathcal{X}^2;\ x \underset{\mathsf{B}}{\sim} y\}$; thus, $\mathcal{E}$ represents all (directed) edges in the graph $(\mathcal{X}, \underset{\mathsf{B}}{\sim})$. The set $\mathbb{Z}^{\mathcal{E}}$ is an abelian group under componentwise addition; an element of $\mathbb{Z}^{\mathcal{E}}$ is called a *chain* on $(\mathcal{X}, \underset{\mathsf{B}}{\sim})$. If $\mathcal{R}$ is a linearly ordered abelian group, and B is an $\mathcal{R}$-valued balance system on $(\mathcal{X}, \mathcal{V})$, then we define a group homomorphism $\mathsf{B} : \mathbb{Z}^{\mathcal{E}} \longrightarrow \mathcal{R}^{\mathcal{V}}$ by setting $\mathsf{B}(\mathbf{c}) := \sum_{(x,y)\in\mathcal{E}} c_{x,y} \mathbf{b}^{x,y}$ for any chain $\mathbf{c} = (c_{x,y})_{(x,y)\in\mathcal{E}}$ in $\mathbb{Z}^{\mathcal{E}}$. For any $\mathbf{c} \in \mathbb{Z}^{\mathcal{E}}$, its *boundary* $\partial\mathbf{c}$ is the element of $\mathbb{Z}^{\mathcal{X}}$ defined by setting $(\partial\mathbf{c})_x := \sum_{(x,y)\in\mathcal{E}} (c_{y,x} - c_{x,y})$, for all $x \in \mathcal{X}$. The chain $\mathbf{c}$ is a *cycle* if $\partial\mathbf{c} = 0$. If $\mathcal{Z}(\mathcal{X}, \underset{\mathsf{B}}{\sim})$ is the set of all cycles, then $\mathcal{Z}(\mathcal{X}, \underset{\mathsf{B}}{\sim})$ is a subgroup of $\mathbb{Z}^{\mathcal{E}}$.

For example, for any $x \in \mathcal{X}$, define $[x] \in \mathbb{Z}^{\mathcal{X}}$ by $[x]_x := 1$, whereas $[x]_y := 0$ for all $y \neq x$. Likewise, for any $x \underset{\mathsf{B}}{\sim} y \in \mathcal{X}$, define $[x, y] \in \mathbb{Z}^{\mathcal{E}}$ by $[x, y]_{x,y} := 1$, whereas $[x, y]_{w,z} = 0$ for all $(w, z) \neq (x, y)$. Then $\partial[x, y] = [y] - [x]$. Next, consider a *path* $x_0 \underset{\mathsf{B}}{\sim} x_1 \underset{\mathsf{B}}{\sim} x_2 \underset{\mathsf{B}}{\sim} \cdots \underset{\mathsf{B}}{\sim} x_N$ (for some $x_0, x_1, x_2, \ldots, x_N \in \mathcal{X}$). This path can be represented by the *path chain* $\mathbf{c} := [x_0, x_1] + [x_1, x_2] + \cdots + [x_{N-1}, x_N]$. (Observe that $-\mathbf{c}$ represents the reversed path $x_N \underset{\mathsf{B}}{\sim} \cdots x_3 \underset{\mathsf{B}}{\sim} x_2 \underset{\mathsf{B}}{\sim} x_1$.) In this case $(\partial\mathbf{c})_x = [x_N] - [x_0]$ for (so $\partial\mathbf{c}$ picks out the *endpoints* of the path). Thus, $\mathbf{c}$ is a cycle if and only if $x_N = x_0$ (i.e. the path is *closed*). Any chain is a sum of path chains, and any cycle is a sum of closed path chains.

Proposition 5.3 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, let B be a nondegenerate, real-valued balance system on $(\mathcal{X}, \mathcal{V})$, and consider the balance rule $F_{\mathsf{B}} : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$. If $\mathsf{B}(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathcal{Z}(\mathcal{X}, \underset{\mathsf{B}}{\sim})$, then F_{B} is a scoring rule.*

Example 5.4. A graph $(\mathcal{X}, \sim)$ is a *tree* if $\mathcal{Z}(\mathcal{X}, \sim)$ is trivial. This means $(\mathcal{X}, \sim)$ contains no nontrivial closed paths. (The graph in Figure 1(a) is a tree, but the graphs in Figures 1(b,c) and 2 are not.) In this case, Proposition 5.3 implies that F_{B} is a scoring rule.

For instance, suppose the elements of $\mathcal{X}$ can be arranged on a line.[15] Then it makes sense to design the voting rule so that $x \sim y$ if and only if x and y are adjacent in the line. Then $(\mathcal{X}, \sim)$ is a tree, so any nondegenerate $\mathbb{R}$-valued perfect balance rule is a scoring rule. $\diamondsuit$

6 Open problems

The conditions of reinforcement and neutrality are sufficient to obtain a scoring rule (Theorem 2.3), but neutrality is not necessary (Proposition 5.3). Are there normatively compelling conditions which are both necessary and sufficient for a scoring rule? Are there normatively compelling balance rules which are *not* scoring rules? Most of our results assume $\mathcal{X}$ is finite —can this hypothesis be eliminated? Many of our results (e.g. Theorems 2.2 and 2.3) allow score systems or balance systems ranging over an abstract linearly ordered abelian group. But in all the natural examples, the score system is real-valued. This is due to the overwhelming majority property, but in some cases, this property may be inappropriate (Example 2.7). Is there a normatively compelling scoring or balance system which is *not* real-valued?

Appendix A: Homogeneous orders on abelian groups

Let $(\mathcal{R}, +)$ be an abelian group, and let $(\succeq)$ be a binary relation on $\mathcal{R}$. We say that $(\succeq)$ is ***homogeneous*** if, for all $r, s \in \mathcal{R}$, we have $(r \succeq s) \iff (r - s \succeq 0)$. The ***positive conoid***[16] of $(\succeq)$ is the set $\mathcal{P}_{\succeq} := \{r \in \mathcal{R};\ r \succeq 0\}$. The conoid $\mathcal{P}_{\succeq}$ completely encodes the relation $(\succeq)$: for any $r, s \in \mathcal{R}$, we have

$$\Big(r \succeq s\Big) \quad \iff \quad \Big(r - s \in \mathcal{P}_{\succeq}\Big). \tag{A1}$$

Conversely, given any subset $\mathcal{P} \subseteq \mathcal{R}$, we can use formula (A1) to define a unique homogeneous binary relation $(\succeq)$ such that $\mathcal{P}_{\succeq} = \mathcal{P}$. Thus, there is a bijective correspondence between homogeneous binary relations on $\mathcal{R}$ and subsets of $\mathcal{R}$.

Let $\mathcal{P} \subseteq \mathcal{R}$. Recall that $\mathcal{P}$ is ***additively closed*** if $p_1 + p_2 \in \mathcal{P}$ whenever $p_1, p_2 \in \mathcal{P}$. We say that $\mathcal{P}$ is ***divisible*** if, for all $r \in \mathcal{R}$ and $n \in \mathbb{N}$, if $n\,r \in \mathcal{P}$, then $r \in \mathcal{P}$ also. Thus, $\mathcal{P}$ is a cone if it is both additively closed and divisible. A ***preorder conoid*** is an additively closed subset of $\mathcal{R}$ which contains 0. A ***partial order conoid*** is an additively closed subset of $\mathcal{R}$ which does *not* contain 0. (Equivalently: for all $r \in \mathcal{R}$, at most one of r or $-r$ is in $\mathcal{P}$). A preorder or partial order conoid $\mathcal{P}$ is ***complete*** if, for all $r \in \mathcal{R} \setminus \{0\}$, at *least* one of r or $-r$ is in $\mathcal{P}$. (Any complete preorder or partial order conoid is divisible.) A ***linear order cone*** is a complete partial order conoid. (Thus, for all $r \in \mathcal{R} \setminus \{0\}$, *exactly* one of $r \in \mathcal{P}$ or $-r \in \mathcal{P}$.) Properties of $(\succeq)$ correspond to properties of the conoid $\mathcal{P}_{\succeq}$ as follows:

[15] For example $\mathcal{X}$ could be a linearly ordered set of 'grades' which we could assign to some object. Or, $\mathcal{X}$ could be the set of possible locations for a facility in a one-dimensional geography, such as along a highway or river. Or $\mathcal{X}$ could be a set of political parties, arranged from 'right' to 'left'. Or $\mathcal{X}$ could represent a one-dimensional policy variable, such as the central bank interest rate.

[16] This object is are often called the positive *cone*. However, we are reserving the term *cone* for a set which is divisible. Hence the ungainly term 'conoid'.

Lemma A.1 *Let $(\mathcal{R},+)$ be an abelian group. Let $(\succeq)$ be a homogeneous binary relation on $\mathcal{R}$.*

(a) *$(\succeq)$ is transitive iff $\mathcal{P}_{\succeq}$ is additively closed. Likewise, $(\succeq)$ is a preorder iff $\mathcal{P}_{\succeq}$ is a preorder conoid. Also, $(\succeq)$ is a partial order iff $\mathcal{P}_{\succeq}$ is a partial order conoid. Finally, $(\succeq)$ is a linear order iff $\mathcal{P}_{\succeq}$ is a linear order cone.*

(b) *Suppose $(\succeq)$ is a homogeneous preorder; let $(\succ)$ be its asymmetric part, and let $(\approx)$ be its symmetric part. The set $\mathcal{O} := \{r \in \mathcal{R};\ \ r \approx 0\}$ is a subgroup. Let $\mathcal{R}' := \mathcal{R}/\mathcal{O}$ be the quotient group, let $\phi : \mathcal{R} \longrightarrow \mathcal{R}'$ be the quotient map, and define $\mathcal{P}' := \phi(\mathcal{P}) \setminus \{0'\}$. Then $\mathcal{P}'$ is a partial order conoid on $\mathcal{R}'$.*

Let $(\succ')$ be the homogeneous partial order on $\mathcal{R}'$ defined by the conoid $\mathcal{P}'$. Then for all $r,s \in \mathcal{R}$, we have $\phi(r) \succ' \phi(s)$ if and only if $r \succ s$.

The proof is straightforward. Next, we extend a classic result about partial orders to the setting of homogeneous orders. If $(\succeq)$ and $(\succeq')$ are two relations on $\mathcal{R}$, we say that $(\succeq')$ *extends* $(\succeq)$ if, for all $r,s \in \mathcal{R}$, we have $(r \succeq s) \Longrightarrow (r \succeq' s)$. An abelian group $(\mathcal{R},+)$ is *torsion free* if $n\,r \neq 0$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathcal{R} \setminus \{0\}$.

Homogeneous Szpilrajn Lemma. *Let $(\mathcal{R},+)$ be a torsion free abelian group. Any homogeneous partial order on $\mathcal{R}$ can be extended to a homogeneous linear order.*

Proof: Let $(\succ)$ be a homogeneous partial order on $\mathcal{R}$, and let $\mathcal{P} := \mathcal{P}_{\succ}$. Then $\mathcal{P} \subseteq \mathcal{R}$ is a partial order conoid. We must construct a linear order cone $\mathcal{L} \subseteq \mathcal{R}$ such that $\mathcal{P} \subseteq \mathcal{L}$. Define $\mathcal{D}_0 := \{r \in \mathcal{R};\ \ n\,r \in \mathcal{P}$ for some $n \in \mathbb{N}\}$.

Claim 1: *$\mathcal{D}_0$ is a divisible partial order conoid.*

Proof: Let $d_1, d_2 \in \mathcal{D}_0$. Then $n_1\,d_1 = p_1 \in \mathcal{P}$ and $n_2\,d_2 = p_2 \in \mathcal{P}$ for some $n_1, n_2 \in \mathbb{N}$. Thus, $(n_1 n_2)\,(d_1 + d_2) = n_2 n_1\,d_1 + n_1 n_2\,d_2 = n_2\,p_1 + n_1\,p_2 \in \mathcal{P}$ also, because $\mathcal{P}$ is closed under addition. Thus, $(d_1 + d_2) \in \mathcal{D}_0$. Thus, $\mathcal{D}_0$ is additively closed. Clearly, $0 \notin \mathcal{D}_0$, because $0 \notin \mathcal{P}$. Thus, $\mathcal{D}_0$ is a partial order conoid. Finally, $\mathcal{D}_0$ is divisible by construction. ◇ `Claim 1`

Let $\mathfrak{D} := \{\mathcal{D} \subseteq \mathcal{R};\ \ \mathcal{D}$ is a divisible partial order conoid and $\mathcal{P} \subseteq \mathcal{D}\}$. Then $\mathfrak{D}$ is nonempty, because Claim 1 says $\mathcal{D}_0 \in \mathfrak{D}$.

Claim 2: *Every ascending chain in $\mathfrak{D}$ has an upper bound.*

Proof: Let $\mathfrak{C} \subseteq \mathfrak{D}$ be a chain. (That is: for all $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$, either $\mathcal{C}_1 \subseteq \mathcal{C}_2$, or $\mathcal{C}_2 \subseteq \mathcal{C}_1$.) Let $\overline{\mathcal{C}} := \bigcup_{\mathcal{C} \in \mathfrak{C}} \mathcal{C}$. It suffices to show that $\overline{\mathcal{C}} \in \mathfrak{D}$ also. Clearly, $\mathcal{P} \subseteq \overline{\mathcal{C}}$. Also, $0 \notin \overline{\mathcal{C}}$ because $0 \notin \mathcal{C}$ for all $\mathcal{C} \in \mathfrak{C}$. It remains to show that $\overline{\mathcal{C}}$ is additively closed and divisible.

Additively Closed. let $c_1, c_2 \in \overline{\mathcal{C}}$. Then there exist some $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$ such that $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$. Without loss of generality, suppose $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Then $c_1 \in \mathcal{C}_2$ also. Thus, $c_1 + c_2 \in \mathcal{C}_2$, because $\mathcal{C}_2$ is additively closed. Thus, $c_1 + c_2 \in \overline{\mathcal{C}}$ also.

Divisible. Let $r \in \mathcal{R}$ and $n \in \mathbb{N}$, and suppose $n\,r \in \overline{\mathcal{C}}$ Then there exist some $\mathcal{C} \in \mathfrak{C}$ such that $n\,r \in \mathcal{C}$. Then $r \in \mathcal{C}$ (because $\mathcal{C}$ is divisible because $\mathcal{C} \in \mathfrak{C} \subseteq \mathfrak{D}$); thus, $r \in \overline{\mathcal{C}}$ also. ◇ `Claim 2`

Zorn's Lemma and Claim 2 imply that $\mathfrak{D}$ has a maximal element —call it $\mathcal{L}$. Clearly, $\mathcal{P} \subseteq \mathcal{L}$. It remains only for us to show that $\mathcal{L}$ is a linear order cone.

Now, $\mathcal{L} \in \mathfrak{D}$, so it is a partial order conoid. Let $s \in \mathcal{R} \setminus \{0\}$. We must show that either $s \in \mathcal{L}$ or $-s \in \mathcal{L}$. By contradiction, suppose neither were true. Then let $\mathcal{Q}_s := \mathcal{L} \cup \{\ell + n\, s;\ \ell \in \mathcal{L} \text{ and } n \in \mathbb{N}\} \cup \{n\, s;\ \ n \in \mathbb{N} \text{ and } n \neq 0\}$. Clearly, $\mathcal{L} \subset \mathcal{Q}_s$. Thus, $\mathcal{P} \subset \mathcal{Q}_s$ (because $\mathcal{P} \subseteq \mathcal{L}$ because $\mathcal{L} \in \mathfrak{D}$). Also, $\mathcal{L} \neq \mathcal{Q}_s$, because $s \in \mathcal{Q}_s$ whereas $s \notin \mathcal{L}$ by hypothesis.

Claim 3: *$\mathcal{Q}_s$ is a partial order conoid.*

Proof: Clearly, $\mathcal{Q}_s$ is additively closed. We must show that $0 \notin \mathcal{Q}_s$. Now, $0 \notin \mathcal{L}$ because $\mathcal{L}$ is a partial order conoid. Also, $n\, s \neq 0$ for any nonzero $n \in \mathbb{N}$, because $s \neq 0$ and $(\mathcal{R}, +)$ is torsion free. It remains to show that $\ell + n\, s \neq 0$, for any $\ell \in \mathcal{L}$ and $n \in \mathbb{N}$. By contradiction, suppose $\ell + n\, s = 0$. Thus, $n\,(-s) = \ell$. But then $-s \in \mathcal{L}$, because $\mathcal{L}$ is divisible, because $\mathcal{L} \in \mathfrak{D}$ by construction. But this contradicts our hypothesis on s. ◇ Claim 3

Now, let $\mathcal{D}'_s := \{r \in \mathcal{R};\ \ n\, r \in \mathcal{Q}_s \text{ for some } n \in \mathbb{N}\}$. Then an argument identical to Claim 1 shows that $\mathcal{D}'_s$ is a divisible partial order conoid containing $\mathcal{Q}_s$, and hence, containing $\mathcal{P}$. Thus, $\mathcal{D}'_s \in \mathfrak{D}$. But clearly, $\mathcal{L} \subsetneq \mathcal{Q}_s \subseteq \mathcal{D}'_s$. But this contradicts the presumed maximality of $\mathcal{L}$ in $\mathfrak{D}$.

By contradiction, either $s \in \mathcal{L}$ or $-s \in \mathcal{L}$. This argument works for all $s \in \mathcal{L}$. Thus, $\mathcal{L}$ is a linear order cone. Since $\mathcal{P} \subseteq \mathcal{L}$, the homogeneous linear order defined by $\mathcal{L}$ extends the homogeneous partial order $(\succ)$. □

Corollary A.2 *Let $(\mathcal{A}, +)$ be an abelian group.*

(a) *If $(\mathcal{A}, +)$ is torsion free, then $\mathcal{A}$ has a homogeneous linear ordering.*

(b) *Let $\mathcal{B} \subset \mathcal{A}$ be a divisible proper subgroup. Then there exists a complete, homogeneous preorder $(\succeq)$ on $\mathcal{A}$ such that $\mathcal{B} = \{a \in \mathcal{A};\ \ a \approx 0\}$.*

Proof: (a) Let $a \in \mathcal{A}$ be arbitrary. Let $\mathcal{P} := \{n\, a;\ \ n \in \mathbb{N}\}$. Then $\mathcal{P}$ is additively closed and $0 \notin \mathcal{P}$ (because $\mathcal{A}$ is torsion free), so $\mathcal{P}$ is a partial order conoid on $\mathcal{A}$. If $(\succ)$ is the homogeneous partial order defined by $\mathcal{P}$, then the Homogeneous Szpilrajn Lemma extends $(\succ)$ to a homogeneous linear order on $\mathcal{A}$.

(b) Let $\mathcal{Q} := \mathcal{A}/\mathcal{B}$, and let $q : \mathcal{A} \longrightarrow \mathcal{Q}$ be the quotient map. Then $\mathcal{Q}$ is a torsion free group, because $\mathcal{B}$ is divisible. Thus, part (a) yields a linear order $(>)$ on $\mathcal{Q}$. Now, define $(\succeq)$ on $\mathcal{A}$ by stipulating that $a_1 \succeq a_2$ if and only if $q(a_1) \geq q(a_2)$. It is easy to check that $(\succeq)$ is a complete, homogeneous preorder, and clearly, $\mathcal{B} = \{a \in \mathcal{A};\ q(a) = 0\} = \{a \in \mathcal{A};\ \ a \approx 0\}$. □

Let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group, and let $r, s \in \mathcal{R}$ be positive. We say r is *infinitesimal* relative to s if $N\, r < s$ for all $N \in \mathbb{N}$. We say $(\mathcal{R}, +, >)$ is *Archimedean* if it has no infinitesimal elements. For example $(\mathbb{R}, +, >)$ (with the standard ordering) is Archimedean. We state the next result for future reference.

Hölder's theorem. $(\mathcal{R}, +, >)$ *is Archimedean if and only if it is isomorphic to a subgroup of* $(\mathbb{R}, +, >)$.

Appendix B: Proofs

Proof of Theorem 2.2. "$\Longleftarrow$" is straightforward.

"$\Longrightarrow$" is an immediate consequence of Lemma B.1, which we prove next. □

Lemma B.1 *Let $\mathcal{D} \subset \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a domain. If $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies reinforcement, then $F = F_{\mathsf{B}}$ for some perfect balance system B.*

Proof: Let $\mathbb{Z}^{\langle\mathcal{V}\rangle} := \{\mathbf{n} \in \mathbb{Z}^{\mathcal{V}};\ \ \|\mathbf{n}\| < \infty\}$. For all $x \in \mathcal{X}$, let $\mathcal{C}_x := \{\mathbf{d} \in \mathcal{D};\ \ x \in F(\mathbf{d})\}$. Then $\mathcal{C}_x$ is a preorder conoid in the abelian group $\mathbb{Z}^{\langle\mathcal{V}\rangle}$. (We have $\mathbf{0} \in \mathcal{C}_x$ because $\mathbf{0} \in \mathcal{D}$ and $F(\mathbf{0}) = \mathcal{X}$ by definition. Meanwhile, $\mathcal{C}_x$ is closed under addition because F satisfies reinforcement).

For any $x, y \in \mathcal{X}$, let $\mathcal{P}_{x,y} := \{\mathbf{c}_x - \mathbf{c}_y;\ \ \mathbf{c}_x \in \mathcal{C}_x \text{ and } \mathbf{c}_y \in \mathcal{C}_y\}$. Then $\mathcal{P}_{x,y}$ is a preorder conoid. ($\mathcal{P}_{x,y}$ is closed under addition because $\mathcal{C}_x$ and $\mathcal{C}_y$ are closed under addition.) Note that $\mathcal{P}_{y,x} = -\mathcal{P}_{x,y}$.

Let $(\succeq)$ be the homogeneous preorder on $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ defined by $\mathcal{P}_{x,y}$. Let $\mathcal{O}_{x,y} := \{\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle};\ \mathbf{z} \approx \mathbf{0}\}$, and let $\mathcal{R}_{x,y} := \mathbb{Z}^{\langle\mathcal{V}\rangle}/\mathcal{O}_{x,y}$, with the homogeneous partial order $(\succ)$ described in Lemma A.1(b). Use the Homogeneous Szpilrajn Lemma to extend $(\succ)$ to a homogeneous linear order $(\underset{x,y}{>})$ on $\mathcal{R}_{x,y}$. Let $\mathbf{b}^{x,y} : \mathbb{Z}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}_{x,y}$ be the quotient map (i.e. $\mathbf{b}^{x,y}(\mathbf{z}) := \mathbf{z} + \mathcal{O}_{x,y}$ for all $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$). Note that $\mathcal{O}_{y,x} = \mathcal{O}_{x,y}$, so $\mathcal{R}_{y,x} = \mathcal{R}_{x,y}$ as groups, and $\mathbf{b}^{y,x} = \mathbf{b}^{x,y}$. However, $(\underset{x,y}{>})$ is the negative ordering to $(\underset{y,x}{>})$ (i.e. $(r \underset{x,y}{>} r') \iff (-r \underset{y,x}{>} -r')$). Thus, without loss of generality, we can redefine $(\underset{y,x}{>})$ to be identical with $(\underset{x,y}{>})$, and redefine $\mathbf{b}^{y,x}$ to be $-\mathbf{b}^{x,y}$ (or vice versa; it doesn't matter whether we reverse $\mathbf{b}^{x,y}$ or $\mathbf{b}^{y,x}$, as long as we reverse only one of them). Finally, define $\mathbf{b}^{x,x} := 0$ for all $x \in \mathcal{X}$.

Claim 1: *Let $\mathbf{d} \in \mathcal{D}$ and let $x \in F(\mathbf{d})$. Then:*

(a) $\mathbf{b}^{x,y}(\mathbf{d}) \geq 0$ *for all* $y \in \mathcal{X}$ (hence, $x \in F_{\mathsf{B}}(\mathbf{d})$).

(b) Furthermore, if $y \notin F(\mathbf{d})$, *then* $\mathbf{b}^{x,y}(\mathbf{d}) > 0$ (hence, $y \notin F_{\mathsf{B}}(\mathbf{d})$).

Proof: (a) If $x \in F(\mathbf{d})$, then $\mathbf{d} \in \mathcal{C}_x$. Meanwhile $\mathbf{0} \in \mathcal{C}_y$; thus, $\mathbf{d} = \mathbf{d} - \mathbf{0} \in \mathcal{P}_{x,y}$, so $\mathbf{d} \succeq \mathbf{0}$, so $\mathbf{b}^{x,y}(\mathbf{d}) \geq 0$.

(b) (by contradiction) Suppose $\mathbf{b}^{x,y}(\mathbf{d}) = 0$. Then $\mathbf{d} \in \mathcal{O}_{x,y}$. Thus, $\mathbf{d} \preceq \mathbf{0}$, so $-\mathbf{d} \succeq \mathbf{0}$, so $-\mathbf{d} \in \mathcal{P}_{x,y}$. Thus, $-\mathbf{d} = \mathbf{c}_x - \mathbf{c}_y$ for some $\mathbf{c}_x \in \mathcal{C}_x$ and $\mathbf{c}_y \in \mathcal{C}_y$. But then $\mathbf{c}_y = \mathbf{c}_x + \mathbf{d}$. Now, $x \in F(\mathbf{d})$ and $x \in F(\mathbf{c}_x)$, so $F(\mathbf{d}) \cap F(\mathbf{c}_x) \neq \emptyset$; thus $F(\mathbf{c}_y) = F(\mathbf{d} + \mathbf{c}_x) = F(\mathbf{d}) \cap F(\mathbf{c}_x)$, by reinforcement. But $y \notin F(\mathbf{d})$ by hypothesis, so $y \notin F(\mathbf{d}) \cap F(\mathbf{c}_x)$, so $y \notin F(\mathbf{c}_y)$. But this contradicts the fact that $\mathbf{c}_y \in \mathcal{C}_y$. By contradiction, we cannot have $\mathbf{b}^{x,y}(\mathbf{d}) = 0$. Thus, $\mathbf{b}^{x,y}(\mathbf{d}) > 0$. ◇ Claim 1

For all $\mathbf{d} \in \mathcal{D}$, Claim 1(a) implies that $F(\mathbf{d}) \subseteq F_{\mathsf{B}}(\mathbf{d})$, while Claim 1(b) implies that $F(\mathbf{d}) \supseteq F_{\mathsf{B}}(\mathbf{d})$. We conclude that $F = F_{\mathsf{B}}$. Furthermore, Claim 1(b) implies that the balance system B is perfect.

Now we must construct a single linearly ordered abelian group $\mathcal{R}$ and a collection of functions $\widetilde{\mathbf{b}}_{x,y} : \mathcal{V} \longrightarrow \mathcal{R}$ (for all $x, y \in \mathcal{X}$) such that $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$. Let $\mathcal{R} := \prod_{x,y \in \mathcal{X}} \mathcal{R}_{x,y}$ with the 'Pareto' order it gets from the linear orders on the components. Then extend this to a homogeneous linear order on $\mathcal{R}$ using the Homogeneous Szpilrajn Lemma. □

The proof of Theorem 2.3 depends upon Propositions 3.1 and 3.2, so it is deferred.

Lemma B.2 *Let* $\mathbf{b} \in \mathcal{R}^{\mathcal{V}}$.

(a) *For any* $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ *and* $\pi \in \Pi_{\mathcal{V}}$, *we have* $(\mathbf{b}\pi)(\mathbf{n}) \; = \; \mathbf{b}\,(\pi(\mathbf{n}))$.

(b) *For any* $\pi, \phi \in \Pi_{\mathcal{V}}$, *we have* $(\mathbf{b}\,\pi)\,\phi \; = \; \mathbf{b}\;(\pi\phi)$.

Proof: (a) $(\mathbf{b}\pi)(\mathbf{n}) \; = \; \sum_{v \in \mathcal{V}} n_v (\mathbf{b}\pi)_v \; = \; \sum_{v \in \mathcal{V}} n_v b_{\pi(v)} \; \underset{(*)}{=} \; \sum_{v' \in \mathcal{V}} n_{\pi^{-1}(v')} b_{v'} \; = \; \sum_{v \in \mathcal{V}} \pi(\mathbf{n})_{v'} b_{v'} \; = \; \mathbf{b}(\pi(\mathbf{n}))$. Here, $(*)$ is the change of variables $v' := \pi(v)$.

(b) For any $v \in \mathcal{V}$, if $w := \phi(v)$, then $((\mathbf{b}\pi)\phi)_v = (\mathbf{b}\pi)_{\phi(v)} = (\mathbf{b}\pi)_w = b_{\pi(w)} = b_{\pi\phi(v)} = (\mathbf{b}\,(\pi\phi))_v$, as desired. □

Proof of Proposition 3.1. "$\Longleftarrow$" is obvious.

"$\Longrightarrow$" Let $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ be a score system such that $F = F_{\mathsf{S}}$. Define the score system $\overline{\mathsf{S}} = \{\overline{\mathbf{s}}^x\}_{x \in \mathcal{X}}$ by setting

$$\overline{\mathbf{s}}^x \quad := \quad \sum_{\pi \in \Pi'_{\mathcal{X}}} \mathbf{s}^{\pi(x)}\,\widetilde{\pi}, \qquad \text{for all } x \in \mathcal{X}.$$

Claim 1: $F_{\overline{\mathsf{S}}} = F$.

Proof: "$\supseteq$" Let $\mathbf{d} \in \mathcal{D}$ and $x \in \mathcal{X}$. Suppose $x \in F(\mathbf{d})$; we will show that $x \in F_{\overline{\mathsf{S}}}(\mathbf{d})$. For any $\pi \in \Pi'_{\mathcal{X}}$, neutrality implies that $\pi(x) \in F(\widetilde{\pi}(\mathbf{d}))$. Thus, $\mathbf{s}^{\pi(x)}(\widetilde{\pi}(\mathbf{d})) \geq \mathbf{s}^{y}(\widetilde{\pi}(\mathbf{d}))$ for all $y \in \mathcal{X}$ and $\pi \in \Pi'_{\mathcal{X}}$. Equivalently, $\mathbf{s}^{\pi(x)}(\widetilde{\pi}(\mathbf{d})) \geq \mathbf{s}^{\pi(z)}(\widetilde{\pi}(\mathbf{d}))$ for all $z \in \mathcal{X}$ and $\pi \in \Pi'_{\mathcal{X}}$. Thus, Lemma B.2(a) says $(\mathbf{s}^{\pi(x)}\widetilde{\pi})(\mathbf{d}) \geq (\mathbf{s}^{\pi(z)}\widetilde{\pi})(\mathbf{d})$ for all $z \in \mathcal{X}$ and $\pi \in \Pi'_{\mathcal{X}}$. Thus,

$$\overline{\mathbf{s}}^x(\mathbf{d}) \quad = \quad \sum_{\pi \in \Pi'_{\mathcal{X}}} (\mathbf{s}^{\pi(x)}\,\widetilde{\pi})(\mathbf{d}) \quad \geq \quad \sum_{\pi \in \Pi'_{\mathcal{X}}} (\mathbf{s}^{\pi(z)}\,\widetilde{\pi})(\mathbf{d}) \quad = \quad \overline{\mathbf{s}}^z(\mathbf{d}),$$

for all $z \in \mathcal{X}$. Thus, $x \in F_{\overline{\mathsf{S}}}(\mathbf{d})$, as desired.

"$\subseteq$" (by contrapositive) Now suppose $x \notin F(\mathbf{d})$; we will show that $x \notin F_{\overline{\mathsf{S}}}(\mathbf{d})$. Let $y \in F(\mathbf{d})$. Then for any $\pi \in \Pi'_{\mathcal{X}}$, neutrality implies that $\pi(x) \notin F(\widetilde{\pi}(\mathbf{d}))$ and

$\pi(y) \in F(\widetilde{\pi}(\mathbf{d}))$. Thus, $\mathbf{s}^{\pi(x)}(\widetilde{\pi}(\mathbf{d})) < \mathbf{s}^{\pi(y)}(\widetilde{\pi}(\mathbf{d}))$ for all $\pi \in \Pi'_{\mathcal{X}}$. Thus, Lemma B.2(a) says $(\mathbf{s}^{\pi(x)}\widetilde{\pi})(\mathbf{d}) < (\mathbf{s}^{\pi(y)}\widetilde{\pi})(\mathbf{d})$ for all $\pi \in \Pi'_{\mathcal{X}}$. Thus,

$$\overline{\mathbf{s}}^{x}(\mathbf{d}) \quad = \quad \sum_{\pi\in\Pi'_{\mathcal{X}}} \left(\mathbf{s}^{\pi(x)}\,\widetilde{\pi}\right)(\mathbf{d}) \quad < \quad \sum_{\pi\in\Pi'_{\mathcal{X}}} \left(\mathbf{s}^{\pi(y)}\,\widetilde{\pi}\right)(\mathbf{d}) \quad = \quad \overline{\mathbf{s}}^{y}(\mathbf{d}),$$

Thus, $x \notin F_{\overline{\mathsf{S}}}(\mathbf{d})$, as desired. $\diamond$ `Claim 1`

It remains to check that the system $\overline{\mathsf{S}} := \{\overline{\mathbf{s}}^x\}_{x\in\mathcal{X}}$ is ν-neutral. To see this, let $x, y \in \mathcal{X}$, let $\phi \in \Pi'_{\mathcal{X}}$, and suppose $\phi(y) = x$. Then

$$\begin{aligned}\overline{\mathbf{s}}^{x}\,\widetilde{\phi} \quad &= \quad \left(\sum_{\pi\in\Pi'_{\mathcal{X}}} \mathbf{s}^{\pi(x)}\,\widetilde{\pi}\right)\widetilde{\phi} \quad \underset{(*)}{=} \quad \sum_{\pi\in\Pi'_{\mathcal{X}}} \mathbf{s}^{\pi(x)}\,(\widetilde{\pi}\widetilde{\phi}) \quad \underset{(\diamond)}{=} \quad \sum_{\pi\in\Pi'_{\mathcal{X}}} \mathbf{s}^{\pi(x)}\,\widetilde{\pi\phi} \\ &\underset{(\dagger)}{=} \quad \sum_{\pi'\in\Pi'_{\mathcal{X}}} \mathbf{s}^{\pi'\phi^{-1}(x)}\,\widetilde{\pi}' \quad = \quad \sum_{\pi'\in\Pi'_{\mathcal{X}}} \mathbf{s}^{\pi'(y)}\,\widetilde{\pi}' \quad = \quad \overline{\mathbf{s}}^{y},\end{aligned}$$

as desired. Here, $(*)$ is by linearity and Lemma B.2(b), $(\diamond)$ is because ν is a group homomorphism, and $(\dagger)$ is the change of variables $\pi' := \pi\phi$. $\square$

Proposition 3.2 is a consequence of the following more technical result.

Proposition B.3 *Let $\mathcal{X}$, $\mathcal{V}$, $\nu : \Pi'_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$, and $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ be as in Proposition 3.2. Let $\mathcal{R}$ be a linearly ordered abelian group, and let $\widetilde{\mathsf{B}}$ be a $\mathcal{R}$-valued perfect balance system on $(\mathcal{X}, \mathcal{V})$. The balance rule $F_{\widetilde{\mathsf{B}}} : \mathcal{D} \rightrightarrows \mathcal{X}$ is ν-neutral if and only if $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$ for some $\mathcal{R}$-valued, ν-neutral, perfect balance system B on $(\mathcal{X}, \mathcal{V})$.*

Proof: "$\Longleftarrow$" Let B be a ν-neutral balance system. Let $\mathbf{d} \in \mathcal{D}$ and let $\pi \in \Pi'_{\mathcal{X}}$. We must show that $F_{\mathsf{B}}(\widetilde{\pi}(\mathbf{d})) = \pi(F_{\mathsf{B}}(\mathbf{d}))$.

"$\supseteq$" Let $x \in F_{\mathsf{B}}(\mathbf{d})$, and let $y = \pi(x)$. For all $z \in \mathcal{X}$, if $w = \pi^{-1}(z)$, then

$$\mathbf{b}^{y,z}\left(\widetilde{\pi}(\mathbf{d})\right) \quad \underset{(*)}{=} \quad \left(\mathbf{b}^{y,z}\widetilde{\pi}\right)(\mathbf{d}) \quad \underset{(\dagger)}{=} \quad \mathbf{b}^{x,w}(\mathbf{d}) \quad \underset{(\diamond)}{\geq} \quad 0. \tag{B1}$$

Here, $(*)$ is by Lemma B.2(a), $(\dagger)$ is because B is ν-neutral, and $(\diamond)$ is because $x \in F_{\mathsf{B}}(\mathbf{d})$. Since this holds for all $z \in \mathcal{X}$, we conclude that $y \in F_{\mathsf{B}}\left(\widetilde{\pi}(\mathbf{d})\right)$.

"$\subseteq$" (by contrapositive) Now suppose $x \notin F_{\mathsf{B}}(\mathbf{d})$. Then there is some $w \in \mathcal{X}$ such that $\mathbf{b}^{x,w}(\mathbf{d}) < 0$. Thus, if $y = \pi(x)$ and $z = \pi(w)$, then the inequality "$\underset{(\diamond)}{\geq}$" in equation (B1) changes to "$<$"; thus, $y \notin F_{\mathsf{B}}\left(\widetilde{\pi}(\mathbf{d})\right)$.

Thus $F_{\mathsf{B}}\left(\widetilde{\pi}(\mathbf{d})\right) = \pi\left(F_{\mathsf{B}}(\mathbf{d})\right)$. This holds for all $\mathbf{d} \in \mathcal{D}$ and $\pi \in \Pi'_{\mathcal{X}}$. Thus, F_{B} is ν-neutral.

"$\Longrightarrow$" Fix $x, y \in \mathcal{X}$ with $x \neq y$. Double transitivity yields some $\alpha \in \Pi'_{\mathcal{X}}$ such that $\alpha(x) = y$ and $\alpha(y) = x$. Let N be the order of α (so $\alpha^N = \mathrm{Id}$). Then N is finite, because $\mathcal{X}$ is finite. Also, N is even, because $\alpha(x) = y$ and $\alpha^2(x) = x$. Let $M = N/2$. Define

$$\overline{\mathbf{b}}^{x,y} \quad := \quad \sum_{m=0}^{M-1} \left(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m} - \widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1}\right). \tag{B2}$$

Claim 1: $\overline{\mathbf{b}}^{x,y}\widetilde{\alpha} = -\overline{\mathbf{b}}^{x,y}$.

Proof: We have

$$\overline{\mathbf{b}}^{x,y}\widetilde{\alpha} \underset{\text{(B2)}}{=\!=} \left(\sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m} - \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1}\right)\widetilde{\alpha} \quad = \quad \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1} - \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+2}$$

$$\underset{(*)}{=\!=} \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1} - \sum_{m'=1}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m'} - \widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2M} \underset{(\dagger)}{=\!=} \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1} - \sum_{m'=1}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m'} - \widetilde{\mathbf{b}}^{x,y}$$

$$= \sum_{m=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1} - \sum_{m'=0}^{M-1}\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m'} \underset{\text{(B2)}}{=\!=} -\overline{\mathbf{b}}^{x,y}, \qquad \text{as desired.}$$

$(*)$ is the change of variables $m' := m+1$, while $(\dagger)$ is because $\widetilde{\alpha}^{2M} = \text{Id}$. ◇ `Claim 1`

Claim 2: *Let* $\mathbf{d} \in \mathcal{D}$.

(a) *If* $x \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$, *then* $\overline{\mathbf{b}}^{x,y}(\mathbf{d}) \geq 0$.

(b) *If* $x \notin F_{\widetilde{\mathsf{B}}}(\mathbf{d})$, *but* $y \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$, *then* $\overline{\mathbf{b}}^{x,y}(\mathbf{d}) < 0$.

Proof: (a) If $x \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$, then for all $m \in [0 \ldots M)$, the neutrality of $F_{\widetilde{\mathsf{B}}}$ implies that $x \in F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m}(\mathbf{d}))$, while $y \in F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m+1}(\mathbf{d}))$ (because $\alpha^{2m}(x) = x$, while $\alpha^{2m+1}(x) = y$). Thus, $\widetilde{\mathbf{b}}^{x,y}(\widetilde{\alpha}^{2m}(\mathbf{d})) \geq 0$, while $\widetilde{\mathbf{b}}^{x,y}(\widetilde{\alpha}^{2m+1}(\mathbf{d})) \leq 0$. Thus, Lemma B.2(a) says $(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m})(\mathbf{d}) \geq 0$, while $(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1})(\mathbf{d}) \leq 0$. Thus,

$$\overline{\mathbf{b}}^{x,y}(\mathbf{d}) \underset{\text{(B2)}}{=\!=} \sum_{m=0}^{M-1}\left(\left(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m}\right)(\mathbf{d}) - \left(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1}\right)(\mathbf{d})\right) \quad \geq \quad 0.$$

(b) If $x \notin F_{\widetilde{\mathsf{B}}}(\mathbf{d})$ and $y \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$, then for all $m \in [0 \ldots M)$, neutrality implies that $x \notin F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m}(\mathbf{d}))$ and $y \in F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m}(\mathbf{d}))$, while $y \notin F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m+1}(\mathbf{d}))$ and $x \in F_{\widetilde{\mathsf{B}}}(\widetilde{\alpha}^{2m+1}(\mathbf{d}))$ (because $\alpha^{2m}(x) = x$ and $\alpha^{2m}(y) = y$, while $\alpha^{2m+1}(x) = y$ and $\alpha^{2m+1}(y) = x$). Thus, $\widetilde{\mathbf{b}}^{x,y}(\widetilde{\alpha}^{2m}(\mathbf{d})) < 0$, while $\widetilde{\mathbf{b}}^{x,y}(\widetilde{\alpha}^{2m+1}(\mathbf{d})) > 0$, because $\widetilde{\mathsf{B}}$ is perfect. Thus, Lemma B.2(a) says $(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m})(\mathbf{d}) < 0$, while $(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1})(\mathbf{d}) > 0$. Thus,

$$\overline{\mathbf{b}}^{x,y}(\mathbf{d}) \underset{\text{(B2)}}{=\!=} \sum_{m=0}^{M-1}\left(\left(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m}\right)(\mathbf{d}) - \left(\widetilde{\mathbf{b}}^{x,y}\widetilde{\alpha}^{2m+1}\right)(\mathbf{d})\right) \quad < \quad 0,$$

because every summand is negative. ◇ `Claim 2`

Let $\Pi_{x,y} := \{\pi \in \Pi'_{\mathcal{X}};\ \pi(x) = x \text{ and } \pi(y) = y\}$ (a subgroup of $\Pi'_{\mathcal{X}}$). Then $\Pi_{x,y}$ is finite, because $\Pi'_{\mathcal{X}}$ is finite, because $\mathcal{X}$ is finite. Define

$$\mathbf{b}^{x,y} \quad := \quad \sum_{\pi \in \Pi_{x,y}} \left(\overline{\mathbf{b}}^{x,y}\widetilde{\pi}\right). \tag{B3}$$

Claim 3: $\mathbf{b}^{x,y}\widetilde{\alpha} = -\mathbf{b}^{x,y}$.

Proof:

$$\begin{aligned}
\mathbf{b}^{x,y}\widetilde{\alpha} &\underset{\text{(B3)}}{=\!=} \left(\sum_{\pi\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}\right)\widetilde{\alpha} \;=\; \sum_{\pi\in\Pi_{x,y}} \left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}\right)\widetilde{\alpha} \underset{(\diamond)}{=\!=} \sum_{\pi\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi\alpha} \\
&= \sum_{\pi\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,(\widetilde{\alpha\alpha^{-1}\pi\alpha}) \underset{(\diamond)}{=\!=} \sum_{\pi\in\Pi_{x,y}} \left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\alpha}\right)\widetilde{\alpha^{-1}\pi\alpha} \underset{(*)}{=\!=} \sum_{\pi'\in\Pi_{x,y}} \left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\alpha}\right)\widetilde{\pi}' \\
&\underset{(\dagger)}{=\!=} \sum_{\pi'\in\Pi_{x,y}} -\overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}' \;=\; -\sum_{\pi'\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}' \underset{\text{(B3)}}{=\!=} -\mathbf{b}^{x,y}.
\end{aligned}$$

Here, both $(\diamond)$ are by Lemma B.2(b) (and the fact that ν is a homomorphism). Meanwhile, $(*)$ is by the change of variables $\pi' := \alpha\pi\alpha^{-1}$ for all $\pi \in \Pi_{x,y}$ (observe that the map $(\pi \mapsto \alpha\pi\alpha^{-1})$ is a permutation of $\Pi_{x,y}$, because $\alpha(x) = y$ and $\alpha(y) = x$). Finally, $(\dagger)$ is by Claim 1. ◇ `Claim 3`

Claim 4: *For any $\phi \in \Pi_{x,y}$, we have $\mathbf{b}^{x,y}\,\widetilde{\phi} = \mathbf{b}^{x,y}$.*

Proof: $\mathbf{b}^{x,y}\,\widetilde{\phi} \underset{\text{(B3)}}{=\!=} \left(\sum_{\pi\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}\right)\widetilde{\phi} \underset{(\diamond)}{=} \sum_{\pi\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi\phi} \underset{(*)}{=} \sum_{\pi'\in\Pi_{x,y}} \overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}' \underset{\text{(B3)}}{=\!=} \mathbf{b}^{x,y}$.

Here, $(\diamond)$ is by linearity and Lemma B.2(b), while $(*)$ is the change of variables $\pi' := \pi\phi$ (the map $(\pi \mapsto \pi\phi)$ is a permutation of $\Pi_{x,y}$, because $\phi \in \Pi_{x,y}$). ◇ `Claim 4`

Now, for all $w, z \in \mathcal{X}$, define $\Pi_{w,z} := \{\pi \in \Pi'_{\mathcal{X}};\ \pi(w) = x \text{ and } \pi(z) = y\}$; then $\Pi_{w,z} \neq \emptyset$ by the double-transitivity of $\Pi'_{\mathcal{X}}$. Define $\mathbf{b}^{w,z} := \mathbf{b}^{x,y}\,\widetilde{\pi}_{w,z}$, for any $\pi_{w,z} \in \Pi_{w,z}$

Claim 5: $\mathbf{b}^{w,z}$ *is well-defined independent of the choice of $\pi_{w,z} \in \Pi_{w,z}$.*

Proof: Suppose $\pi_{w,z} \in \Pi_{w,z}$ and $\phi_{w,z} \in \Pi_{w,z}$. Then $\pi_{w,z}\,\phi_{w,z}^{-1} \in \Pi_{x,y}$. Thus,

$$\left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}_{w,z}\right)\widetilde{\phi}_{w,z}^{-1} \underset{(*)}{=\!=} \overline{\mathbf{b}}^{x,y}\,(\widetilde{\pi_{w,z}\,\phi_{w,z}^{-1}}) \underset{(\dagger)}{=\!=} \overline{\mathbf{b}}^{x,y}, \tag{B4}$$

where $(*)$ is by Lemma B.2(b), and $(\dagger)$ is by Claim 4. Multiplying both ends of (B4) by $\widetilde{\phi}_{w,z}$, we obtain: $\overline{\mathbf{b}}^{x,y}\,\widetilde{\pi}_{w,z} = \overline{\mathbf{b}}^{x,y}\,\widetilde{\phi}_{w,z}$, as desired. ◇ `Claim 5`

Claim 6: *For any $w, z, w', z' \in \mathcal{X}$ and $\phi \in \Pi'_{\mathcal{X}}$, if $\phi^{-1}(w) = w'$ and $\phi^{-1}(z) = z'$, then $\mathbf{b}^{w,z}\widetilde{\phi} = \mathbf{b}^{w',z'}$.*

Proof: Let $\pi_{w,z} \in \Pi_{w,z}$. Then $\mathbf{b}^{w,z} = \mathbf{b}^{x,y}\widetilde{\pi}_{w,z}$ by definition (and Claim 5). But $\pi_{w,z}\phi \in \Pi_{w',z'}$ (because $\pi_{w,z}\phi(w') = \pi_{w,z}(w) = x$ and $\pi_{w,z}\phi(z') = \pi_{w,z}(z) = y$). Thus,

$$\mathbf{b}^{w,z}\widetilde{\phi} \;=\; (\mathbf{b}^{x,y}\,\widetilde{\pi}_{w,z})\,\widetilde{\phi} \underset{(*)}{=\!=} \mathbf{b}^{x,y}\,(\widetilde{\pi_{w,z}\phi}) \underset{(\dagger)}{=\!=} \mathbf{b}^{w',z'},$$

as desired. Here, $(*)$ is by Lemma B.2(b) and $(\dagger)$ is by definition (and Claim 5). ◇ `Claim 6`

Claim 7: $\mathbf{b}^{z,w} = -\mathbf{b}^{w,z}$ *for all $w, z \in \mathcal{X}$.*

Proof: Let $\pi_{w,z} \in \Pi_{w,z}$. Then $\alpha\pi_{w,z} \in \Pi_{z,w}$ (because $\alpha\pi_{w,z}(z) = \alpha(y) = x$, etc.). Thus,

$$\mathbf{b}^{z,w} \underset{(*)}{=} \mathbf{b}^{x,y}\left(\widetilde{\alpha\pi_{w,z}}\right) \underset{(\ddagger)}{=} (\mathbf{b}^{x,y}\widetilde{\alpha})\widetilde{\pi}_{w,z} \underset{(\diamond)}{=} -\mathbf{b}^{x,y}\widetilde{\pi}_{w,z} \underset{(\dagger)}{=} -\mathbf{b}^{w,z}.$$

Here, $(*)$ is by definition of $\mathbf{b}^{z,w}$ (and Claim 5). Next, $(\ddagger)$ is by Lemma B.2(b), $(\diamond)$ is by Claim 3, and $(\dagger)$ is by definition of $\mathbf{b}^{w,z}$ (and Claim 5). ◇ Claim 7

Now define $\mathsf{B} := \{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}}$. Then B is a balance system by Claim 7, and B is ν-neutral by Claim 6. The "$\Longleftarrow$" direction of the theorem (already established) then implies that the balance rule F_{B} is ν-neutral.

Claim 8: $F_{\mathsf{B}}(\mathbf{d}) = F_{\widetilde{\mathsf{B}}}(\mathbf{d})$ *for all* $\mathbf{d} \in \mathcal{D}$*. Also,* B *is a perfect balance rule on* $\mathcal{D}$*.*

Proof: "$\supseteq$" Let $\mathbf{d} \in \mathcal{D}$ and let $w \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$. Fix $z \in \mathcal{X}$ and let $\pi_{w,z} \in \Pi_{w,z}$. Then $\pi(w) = x$ and $\pi(z) = y$. Since $F_{\widetilde{\mathsf{B}}}$ is ν-neutral, we have $F_{\widetilde{\mathsf{B}}}\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right) = \pi_{w,z}(F_{\widetilde{\mathsf{B}}}(\mathbf{d}))$; thus $x \in F_{\widetilde{\mathsf{B}}}\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right)$. Now, for any $\phi \in \Pi_{x,y}$, neutrality implies that $x \in F_{\widetilde{\mathsf{B}}}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right)$; thus, Claim 2(a) says that $\overline{\mathbf{b}}^{x,y}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right) \geq 0$. But Lemma B.2(a) says $\overline{\mathbf{b}}^{x,y}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right) = \left(\overline{\mathbf{b}}^{x,y}\widetilde{\phi}\right)\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right)$. Thus, we have

$$\left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\phi}\right)\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right) \geq 0, \qquad \text{for all } \phi \in \Pi_{x,y}. \tag{B5}$$

$$\text{Thus,} \quad \mathbf{b}^{w,z}(\mathbf{d}) \underset{(*)}{=} (\mathbf{b}^{x,y}\widetilde{\pi}_{w,z})(\mathbf{d}) \underset{(\dagger)}{=} \mathbf{b}^{x,y}\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right)$$
$$\underset{\text{(B3)}}{=\!=} \sum_{\phi\in\Pi_{x,y}} \left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\phi}\right)\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right) \underset{\text{(B5)}}{\geq} 0. \tag{B6}$$

Here, $(*)$ is by definition of $\mathbf{b}^{w,z}$ (and Claim 5), while $(\dagger)$ is by Lemma B.2(a).

Inequality (B6) holds for any $z \in \mathcal{X}$. Thus, $w \in F_{\mathsf{B}}(\mathbf{d})$.

"$\subseteq$" (by contrapositive) Suppose $w \notin F_{\widetilde{\mathsf{B}}}(\mathbf{d})$. Let $z \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})$. We will show that $\mathbf{b}^{w,z}(\mathbf{d}) < 0$, thereby showing both that $w \notin F_{\mathsf{B}}(\mathbf{d})$ and that B is perfect.

If $\pi_{w,z} \in \Pi_{w,z}$, then the neutrality of $F_{\widetilde{\mathsf{B}}}$ implies that $x \notin F_{\widetilde{\mathsf{B}}}\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right)$, while $y \in F_{\widetilde{\mathsf{B}}}\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right)$. Thus, for all $\phi \in \Pi_{x,y}$, we have $x \notin F_{\widetilde{\mathsf{B}}}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right)$, while $y \in F_{\widetilde{\mathsf{B}}}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right)$. Thus, Claim 2(b) says that $\overline{\mathbf{b}}^{x,y}\left(\widetilde{\phi}\,\widetilde{\pi}_{w,z}(\mathbf{d})\right) < 0$; hence Lemma B.2(a) says $\left(\overline{\mathbf{b}}^{x,y}\,\widetilde{\phi}\right)\left(\widetilde{\pi}_{w,z}(\mathbf{d})\right) < 0$. This holds for all $\phi \in \Pi_{x,y}$; thus, an argument similar to equation (B6) yields $\mathbf{b}^{w,z}(\mathbf{d}) < 0$. Thus, $w \notin F_{\mathsf{B}}(\mathbf{d})$. ◇ Claim 8

□

Proof of Proposition 3.2. "$\Longleftarrow$" follows immediately from Proposition B.3 "$\Longleftarrow$".

"$\Longrightarrow$" If F is a balance rule, then F satisfies reinforcement (by Theorem 2.2). Then $F = F_{\widetilde{\mathsf{B}}}$ for some *perfect* balance system $\widetilde{\mathsf{B}}$ (by Lemma B.1). Now apply Proposition B.3 "$\Longrightarrow$". □

The proof of Theorem 2.3 requires Lemma 5.1, so we prove that first.

Proof of Lemma 5.1 "$\Longrightarrow$" follows from Example 2.1(a).

"$\Longleftarrow$" Let $\mathsf{B} = \{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}}$ be a balance system on $\mathcal{V}$ satisfying (3), and such that $F_{\mathsf{B}}(\mathbf{d}) = F(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$. Fix $o \in \mathcal{X}$, and define $\mathbf{s}^o := \underline{0}$. Then for all $x \in \mathcal{X}$, define $\mathbf{s}^x := \mathbf{b}^{x,o}$. This yields a score system $\mathsf{S} = \{\mathbf{s}^x\}_{x\in\mathcal{X}}$ on $\mathcal{V}$. Let $\widetilde{\mathsf{B}} = \nabla\mathsf{S}$ (i.e. $\widetilde{\mathbf{b}}^{x,y} := \mathbf{s}^x - \mathbf{s}^y$ for all $x, y \in \mathcal{X}$). Thus, Example 2.1(a) implies that $F_{\mathsf{S}}(\mathbf{n}) = F_{\widetilde{\mathsf{B}}}(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$. For any $\mathbf{d} \in \mathcal{D}$, and all $x, y \in \mathcal{X}$, we have

$$\widetilde{\mathbf{b}}^{x,y}(\mathbf{d}) \quad = \quad \mathbf{s}^x(\mathbf{d}) - \mathbf{s}^y(\mathbf{d}) \quad = \quad \mathbf{b}^{x,o}(\mathbf{d}) - \mathbf{b}^{y,o}(\mathbf{d}) \underset{(*)}{=} \mathbf{b}^{x,y}(\mathbf{d}),$$

where $(*)$ is by condition (3). Thus, $F_{\widetilde{\mathsf{B}}}(\mathbf{d}) = F_{\mathsf{B}}(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$. But $F_{\mathsf{B}}(\mathbf{d}) = F(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$, by hypothesis. We conclude that $F_{\mathsf{S}}(\mathbf{d}) = F(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$. □

The proof of Theorem 2.3 requires some preliminary lemmas. Let $(\mathcal{A}, +)$ be an abelian group, and let $\mathcal{S} \subseteq \mathcal{A}$. Let $\langle\mathcal{S}\rangle$ be the divisible subgroup generated by $\mathcal{S}$ —that is, $\langle\mathcal{S}\rangle$ is the smallest divisible subgroup of $\mathcal{A}$ which contains $\mathcal{S}$.

Lemma B.4 *Let $(\mathcal{A}, +)$ be an abelian group, let $(\mathcal{R}, +)$ be a torsion-free abelian group, and let $\phi : \mathcal{A}\longrightarrow\mathcal{R}$ be a group homomorphism. Let $\mathcal{S} \subset \mathcal{A}$. If $\phi(s) = 0$ for all $s \in \mathcal{S}$, then $\phi(a) = 0$ for all $a \in \langle\mathcal{S}\rangle$.*

Proof: Let $\mathcal{K} := \{a \in \mathcal{A};\ \ \phi(a) = 0\}$. Then $\mathcal{S} \subseteq \mathcal{K}$. We must show that $\langle\mathcal{S}\rangle \subseteq \mathcal{K}$. To do this, it suffices to show that $\mathcal{K}$ is a divisible subgroup. Clearly, $\mathcal{K}$ is a subgroup. To see that $\mathcal{K}$ is divisible, let $a \in \mathcal{A}$ and $n \in \mathbb{N}$ and suppose $n\,a \in \mathcal{K}$. Then we have $0 = \phi(n\,a) = n\,\phi(a)$. But $\mathcal{R}$ is torsion free, so this means $\phi(a) = 0$; thus $a \in \mathcal{K}$. □

Lemma B.5 *Let $(\mathcal{A}, +)$ be an abelian group. Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{A}$ be arbitrary subsets, and suppose that $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ is additively closed. Then either $\langle\mathcal{S}\rangle = \langle\mathcal{S}_1\rangle$, or $\langle\mathcal{S}\rangle = \langle\mathcal{S}_2\rangle$.*

Proof: Suppose $\langle\mathcal{S}_1\rangle \neq \langle\mathcal{S}\rangle$; we will show that $\langle\mathcal{S}_2\rangle = \langle\mathcal{S}\rangle$. Clearly, $\langle\mathcal{S}_1\rangle \subseteq \langle\mathcal{S}\rangle$. Thus, if $\langle\mathcal{S}\rangle \neq \langle\mathcal{S}_1\rangle$, then $\langle\mathcal{S}\rangle \not\subseteq \langle\mathcal{S}_1\rangle$, which means that $\mathcal{S} \not\subseteq \langle\mathcal{S}_1\rangle$. Thus, $\mathcal{S} \setminus \langle\mathcal{S}_1\rangle \neq \emptyset$. Fix some element $s_2 \in \mathcal{S} \setminus \langle\mathcal{S}_1\rangle$. Thus, $s_2 \in \mathcal{S} \setminus \mathcal{S}_1 \subseteq \mathcal{S}_2$.

Claim 1: $\mathcal{S}_1 \subseteq \langle\mathcal{S}_2\rangle$.

Proof: Let $s_1 \in \mathcal{S}_1$. Let $t := s_1 + s_2$. Then $t \in \mathcal{S}$, because $s_2 \in \mathcal{S}$ and $s_1 \in \mathcal{S}_1 \subset \mathcal{S}$ and $\mathcal{S}$ is additively closed.

Note that $t \notin \mathcal{S}_1$ (*Proof:* $s_2 = t - s_1$, and $s_1 \in \mathcal{S}_1$. So if $t \in \mathcal{S}_1$, then $s_2 \in \langle\mathcal{S}_1\rangle$. Contradiction.) Thus, $t \in \mathcal{S} \setminus \mathcal{S}_1 \subseteq \mathcal{S}_2$. But $s_1 = t - s_2$. Thus, $s_1 \in \langle\mathcal{S}_2\rangle$. This argument works for all $s_1 \in \mathcal{S}_1$. The claim follows. ◇ Claim 1

Now,

$$\langle\mathcal{S}_2\rangle \underset{(*)}{\subseteq} \langle\mathcal{S}\rangle \underset{(\dagger)}{=} \langle\mathcal{S}_1 \cup \mathcal{S}_2\rangle \underset{(\diamond)}{\subseteq} \langle\langle\mathcal{S}_2\rangle \cup \mathcal{S}_2\rangle \quad = \quad \langle\mathcal{S}_2\rangle,$$

where $(*)$ is because $\mathcal{S}_2 \subseteq \mathcal{S}$, $(\dagger)$ is because $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, and $(\diamond)$ is by Claim 1. We conclude that $\langle\mathcal{S}\rangle = \langle\mathcal{S}_2\rangle$, as desired. □

Lemma B.6 *Let $(\mathcal{A},+)$ be an abelian group. Let $\mathcal{S}_1,\ldots,\mathcal{S}_N \subseteq \mathcal{A}$ be cones, and suppose $\mathcal{S} := \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_N$ is additively closed. Then there exists $n \in [1\ldots N]$ such that $\langle \mathcal{S} \rangle = \langle \mathcal{S}_n \rangle$.*

Proof: (by induction on N) If $N = 1$, then $\mathcal{S} = \mathcal{S}_1$, and the result is trivially true.

Now suppose $N = K \geq 2$. By induction, suppose the theorem statement is true for $N = K - 1$. If $\langle \mathcal{S} \rangle = \langle \mathcal{S}_N \rangle$, then we're done. So, suppose $\langle \mathcal{S}_N \rangle \subsetneq \langle \mathcal{S} \rangle$. Then $\langle \mathcal{S}_N \rangle$ is a divisible proper subgroup of $\mathcal{A}$. Corollary A.2(b) yields a complete, homogeneous preorder $(\succeq)$ on $\mathcal{A}$ such that $\langle \mathcal{S}_N \rangle = \{a \in \mathcal{A};\ a \approx 0\}$.

Define $\mathcal{S}^{\succeq} := \{s \in \mathcal{S};\ s \succeq 0\}$ and $\mathcal{S}^{\preceq} := \{s \in \mathcal{S};\ s \preceq 0\}$. Then $\mathcal{S} = \mathcal{S}^{\succeq} \cup \mathcal{S}^{\preceq}$, and $\mathcal{S}$ is additively closed, so Lemma B.5 says that either $\langle \mathcal{S} \rangle = \left\langle \mathcal{S}^{\succeq} \right\rangle$ or $\langle \mathcal{S} \rangle = \left\langle \mathcal{S}^{\preceq} \right\rangle$. Without loss of generality, suppose $\langle \mathcal{S} \rangle = \left\langle \mathcal{S}^{\succeq} \right\rangle$.

Define $\mathcal{S}^{\succ} := \{s \in \mathcal{S};\ s \succ 0\}$ and $\mathcal{S}^0 := \{s \in \mathcal{S};\ s \approx 0\} = \mathcal{S} \cap \langle \mathcal{S}_N \rangle$. Then $\mathcal{S}^{\succeq} = \mathcal{S}^{\succ} \cup \mathcal{S}^0$. Also $\mathcal{S}^{\succeq}$ is itself additively closed (because it is an intersection of two additively closed sets) so Lemma B.5 says that either $\left\langle \mathcal{S}^{\succeq} \right\rangle = \langle \mathcal{S}^{\succ} \rangle$ or $\left\langle \mathcal{S}^{\succeq} \right\rangle = \langle \mathcal{S}^0 \rangle$. However, by hypothesis, $\left\langle \mathcal{S}^{\succeq} \right\rangle = \langle \mathcal{S} \rangle \supsetneq \langle \mathcal{S}_N \rangle = \langle \langle \mathcal{S}_N \rangle \rangle \supseteq \langle \mathcal{S} \cap \langle \mathcal{S}_N \rangle \rangle = \langle \mathcal{S}^0 \rangle$. Thus, $\left\langle \mathcal{S}^{\succeq} \right\rangle \neq \langle \mathcal{S}^0 \rangle$. Thus, $\left\langle \mathcal{S}^{\succeq} \right\rangle = \langle \mathcal{S}^{\succ} \rangle$. Thus, $\langle \mathcal{S} \rangle = \langle \mathcal{S}^{\succ} \rangle$.

For all $n \in [1\ldots N]$, define $\mathcal{S}_n^{\succ} := \{s \in \mathcal{S}_n;\ s \succ 0\}$. Then $\mathcal{S}^{\succ} = \mathcal{S}_1^{\succ} \cup \cdots \cup \mathcal{S}_{N-1}^{\succ} \cup \mathcal{S}_N^{\succ}$, (because $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{N-1} \cup \mathcal{S}_N$). But $\mathcal{S}_N^{\succ} = \emptyset$ (because $\mathcal{S}_N \subseteq \mathcal{S}^0$). Thus, $\mathcal{S}^{\succ} = \mathcal{S}_1^{\succ} \cup \cdots \cup \mathcal{S}_{N-1}^{\succ}$. Furthermore, $\mathcal{S}^{\succ}$ is additively closed and $\mathcal{S}_1^{\succ},\ldots,\mathcal{S}_{N-1}^{\succ}$ are all cones (because $\mathcal{S}$ is additively closed and $\mathcal{S}_1,\ldots,\mathcal{S}_{N-1}$ are cones, and the set $\{a \in \mathcal{A};\ a \succ 0\}$ is itself a cone). Thus, the induction hypothesis yields some $n \in [1\ldots\ N-1]$ such that $\langle \mathcal{S}^{\succ} \rangle = \langle \mathcal{S}_n^{\succ} \rangle$. But then we have:

$$\langle \mathcal{S} \rangle \quad = \quad \langle \mathcal{S}^{\succ} \rangle \quad = \quad \langle \mathcal{S}_n^{\succ} \rangle \quad \subseteq \quad \langle \mathcal{S}_n \rangle \quad \subseteq \quad \langle \mathcal{S} \rangle.$$

We conclude that $\langle \mathcal{S}_n \rangle = \langle \mathcal{S} \rangle$, as desired. □

Theorem 2.3 follows from Theorem 2.2 and the next result:

Proposition B.7 *Let $\mathcal{R}$ be a linearly ordered abelian group. Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be any set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a cone, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a balance rule determined by a perfect $\mathcal{R}$-valued balance system. Then F is neutral if and only if F is a scoring rule with an $\mathcal{R}$-valued neutral score system.*

Proof: "$\Longleftarrow$" If the score function of F is neutral, then F itself is neutral.

"$\Longrightarrow$" Suppose F is a perfect $\mathcal{R}$-valued balance system. If F is $\Pi_{\mathcal{X}}$-neutral, then Proposition B.3 says that $F = F_{\mathsf{B}}$, where B is a neutral, $\mathcal{R}$-valued balance system.

First suppose $|\mathcal{X}| = 2$. If $\mathcal{X} = \{x, y\}$, then $\mathsf{B} = \{\mathbf{b}^{x,y}, \mathbf{b}^{y,x}\}$, where $\mathbf{b}^{y,x} = -\mathbf{b}^{x,y}$. Thus, if we define $\mathbf{s}^x := \mathbf{b}^{x,y}$ and $\mathbf{s}^y := \mathbf{0}$, then $F = F_{\mathsf{S}}$.

Now suppose $|\mathcal{X}| \geq 3$. There are three cases: either $|\mathcal{X}| \equiv 0$ or $|\mathcal{X}| \equiv 1$ or $|\mathcal{X}| \equiv 2$, mod 3. Refer to these as Cases 0, 1, and 2. Partition $\mathcal{X}$ as $\mathcal{X} = \mathcal{X}_0 \sqcup \mathcal{X}_1 \sqcup \mathcal{X}_2 \sqcup \cdots \sqcup \mathcal{X}_N$, (for some $N \geq 0$), where $|\mathcal{X}_n| = 3$ for all $n \in [1\ldots N]$, and where $\mathcal{X}_0 = \emptyset$ in Case 0, $|\mathcal{X}_0| = 4$ in Case 1, and $|\mathcal{X}_0| = 5$ in Case 2.

Let $\widetilde{\Pi}_{\mathcal{X}} := \{\widetilde{\pi};\ \pi \in \Pi_{\mathcal{X}}\} \subseteq \Pi_{\mathcal{V}}$. Then $\mathcal{D}$ is a $\widetilde{\Pi}_{\mathcal{X}}$-invariant subset of $\mathbb{N}^{\langle\mathcal{V}\rangle}$, because F is neutral. Let $\Phi \subset \Pi_{\mathcal{X}}$ be the set of all permutations whose orbits are $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_N$. All elements of Φ have the same order m.[17] Let $M := m - 1$. For any $\phi \in \Phi$ and $\mathbf{d} \in \mathcal{D}$, define $\mathbf{d}^\phi := \mathbf{d} + \widetilde{\phi}(\mathbf{d}) + \widetilde{\phi}^2(\mathbf{d}) + \cdots + \widetilde{\phi}^M(\mathbf{d})$; then $\mathbf{d}^\phi \in \mathcal{D}$ also, because $\mathcal{D}$ is $\widetilde{\phi}$-invariant and additively closed. Thus, $F_{\mathsf{B}}(\mathbf{d}^\phi) \neq \emptyset$. For all $n \in [0 \ldots N]$, let $\mathcal{D}_n^\phi := \{\mathbf{d} \in \mathcal{D};\ \mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{d}^\phi)\}$.

Claim 1: *For any $\phi \in \Phi$ and any $n \in [0 \ldots N]$, the set $\mathcal{D}_n^\phi$ is a cone.*

Proof: (*Additively closed*) Let $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}_n^\phi$. We must show that $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}_n^\phi$. By hypothesis, $\mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{d}_1^\phi)$ and $\mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{d}_1^\phi)$. Thus, reinforcement implies that $\mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{d}_1^\phi + \mathbf{d}_2^\phi)$. But $\mathbf{d}_1^\phi + \mathbf{d}_2^\phi = (\mathbf{d}_1 + \mathbf{d}_2)^\phi$; thus $\mathcal{X}_n \subseteq F_{\mathsf{B}}((\mathbf{d}_1 + \mathbf{d}_2)^\phi)$, which means $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}_n^\phi$.

(*Divisible*) Let $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$ and $m \in \mathbb{N}$, and suppose $m\,\mathbf{z} \in \mathcal{D}_n^\phi$. We must show that $\mathbf{z} \in \mathcal{D}_n^\phi$. By hypothesis, $\mathcal{X}_n \subseteq F_{\mathsf{B}}((m\,\mathbf{z})^\phi)$. But $(m\,\mathbf{z})^\phi = m\,\mathbf{z}^\phi$. Thus, $\mathbf{z}^\phi \in \mathcal{D}$ (because $\mathcal{D}$ is a cone), and $F_{\mathsf{B}}(m\,\mathbf{z}^\phi) = F_{\mathsf{B}}(\mathbf{z}^\phi)$, by reinforcement. Thus, $\mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{z}^\phi)$. Thus, $\mathbf{z} \in \mathcal{D}_n^\phi$. ◇ `Claim 1`

Claim 2: *For any $\phi \in \Phi$, we have $\mathcal{D} = \mathcal{D}_0^\phi \cup \mathcal{D}_1^\phi \cup \cdots \cup \mathcal{D}_N^\phi$.*

Proof: For any $\mathbf{d} \in \mathcal{D}$, we have $\widetilde{\phi}(\mathbf{d}^\phi) = \mathbf{d}^\phi$. Thus neutrality implies that $\phi[F_{\mathsf{B}}(\mathbf{d}^\phi)] = F_{\mathsf{B}}(\mathbf{d}^\phi)$. Thus, $F_{\mathsf{B}}(\mathbf{d}^\phi)$ must be a union of ϕ-orbits. Thus, $F_{\mathsf{B}}(\mathbf{d}^\phi)$ contains $\mathcal{X}_n$ for some $n \in [0 \ldots N]$. Thus, $\mathbf{d} \in \mathcal{D}_n^\phi$ for some $n \in [0 \ldots N]$. ◇ `Claim 2`

Claim 3: *Suppose there exists some $\phi \in \Phi$ and some $n \in [1 \ldots N]$ such that $\langle \mathcal{D}_n^\phi \rangle = \langle \mathcal{D} \rangle$. For simplicity, suppose $\mathcal{X}_n = \{1, 2, 3\}$, where $\phi(1) = 3$, $\phi(2) = 1$, and $\phi(3) = 2$. Then $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) = \mathbf{b}^{1,3}(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$.*

Proof: For any $\mathbf{d} \in \mathcal{D}_n^\phi$, we have $\mathcal{X}_n \subseteq F_{\mathsf{B}}(\mathbf{d}^\phi)$. Then we must have

$$\begin{aligned} 0 \;=\; \mathbf{b}^{1,2}(\mathbf{d}^\phi) \;&=\; \mathbf{b}^{1,2}\left(\mathbf{d} + \widetilde{\phi}(\mathbf{d}) + \widetilde{\phi}^2(\mathbf{d}) + \cdots + \widetilde{\phi}^M(\mathbf{d})\right) \\ &\underset{(\diamond)}{=} \left(\mathbf{b}^{1,2} + \mathbf{b}^{1,2}\widetilde{\phi} + \mathbf{b}^{1,2}\widetilde{\phi}^2 + \cdots + \mathbf{b}^{1,2}\widetilde{\phi}^M\right)(\mathbf{d}) \\ &\underset{(*)}{=} \left(\mathbf{b}^{1,2} + \mathbf{b}^{\phi^{-1}(1),\phi^{-1}(2)} + \cdots + \mathbf{b}^{\phi^{-M}(1),\phi^{-M}(2)}\right)(\mathbf{d}) \\ &= \frac{M+1}{3}\left(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,1}\right)(\mathbf{d}), \end{aligned}$$

where $(\diamond)$ is by Lemma B.2(a), and $(*)$ is because B is neutral. But $\langle \mathcal{D}_n^\phi \rangle = \langle \mathcal{D} \rangle$, so Lemma B.4 then implies that $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) + \mathbf{b}^{3,1}(\mathbf{d}) = 0$ for all $\mathbf{d} \in \mathcal{D}$. In other words: $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) = -\mathbf{b}^{3,1}(\mathbf{d}) = \mathbf{b}^{1,3}(\mathbf{d})$, as desired. ◇ `Claim 3`

Claim 4: *Suppose there exists no $\phi \in \Phi$ and $n \in [1 \ldots N]$ such that $\langle \mathcal{D}_n^\phi \rangle = \langle \mathcal{D} \rangle$. Suppose that either $\mathcal{X}_0 = \{1, 2, 3, 4\}$* (in Case 1), *or $\mathcal{X}_0 = \{1, 2, 3, 4, 5\}$* (in Case 2). *Then $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) = \mathbf{b}^{1,3}(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$.*

[17] $m = 3$ in Case 0. If $N = 0$, then $m = 4$ in Case 1 and $m = 5$ in Case 2. If $N \geq 1$, then $m = 12$ in Case 1 and $m = 15$ in Case 2.

Proof: For all $\phi \in \Phi$, if there is no $n \in [1 \ldots N]$ such that $\left\langle \mathcal{D}_n^\phi \right\rangle = \langle \mathcal{D} \rangle$, then Claims 1 and 2 and Lemma B.6 imply that $\left\langle \mathcal{D}_0^\phi \right\rangle = \langle \mathcal{D} \rangle$.

(*Case 1*) Let $\phi \in \Phi$, and suppose $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 3$, and $\phi(1) = 4$. Then by an argument similar to Claim 3, we conclude that

$$(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,4} + \mathbf{b}^{4,1})(\mathbf{d}) \quad = \quad 0, \tag{B7}$$

for all $\mathbf{d} \in \mathcal{D}$. By choosing other permutations in Φ, we can similarly derive the following five equations. For all $\mathbf{d} \in \mathcal{D}$,

$$\begin{aligned}
(\mathbf{b}^{1,2} + \mathbf{b}^{2,4} + \mathbf{b}^{4,3} + \mathbf{b}^{3,1})(\mathbf{d}) &= 0, && \text{(B8)}\\
(\mathbf{b}^{2,3} + \mathbf{b}^{3,1} + \mathbf{b}^{1,4} + \mathbf{b}^{4,2})(\mathbf{d}) &= 0, && \text{(B9)}\\
(\mathbf{b}^{2,3} + \mathbf{b}^{3,4} + \mathbf{b}^{4,1} + \mathbf{b}^{1,2})(\mathbf{d}) &= 0, && \text{(B10)}\\
(\mathbf{b}^{3,1} + \mathbf{b}^{1,2} + \mathbf{b}^{2,4} + \mathbf{b}^{4,3})(\mathbf{d}) &= 0, && \text{(B11)}\\
\text{and} \qquad (\mathbf{b}^{3,1} + \mathbf{b}^{1,4} + \mathbf{b}^{4,2} + \mathbf{b}^{2,3})(\mathbf{d}) &= 0. && \text{(B12)}
\end{aligned}$$

Recall that $\mathbf{b}^{2,4} = -\mathbf{b}^{4,2}$, and $\mathbf{b}^{3,4} = -\mathbf{b}^{4,3}$, etc. Thus, by adding equations (B7)-(B12) together and cancelling, we obtain $4(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,1})(\mathbf{d}) = 0$, for all $\mathbf{d} \in \mathcal{D}$. Thus, $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) = -\mathbf{b}^{3,1}(\mathbf{d}) = \mathbf{b}^{1,3}(\mathbf{d})$, as desired.

(*Case 2*) Let $\phi \in \Phi$, and suppose $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 3$, $\phi(5) = 4$, and $\phi(1) = 5$. Then by an argument similar to Claim 3, we conclude that

$$(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,4} + \mathbf{b}^{4,5} + \mathbf{b}^{5,1})(\mathbf{d}) \quad = \quad 0, \tag{B13}$$

for all $\mathbf{d} \in \mathcal{D}$. By choosing other permutations in Φ, we can similarly derive the following five equations. For all $\mathbf{d} \in \mathcal{D}$,

$$\begin{aligned}
(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,5} + \mathbf{b}^{5,4} + \mathbf{b}^{4,1})(\mathbf{d}) &= 0, && \text{(B14)}\\
(\mathbf{b}^{2,3} + \mathbf{b}^{3,1} + \mathbf{b}^{1,4} + \mathbf{b}^{4,5} + \mathbf{b}^{5,2})(\mathbf{d}) &= 0, && \text{(B15)}\\
(\mathbf{b}^{2,3} + \mathbf{b}^{3,1} + \mathbf{b}^{1,5} + \mathbf{b}^{5,4} + \mathbf{b}^{4,2})(\mathbf{d}) &= 0, && \text{(B16)}\\
(\mathbf{b}^{3,1} + \mathbf{b}^{1,2} + \mathbf{b}^{2,4} + \mathbf{b}^{4,5} + \mathbf{b}^{5,3})(\mathbf{d}) &= 0, && \text{(B17)}\\
\text{and} \qquad (\mathbf{b}^{3,1} + \mathbf{b}^{1,2} + \mathbf{b}^{2,5} + \mathbf{b}^{5,4} + \mathbf{b}^{4,3})(\mathbf{d}) &= 0, && \text{(B18)}
\end{aligned}$$

By adding equations (B13)-(B18) together and cancelling, we obtain $4(\mathbf{b}^{1,2} + \mathbf{b}^{2,3} + \mathbf{b}^{3,1})(\mathbf{d}) = 0$, for all $\mathbf{d} \in \mathcal{D}$, Thus, $\mathbf{b}^{1,2}(\mathbf{d}) + \mathbf{b}^{2,3}(\mathbf{d}) = \mathbf{b}^{1,3}(\mathbf{d})$, as desired. ◇ `Claim 4`

In Case 0, we have $\mathcal{D}_0 = \emptyset$, so $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_N$. Thus, Claims 1 and 2 and Lemma B.6 yields some $n \in [1 \ldots N]$ such that $\langle \mathcal{D}_n \rangle = \langle \mathcal{D} \rangle$. Thus, Claim 3 applies. In Cases 1 or 2, either the hypothesis of Claim 3 holds or the hypothesis of Claim 4 holds. In any of these cases, we obtain some distinct elements $x, y, z \in \mathcal{X}$ such that

$$\mathbf{b}^{x,y}(\mathbf{d}) + \mathbf{b}^{y,z}(\mathbf{d}) \quad = \quad \mathbf{b}^{x,z}(\mathbf{d}), \quad \text{for all } \mathbf{d} \in \mathcal{D}. \tag{B19}$$

Claim 5: *For any $x', y', z' \in \mathcal{X}$, and all $\mathbf{d} \in \mathcal{D}$, we have $\mathbf{b}^{x',y'}(\mathbf{d}) + \mathbf{b}^{y',z'}(\mathbf{d}) = \mathbf{b}^{x',z'}(\mathbf{d})$.*

Proof: There exists some $\alpha \in \Pi_{\mathcal{X}}$ such that $\alpha(x') = x$, $\alpha(y') = y$, and $\alpha(z') = z$. Thus,

$$\begin{aligned}\mathbf{b}^{x',y'}(\mathbf{d}) + \mathbf{b}^{y',z'}(\mathbf{d}) \underset{(\diamond)}{=} (\mathbf{b}^{x,y}\widetilde{\alpha})(\mathbf{d}) + (\mathbf{b}^{y,z}\widetilde{\alpha})(\mathbf{d}) \underset{(\dagger)}{=} (\mathbf{b}^{x,y} + \mathbf{b}^{y,z})\,(\widetilde{\alpha}(\mathbf{d})) \\ \underset{(*)}{=} \mathbf{b}^{x,z}\,(\widetilde{\alpha}(\mathbf{d})) \underset{(\diamond)}{=} \mathbf{b}^{x',z'}(\mathbf{d}),\end{aligned}$$

as desired. Here, $(*)$ is by eqn.(B19) (because $\widetilde{\alpha}(\mathbf{d}) \in \mathcal{D}$ because $\mathcal{D}$ is $\widetilde{\Pi}_{\mathcal{X}}$-invariant), and $(\dagger)$ is by Lemma B.2(a), while both $(\diamond)$ are because B is neutral. $\diamond$ `Claim 5`

Now, Claim 5 and Lemma 5.1 imply that F_{B} is a scoring rule. Finally, if F_{B} is neutral, then Proposition 3.1 says that F_{B} has a neutral score function. $\square$

Proof of Theorem 2.3. "$\Longleftarrow$" It is easy to check that any scoring rule satisfies reinforcement. If the score system of F is neutral, then F itself is neutral.

"$\Longrightarrow$" If F satisfies reinforcement, then Theorem 2.2 says F is a balance rule. If F is neutral, then Proposition B.7 says F is scoring rule with a neutral score system. $\square$

Proof of Proposition 2.4. "$\Longleftarrow$" is straightforward.

"$\Longrightarrow$" By hypothesis, $F = F_{\widetilde{\mathsf{B}}}$ for some balance system $\widetilde{\mathsf{B}} := \{\widetilde{\mathbf{b}}^{x,y}\}_{x,y\in\mathcal{X}}$ taking values in some linearly ordered abelian group $\widetilde{\mathcal{R}}$. Fix $x, y \in \mathcal{X}$. Let $\mathcal{R}'_{x,y} := \widetilde{\mathbf{b}}^{x,y}(\mathbb{Z}^{\langle\mathcal{V}\rangle}) \subseteq \widetilde{\mathcal{R}}$. Then $\mathcal{R}'_{x,y}$ is also a linearly ordered abelian group, and we can treat $\widetilde{\mathbf{b}}^{x,y}$ as a homomorphism from $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ into $\mathcal{R}'_{x,y}$.

Claim 1: $\mathcal{R}'_{x,y} := \widetilde{\mathbf{b}}^{x,y}(\mathbb{N}^{\langle\mathcal{V}\rangle})$.

Proof: Let $r \in \mathcal{R}'_{x,y}$; then $r = \widetilde{\mathbf{b}}^{x,y}(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$. Let $\mathcal{W} := \{v \in \mathcal{V};\ z_v \neq 0\}$ (a finite set). Condition (TC2) yields some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $n_w > 0$ for all $w \in \mathcal{W}$, such that $F(\mathbf{n}) \supseteq \{x, y\}$. Thus, $\mathbf{b}^{x,y}(\mathbf{n}) = 0$, so $\mathbf{n} \in \ker(\widetilde{\mathbf{b}}^{x,y})$. Thus, $M\mathbf{n} \in \ker(\widetilde{\mathbf{b}}^{x,y})$ for all $M \in \mathbb{N}$.

Let $M = 1 + \max\{|z_w|/n_w;\ \ w \in \mathcal{W}\}$ (so M is finite, because $|\mathcal{W}| < \infty$). Thus, $Mn_w + z_w > 0$ for all $w \in \mathcal{W}$. Thus, $M\mathbf{n} + \mathbf{z} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, and clearly, $\widetilde{\mathbf{b}}^{x,y}(M\mathbf{n} + \mathbf{z}) = M \cdot \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}) + \widetilde{\mathbf{b}}^{x,y}(\mathbf{z}) = M \cdot 0 + r = r$, as desired. $\diamond$ `Claim 1`

Claim 2: *$\mathcal{R}'_{x,y}$ is Archimedean for all $x, y \in \mathcal{X}$.*

Proof: Let $r_1, r_2 \in \mathcal{R}_{x,y}$, with $r_1 > 0$. We must find some $N \in \mathbb{N}$ such that $N \cdot r_1 > -r_2$. By Claim 1, there exist $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^{\mathcal{V}}$ such that $r_1 = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_1)$ and $r_2 = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_2)$. Condition (TC1) yields some $\mathbf{n}_0 \in \mathbb{N}^{\mathcal{V}}$ such that $F(\mathbf{n}_0) = \{x, y\}$. By overwhelming majority, there exists some $M_1, M_2 \in \mathbb{N}$ such that $F(\mathbf{n}_1 + M_1\mathbf{n}_0) \subseteq \{x, y\}$ and $F(\mathbf{n}_2 + M_2\mathbf{n}_0) \subseteq \{x, y\}$. For both $j \in \{1, 2\}$, define $\mathbf{n}'_j := \mathbf{n}_j + M_j\mathbf{n}_0$. Then $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_j) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_j) + M_j \cdot \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_0) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_j) = r_j$, because $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}_0) = 0$ because $F(\mathbf{n}_0) = \{x, y\}$. Thus, $F(\mathbf{n}'_1) = \{x\}$, because $\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_1) = r_1 > 0$. By overwhelming majority, there exists some $N \in \mathbb{N}$ such that $F(N\,\mathbf{n}'_1 + \mathbf{n}_2) = \{x\}$. But this means that $0 < \widetilde{\mathbf{b}}^{x,y}(N\,\mathbf{n}'_1 + \mathbf{n}_2) = N\,\widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_1) + \widetilde{\mathbf{b}}^{x,y}(\mathbf{n}'_2) = N'r_1 + r_2$. Thus, $N \cdot r_1 > -r_2$, as desired. $\diamond$ `Claim 2`

For all $x, y \in \mathcal{X}$, Hölder's theorem and Claim 2 imply that $\mathcal{R}'_{x,y}$ is isomorphic to some ordered subgroup of $\mathbb{R}$; thus, we can regard $\widetilde{\mathbf{b}}^{x,y}$ as a real-valued function, so that $\widetilde{\mathsf{B}}$ is a real-valued balance system. $\Box$

The proofs of Propositions 2.5 and 2.6 require three lemmas. A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is *trivial* if $F(\mathbf{d}) = \mathcal{X}$ for all $\mathbf{d} \in \mathcal{D}$; otherwise F is *nontrivial*. Recall that F is *nondegenerate* if, for all $x \in \mathcal{X}$, there is some $\mathbf{d} \in \mathcal{D}$ with $F(\mathbf{d}) = \{x\}$.

Lemma B.8 *Suppose $\mathcal{X}$ is finite, and let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain. If a voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is neutral, nontrivial, and satisfies reinforcement, then F is nondegenerate.*

Proof: Let $N := |\mathcal{X}|$ (finite).

Claim 1: *Let $\mathbf{d} \in \mathcal{D}$, and let $M := |\mathcal{X} \setminus F(\mathbf{d})|$* (so $|F(\mathbf{d})| = N - M$).

(a) *If $|F(\mathbf{d})| \leq N/2$, then there exists $\mathbf{d}' \in \mathcal{D}$ with $|F(\mathbf{d})| = 1$.*

(b) *If $|F(\mathbf{d})| > N/2$, then there exists $\mathbf{d}' \in \mathcal{D}$ with $|F(\mathbf{d}')| = N - 2M$.*

Proof: Let $\mathcal{Y} := F(\mathbf{d}) \subseteq \mathcal{X}$. If $|\mathcal{Y}| \leq |\mathcal{X}|/2$, then there exists some $\mathcal{Y}' \in \mathcal{X}$ such that $|\mathcal{Y}'| = |\mathcal{Y}|$ and $|\mathcal{Y} \cap \mathcal{Y}'| = 1$.

If $|\mathcal{Y}| > |\mathcal{X}|/2$, then there exists some $\mathcal{Y}' \in \mathcal{X}$ such that $|\mathcal{Y}'| = |\mathcal{Y}|$ and $\mathcal{X} \setminus \mathcal{Y}'$ is disjoint from $\mathcal{X} \setminus \mathcal{Y}$, and thus,

$$|\mathcal{X} \setminus (\mathcal{Y}' \cap \mathcal{Y})| \quad = \quad |(\mathcal{X} \setminus \mathcal{Y}') \sqcup (\mathcal{X} \setminus \mathcal{Y})| \quad = \quad |\mathcal{X} \setminus \mathcal{Y}'| + |\mathcal{X} \setminus \mathcal{Y}| \quad = \quad 2|\mathcal{X} \setminus \mathcal{Y}| \quad = \quad 2M.$$

Thus, $|\mathcal{Y}' \cap \mathcal{Y}| = |\mathcal{X}| - 2M = N - 2M$.

In either case, $|\mathcal{Y}'| = |\mathcal{Y}|$, so there exists $\pi \in \Pi_{\mathcal{X}}$ with $\pi(\mathcal{Y}) = \mathcal{Y}'$. Thus, $F(\widetilde{\pi}(\mathbf{d})) = \mathcal{Y}'$, by neutrality. Let $\mathbf{d}' := \mathbf{d} + \pi(\mathbf{d})$. By construction $\mathcal{Y}' \cap \mathcal{Y} \neq \emptyset$; thus, $\mathbf{d}' \in \mathcal{D}$ and $F(\mathbf{d}') = \mathcal{Y}' \cap \mathcal{Y}$, by reinforcement. $\diamond$ `Claim 1`

Claim 2: *There exists $\mathbf{d}' \in \mathcal{D}$ with $|F(\mathbf{d}')| = 1$.*

Proof: Since F is nontrivial, there exists some $\mathbf{d} \in \mathcal{D}$ with $|F(\mathbf{d})| < N$. Now applying Claim 1(b) repeatedly, we can obtain some $\mathbf{d}'' \in \mathcal{D}$ with $|F(\mathbf{d}'')| \leq N/2$. Then apply Claim 1(a) to obtain some $\mathbf{d}' \in \mathcal{D}$ with $|F(\mathbf{d}')| = 1$. $\diamond$ `Claim 2`

Now, let $\mathbf{d}' \in \mathcal{D}$ be from Claim 2. Thus, $F(\mathbf{d}') = \{x\}$ for some $x \in \mathcal{X}$. Let $y \in \mathcal{X}$. Find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = y$. Then $F(\widetilde{\pi}(\mathbf{d}')) = \pi(F(\mathbf{d})) = \{y\}$, by neutrality. This works for any $y \in \mathcal{X}$; thus, F is nondegenerate. $\Box$

Lemma B.9 *Suppose $\mathcal{X}$ is finite, and let $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ be a voting rule.*

(a) *If F is neutral, then F satisfies* (TC2) *of the tie condition.*

(b) *If F is neutral, nontrivial, and satisfies reinforcement, then F satisfies* (TC1).

Proof: (a) Fix $x, y \in \mathcal{X}$. Let $\mathcal{W} \subseteq \mathcal{V}$ be finite, and define $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ by $n_w := 1$ for all $w \in \mathcal{W}$, while $n_v := 0$ for all $v \in \mathcal{V} \setminus \mathcal{W}$. Define $\overline{\mathbf{n}} := \sum_{\pi \in \Pi_{\mathcal{X}}} \widetilde{\pi}(\mathbf{n})$. Then $\overline{\mathbf{n}}$ is $\widetilde{\Pi}_{\mathcal{X}}$-fixed, so neutrality implies that $F(\overline{\mathbf{n}})$ is a $\Pi_{\mathcal{X}}$-invariant subset of $\mathcal{X}$. Since $F(\overline{\mathbf{n}}) \neq \emptyset$, this means that $F(\overline{\mathbf{n}}) = \mathcal{X}$. In particular, $\{x, y\} \subseteq F(\overline{\mathbf{n}})$. Finally, for all $w \in \mathcal{W}$, we have $\overline{n}_w \geq n_w = 1 \geq 0$, as required by (TC2).

(b) Theorem 2.3 says that F is a scoring rule with a neutral scoring system S. Lemma B.8 yields some $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ with $|F(\mathbf{n})| = 1$. Let $F(\mathbf{n}) = \{x\}$. Since $\mathcal{X}$ is finite, there exists $y \in \mathcal{X}$ such that

$$\mathbf{s}^x(\mathbf{n}) \quad > \quad \mathbf{s}^y(\mathbf{n}) \quad \geq \quad \mathbf{s}^z(\mathbf{n}), \quad \text{for all } z \in \mathcal{X} \setminus \{x, y\}. \tag{B20}$$

Let $\pi \in \Pi_{\mathcal{X}}$ be a permutation such that $\pi(x) = y$ and $\pi(y) = x$. Since $\mathcal{X}$ is finite, there is some $M \in \mathbb{N}$ such that $\pi^M = \mathrm{Id}$. Define $\overline{\mathbf{n}} := \sum_{m=0}^{M-1} \widetilde{\pi}^m(\mathbf{n})$.

Claim 1: $F(\overline{\mathbf{n}}) = \{x, y\}$.

Proof: Let $z \in \mathcal{X} \setminus \{x, y\}$. If m is even, then $\pi^m(x) = x$ and $\pi^m(y) = y$. Thus,

$$\left.\begin{array}{rrcccccc} & \mathbf{s}^x(\widetilde{\pi}^m(\mathbf{n})) & \underset{(*)}{=} & (\mathbf{s}^x\widetilde{\pi}^m)(\mathbf{n}) & \underset{(\dagger)}{=} & \mathbf{s}^{\pi^{-m}(x)}(\mathbf{n}) & = & \mathbf{s}^x(\mathbf{n}) \\ & & \underset{(\diamond)}{>} & \mathbf{s}^{\pi^{-m}(z)}(\mathbf{n}) & \underset{(\dagger)}{=} & (\mathbf{s}^z\widetilde{\pi}^m)(\mathbf{n}) & \underset{(*)}{=} & \mathbf{s}^z(\widetilde{\pi}^m(\mathbf{n})), \\ \text{and} & \mathbf{s}^y(\widetilde{\pi}^m(\mathbf{n})) & \underset{(*)}{=} & (\mathbf{s}^y\widetilde{\pi}^m)(\mathbf{n}) & \underset{(\dagger)}{=} & \mathbf{s}^{\pi^{-m}(y)}(\mathbf{n}) & = & \mathbf{s}^y(\mathbf{n}) \\ & & \underset{(\diamond)}{\geq} & \mathbf{s}^{\pi^{-m}(z)}(\mathbf{n}) & \underset{(\dagger)}{=} & (\mathbf{s}^z\widetilde{\pi}^m)(\mathbf{n}) & \underset{(*)}{=} & \mathbf{s}^z(\widetilde{\pi}^m(\mathbf{n})). \end{array}\right\} \tag{B21}$$

Here, all four $(*)$ are by Lemma B.2(a), all four $(\dagger)$ are by neutrality, and both $(\diamond)$ are by the inequalities (B20).

If m is odd, then $\pi^m(x) = y$ and $\pi^m(y) = x$. Then by an argument very similar to (B21), we have

$$\left.\begin{array}{rrcccc} & \mathbf{s}^x(\widetilde{\pi}^m(\mathbf{n})) & = & \mathbf{s}^y(\mathbf{n}) & \geq & \mathbf{s}^z(\widetilde{\pi}^m(\mathbf{n})), \\ \text{and} & \mathbf{s}^y(\widetilde{\pi}^m(\mathbf{n})) & = & \mathbf{s}^x(\mathbf{n}) & > & \mathbf{s}^z(\widetilde{\pi}^m(\mathbf{n})). \end{array}\right\} \tag{B22}$$

Note that M must be even. Observe that

$$\left.\begin{array}{rrcccc} & \mathbf{s}^x(\overline{\mathbf{n}}) & = & \mathbf{s}^x\left(\displaystyle\sum_{m=0}^{M-1} \widetilde{\pi}^m(\mathbf{n})\right) & = & \displaystyle\sum_{m=0}^{M-1} \mathbf{s}^x(\widetilde{\pi}^m(\mathbf{n})), \\ & \mathbf{s}^y(\overline{\mathbf{n}}) & = & \mathbf{s}^y\left(\displaystyle\sum_{m=0}^{M-1} \widetilde{\pi}^m(\mathbf{n})\right) & = & \displaystyle\sum_{m=0}^{M-1} \mathbf{s}^y(\widetilde{\pi}^m(\mathbf{n})), \\ \text{and} & \mathbf{s}^z(\overline{\mathbf{n}}) & = & \mathbf{s}^z\left(\displaystyle\sum_{m=0}^{M-1} \widetilde{\pi}^m(\mathbf{n})\right) & = & \displaystyle\sum_{m=0}^{M-1} \mathbf{s}^z(\widetilde{\pi}^m(\mathbf{n})). \end{array}\right\} \tag{B23}$$

Applying the inequalities in (B21) and (B22) to the right hand side of the three equations in (B23), we conclude that $\mathbf{s}^x(\overline{\mathbf{n}}) = \mathbf{s}^y(\overline{\mathbf{n}}) > \mathbf{s}^z(\overline{\mathbf{n}})$, for all $z \in \mathcal{X} \setminus \{x, y\}$. Thus, $F(\overline{\mathbf{n}}) = \{x, y\}$, as desired. ◇ `Claim 1`

Now, let $x', y' \in \mathcal{X}$. Find $\pi \in \Pi_{\mathcal{X}}$ such that $\pi(x) = x'$ and $\pi(y) = y'$. Then

$$F(\pi(\overline{\mathbf{n}})) \underset{(N)}{=} \pi[F(\overline{\mathbf{n}})] \underset{(*)}{=} \pi\{x, y\} \quad = \quad \{x', y'\},$$

as desired. Here, (N) is by neutrality, and $(*)$ is by Claim 1. □

The next lemma just extends Myerson's (1995) result to the case when $\mathcal{V}$ is infinite.

Lemma B.10 *Let $\mathcal{X}$ be a finite set, and let $\mathcal{V}$ be an arbitrary set. If a voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies reinforcement and overwhelming majority, and is neutral and nontrivial, then $F = F_{\mathsf{S}}$, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$.*

Proof: If $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies reinforcement and is neutral and nontrivial, then Lemma B.9 says that F satisfies the tie condition. If F also satisfies overwhelming majority, then Proposition 2.4 says F is a real-valued balance rule. Since F is neutral, Proposition B.7 says that $F = F_{\mathsf{S}}$, where S is a neutral, real-valued score system. □

Proof of Proposition 2.5. Lemma B.10 says that any neutral, nontrivial, $\mathcal{X}$-valued voting rule which satisfies reinforcement and overwhelming majority must be a scoring rule F_{S}, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$ (for some signal set $\mathcal{V}$). For any $v \in \mathcal{V}$, let $\mathbf{s}_v := (s_v^x)$, a vector in $\mathbb{R}^{\mathcal{X}}$. Let $\mathsf{S}^\dagger := \{\mathbf{s}_v\}_{v \in \mathcal{V}}$; then $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$.

Claim 1: *If F_{S} is the most expressive scoring rule with $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$, then $\mathsf{S}^\dagger = \mathbb{R}^{\mathcal{X}}$.*

Proof: (by contradiction) Suppose that $\mathsf{S}^\dagger \subsetneq \mathbb{R}^{\mathcal{X}}$. Let $\mathbf{r} \in \mathbb{R}^{\mathcal{X}} \setminus \mathsf{S}^\dagger$. Let w be some new signal not in $\mathcal{V}$. Define $\mathcal{W} := \mathcal{V} \cup \{w\}$. Define the score system $\widetilde{\mathsf{S}} \subset \mathbb{R}^{\mathcal{W}}$ by setting $\widetilde{s}_v^x := s_v^x$ for all $v \in \mathcal{V}$ and $x \in \mathcal{X}$, whereas $\widetilde{s}_w^x := r_w^x$ for all $x \in \mathcal{X}$. Let $\alpha : \mathcal{V} \longrightarrow \mathcal{W}$ be the inclusion map; then for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, and any $x \in \mathcal{X}$ it is clear that $\widetilde{\mathbf{s}}^x[\alpha_*(\mathbf{n})] = \mathbf{s}^x(\mathbf{n})$. Thus, $F_{\widetilde{\mathsf{S}}}[\alpha_*(\mathbf{n})] = F_{\mathsf{S}}(\mathbf{n})$. Thus, $F_{\widetilde{\mathsf{S}}}$ is at least as expressive as F_{S}. But this contradicts our hypothesis that F_{S} is the most expressive rule with $\mathsf{S}^\dagger \subseteq \mathbb{R}^{\mathcal{X}}$. ◇ `Claim 1`

So, suppose $\mathsf{S}^\dagger = \mathbb{R}^{\mathcal{X}}$. For any $v, w \in \mathcal{V}$, if $\mathbf{s}_v = \mathbf{s}_w$, then a vote for v has the same effect as a vote for w, when added to any profile. Thus, we can regard v and w as the same. Thus, for each $\mathbf{r} \in \mathbb{R}^{\mathcal{X}}$, there exists a unique $v \in \mathcal{V}$ with $\mathbf{s}_v = \mathbf{r}$. At this point it is clear that F_{S} is the formally utilitarian voting rule. □

The proof of Proposition 2.6 also requires the next lemma.

Lemma B.11 *Let $\mathcal{X}$ be a finite set, and let S be a real-valued scoring system on $\mathcal{X}$.*

(a) *Suppose there exists $R \in \mathbb{R}_+$ such that $|s_v^x - s_v^y| \leq R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$. Then F_{S} does not admit minority overrides.*

(b) *Let $\Pi'_{\mathcal{X}}$ be a transitive group of permutations on $\mathcal{X}$, and suppose S is $\Pi'_{\mathcal{X}}$-neutral and nontrivial. If S does not admit minority overrides, then there is some $R \in \mathcal{R}$ such that $|s_v^x - s_v^y| \leq R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$.*

Proof: (a) Let $M := \min\{|F_{\mathsf{S}}(\mathbf{n})|;\ \mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}\}$ (so $M \geq 1$). Find $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $|F_{\mathsf{S}}(\mathbf{n})| = M$. Fix $x \in F_{\mathsf{S}}(\mathbf{n})$, and let $\delta := \min\{\mathbf{s}^x(\mathbf{n}) - \mathbf{s}^y(\mathbf{n});\ y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})\}$; then $\delta > 0$ and is well-defined because $\mathcal{X}$ is finite. Let $K := \lceil R/\delta \rceil + 1$, and let $\mathbf{n}' := K\,\mathbf{n}$; then $F_{\mathsf{S}}(\mathbf{n}') = F_{\mathsf{S}}(\mathbf{n})$, by reinforcement.

Claim 1: *For any $v \in \mathcal{V}$, we have $F(\mathbf{n}' + \mathbf{1}^v) = F(\mathbf{n})$.*

Proof: For any $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})\}$, we have

$$\begin{aligned}
\mathbf{s}^y(\mathbf{n}' + \mathbf{1}^v) &= K\mathbf{s}^y(\mathbf{n}) + \mathbf{s}^y(\mathbf{1}^v) \quad = \quad K\mathbf{s}^y(\mathbf{n}) + s_v^y \\
&\underset{(\dagger)}{\leq} K\mathbf{s}^x(\mathbf{n}) - K\delta + s_v^x + R \quad \underset{(*)}{<} \quad K\mathbf{s}^x(\mathbf{n}) + s_v^x \quad = \quad K\mathbf{s}^x(\mathbf{n}) + \mathbf{s}^x(\mathbf{1}^v) \\
&= \mathbf{s}^x(\mathbf{n}' + \mathbf{1}^v).
\end{aligned}$$

(Here, $(\dagger)$ is because $\mathbf{s}^y(\mathbf{n}) \leq \mathbf{s}^x(\mathbf{n}) - \delta$ (because $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})$), while $s_v^y \leq s_v^x + R$, by definition of R. Next, $(*)$ is because $K\delta > R$, by definition of K.) Thus, $y \notin F(\mathbf{n}' + \mathbf{1}^v)$. This holds for all $y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})$, so we conclude that $F(\mathbf{n}' + \mathbf{1}^v) \subseteq F(\mathbf{n}')$. But then $F(\mathbf{n}' + \mathbf{1}^v) = F(\mathbf{n}')$, because $F(\mathbf{n}') = M$ is already of minimal size. $\diamond$ `Claim 1`

Claim 1 shows that F does not admit minority overrides.

(b) (by contrapositive) Suppose that, for all $R \in \mathcal{R}$, there exist $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$ with $|s_v^x - s_v^y| > R$. Since $\Pi'_{\mathcal{X}}$ is transitive and S is $\Pi'_{\mathcal{X}}$-neutral, this means that, for all $R \in \mathcal{R}$ and all $x \in \mathcal{X}$, there exist $v \in \mathcal{V}$ and $y \in \mathcal{X}$ with $s_v^y - s_v^x > R$. We will show that F admits minority overrides.

Let $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ be any profile with $F_{\mathsf{S}}(\mathbf{n}) \neq \mathcal{X}$. Fix $x \in F_{\mathsf{S}}(\mathbf{n})$. Let $R := \max\{\mathbf{s}^x(\mathbf{n}) - \mathbf{s}^y(\mathbf{n});\ y \in \mathcal{X} \setminus F_{\mathsf{S}}(\mathbf{n})\}$; then $R > 0$ and is well-defined because $\mathcal{X}$ is finite. Now, find some $v \in \mathcal{V}$ and $y \in \mathcal{X}$ with $s_v^y - s_v^x > R$. There are now two cases: either $y \in F_{\mathsf{S}}(\mathbf{n})$, or $y \notin F_{\mathsf{S}}(\mathbf{n})$.

Case 1. If $y \in F_{\mathsf{S}}(\mathbf{n})$, then

$$\begin{aligned}
\mathbf{s}^y(\mathbf{n} + \mathbf{1}^v) &= \mathbf{s}^y(\mathbf{n}) + \mathbf{s}^y(\mathbf{1}^v) \quad = \quad \mathbf{s}^y(\mathbf{n}) + s_v^y \quad \underset{(*)}{=} \quad \mathbf{s}^x(\mathbf{n}) + s_v^y \\
&\underset{(\dagger)}{>} \mathbf{s}^x(\mathbf{n}) + s_v^x \quad = \quad \mathbf{s}^x(\mathbf{n} + \mathbf{1}^v).
\end{aligned}$$

Thus, $x \notin F_{\mathsf{S}}(\mathbf{n}+\mathbf{1}^v)$, so $F_{\mathsf{S}}(\mathbf{n}+\mathbf{1}^v) \neq F_{\mathsf{S}}(\mathbf{n})$. (Here $(*)$ is because $\mathbf{s}^y(\mathbf{n}) = \mathbf{s}^x(\mathbf{n})$ because $\{x, y\} \in F_{\mathsf{S}}(\mathbf{n})$. Meanwhile, $(\dagger)$ is because $s_v^y > s_v^x + R > s_v^x$.)

Case 2. If $y \notin F(\mathbf{n})$, then

$$\begin{aligned}
\mathbf{s}^y(\mathbf{n} + \mathbf{1}^v) &= \mathbf{s}^y(\mathbf{n}) + s_v^y \quad \geq \quad \mathbf{s}^x(\mathbf{n}) - R + s_v^y \\
&\underset{(\dagger)}{>} \mathbf{s}^x(\mathbf{n}) - R + R + s_v^x \quad = \quad \mathbf{s}^x(\mathbf{n}) + s_v^x \quad = \quad \mathbf{s}^x(\mathbf{n} + \mathbf{1}^v).
\end{aligned}$$

Thus, again $x \notin F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v)$, so $F_{\mathsf{S}}(\mathbf{n} + \mathbf{1}^v) \neq F_{\mathsf{S}}(\mathbf{n})$. (Here, $(\dagger)$ is because $s_v^y > s_v^x + R$.)

This construction works for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$; thus, F_{S} admits minority overrides. $\square$

Proof of Proposition 2.6. Lemma B.10 says that any neutral, nontrivial, $\mathcal{X}$-valued voting rule which satisfies reinforcement and overwhelming majority must be a scoring rule F_{S}, where S is a neutral, real-valued score system on $(\mathcal{X}, \mathcal{V})$ (for some signal set $\mathcal{V}$).

Claim 1: *If F_{S} does not admit minority overrides, then there is a real-valued scoring system $\widetilde{\mathsf{S}} = \{\widetilde{\mathbf{s}}^x\}_{x\in\mathcal{X}}$ with $0 \leq \widetilde{s}_v^x \leq 1$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$, such that $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$.*

Proof: Lemma B.11 says that F does not admit minority overrides if and only if there is some $R \in \mathbb{R}_+$ such that $|s_v^x - s_v^y| \leq R$ for all $v \in \mathcal{V}$ and $x, y \in \mathcal{X}$. Let $r := 1/R$; thus, $|r\, s_v^x - r\, s_v^y| \leq 1$ for all $x, y \in \mathcal{X}$ and $v \in \mathcal{V}$. Now, for each $v \in \mathcal{V}$, let $t_v := \min\{r\, s_v^x;\ x \in \mathcal{X}\}$, to obtain a vector $\mathbf{t} := (t_v)_{v\in\mathcal{V}}$ (this is well-defined because $\mathcal{X}$ is finite). Now define $\widetilde{\mathbf{s}}^x := r\,\mathbf{s}^x - \mathbf{t}$, for all $x \in \mathcal{X}$; then $\widetilde{\mathsf{S}}$ is an affine transform of S, so $F_{\widetilde{\mathsf{S}}} = F_{\mathsf{S}}$.

Now, for any $v \in \mathcal{V}$, we have $\min\{\widetilde{s}_v^x;\ x \in \mathcal{X}\} = 0$, by construction. Also, $\max\{\widetilde{s}_v^x - \widetilde{s}_v^y;\ x, y \in \mathcal{X}\} \leq 1$, which implies that $\max\{\widetilde{s}_v^x;\ x \in \mathcal{X}\} \leq 1$. Thus, $0 \leq \widetilde{s}_v^x \leq 1$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. ◇ `Claim 1`

By replacing S by $\widetilde{\mathsf{S}}$ from Claim 1 if necessary, we can assume without loss of generality that $\widetilde{s}_v^x \in [0,1]$ for all $x \in \mathcal{X}$ and $v \in \mathcal{V}$. For any $v \in \mathcal{V}$, define $\mathbf{s}_v := (s_v^x)$, a vector in $[0,1]^{\mathcal{X}}$. Let $\mathsf{S}^\dagger := \{\mathbf{s}_v\}_{v\in\mathcal{V}}$; then $\mathsf{S}^\dagger \subseteq [0,1]^{\mathcal{X}}$.

At this point, the argument is very similar to the proof of Proposition 2.5. By an argument identical to Claim 1 in that proof, one can show: *If F_{S} is the most expressive rule with $\mathsf{S}^\dagger \subseteq [0,1]^{\mathcal{X}}$, then $\mathsf{S}^\dagger = [0,1]^{\mathcal{X}}$.* So, suppose $\mathsf{S}^\dagger = [0,1]^{\mathcal{X}}$. For any $v, w \in \mathcal{V}$, if $\mathbf{s}_v = \mathbf{s}_w$, then a vote for v has the same effect as a vote for w, when added to any profile. Thus, we can regard v and w as the same. Thus, for each $\mathbf{t} \in [0,1]^{\mathcal{X}}$, there exists a unique $v \in \mathcal{V}$ with $\mathbf{s}_v = \mathbf{t}$. At this point it is clear that F_{S} is the range voting rule. □

The proof of Proposition 4.1 depends on Proposition 4.3, so we prove that first. The proofs of Propositions 4.3 and 5.3 require some preliminaries. Let $\mathsf{B} = \{\mathbf{b}^{x,y}\}_{x,y\in\mathcal{X}}$ be a real-valued balance system on $(\mathcal{X}, \mathcal{V})$. For any $x \in \mathcal{X}$, recall equation (2) defining ${}_{\mathbb{R}}\mathcal{C}_x$. Let $\mathbb{Q}_+ := \{q \in \mathbb{Q};\ q \geq 0\}$. Let ${}_{\mathbb{Q}}\mathcal{C}_x := \mathbb{Q}_+^{\mathcal{V}} \cap {}_{\mathbb{R}}\mathcal{C}_x$, and let $\mathbb{Q}_+[\mathcal{D}] := \{$be all $\mathbb{Q}_+$-linear combinations of elements of $\mathcal{D}\}$. We also define

$$
{}_{\mathbb{R}}\mathcal{C}_x^o \quad := \quad \mathbb{R}_+[\mathcal{D}] \setminus \bigcup_{y\in\mathcal{X}\setminus\{x\}} {}_{\mathbb{R}}\mathcal{C}_y. \tag{B24}
$$

Lemma B.12 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite, and let $\mathcal{D} \subset \mathbb{N}^{\mathcal{V}}$ be a thick domain. Let $x \in \mathcal{X}$.*

(a) *${}_{\mathbb{R}}\mathcal{C}_x^o$ is a relatively open subset of $\mathbb{R}_+[\mathcal{D}]$, and is closed under multiplication by positive real numbers.*

(b) *For any $\mathbf{d} \in \mathcal{D}$, we have* (b1) $(\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x) \Leftrightarrow (x \in F_{\mathsf{S}}(\mathbf{d}))$ *and* (b2) $(\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x^o) \Leftrightarrow (F_{\mathsf{S}}(\mathbf{d}) = \{x\})$.

(c) *If ${}_{\mathbb{R}}\mathcal{C}_x^o \neq \emptyset$, then* (c1) ${}_{\mathbb{R}}\mathcal{C}_x = \mathrm{cl}({}_{\mathbb{R}}\mathcal{C}_x^o)$ *and* (c2) ${}_{\mathbb{R}}\mathcal{C}_x = \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x)$. *(Here, "cl" denotes the relative closure inside $\mathbb{R}_+[\mathcal{D}]$).*

Proof: (a) For all $y \in \mathcal{X}$, it is clear from defining equation (2) that ${}_{\mathbb{R}}\mathcal{C}_y$ is a relatively closed subset of $\mathbb{R}_+[\mathcal{D}]$, and closed under positive scalar multiplication. Thus, the set $({}_{\mathbb{R}}\mathcal{C}_x^o)^{\complement} = \bigcup_{y \in \mathcal{X} \setminus \{x\}} {}_{\mathbb{R}}\mathcal{C}_y$ is relatively closed (because $\mathcal{X}$ is finite), and also closed under positive scalar multiplication. Thus, its complement ${}_{\mathbb{R}}\mathcal{C}_x^o$ is relatively open, and closed under positive scalar multiplication.

(b1) $(\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x) \iff (\mathbf{b}^{x,y}(\mathbf{d}) \geq 0$, for all $y \in \mathcal{X}) \iff (x \in F_{\mathsf{B}}(\mathbf{d}))$.

(b2) "$\Longrightarrow$" If $\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x^o$, then $\mathbf{d} \notin {}_{\mathbb{R}}\mathcal{C}_y$ for any $y \neq x$. Thus, for all $y \neq x$, (b1) implies that $y \notin F_{\mathsf{B}}(\mathbf{d})$. But $F_{\mathsf{B}}(\mathbf{d}) \neq \emptyset$, so we must have $F_{\mathsf{B}}(\mathbf{d}) = \{x\}$.

"$\Longleftarrow$" Suppose $F_{\mathsf{B}}(\mathbf{d}) = \{x\}$. Then for all $y \neq x$, we have $y \notin F_{\mathsf{B}}(\mathbf{d})$, so (b1) implies that $\mathbf{d} \notin {}_{\mathbb{R}}\mathcal{C}_y$. Thus, $\mathbf{d} \notin ({}_{\mathbb{R}}\mathcal{C}_x^o)^{\complement}$. But $\mathbf{d} \in \mathbb{R}_+[\mathcal{D}]$. Thus, $\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x^o$.

(c1) "$\supseteq$" Defining equation (2) implies that ${}_{\mathbb{R}}\mathcal{C}_x$ is relatively closed in $\mathbb{R}_+[\mathcal{D}]$. Thus, it suffices to show that ${}_{\mathbb{R}}\mathcal{C}_x \supseteq {}_{\mathbb{R}}\mathcal{C}_x^o$. This follows immediately from defining equation (B24) and the next claim.

Claim 1: $\mathbb{R}_+[\mathcal{D}] = \bigcup_{x \in \mathcal{X}} {}_{\mathbb{R}}\mathcal{C}_x$.

Proof: $\mathcal{D}$ is additively closed (because it is thick); it is then easy to check that $\mathbb{Q}_+[\mathcal{D}] = \mathbb{Q}_+ \cdot \mathcal{D}$. For all $x \in \mathcal{X}$, let $\mathcal{C}_x := \{\mathbf{d} \in \mathcal{D};\ x \in F_{\mathsf{B}}(\mathbf{d})\}$; then $\mathcal{C}_x$ is also additively closed, so it follows from (b1) that ${}_{\mathbb{Q}}\mathcal{C}_x = \mathbb{Q}_+ \cdot \mathcal{C}_x$. Condition (1) implies $\mathcal{D} = \bigcup_{x \in \mathcal{X}} \mathcal{C}_x$. Thus,

$$\mathbb{Q}_+[\mathcal{D}] \;=\; \mathbb{Q}_+ \cdot \mathcal{D} \;=\; \mathbb{Q}_+ \cdot \bigcup_{x \in \mathcal{X}} \mathcal{C}_x \;=\; \bigcup_{x \in \mathcal{X}} (\mathbb{Q}_+ \cdot \mathcal{C}_x) \;=\; \bigcup_{x \in \mathcal{X}} {}_{\mathbb{Q}}\mathcal{C}_x.$$

$$\text{Thus,}\quad \mathbb{R}_+[\mathcal{D}] \;=\; \mathrm{cl}(\mathbb{Q}_+[\mathcal{D}]) \;=\; \mathrm{cl} \bigcup_{x \in \mathcal{X}} {}_{\mathbb{Q}}\mathcal{C}_x \;\underset{(*)}{=}\; \bigcup_{x \in \mathcal{X}} \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x) \;\underset{(\dagger)}{\subseteq}\; \bigcup_{x \in \mathcal{X}} {}_{\mathbb{R}}\mathcal{C}_x \;\subseteq\; \mathbb{R}_+[\mathcal{D}],$$

where $(*)$ is because $\mathcal{X}$ is finite, and $(\dagger)$ is because $\mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x) \subseteq {}_{\mathbb{R}}\mathcal{C}_x$ because ${}_{\mathbb{Q}}\mathcal{C}_x \subset {}_{\mathbb{R}}\mathcal{C}_x$ and ${}_{\mathbb{R}}\mathcal{C}_x$ is relatively closed in $\mathbb{R}_+[\mathcal{D}]$. ◇ `Claim 1`

"$\subseteq$" The set $\mathbb{R}_+[\mathcal{D}]$ is convex with nonempty interior in $\mathbb{R}^{\mathcal{V}}$ (because $\mathcal{D}$ is thick). Part (a) says that ${}_{\mathbb{R}}\mathcal{C}_x^o$ is relatively open in $\mathbb{R}_+[\mathcal{D}]$. Thus, ${}_{\mathbb{R}}\mathcal{C}_x^o$ has nonempty interior in $\mathbb{R}^{\mathcal{V}}$. Let $\mathbf{q} \in {}_{\mathbb{R}}\mathcal{C}_x^o$ be an element of this interior.

Claim 2: *For all $y \in \mathcal{X} \setminus \{x\}$, we have $\mathbf{b}^{x,y}(\mathbf{q}) > 0$.*

Proof: $\mathbf{q}$ is in the $\mathbb{R}^{\mathcal{V}}$-interior of ${}_{\mathbb{R}}\mathcal{C}_x^o$. Thus, if $\epsilon > 0$ is small enough, then $\mathbf{q} - \epsilon \mathbf{b}^{x,y} \in {}_{\mathbb{R}}\mathcal{C}_x^o$. Thus, Claim 1 implies that $\mathbf{q} - \epsilon\, \mathbf{b}^{x,y} \in {}_{\mathbb{R}}\mathcal{C}_x$. Thus, equation (2) says $0 \leq \mathbf{b}^{x,y}(\mathbf{q} - \epsilon\, \mathbf{b}^{x,y}) = \mathbf{b}^{x,y}(\mathbf{q}) - \epsilon\, \mathbf{b}^{x,y}(\mathbf{b}^{x,y}) = \mathbf{b}^{x,y}(\mathbf{q}) - \epsilon\, \|\mathbf{b}^{x,y}\|^2$. Thus, $\mathbf{b}^{x,y}(\mathbf{q}) \geq \epsilon\, \|\mathbf{b}^{x,y}\|^2 > 0$, as desired. ◇ `Claim 2`

Now, let $\mathbf{r} \in {}_{\mathbb{R}}\mathcal{C}_x$. For all $n \in \mathbb{N}$, let $\mathbf{r}_n := (\frac{n-1}{n})\mathbf{r} + \frac{1}{n}\mathbf{q}$; then $\mathbf{r}_n \in \mathbb{R}_+[\mathcal{D}]$ because $\mathbb{R}_+[\mathcal{D}]$ is convex. For all $y \in \mathcal{X}$, equation (2) says that $\mathbf{b}^{x,y}(\mathbf{r}) \geq 0$, while Claim 2 says $\mathbf{b}^{x,y}(\mathbf{q}) > 0$. Thus, $\mathbf{b}^{x,y}(\mathbf{r}_n) = (\frac{n-1}{n})\mathbf{b}^{x,y}(\mathbf{r}) + \frac{1}{n}\mathbf{b}^{x,y}(\mathbf{q}) > 0$. Thus, equation (2) says

$\mathbf{r}_n \notin {}_{\mathbb{R}}\mathcal{C}_y$ for all $y \in \mathcal{X} \setminus \{x\}$; thus, equation (B24) says $\mathbf{r}_n \in {}_{\mathbb{R}}\mathcal{C}^o_x$. But clearly, $\mathbf{r}_n \xrightarrow[n\to\infty]{} \mathbf{r}$. Thus, $\mathbf{r}$ is a cluster point of ${}_{\mathbb{R}}\mathcal{C}^o_x$.

This argument works for any $\mathbf{r} \in {}_{\mathbb{R}}\mathcal{C}_x$. Thus, ${}_{\mathbb{R}}\mathcal{C}_x \subseteq \mathrm{cl}({}_{\mathbb{R}}\mathcal{C}^o_x)$.

(c2) "$\supseteq$" holds because ${}_{\mathbb{R}}\mathcal{C}_x$ is relatively closed in $\mathbb{R}_+[\mathcal{D}]$, and ${}_{\mathbb{R}}\mathcal{C}_x \supseteq {}_{\mathbb{Q}}\mathcal{C}_x$.

"$\subseteq$" $\mathbb{Q}_+[\mathcal{D}]$ is dense in $\mathbb{R}_+[\mathcal{D}]$. Part (a) says that ${}_{\mathbb{R}}\mathcal{C}^o_x$ is relatively open in $\mathbb{R}_+[\mathcal{D}]$. Thus, ${}_{\mathbb{Q}}\mathcal{C}^o_x := \mathbb{Q}_+[\mathcal{D}] \cap {}_{\mathbb{R}}\mathcal{C}^o_x$ is dense in ${}_{\mathbb{R}}\mathcal{C}^o_x$. Thus, $\mathrm{cl}({}_{\mathbb{R}}\mathcal{C}^o_x) = \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}^o_x)$. But $\mathrm{cl}({}_{\mathbb{R}}\mathcal{C}^o_x) = {}_{\mathbb{R}}\mathcal{C}_x$ by (c1), while $\mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}^o_x) \subseteq \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x)$. Thus, ${}_{\mathbb{R}}\mathcal{C}_x \subseteq \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x)$. □

Lemma B.13 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, and let $\mathcal{D} \subseteq \mathbb{N}^{\mathcal{V}}$ be a thick domain. Let B and $\widetilde{\mathsf{B}}$ be real-valued balance systems on $\mathcal{X}$. Suppose F_{B} is nondegenerate. For every $x \underset{\mathsf{B}}{\sim} y \in \mathcal{X}$, suppose there is some scalar $r_{x,y} > 0$ such that $\widetilde{\mathbf{b}}^{x,y} = r_{x,y} \cdot \mathbf{b}^{x,y}$. Then $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$ on $\mathcal{D}$.*

Proof:

Claim 1: *For all $x \in \mathcal{X}$, we have ${}_{\mathbb{R}}\mathcal{C}_x := \{\mathbf{r} \in \mathbb{R}_+[\mathcal{D}];\ \mathbf{b}^{x,y}(\mathbf{r}) \geq 0$ for all $y \underset{\mathsf{B}}{\sim} x\}$.*

Proof: Let $\mathcal{C}' := \{\mathbf{r} \in \mathbb{R}_+[\mathcal{D}];\ \mathbf{b}^{x,y}(\mathbf{r}) \geq 0$ for all $y \underset{\mathsf{B}}{\sim} x\}$. Let $\partial_{\mathbb{R}}\mathcal{C}_x$ be the topological boundary of ${}_{\mathbb{R}}\mathcal{C}_x$ as a relatively closed subset of $\mathbb{R}_+[\mathcal{D}]$. Then

$$\partial_{\mathbb{R}}\mathcal{C}_x \quad = \quad {}_{\mathbb{R}}\mathcal{C}_x \cap \bigcup_{y \in \mathcal{X}\setminus\{x\}} {}_{\mathbb{R}}\mathcal{C}_y \quad = \quad \bigcup_{y \in \mathcal{X}\setminus\{x\}} \mathcal{B}_{x,y} \quad = \quad \overbrace{\bigcup_{y \underset{\mathsf{B}}{\sim} x} \mathcal{B}_{x,y}}^{(A)} \cup \overbrace{\bigcup_{y \underset{\mathsf{B}}{\not\sim} x} \mathcal{B}_{x,y}}^{(B)}.$$

Now, the union (B) is nowhere dense in $\partial_{\mathbb{R}}\mathcal{C}_x$, because it is a finite union of sets of codimension 2 or more. Thus, the union (A) must be dense in $\partial_{\mathbb{R}}\mathcal{C}_x$. Since it is a finite union of closed sets, (A) is also closed. Thus, (A) must be all of $\partial_{\mathbb{R}}\mathcal{C}_x$. Thus, ${}_{\mathbb{R}}\mathcal{C}_x$ is the intersection of $\mathbb{R}_+[\mathcal{D}]$ with all halfspaces which are bounded by a hyperplane tangent to one of the faces in (A). But this set is just $\mathcal{C}'$. ◇ `Claim 1`

Define ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x := \{\mathbf{r} \in \mathbb{R}^{\mathcal{V}}_+;\ \widetilde{\mathbf{b}}_{x,y}(\mathbf{r}) \geq 0$, for all $y \in \mathcal{X}\}$, and ${}_{\mathbb{R}}\widetilde{\mathcal{C}}^o_x := \mathbb{R}_+[\mathcal{D}] \setminus \bigcup_{y \in \mathcal{X}\setminus\{x\}} {}_{\mathbb{R}}\widetilde{\mathcal{C}}_y$.

Claim 2: *For any $x \in \mathcal{X}$, we have ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x \subseteq {}_{\mathbb{R}}\mathcal{C}_x$.*

Proof: Let $\mathbf{r} \in \mathbb{R}_+[\mathcal{D}]$. If $\mathbf{r} \in {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x$, then $\widetilde{\mathbf{b}}^{x,y}(\mathbf{r}) \geq 0$ for all $y \in \mathcal{X}$, and in particular, for all $y \underset{\mathsf{B}}{\sim} x$. This means that $\mathbf{b}^{x,y}(\mathbf{r}) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{r})/r_{x,y} \geq 0$ for all $y \underset{\mathsf{B}}{\sim} x$ (because $r_{x,y} > 0$); thus Claim 1 says that $\mathbf{r} \in {}_{\mathbb{R}}\mathcal{C}_{\mathbf{x}}$. ◇ `Claim 2`

Claim 3: *For any $x \in \mathcal{X}$, we have ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x \supseteq {}_{\mathbb{R}}\mathcal{C}_x$.*

Proof: We have

$${}_{\mathbb{R}}\widetilde{\mathcal{C}}^o_x \quad := \quad \mathbb{R}_+[\mathcal{D}] \setminus \bigcup_{y \in \mathcal{X}\setminus\{x\}} {}_{\mathbb{R}}\widetilde{\mathcal{C}}_y \quad \underset{(\diamond)}{\supseteq} \quad \mathbb{R}_+[\mathcal{D}] \setminus \bigcup_{y \in \mathcal{X}\setminus\{x\}} {}_{\mathbb{R}}\mathcal{C}_y \quad =: \quad {}_{\mathbb{R}}\mathcal{C}^o_x. \qquad \text{(B25)}$$

(Here, $(\diamond)$ is by Claim 2.) By hypothesis, F_{B} is nondegenerate. Thus, Lemma B.12(b2) implies that ${}_{\mathbb{R}}\mathcal{C}_x^o \neq \emptyset$. Thus, (B25) implies that ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x^o \neq \emptyset$. Thus, Lemma B.12(c1) says that ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x = \mathrm{cl}({}_{\mathbb{R}}\widetilde{\mathcal{C}}_x^o)$ and ${}_{\mathbb{R}}\mathcal{C}_x = \mathrm{cl}({}_{\mathbb{R}}\mathcal{C}_x^o)$. But then (B25) implies that ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x \supseteq {}_{\mathbb{R}}\mathcal{C}_x$. $\diamondsuit$ Claim 3

For all $x \in \mathcal{X}$, Claims 2 and 3 imply that ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x = {}_{\mathbb{R}}\mathcal{C}_x$. Thus, for all $\mathbf{d} \in \mathcal{D}$ we have:

$$\Big(x \in F_{\mathsf{B}}(\mathbf{d})\Big) \underset{(*)}{\Longleftrightarrow} \Big(\mathbf{d} \in {}_{\mathbb{R}}\mathcal{C}_x\Big) \Longleftrightarrow \Big(\mathbf{d} \in {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x\Big) \underset{(*)}{\Longleftrightarrow} \Big(x \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})\Big),$$

where both $(*)$ are by Lemma B.12(b1). Thus, $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$, as desired. $\Box$

Proof of Proposition 4.3. "$\Longleftarrow$" follows immediately from Lemma B.13.

"$\Longrightarrow$" For any $x \in \mathcal{X}$, let ${}_{\mathbb{Q}}\widetilde{\mathcal{C}}_x := \mathbb{Q}_+^{\mathcal{V}} \cap {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x$. For all $y \in \mathcal{X}$, define $\widetilde{\mathcal{B}}_{x,y} := {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x \cap {}_{\mathbb{R}}\widetilde{\mathcal{C}}_y$.

Claim 1: *For all $x \in \mathcal{X}$, ${}_{\mathbb{Q}}\mathcal{C}_x = {}_{\mathbb{Q}}\widetilde{\mathcal{C}}_x$.*

Proof: Let $\mathbf{q} \in \mathbb{Q}_+[\mathcal{D}]$. Since $\mathcal{D}$ is additively closed, there exists $N \in \mathbb{N}$ with $N\mathbf{q} \in \mathcal{D}$. Then

$$\begin{aligned}\Big(\mathbf{q} \in {}_{\mathbb{R}}\mathcal{C}_x\Big) \underset{(*)}{\Longleftrightarrow} \Big(N\mathbf{q} \in {}_{\mathbb{R}}\mathcal{C}_x\Big) &\underset{(\diamond)}{\Longleftrightarrow} \Big(x \in F_{\mathsf{B}}(N\mathbf{q})\Big) \\ \underset{(\dagger)}{\Longleftrightarrow} \Big(x \in F_{\widetilde{\mathsf{B}}}(N\mathbf{q})\Big) &\underset{(\diamond)}{\Longleftrightarrow} \Big(N\mathbf{q} \in {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x\Big) \underset{(*)}{\Longleftrightarrow} \Big(\mathbf{q} \in {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x\Big).\end{aligned}$$

Here, both $(*)$ is by Lemma B.12(a), while both $(\diamond)$ are by Lemma B.12(b1). Finally $(\dagger)$ is because $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$ on $\mathcal{D}$. $\diamondsuit$ Claim 1

Now, F_{B} is nondegenerate, and $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$ by hypothesis, so $F_{\widetilde{\mathsf{B}}}$ also is nondegenerate. Thus, for all $x \in \mathcal{X}$, Lemma B.12(b2) implies that ${}_{\mathbb{R}}\mathcal{C}_x^o \neq \emptyset \neq {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x^o$. Thus, Lemma B.12(c2) says that ${}_{\mathbb{R}}\mathcal{C}_x = \mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x)$, and ${}_{\mathbb{R}}\widetilde{\mathcal{C}}_x = \mathrm{cl}({}_{\mathbb{Q}}\widetilde{\mathcal{C}}_x)$. Thus, Claim 1 implies that ${}_{\mathbb{R}}\mathcal{C}_x = {}_{\mathbb{R}}\widetilde{\mathcal{C}}_x$ for all $x \in \mathcal{X}$. This implies that $\widetilde{\mathcal{B}}_{x,y} = \mathcal{B}_{x,y}$ for all $x, y \in \mathcal{X}$.

Now suppose $x \underset{\mathsf{B}}{\approx} y$. Then $\mathbf{b}^{x,y} \neq \mathbf{0}$ and $\widetilde{\mathbf{b}}^{x,y} \neq \mathbf{0}$, because B and $\widetilde{\mathsf{B}}$ have no zeros. Let $\mathcal{H}_{x,y} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{V}};\ \mathbf{b}^{x,y}(\mathbf{r}) = 0\}$ and $\widetilde{\mathcal{H}}_{x,y} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{V}};\ \widetilde{\mathbf{b}}^{x,y}(\mathbf{r}) = 0\}$; then $\mathcal{H}_{x,y}$ and $\widetilde{\mathcal{H}}_{x,y}$ are hyperplanes, and clearly $\mathcal{B}_{x,y} \subseteq \mathcal{H}_{x,y}$ and $\widetilde{\mathcal{B}}_{x,y} \subseteq \widetilde{\mathcal{H}}_{x,y}$. But $x \underset{\mathsf{B}}{\approx} y$, so $\mathcal{B}_{x,y}$ is hyperplanar; thus, $\mathcal{H}_{x,y}$ must be the *unique* hyperplane containing $\mathcal{B}_{x,y}$. As we have established that $\mathcal{B}_{x,y} = \widetilde{\mathcal{B}}_{x,y} \subseteq \widetilde{\mathcal{H}}_{x,y}$, it follows that $\widetilde{\mathcal{H}}_{x,y} = \mathcal{H}_{x,y}$, which means that there is some $r_{x,y} > 0$ such that $\widetilde{\mathbf{b}}^{x,y} = r_{x,y}\mathbf{b}^{x,y}$. $\Box$

Proof of Proposition 4.4. "$\Longleftarrow$" is a straightforward computation.

"$\Longrightarrow$" Let $\mathsf{B} := \nabla\mathsf{S}$ and $\widetilde{\mathsf{B}} := \nabla\widetilde{\mathsf{S}}$. Then $F_{\mathsf{B}} = F_{\mathsf{S}}$ and $F_{\widetilde{\mathsf{B}}} = F_{\widetilde{\mathsf{S}}}$ (see Example 2.1(a)).

Claim 1: *Suppose F_{S} is nondegenerate. Then for all $x, y, z \in \mathcal{X}$, the vectors $\mathbf{b}^{x,y}$ and $\mathbf{b}^{y,z}$ are linearly independent in $\mathbb{R}^{\mathcal{V}}$.*

Proof: (by contradiction) Since $\mathsf{B} := \nabla\mathsf{S}$, it is a perfect balance system, and satisfies equation (3) for all $\mathbf{d} \in \mathcal{D}$. Suppose $\mathbf{b}^{x,y} = r\mathbf{b}^{y,z}$ for some $r \in \mathbb{R}$. We will consider three cases: either $r \geq 0$, or $r \leq -1$, or $-1 < r < 0$.

Case 1. Suppose $r \geq 0$. Then for all $\mathbf{d} \in \mathcal{D}$, we have

$$\Big(y \in F_{\mathsf{B}}(\mathbf{d})\Big) \implies \Big(\mathbf{b}^{y,z}(\mathbf{d}) \geq 0\Big) \implies \Big(\mathbf{b}^{x,y}(\mathbf{d}) \geq 0)\Big) \underset{(*)}{\implies} \Big(x \in F_{\mathsf{B}}(\mathbf{d}) \text{ also}\Big).$$

(Here, $(*)$ is because B is perfect.) Thus, there does not exist any $\mathbf{d} \in \mathcal{D}$ with $F_{\mathsf{B}}(\mathbf{d}) = \{y\}$, contradicting nondegeneracy.

Case 2. Suppose $r \leq -1$. Then $(1+r) \leq 0$, and

$$\mathbf{b}^{x,z} \underset{(*)}{=} \mathbf{b}^{x,y} + \mathbf{b}^{y,z} = r\mathbf{b}^{y,z} + \mathbf{b}^{y,z} = (1+r)\mathbf{b}^{y,z},$$

where $(*)$ is by equation (3). Thus, for all $\mathbf{d} \in \mathcal{D}$, we have

$$\Big(z \in F_{\mathsf{B}}(\mathbf{d})\Big) \implies \Big(\mathbf{b}^{y,z}(\mathbf{d}) \leq 0\Big) \implies \Big(\mathbf{b}^{x,z}(\mathbf{d}) \geq 0)\Big) \underset{(*)}{\implies} \Big(x \in F_{\mathsf{B}}(\mathbf{d}) \text{ also}\Big).$$

(Here, $(*)$ is because B is perfect.) Thus, there does not exist any $\mathbf{d} \in \mathcal{D}$ with $F_{\mathsf{B}}(\mathbf{d}) = \{z\}$, contradicting nondegeneracy.

Case 3. Now suppose $-1 < r < 0$. Then $\frac{1}{r} < -1$, so $1 + \frac{1}{r} < 0$. Also, $\mathbf{b}^{y,z} = \frac{1}{r}\mathbf{b}^{x,y}$; thus,

$$\mathbf{b}^{x,z} \underset{(*)}{=} \mathbf{b}^{x,y} + \mathbf{b}^{y,z} = \mathbf{b}^{x,y} + \frac{1}{r}\mathbf{b}^{x,y} = \left(1 + \frac{1}{r}\right)\mathbf{b}^{x,y},$$

where $(*)$ is by equation (3). Thus, for all $\mathbf{d} \in \mathcal{D}$, we have

$$\Big(x \in F_{\mathsf{B}}(\mathbf{d})\Big) \implies \Big(\mathbf{b}^{x,y}(\mathbf{d}) \geq 0\Big) \implies \Big(\mathbf{b}^{x,z}(\mathbf{d}) \leq 0)\Big) \underset{(*)}{\implies} \Big(z \in F_{\mathsf{B}}(\mathbf{d}) \text{ also}\Big).$$

(Here, $(*)$ is because B is perfect.) Thus, there does not exist any $\mathbf{d} \in \mathcal{D}$ with $F_{\mathsf{B}}(\mathbf{d}) = \{x\}$, contradicting nondegeneracy. ◇ `Claim 1`

The rule F_{S} $(= F_{\widetilde{\mathsf{S}}})$ is nondegenerate, so $\mathbf{s}^x \neq \mathbf{s}^y$ and $\widetilde{\mathbf{s}}^x \neq \widetilde{\mathbf{s}}^y$ for all distinct $x, y \in \mathcal{X}$. This implies that B and $\widetilde{\mathsf{B}}$ have no zeros. Thus, if $F_{\mathsf{S}} = F_{\widetilde{\mathsf{S}}}$, then Proposition 4.3 says that, for all $x \underset{\mathsf{S}}{\sim} y$ in $\mathcal{X}$, there is some $r_{x,y} > 0$ such that $\widetilde{\mathbf{b}}^{x,y} = r_{x,y}\, \mathbf{b}^{x,y}$

Claim 2: *Let $x, y, z \in \mathcal{X}$. If $(x \underset{\mathsf{S}}{\sim} y) \equiv (y \underset{\mathsf{S}}{\sim} z)$, then $r_{x,y} = r_{y,z} = r_{x,z}$.*

Proof: By hypothesis, $x \underset{\mathsf{S}}{\sim} y \underset{\mathsf{S}}{\sim} z \underset{\mathsf{S}}{\sim} x$. Thus,

$$r_{x,y}\mathbf{b}^{x,y} + r_{y,z}\mathbf{b}^{y,z} = \widetilde{\mathbf{b}}^{x,y} + \widetilde{\mathbf{b}}^{y,z} \underset{(*)}{=} \widetilde{\mathbf{b}}^{x,z} = r_{x,z}\mathbf{b}^{x,z} \underset{(*)}{=} r_{x,z}(\mathbf{b}^{x,y} + \mathbf{b}^{y,z}),$$

where both $(*)$ are by equation (3). Thus, $(r_{x,y} - r_{x,z})\mathbf{b}^{x,y} + (r_{y,z} - r_{x,z})\mathbf{b}^{y,z} = 0$. But Claim 1 says that $\mathbf{b}^{x,y}$ and $\mathbf{b}^{y,z}$ are linearly independent. Thus, we must have $(r_{x,y} - r_{x,z}) = 0$ and $(r_{y,z} - r_{x,z}) = 0$. ◇ `Claim 2`

Claim 3: *There is some $r > 0$ such that $(\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^z) = r\,(\mathbf{s}^x - \mathbf{s}^z)$ for all $x, z \in \mathcal{X}$.*

Proof: For any $w, x, y, z \in \mathcal{X}$, if $(w \underset{\mathsf{S}}{\sim} x) \cong (y \underset{\mathsf{S}}{\sim} z)$, then iterating Claim 2 yields $r_{w,x} = r_{y,z}$. But S is simple; thus, there exists $r > 0$ such that $r_{y,y'} = r$ for all $y \underset{\mathsf{S}}{\sim} y' \in \mathcal{X}$. Now, let $x, z \in \mathcal{X}$, and let $x = y_0 \underset{\mathsf{S}}{\sim} y_1 \underset{\mathsf{S}}{\sim} y_2 \underset{\mathsf{S}}{\sim} \cdots \underset{\mathsf{S}}{\sim} y_N = z$ be a path connecting x to z. For all $n \in [1 \ldots N]$, we have $(\widetilde{\mathbf{s}}^{y_{n-1}} - \widetilde{\mathbf{s}}^{y_n}) = r\,(\mathbf{s}^{y_{n-1}} - \mathbf{s}^{y_n})$. Thus,

$$\begin{aligned}
\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^z &= \widetilde{\mathbf{s}}^{y_0} - \widetilde{\mathbf{s}}^{y_N} = \widetilde{\mathbf{s}}^{y_0} - \widetilde{\mathbf{s}}^{y_1} + \widetilde{\mathbf{s}}^{y_1} - \widetilde{\mathbf{s}}^{y_2} + \widetilde{\mathbf{s}}^{y_2} - \widetilde{\mathbf{s}}^{y_3} + \cdots + \widetilde{\mathbf{s}}^{y_{N-1}} - \widetilde{\mathbf{s}}^{y_N} \\
&= r(\mathbf{s}^{y_0} - \mathbf{s}^{y_1}) + r(\mathbf{s}^{y_1} - \mathbf{s}^{y_2}) + r(\mathbf{s}^{y_2} - \mathbf{s}^{y_3}) + \cdots + r(\mathbf{s}^{y_{N-1}} - \mathbf{s}^{y_N}) \\
&= r\left(\mathbf{s}^{y_0} - \mathbf{s}^{y_1} + \mathbf{s}^{y_1} - \mathbf{s}^{y_2} + \mathbf{s}^{y_2} - \mathbf{s}^{y_3} + \cdots + \mathbf{s}^{y_{N-1}} - \mathbf{s}^{y_N}\right) = r(\mathbf{s}^{y_0} - \mathbf{s}^{y_N}) \\
&= r(\mathbf{s}^x - \mathbf{s}^z),
\end{aligned}$$

as desired. ◇ `Claim 3`

Now, let $y \in \mathcal{X}$ be arbitrary, and define $\mathbf{t} := \widetilde{\mathbf{s}}^y - r\,\mathbf{s}^y$. Then for all $x \in \mathcal{X}$, we have:

$$\widetilde{\mathbf{s}}^x \;=\; \widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^y + \widetilde{\mathbf{s}}^y \underset{(*)}{=} \widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^y + r\,\mathbf{s}^y + \mathbf{t} \underset{(\dagger)}{=} r\,(\mathbf{s}^x - \mathbf{s}^y) + r\,\mathbf{s}^y + \mathbf{t} \;=\; r\,\mathbf{s}^x + \mathbf{t}.$$

Thus, $\widetilde{\mathsf{S}}$ is an affine transform of S. Here, $(*)$ is because $\widetilde{\mathbf{s}}^y = r\,\mathbf{s}^y + \mathbf{t}$, and $(\dagger)$ is by Claim 3. □

The proof of Proposition 4.1 requires three more preliminary lemmas. Let $(\mathcal{X}, \sim)$ be a graph (that is: $\mathcal{X}$ is a set of 'vertices', and $\sim$ is a binary relation on $\mathcal{X}$ —e.g. the relation defined prior to Proposition 4.1.) A pair $x \sim y$ is called an ***edge***. A ***circuit*** in $(\mathcal{X}, \sim)$ is a closed path $x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_N \sim x_0$, where $x_0, x_1, x_2, \ldots, x_N \in \mathcal{X}$ and $N \geq 3$, and no edge appears twice. A ***cut edge*** is an edge whose removal disconnects the graph.

Lemma B.14 *Let $(\mathcal{X}, \sim)$ be a connected graph. The following are equivalent.*

(a) *$(\mathcal{X}, \sim)$ has no cut edges.*

(b) *Every edge is part of a circuit.*

(c) *Any two edges in the graph are part of a common circuit.*

A connected graph is ***multiply connected*** if it satisfies any (hence all) of the conditions of Lemma B.14. For example: the graphs of the voting rules in Figures 2(a,b,c) and 1(b,c) are multiply connected, but the one in Figure 1(a) is not.

Proof of Lemma B.14. "(c) $\Longrightarrow$ (b)" is immediate.

"(a) $\Longrightarrow$ (c)" Let $w, x, y, z \in \mathcal{X}$ and suppose $(w \sim x)$ and $(y \sim z)$. We must construct a circuit containing both edges. First, remove the edge $(w \sim x)$ from the graph. By hypothesis, the graph is still connected. Thus, there exists a path $\mathbf{p}_{x,y}$ from x to y which avoids $(w \sim x)$. There also exists a path $\mathbf{p}_{z,w}$ from z to w which avoids $(w \sim x)$. Now, join together $(w \sim x)$, $\mathbf{p}_{x,y}$, $(y \sim z)$, and $\mathbf{p}_{z,w}$ to get a circuit which starts and ends at w and includes the edges $(w \sim x)$ and $(y \sim z)$.

"(b) $\Longrightarrow$ (a)" (by contradiction) Suppose $(x \sim y)$ is a cut edge. If we remove it, then the remaining graph is disconected; x must be in one connected component, and y in another. This means there is no path from x to y that does not go through $(x \sim y)$. Thus, there is no circuit containing $(x \sim y)$. Contradiction. □

Lemma B.15 *Let $(\mathcal{X}, \sim)$ be a connected graph, with $|\mathcal{X}| \geq 3$. Let $\Pi'_{\mathcal{X}}$ be a group of automorphisms of the graph such that every element of $\Pi'_{\mathcal{X}}$ has finite order, and $\Pi'_{\mathcal{X}}$ acts transitively on $\mathcal{X}$. Then $(\mathcal{X}, \sim)$ is multiply connected.*

Proof: First, observe that every vertex has degree 2 or more. (*Proof:* Suppose some vertex had degree 1. Then the transitive action of $\Pi'_{\mathcal{X}}$ implies that every vertex has degree 1. But then $(\mathcal{X}, \sim)$ cannot be connected, because $|\mathcal{X}| \geq 3$. Contradiction.)

Now, let $(x \sim y)$ be an edge in $(\mathcal{X}, \sim)$. We will show that $(x \sim y)$ is part of a circuit. Since y has degree at least 2, we have $y \sim z$ for some $z \neq x$. Find some $\pi \in \Pi'_{\mathcal{X}}$ with $\pi(x) = z$. Define $x_n := \pi^n(x)$ and $y_n := \pi^n(y)$ for all $n \in \mathbb{N}$. (Thus, $x_0 = x$ and $x_1 = z$.) Then for all $n \in \mathbb{N}$, we have $x_n \sim y_n \sim x_{n+1}$, because π is an automorphism of $(\mathcal{X}, \sim)$ by Claim 2. Since π has finite order, there exists some n such that $\pi^n = \mathrm{Id}$, which means $x_n = x_0$. Thus, we have a circuit.

Thus, every edge is part of a circuit. Thus, Lemma B.14 says $(\mathcal{X}, \sim)$ is multiply connected. □

Lemma B.16 *Let $\mathcal{X}$ and $\mathcal{V}$ be finite sets, and let S be a nondegenerate real-valued score system on $(\mathcal{X}, \mathcal{V})$, such that the score vectors $\{\mathbf{s}^x\}_{x \in \mathcal{X}}$ are linearly independent in $\mathbb{R}^{\mathcal{V}}$, and the graph of F_{S} is multiply connected. Let $\widetilde{\mathsf{S}}$ be another real-valued score system on $(\mathcal{X}, \mathcal{V})$. Then $F_{\mathsf{S}} = F_{\widetilde{\mathsf{S}}}$ if and only if $\widetilde{\mathsf{S}}$ is an affine transform of S.*

Proof of Lemma B.16. "$\Longleftarrow$" is a straightforward computation.

"$\Longrightarrow$" Suppose $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}} \subset \mathbb{R}^{\mathcal{V}}$ and $\widetilde{\mathsf{S}} = \{\widetilde{\mathbf{s}}^x\}_{x \in \mathcal{X}} \subset \mathbb{R}^{\mathcal{V}}$. Define $\nabla\mathsf{S}$ and $\nabla\widetilde{\mathsf{S}}$ as in Example 2.1(a); then $\nabla\mathsf{S}$ and $\nabla\widetilde{\mathsf{S}}$ are real-valued perfect balance rules. Also, F_{S} $(= F_{\widetilde{\mathsf{S}}})$ is nondegenerate, so $\mathbf{s}^x \neq \mathbf{s}^y$ and $\widetilde{\mathbf{s}}^x \neq \widetilde{\mathbf{s}}^y$ for all distinct $x, y \in \mathcal{X}$. Thus, $\nabla\mathsf{S}$ and $\nabla\widetilde{\mathsf{S}}$ have no zeros, and are nondegenerate. Thus, for any $x, y \in \mathcal{X}$, if $x \underset{\mathsf{S}}{\sim} y$, then Proposition 4.3 yields some real number $r_{x,y} > 0$ such that $(\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^y) = r_{x,y}\,(\mathbf{s}^x - \mathbf{s}^y)$.

Claim 1: *If $x_0 \underset{\mathsf{S}}{\sim} x_1 \underset{\mathsf{S}}{\sim} x_2 \underset{\mathsf{S}}{\sim} \cdots \underset{\mathsf{S}}{\sim} x_N \underset{\mathsf{S}}{\sim} x_0$ is any circuit in $(\mathcal{X}, \underset{\mathsf{S}}{\sim})$, then $r_{x_0,x_1} = r_{x_1,x_2} = \cdots = r_{x_{N-1},x_N} = r_{x_N,x_0}$.*

Proof: We have

$$\begin{aligned}
&(r_{x_0,x_1} - r_{x_N,x_0})\,\mathbf{s}^{x_0} + (r_{x_1,x_2} - r_{x_0,x_1})\,\mathbf{s}^{x_1} + (r_{x_2,x_3} - r_{x_1,x_2})\,\mathbf{s}^{x_2} + \cdots + (r_{x_N,x_0} - r_{x_{N-1},x_N})\,\mathbf{s}^{x_N} \\
&\quad = \quad r_{x_0,x_1}\,\mathbf{s}^{x_0} + (r_{x_1,x_2} - r_{x_0,x_1})\,\mathbf{s}^{x_1} + (r_{x_2,x_3} - r_{x_1,x_2})\,\mathbf{s}^{x_2} + \cdots \\
&\qquad\qquad \cdots + (r_{x_N,x_0} - r_{x_{N-1},x_N})\,\mathbf{s}^{x_N} - r_{x_N,x_0}\,\mathbf{s}^{x_0} \\
&\quad = \quad r_{x_0,x_1}\,(\mathbf{s}^{x_0} - \mathbf{s}^{x_1}) + r_{x_1,x_2}\,(\mathbf{s}^{x_1} - \mathbf{s}^{x_2}) + r_{x_2,x_3}\,(\mathbf{s}^{x_2} - \mathbf{s}^{x_3}) + \cdots + r_{x_N,x_0}\,(\mathbf{s}^{x_N} - \mathbf{s}^{x_0}) \\
&\quad = \quad (\widetilde{\mathbf{s}}^{x_0} - \widetilde{\mathbf{s}}^{x_1}) + (\widetilde{\mathbf{s}}^{x_1} - \widetilde{\mathbf{s}}^{x_2}) + (\widetilde{\mathbf{s}}^{x_2} - \widetilde{\mathbf{s}}^{x_3}) + \cdots + (\widetilde{\mathbf{s}}^{x_{N-1}} - \widetilde{\mathbf{s}}^{x_N}) + (\widetilde{\mathbf{s}}^{x_N} - \widetilde{\mathbf{s}}^{x_0}) \\
&\quad = \quad (\widetilde{\mathbf{s}}^{x_0} - \widetilde{\mathbf{s}}^{x_0}) + (\widetilde{\mathbf{s}}^{x_1} - \widetilde{\mathbf{s}}^{x_1}) + (\widetilde{\mathbf{s}}^{x_2} - \widetilde{\mathbf{s}}^{x_2}) + \cdots + (\widetilde{\mathbf{s}}^{x_{N-1}} - \widetilde{\mathbf{s}}^{x_{N-1}}) + (\widetilde{\mathbf{s}}^{x_N} - \widetilde{\mathbf{s}}^{x_N}) \;=\; 0.
\end{aligned}$$

But the set $\{\mathbf{s}^{x_0}, \mathbf{s}^{x_1}, \ldots, \mathbf{s}^{x_N}\}$ is linearly independent by hypothesis. Thus, we must have $(r_{x_0,x_1} - r_{x_N,x_0}) = (r_{x_1,x_2} - r_{x_0,x_1}) = \cdots = (r_{x_N,x_0} - r_{x_{N-1},x_N}) = 0$. The claim follows. ◇ `Claim 1`

Claim 2: *For all $w, x, y, z \in \mathcal{X}$, if $w \underset{\mathsf{S}}{\sim} x$ and $y \underset{\mathsf{S}}{\sim} z$, then $r_{w,x} = r_{y,z}$.*

Proof: The graph $(\mathcal{X}, \underset{\mathsf{S}}{\sim})$ is multiply connected, so Lemma B.14 yields a circuit containing the edges $(w \underset{\mathsf{S}}{\sim} x)$ and $(y \underset{\mathsf{S}}{\sim} z)$. Thus, Claim 1 implies that $r_{w,x} = r_{y,z}$. ◇ Claim 2

Claim 2 implies that there is some $r > 0$ such that $r_{y,y'} = r$ for all $y \underset{\mathsf{S}}{\sim} y' \in \mathcal{X}$. Now, by an argument very similar to the proof of Claim 3 in the proof of Proposition 4.4, we deduce that $(\widetilde{\mathbf{s}}^x - \widetilde{\mathbf{s}}^z) = r\,(\mathbf{s}^x - \mathbf{s}^z)$ for all $x, z \in \mathcal{X}$. From this point, the proof is the same as the proof of Proposition 4.4. □

Proof of Proposition 4.1. (a) "$\Longleftarrow$" is a straightforward computation.

"$\Longrightarrow$" Recall that F_{S} is $\Pi'_{\mathcal{X}}$-neutral.

Claim 1: *If $\pi \in \Pi'_{\mathcal{X}}$, then π is an automorphism of the graph $(\mathcal{X}, \underset{\mathsf{S}}{\sim})$.*

Proof: Define $\mathsf{B} := \nabla \mathsf{S}$ in Example 2.1(a); then B is a real-valued, perfect balance rule. For any $x \in \mathcal{X}$ define ${}_{\mathbb{R}}\mathcal{C}_x$ as in equation (2). The domain $\mathbb{N}^{\mathcal{V}}$ is thick. Thus, we can apply Lemma B.12(c2).

Now, let $x \in \mathcal{X}$. Then for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, neutrality implies that $x \in F(\mathbf{n})$ if and only if $\pi(x) \in F(\widetilde{\pi}(\mathbf{n}))$. Thus, $\widetilde{\pi}(\mathcal{C}_x) = \mathcal{C}_{\pi(x)}$. By an argument similar to the proof of Claim 1 in the proof of Proposition 4.3, we deduce that $\widetilde{\pi}({}_{\mathbb{Q}}\mathcal{C}_x) = {}_{\mathbb{Q}}\mathcal{C}_{\pi(x)}$. By hypothesis, F_{S} is nondegenerate. Thus, Lemma B.12(b2) implies that ${}_{\mathbb{R}}\mathcal{C}_x^o \neq \emptyset$. Then Lemma B.12(c2) says $\mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_x) = {}_{\mathbb{R}}\mathcal{C}_x$, Thus, $\mathrm{cl}[\widetilde{\pi}({}_{\mathbb{Q}}\mathcal{C}_x)] = \pi({}_{\mathbb{R}}\mathcal{C}_x)$, because $\widetilde{\pi}$ is continuous. Likewise Lemma B.12(b2,c2) also imply that $\mathrm{cl}({}_{\mathbb{Q}}\mathcal{C}_{\pi(x)}) = {}_{\mathbb{R}}\mathcal{C}_{\pi(x)}$. Thus, we conclude that $\pi({}_{\mathbb{R}}\mathcal{C}_x) = {}_{\mathbb{R}}\mathcal{C}_{\pi(x)}$.

Thus, for any $y \in \mathcal{X}$, we have $\widetilde{\pi}(\mathcal{B}_{x,y}) = \widetilde{\pi}({}_{\mathbb{R}}\mathcal{C}_x \cap {}_{\mathbb{R}}\mathcal{C}_y) = {}_{\mathbb{R}}\mathcal{C}_{\pi(x)} \cap {}_{\mathbb{R}}\mathcal{C}_{\pi(y)} = \mathcal{B}_{\pi(x),\pi(y)}$. Thus, $\mathcal{B}_{\pi(x),\pi(y)}$ is hyperplanar if and only if $\mathcal{B}_{x,y}$ is hyperplanar (because $\widetilde{\pi}$ is a linear map). In other words, $x \underset{\mathsf{S}}{\sim} y$ if and only if $\pi(x) \underset{\mathsf{S}}{\sim} \pi(y)$. ◇ Claim 1

By hypothesis, $\Pi'_{\mathcal{X}}$ is transitive. Thus, Claim 1 and Lemma B.15 imply that the graph $(\mathcal{X}, \underset{\mathsf{S}}{\sim})$ is multiply connected. Then Lemma B.16 says that $\widetilde{\mathsf{S}}$ is an affine transform of S.

(b) Suppose S and $\widetilde{\mathsf{S}}$ are $\Pi'_{\mathcal{X}}$-neutral. Let $\pi \in \Pi'_{\mathcal{X}}$. Let $y \in \mathcal{X}$ and let $x := \pi(y)$. Then

$$r\mathbf{s}^y + \mathbf{t} \underset{(*)}{=} \widetilde{\mathbf{s}}^y \underset{(\dagger)}{=} \widetilde{\mathbf{s}}^x \widetilde{\pi} \underset{(*)}{=} (r\mathbf{s}^x + \mathbf{t})\widetilde{\pi} = r\mathbf{s}^x\widetilde{\pi} + \mathbf{t}\widetilde{\pi} \underset{(\diamond)}{=} r\mathbf{s}^y + \mathbf{t}\widetilde{\pi}.$$

Cancelling $r\mathbf{s}^y$ from both sides, we get $\mathbf{t} = \mathbf{t}\widetilde{\pi}$. This holds for all $\pi \in \Pi'_{\mathcal{X}}$. (Here, $(*)$ is because $\widetilde{\mathsf{S}}$ is an affine transform of S, while $(\dagger)$ is because $\widetilde{\mathsf{S}}$ is $\Pi'_{\mathcal{X}}$-neutral, and $(\diamond)$ is because S is $\Pi'_{\mathcal{X}}$-neutral.) □

Proof of Proposition 5.3. Fix $o \in \mathcal{X}$. For any $x \in \mathcal{X}$, a ***path*** from o to x is a chain $\mathbf{c} \in \mathbb{Z}^{\mathcal{E}}$ such that $(\partial \mathbf{c})_o = -1$, $(\partial \mathbf{c})_x = 1$, and $(\partial \mathbf{c})_z = 0$ for all $z \in \mathcal{X} \setminus \{o, x\}$.

Claim 1: *For any $x \in \mathcal{X}$ and any two paths $\mathbf{c}_1, \mathbf{c}_2$ from o to x, we have $\mathsf{B}(\mathbf{c}_1) = \mathsf{B}(\mathbf{c}_2)$.*

Proof: Observe that $\mathbf{c}_1 - \mathbf{c}_2$ is a cycle. Thus, $\mathsf{B}(\mathbf{c}_1) - \mathsf{B}(\mathbf{c}_2) = \mathsf{B}(\mathbf{c}_1 - \mathbf{c}_2) = 0$ by hypothesis. Thus, $\mathsf{B}(\mathbf{c}_1) = \mathsf{B}(\mathbf{c}_2)$. $\diamond$ `Claim 1`

Now, for any $x \in \mathcal{X}$, let $\mathbf{c}$ be a path from o to x. Define $\mathbf{s}^x := \mathsf{B}(\mathbf{c})$. (Claim 1 implies that $\mathbf{s}^x$ is well-defined, independent of the path $\mathbf{c}$). This yields a score system $\mathsf{S} := \{\mathbf{s}^x\}_{x\in\mathcal{X}}$. Define $\nabla\mathsf{S}$, as in Example 2.1(a); then $F_{\mathsf{S}} = F_{\nabla\mathsf{S}}$, and $\nabla\mathsf{S}$ is a real-valued balance system.

Claim 2: *For any $x \underset{\mathsf{B}}{\sim} y \in \mathcal{X}$, we have $\nabla^{x,y}\mathsf{S} = \mathbf{b}^{x,y}$.*

Proof: Let $\mathbf{c}_1$ be a path from o to x, and let $\mathbf{c}_2$ be a path from o to y. Then the chain $\mathbf{c} := \mathbf{c}_2 + [y,x] - \mathbf{c}_1$ is a cycle. Thus, $\mathsf{B}(\mathbf{c}) = 0$ by hypothesis. Then we have

$$\begin{aligned} 0 \;=\; \mathsf{B}(\mathbf{c}) \;&=\; \mathsf{B}(\mathbf{c}_2) - \mathsf{B}(\mathbf{c}_1) + \mathsf{B}([y,x]) \\ =\; \mathbf{s}^x - \mathbf{s}^y + \mathbf{b}^{y,x} \;&=\; \nabla^{x,y}\mathsf{S} - \mathbf{b}^{x,y}. \end{aligned}$$

Thus, $\nabla^{x,y}\mathsf{S} = \mathbf{b}^{x,y}$, as desired. $\diamond$ `Claim 2`

By hypothesis, B is a nondegenerate, real-valued balance system. Thus, Lemma B.13 and Claim 2 imply that $F_{\mathsf{B}} = F_{\nabla\mathsf{S}}$. Meanwhile, Example 2.1(a) observes that $F_{\nabla\mathsf{S}} = F_{\mathsf{S}}$. Thus, $F_{\mathsf{B}} = F_{\mathsf{S}}$, so F_{B} is a scoring rule. $\Box$

Appendix C: Uniqueness of arbitrary balance rules

Let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be two linearly ordered abelian groups. A function $\alpha : \mathcal{R} \longrightarrow \widetilde{\mathcal{R}}$ is an ***order-preserving group homomorphism*** if, for all $r, s \in \mathcal{R}$, we have $\alpha(r+s) = \alpha(r) + \alpha(s)$, and also, $\alpha(r) \geq \alpha(s)$ if and only if $r \geq s$. We say α is ***strictly*** order-preserving if we have $\alpha(r) > \alpha(s)$ if and only if $r > s$. (Equivalently: α is injective.)

Example C.1. (a) Suppose $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are subgroups of $\mathbb{R}$, with the standard ordering. Then a map $\alpha : \mathcal{R} \longrightarrow \widetilde{\mathcal{R}}$ is an order-preserving group homomorphism if and only if α is multiplication by a nonnegative real number. (See below for proof.)

(b) Let $\mathbb{R}^2$ have the lexicographical order from Example 2.7. Define $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\alpha(x_1, x_2) = x_1$. Then α is a *non*-strictly order-preserving group homomorphism. $\diamond$

Let $\mathbf{b} \in \mathcal{R}^{\mathcal{V}}$, and let $\mathcal{R}_{\mathbf{b}} \subseteq \mathcal{R}$ be the subgroup of $\mathcal{R}$ generated by the elements $\{b_v\}_{v\in\mathcal{V}}$; then $\mathcal{R}_{\mathbf{b}}$ is also a linearly ordered abelian group. A vector $\widetilde{\mathbf{b}} \in \widetilde{\mathcal{R}}^{\mathcal{V}}$ is called a ***(strict) rescaling*** of $\mathbf{b}$ if there exists a (strictly) order-preserving group homomorphism $\alpha : \mathcal{R}_{\mathbf{b}} \longrightarrow \widetilde{\mathcal{R}}$ such that $\widetilde{b}_v = \alpha(b_v)$ for all $v \in \mathcal{V}$. (For example: if $\mathbf{b}, \widetilde{\mathbf{b}} \in \mathbb{R}^{\mathcal{V}}$, then $\widetilde{\mathbf{b}}$ is a rescaling of $\mathbf{b}$ if and only if $\widetilde{\mathbf{b}} = r\,\mathbf{b}$ for some nonnegative $r \in \mathbb{R}$.) If B and $\widetilde{\mathsf{B}}$ are two balance systems, and $\widetilde{\mathbf{b}}^{x,y}$ is a strict rescaling of $\mathbf{b}^{x,y}$ for all $x, y \in \mathcal{X}$, then it easy to see that $F_{\mathsf{B}} = F_{\widetilde{\mathsf{B}}}$.

Suppose a voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies reinforcement. Then for any $x \in \mathcal{X}$, the set $\mathcal{C}_x := \{\mathbf{d} \in \mathcal{D};\ x \in F(\mathbf{d})\}$ is closed under addition. For any $y \in \mathcal{X}$, we define $\mathcal{P}_{x,y}$ to be the smallest divisible subset of $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ which contains $\mathcal{C}_x - \mathcal{C}_y$. Observe that $\mathcal{C}_x \subseteq \mathcal{P}_{x,y}$ and $-\mathcal{C}_y \subseteq \mathcal{P}_{x,y}$ (because $\mathbf{0} \in \mathcal{C}_y$ and $\mathbf{0} \in \mathcal{C}_x$). Also, $\mathcal{P}_{x,y}$ is closed under addition (because $\mathcal{C}_x$

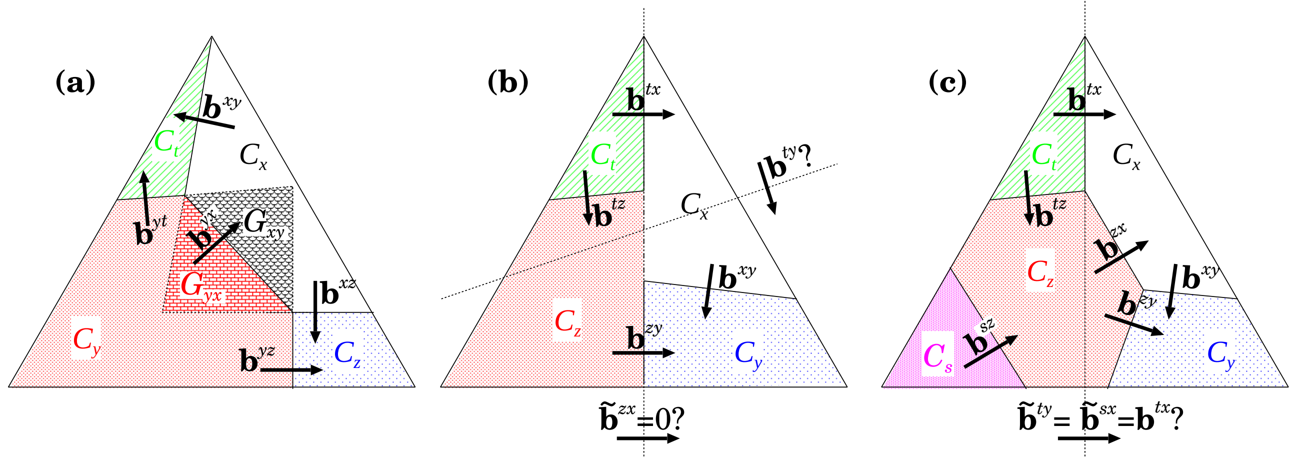

Figure 2: Here, as in Figure 1, we project $\mathbb{R}^3_+$ onto the unit simplex. (a) $\mathcal{G}_{x,y}$ and $\mathcal{G}_{y,x}$. (b,c) Balance rules without unique representations; see Example C.4.

and $\mathcal{C}_y$ are), but not necessarily under negation. Indeed, $-\mathcal{P}_{x,y} = \mathcal{P}_{y,x}$. We write "$x \underset{F}{\asymp} y$" if $\mathbb{Z}^{\langle\mathcal{V}\rangle} = \mathcal{P}_{x,y} \cup \mathcal{P}_{y,x}$. For any $x \in \mathcal{X}$, let $\mathcal{X}_F(x) := \{y \in \mathcal{X};\ \ x \underset{F}{\asymp} y\}$. A balance system $\widetilde{\mathsf{B}}$ representing F is *fine* if, for all $\mathbf{d} \in \mathcal{D}$ and all $x \in \mathcal{X}$, we have

$$\left(x \in F_{\widetilde{\mathsf{B}}}(\mathbf{d})\right) \quad \Longleftrightarrow \quad \left(\widetilde{\mathbf{b}}^{x,y}(\mathbf{d}) \geq 0 \text{ for all } y \in \mathcal{X}_F(x)\right).$$

(In particular, if $x \underset{F}{\asymp} y$ for all $x, y \in \mathcal{X}$, then $\widetilde{\mathsf{B}}$ is fine.)

Example C.2. Let $\mathcal{B}_{x,y} := \mathcal{C}_x \cap \mathcal{C}_y = \{\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle};\ \ \{x,y\} \subseteq F(\mathbf{n})\}$. Let $\langle\mathcal{B}_{x,y}\rangle$ be the smallest divisible subgroup of $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ which contains $\mathcal{B}_{x,y}$. Suppose $\mathbb{Z}^{\langle\mathcal{V}\rangle}/\langle\mathcal{B}_{x,y}\rangle$ is isomorphic to a subgroup of $\mathbb{Q}$ (heuristically, this means $\mathcal{B}_{x,y}$ spans a 'hyperplane' in $\mathbb{Z}^{\langle\mathcal{V}\rangle}$), and also suppose that $(\mathcal{C}_x \cup \mathcal{C}_y) \not\subseteq \langle\mathcal{B}_{x,y}\rangle$. Then $x \underset{F}{\asymp} y$. (See below for proof.) $\diamondsuit$

For any $x, y \in \mathcal{X}$, define $\mathcal{G}_{x,y} := \{\mathbf{c} \in \mathcal{C}_x \setminus \mathcal{C}_y;\ \ \mathbf{b}^{y,z}(\mathbf{c}^x) \geq 0 \text{ for all } z \in \mathcal{X}_F(y) \setminus \{x\}\}$. That is: $\mathcal{G}_{x,y}$ is the set of elements $\mathbf{c} \in \mathcal{C}_x$ which are excluded from $\mathcal{C}_y$ *only* because $\mathbf{b}^{x,y}(\mathbf{c}) > 0$ (none of the other balance constraints excludes $\mathbf{c}$ from $\mathcal{C}_y$). See Figure 2(a). Now define $\mathcal{Q}_{x,y} := \mathcal{G}_{x,y} - \mathcal{C}_y$. Note that $\mathcal{Q}_{y,x} = -\mathcal{Q}_{x,y}$. Write "$x \underset{B}{\approx} y$" if $\mathbb{Z}^{\langle\mathcal{V}\rangle} = (\mathcal{P}_{x,y} \cap \mathcal{P}_{y,x}) \cup \mathcal{Q}_{x,y} \cup \mathcal{Q}_{y,x}$. Note that $(x \underset{B}{\approx} y) \Longrightarrow (x \underset{F}{\asymp} y)$ (because $\mathcal{Q}_{x,y} \subseteq \mathcal{P}_{x,y}$, because $\mathcal{G}_{x,y} \subseteq \mathcal{C}_x$).

Proposition C.3 *Let $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a domain, and suppose $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies reinforcement. Then $F = F_{\mathsf{B}}$, for a perfect balance system B with the following properties:*

(a) *For any other balance system $\widetilde{\mathsf{B}}$ with $F = F_{\widetilde{\mathsf{B}}}$, and for all $x, y \in \mathcal{X}$, if $x \underset{F}{\asymp} y$, then $\widetilde{\mathbf{b}}^{x,y}$ is a rescaling of $\mathbf{b}^{x,y}$.*

(b) *Suppose $\widetilde{\mathsf{B}}$ is fine. For any $x, y \in \mathcal{X}$, if $x \underset{B}{\approx} y$, then $\widetilde{\mathbf{b}}^{x,y}$ is a strict rescaling of $\mathbf{b}^{x,y}$.*

(c) *Suppose $x \underset{B}{\approx} y$ for all $x, y \in \mathcal{X}$. Then $F = F_{\widetilde{\mathsf{B}}}$ if and only if $\widetilde{\mathbf{b}}^{x,y}$ is a strict rescaling of $\mathbf{b}^{x,y}$ for all $x, y \in \mathcal{X}$.*

Example C.4. To see the need for the hypotheses of Proposition C.3, let $\mathcal{X} = \{s, t, x, y, z\}$, and $\mathcal{V} = \{1, 2, 3\}$ (so that $\mathbb{N}^{\langle \mathcal{V} \rangle} = \mathbb{N}^3$), and consider the balance rules shown in Figures 2(b,c). In both figures, we have $t \underset{F}{\not\asymp} y$, so the vector $\mathbf{b}^{t,y}$ is *not* uniquely determined up to rescaling. Indeed, if we replaced $\mathbf{b}^{t,y}$ with any other vector $\widetilde{\mathbf{b}}^{t,y}$ such that $\ker(\widetilde{\mathbf{b}}^{t,y}) \cap \mathcal{C}_t = \{\mathbf{0}\}$ and $\ker(\widetilde{\mathbf{b}}^{t,y}) \cap \mathcal{C}_y = \{\mathbf{0}\}$ (as shown in Figure 2(b)), then we would obtain a new balance system $\widetilde{\mathsf{B}}$ with $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$.

On the other hand, both Figures (b) and (c) have $x \underset{F}{\asymp} y \underset{F}{\asymp} z \underset{F}{\asymp} t \underset{F}{\asymp} x \underset{F}{\asymp} z$. Thus, if $\widetilde{\mathsf{B}}$ is any balance system with $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$, then Proposition C.3(a) says that $\widetilde{\mathbf{b}}^{x,y}$ is a rescaling of $\mathbf{b}^{x,y}$, and $\widetilde{\mathbf{b}}^{y,z}$ is a rescaling of $\mathbf{b}^{y,z}$, and $\widetilde{\mathbf{b}}^{z,t}$ is a rescaling of $\mathbf{b}^{z,t}$, and $\widetilde{\mathbf{b}}^{t,x}$ is a rescaling of $\mathbf{b}^{t,x}$, and $\widetilde{\mathbf{b}}^{x,z}$ is a rescaling of $\mathbf{b}^{x,z}$.

However, some rescalings might not to be strict. For example, in Figure 2(b), we have $\mathbf{b}^{t,x} = \mathbf{b}^{z,x} = \mathbf{b}^{z,y}$ (as indicated by the fact the three boundaries $\mathcal{C}_t \cap \mathcal{C}_x$, $\mathcal{C}_z \cap \mathcal{C}_x$, and $\mathcal{C}_z \cap \mathcal{C}_y$ are coplanar). Thus, the boundary between $\mathcal{C}_z$ and $\mathcal{C}_x$ is effectively 'enforced' by $\mathbf{b}^{t,x}$ and $\mathbf{b}^{z,y}$, so the vector $\mathbf{b}^{z,x}$ is redundant. Thus, $x \underset{B}{\not\approxeq} z$, so $\widetilde{\mathbf{b}}^{z,x}$ need not be a strict rescaling of $\mathbf{b}^{z,x}$. Indeed, we could even set $\widetilde{\mathbf{b}}^{z,x} = \mathbf{0}$ (and keep all other entries of $\widetilde{\mathsf{B}}$ the same as B), and we would still obtain $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$. Or, we could set *both* of $\widetilde{\mathbf{b}}^{t,x}$ and $\widetilde{\mathbf{b}}^{z,y}$ to $\mathbf{0}$ at the same time; if we also set $\widetilde{\mathbf{b}}^{t,y} := \widetilde{\mathbf{b}}^{z,x} := \mathbf{b}^{z,x}$ (and keep all other entries in $\widetilde{\mathsf{B}}$ the same as in B), then we will again have $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$.

In Figure 2(c), we could set $\widetilde{\mathbf{b}}^{s,x} := \widetilde{\mathbf{b}}^{t,y} := \mathbf{b}^{t,x}$ and set $\widetilde{\mathbf{b}}^{t,x} := \mathbf{0}$ and keep all other entries of $\widetilde{\mathsf{B}}$ the same as B, and obtain $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$. Or, we could set $\widetilde{\mathbf{b}}^{x,s} := \widetilde{\mathbf{b}}^{w,y} := \mathbf{b}^{x,y}$ and $\widetilde{\mathbf{b}}^{x,y} := \mathbf{0}$ and keep all other entries the same, and obtain $F_{\widetilde{\mathsf{B}}} = F_{\mathsf{B}}$. Thus, either $\widetilde{\mathbf{b}}^{t,x}$ or $\widetilde{\mathbf{b}}^{x,y}$ can be set to $\mathbf{0}$, while still obtaining an equivalent balance rule. Note these 'trivial' rescalings are possible even though $t \underset{B}{\approxeq} x$ and $x \underset{B}{\approxeq} y$; this shows that the " $\underset{B}{\approxeq}$ " relation alone is not enough to guarantee strict rescaling. These trivial rescalings are excluded if $\widetilde{\mathsf{B}}$ is fine. $\diamondsuit$

Proof of Example C.1(a). Without loss of generality, suppose $\widetilde{\mathcal{R}} := f(\mathcal{R})$.

Claim 1: *Either $\mathcal{R}$ is a discrete subgroups of $\mathbb{R}$, or $\mathcal{R}$ is dense in $\mathbb{R}$.*

Proof: For any $r \in \mathcal{R}$, define $\epsilon(r) := \inf\{|r - s|;\ s \in \mathcal{R} \setminus \{r\}\}$. Because $\mathcal{R}$ is a group, it is easy to check that there is some $\epsilon \geq 0$ such that $\epsilon(r) = \epsilon$ for all $r \in \mathcal{R}$. Now, either $\epsilon > 0$ (so $\mathcal{R}$ is discrete) or $\epsilon = 0$ (so $\mathcal{R}$ is dense).) $\diamondsuit$ `Claim 1`

Claim 1 leaves us with two cases.

Case 1. ($\mathcal{R}$ is discrete) In this case, there exists some $g \in \mathbb{R}_+$ such that $\mathcal{R} = \{z\,g;\ z \in \mathbb{Z}\}$. Let $s := f(g)/g$. Then $s > 0$, because $f(g) > 0$ because $g > 0$. For all $z \in \mathbb{Z}$, it is easy to check that $f(z\,g) = z\,f(g) = z\,g\,s$. That is: $f(r) = s\,r$ for all $r \in \mathcal{R}$.

Case 2. ($\mathcal{R}$ is dense) If $f : \mathcal{R} \longrightarrow \widetilde{\mathcal{R}}$ is an order-preserving homomorphism, then f is nondecreasing. For all $s \in \mathbb{R}$, define $\underline{f}(s) := \sup\{f(r);\ \ r \in \mathcal{R} \text{ and } r \leq s\}$ and $\overline{f}(s) := \inf\{f(r);\ r \in \mathcal{R} \text{ and } r \geq s\}$.

Claim 2: *$\underline{f}$ and $\overline{f}$ are group homomorphisms from $\mathbb{R}$ to $\mathbb{R}$.*

Proof: Let $s_1, s_2 \in \mathbb{R}$. Let $s := s_1 + s_2$. For any $r \in \mathcal{R}$, we have $r \leq s$ if and only if $r = r_1 + r_2$ for some $r_1, r_2 \in \mathcal{R}$ with $r_1 \leq s_1$ and $r_2 \leq s_2$. Thus,

$$\begin{aligned}
\underline{f}(s) &= \sup\{f(r)\ ;\ r \in \mathcal{R} \text{ and } r \leq s\} \\
&= \sup\{f(r_1 + r_2)\ ;\ r_1, r_2 \in \mathcal{R} \text{ and } r_1 \leq s_1 \text{ and } r_2 \leq s_2\} \\
&\underset{(*)}{=} \sup\{f(r_1) + f(r_2)\ ;\ r_1, r_2 \in \mathcal{R} \text{ and } r_1 \leq s_1 \text{ and } r_2 \leq s_2\} \\
&= \sup\{f(r_1)\ ;\ r_1 \in \mathcal{R} \text{ and } r_1 \leq s_1\} + \sup\{f(r_2)\ ;\ r_2 \in \mathcal{R} \text{ and } r_2 \leq s_2\} \\
&= \underline{f}(s_1) + \underline{f}(s_2),
\end{aligned}$$

as desired. Here, $(*)$ is because f is a homomorphism. The proof for $\overline{f}$ is similar. $\diamondsuit$ `Claim 2`

Claim 3: *$\underline{f}(s) = \overline{f}(s)$ for all $s \in \mathbb{R}$, and $\underline{f}(r) = f(r) = \overline{f}(r)$ for all $r \in \mathcal{R}$.*

Proof: (by contradiction) Clearly $\overline{f}(r) - \underline{f}(r) \geq 0$ for all $r \in \mathbb{R}$. Suppose $\overline{f}(s) - \underline{f}(s) = \epsilon > 0$ for some $s \in \mathbb{R}$. Then $\overline{f}(s + r) - \underline{f}(s + r) \geq \epsilon$ for all $r \in \mathcal{R}$, by Claim 2. But $\mathcal{R}$ is dense by hypothesis; thus, f has a dense set of ϵ-sized jump discontinuities. But this is impossible, because f is nondecreasing.

It follows that $\underline{f}(s) = \overline{f}(s)$ for all $s \in \mathbb{R}$. From the definition, it is clear that $\underline{f}(r) = f(r) = \overline{f}(r)$ for all $r \in \mathcal{R}$. $\diamondsuit$ `Claim 3`

Thus, we have extended f to a nondecreasing group homomorphism from $\mathbb{R}$ to $\mathbb{R}$. At this point, a well-known result about solutions to the Cauchy functional equation implies that f is multiplication by a positive scalar. $\square$

Proof of Example C.2. Recall that $\langle \mathcal{B}_{x,y} \rangle$ is the smallest divisible subgroup of $\mathbb{Z}^{\langle \mathcal{V} \rangle}$ which contains $\mathcal{B}_{x,y}$.

Claim 1: $\langle \mathcal{B}_{x,y} \rangle = \{\mathbf{z} \in \mathbb{Z}^{\langle \mathcal{V} \rangle};\ \exists\, n \in \mathbb{N}$ *and* $\mathbf{b}_1, \ldots, \mathbf{b}_N, \mathbf{b}'_1, \ldots, \mathbf{b}'_M \in \mathcal{B}_{x,y}$ *such that* $n\,\mathbf{z} = (\mathbf{b}_1 + \cdots + \mathbf{b}_N) - (\mathbf{b}'_1 + \cdots + \mathbf{b}'_M)\}$.

Proof: Let $\mathcal{Z}$ denote the set defined on the right hand side. Clearly, $\mathcal{B}_{x,y} \subseteq \mathcal{Z}$. Also, it is easy to see that $\mathcal{Z}$ is a divisible subgroup of $\mathbb{Z}^{\langle \mathcal{V} \rangle}$. Thus, $\langle \mathcal{B}_{x,y} \rangle \subseteq \mathcal{Z}$.

Conversely, $\mathcal{B}_{x,y} \subseteq \langle \mathcal{B}_{x,y} \rangle$, and $\langle \mathcal{B}_{x,y} \rangle$ is closed under addition, so $\mathbf{b}_1 + \cdots + \mathbf{b}_N - (\mathbf{b}'_1 + \cdots + \mathbf{b}'_M) \in \langle \mathcal{B}_{x,y} \rangle$ for all $\mathbf{b}_1, \ldots, \mathbf{b}_N, \mathbf{b}'_1, \ldots, \mathbf{b}'_M \in \mathcal{B}_{x,y}$. Also, $\langle \mathcal{B}_{x,y} \rangle$ is divisible; thus, $\mathcal{Z} \subseteq \langle \mathcal{B}_{x,y} \rangle$. $\diamondsuit$ `Claim 1`

Claim 2: $\langle \mathcal{B}_{x,y} \rangle \subseteq \mathcal{P}_{x,y}$.

Proof: Let $\mathbf{a} \in \langle \mathcal{B}_{x,y} \rangle$. Claim 1 yields some $n \in \mathbb{N}$ such that $n\,\mathbf{a} = (\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_N) - (\mathbf{b}'_1 + \mathbf{b}'_2 + \cdots + \mathbf{b}'_M)$, for some $\mathbf{b}_1, \ldots, \mathbf{b}_N, \mathbf{b}'_1, \ldots, \mathbf{b}'_M \in \mathcal{B}_{x,y}$. But $(\mathbf{b}_1 + \cdots + \mathbf{b}_N) \in \mathcal{B}_{x,y}$ and $(\mathbf{b}'_1 + \cdots + \mathbf{b}'_M) \in \mathcal{B}_{x,y}$, because $\mathcal{B}_{x,y}$ is additively closed (because $\mathcal{C}_x$ and $\mathcal{C}_y$ are additively closed, because F satisfies reinforcement). Thus, $n\,\mathbf{a} \in \mathcal{B}_{x,y} - \mathcal{B}_{x,y}$. But $\mathcal{B}_{x,y} \subseteq \mathcal{C}_x$, and $\mathcal{B}_{x,y} \subseteq \mathcal{C}_y$; thus, $n\,\mathbf{a} \in \mathcal{C}_x - \mathcal{C}_y$. Thus, $\mathbf{a} \in \mathcal{P}_{x,y}$. $\diamondsuit$ `Claim 2`

Since $\mathcal{P}_{x,y}$ is additively closed, Claim 2 implies that $\mathcal{P}_{x,y}$ is a union of cosets of $\langle \mathcal{B}_{x,y} \rangle$. Likewise $\mathcal{P}_{y,x}$ is a union of cosets of $\langle \mathcal{B}_{x,y} \rangle$.

Let $\mathcal{Q} := \mathbb{Z}^{\langle \mathcal{V} \rangle} / \langle \mathcal{B}_{x,y} \rangle$. By hypothesis, we can regard $\mathcal{Q}$ as a subgroup of $\mathbb{Q}$. Without loss of generality, rescale $\mathcal{Q}$ so that $\mathbb{Z} \subseteq \mathcal{Q}$. Let $\mathcal{Q}^+ := \{q \in \mathcal{Q};\ q \geq 0\}$ and $\mathcal{Q}^- := \{q \in \mathcal{Q};\ q \leq 0\}$.

Claim 3: *If $\mathcal{P} \subset \mathcal{Q}$ is a cone containing 0, then either $\mathcal{P} = \{0\}$ or $\mathcal{P} = \mathcal{Q}$ or $\mathcal{P} = \mathcal{Q}^+$ or $\mathcal{P} = \mathcal{Q}^-$.*

Proof: If $\mathcal{P} \neq \{0\}$, then $\mathcal{P}$ contains some $p \neq 0$. Either $p > 0$ or $p < 0$. Suppose $p > 0$. Now, $p \in \mathbb{Q}$, so there exists some $n \in \mathbb{N}$ such that $n\,p \in \mathbb{N}$. But $n\,p \in \mathcal{P}$ also, and $\mathcal{P}$ is divisible; thus $1 \in \mathcal{P}$. Thus, $\mathbb{N} \subseteq \mathcal{P}$, since $\mathcal{P}$ is additively closed. Now, for any $q \in \mathcal{Q}^+$, there is some $m \in \mathbb{N}$ such that $m\,q \in \mathbb{N}$; thus, $q \in \mathcal{P}$, because $\mathcal{P}$ is divisible.

We conclude: if $\mathcal{P}$ contains any $p > 0$, then $\mathcal{Q}^+ \subseteq \mathcal{P}$. Likewise, if $\mathcal{P}$ contains any $p < 0$, then $\mathcal{Q}^- \subseteq \mathcal{P}$. The claim follows. $\diamond$ `Claim 3`

Let $\phi : \mathbb{Z}^{\langle \mathcal{V} \rangle} \longrightarrow \mathcal{Q}$ be the quotient map.

Claim 4: *$\phi(\mathcal{P}_{x,y})$ is a cone in $\mathcal{Q}$.*

Proof: $\phi(\mathcal{P}_{x,y})$ is additively closed because $\mathcal{P}_{x,y}$ is additively closed, and ϕ is a homomorphism. It remains to show that $\phi(\mathcal{P}_{x,y})$ is divisible.

Let $q \in \mathcal{Q}$ and $n \in \mathbb{N}$, and suppose $n\,q \in \phi(\mathcal{P}_{x,y})$. Now, $q = \phi(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{\langle \mathcal{V} \rangle}$. Thus, we have $\phi(n\,\mathbf{z}) = n\phi(\mathbf{z}) = n\,q \in \phi(\mathcal{P}_{x,y})$. Thus, $n\,\mathbf{z} \in \phi^{-1}(\phi(\mathcal{P}_{x,y}))$. But $\mathcal{P}_{x,y}$ is a union of cosets of $\langle \mathcal{B}_{x,y} \rangle$, so $\phi^{-1}(\phi(\mathcal{P}_{x,y})) = \mathcal{P}_{x,y}$. Thus, $n\,\mathbf{z} \in \mathcal{P}_{x,y}$. Thus, $\mathbf{z} \in \mathcal{P}_{x,y}$, because $\mathcal{P}_{x,y}$ is divisible. Thus, $q = \phi(\mathbf{z}) \in \phi(\mathcal{P}_{x,y})$, as desired. $\diamond$ `Claim 4`

By hypothesis, $(\mathcal{C}_x \cup \mathcal{C}_y) \not\subseteq \langle \mathcal{B}_{x,y} \rangle$. Since $(\mathcal{C}_x \cup \mathcal{C}_y) \subseteq (\mathcal{C}_x - \mathcal{C}_y) \subseteq \mathcal{P}_{x,y}$ this implies that $\phi(\mathcal{P}_{x,y}) \neq \{0\}$. Thus Claims 3 and 4 together imply that either $\phi(\mathcal{P}_{x,y}) = \mathcal{Q}$ or $\phi(\mathcal{P}_{x,y}) = \mathcal{Q}^+$ or $\phi(\mathcal{P}_{x,y}) = \mathcal{Q}^-$.

If $\phi(\mathcal{P}_{x,y}) = \mathcal{Q}^+$, then $\phi(\mathcal{P}_{y,x}) = -\mathcal{Q}^-$ Thus, $\phi(\mathcal{P}_{x,y} \cup \mathcal{P}_{y,x}) = \mathcal{Q}^+ \cup \mathcal{Q}^- = \mathcal{Q}$. Thus, $\phi^{-1}[\phi(\mathcal{P}_{x,y} \cup \mathcal{P}_{y,x})] = \phi^{-1}(\mathcal{Q}) = \mathbb{Z}^{\langle \mathcal{V} \rangle}$. But $\mathcal{P}_{x,y}$ and $\mathcal{P}_{y,x}$ are unions of cosets of $\langle \mathcal{B}_{x,y} \rangle$, so this implies that $\mathcal{P}_{x,y} \cup \mathcal{P}_{y,x} = \mathbb{Z}^{\langle \mathcal{V} \rangle}$, as desired.

Likewise, if $\phi(\mathcal{P}_{x,y}) = -\mathcal{Q}^-$ or $\mathcal{Q}$, then $\mathcal{P}_{x,y} \cup \mathcal{P}_{y,x} = \mathbb{Z}^{\langle \mathcal{V} \rangle}$. $\square$

Proof of Proposition C.3. For all $x, y \in \mathcal{X}$, let $\mathcal{P}_{x,y}$, $\mathcal{O}_{x,y}$ and $\mathbf{b}^{x,y} : \mathbb{Z}^{\langle \mathcal{V} \rangle} \longrightarrow \mathcal{R}_{x,y} := \mathbb{Z}^{\langle \mathcal{V} \rangle} / \mathcal{O}_{x,y}$ be as in the proof of Lemma B.1.

(a) Let $x, y \in \mathcal{X}$, and suppose $x \underset{F}{\succeq} y$. Then $\mathcal{P}_{x,y}$ is a *complete* preorder conoid on $\mathbb{Z}^{\langle \mathcal{V} \rangle}$. Thus, if we apply Lemma A.1(b), then the order on the resulting group $\mathcal{R}_{x,y}$ is already a linear order (so there is no need to apply the Homogeneous Szpilrajn Lemma).

Now, for any $\mathbf{c}_x \in \mathcal{C}_x$ and $\mathbf{c}_y \in \mathcal{C}_y$, we must have $\widetilde{\mathbf{b}}^{x,y}(\mathbf{c}_x) \geq 0 \geq \widetilde{\mathbf{b}}^{x,y}(\mathbf{c}_y)$, and thus, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{c}_x - \mathbf{c}_y) \geq 0$. Thus, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{p}) \geq 0$ for all $\mathbf{p} \in \mathcal{P}_{x,y}$. In particular, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{o}) = 0$ for all $\mathbf{o} \in \mathcal{O}_{x,y}$. Thus, $\ker(\mathbf{b}^{x,y}) \subseteq \ker(\widetilde{\mathbf{b}}^{x,y})$. Thus, the Third Isomorphism Theorem (Dummit and Foote, 2004, Theorem 19, §3.3) yields a group homomorphism $\alpha : \mathcal{R}_{x,y} \longrightarrow \widetilde{\mathcal{R}}_{x,y}$ such that $\widetilde{\mathbf{b}}^{x,y} = \alpha \circ \mathbf{b}^{x,y}$.

It remains to show that α is order-preserving. To see this, let $r \in \mathcal{R}_{x,y}$; then $r = \mathbf{b}^{x,y}(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$. If $r \geq 0$, then $\mathbf{z} \in \mathcal{P}_{x,y}$. Thus, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{p}) \geq 0$, as observed above. But $\widetilde{\mathbf{b}}^{x,y}(\mathbf{p}) = \alpha \circ \mathbf{b}^{x,y}(\mathbf{z}) = \alpha(r)$. Thus, $\alpha(r) \geq 0$.

This holds for all $r \in \mathcal{R}_{x,y}$; thus, α is an order-preserving homomorphism, so $\widetilde{\mathbf{b}}^{x,y}$ is a rescaling of $\mathbf{b}^{x,y}$.

(b) Suppose that $x \underset{B}{\approx} y$, and let $\alpha : \mathcal{R}_{x,y} \longrightarrow \widetilde{\mathcal{R}}_{x,y}$ be the order-preserving homomorphism constructed in part (a). We must show that α is strictly order preserving.

Claim 1: *For all $\mathbf{q} \in \mathcal{Q}_{x,y}$, we have $\widetilde{\mathbf{b}}^{x,y}(\mathbf{q}) > 0$.*

Proof: If $\mathbf{q} \in \mathcal{Q}_{x,y}$, then $\mathbf{q} = \mathbf{g}_x - \mathbf{c}_y$, for some $\mathbf{g}_x \in \mathcal{G}_{x,y}$ and $\mathbf{c}_y \in \mathcal{C}_y$. Thus, $\mathbf{g}_x \notin \mathcal{C}_y$, but $\mathbf{b}^{y,z}(\mathbf{g}_x) \geq 0$ for all $z \in \mathcal{X}(y) \setminus \{x\}$ Then $\widetilde{\mathbf{b}}^{y,z}(\mathbf{g}_x) \geq 0$ for all $z \in \mathcal{X}(y) \setminus \{x\}$ (because part (a) says that $\widetilde{\mathbf{b}}^{y,z}$ is a rescaling of $\mathbf{b}^{y,z}$). Thus, if $\widetilde{\mathbf{b}}^{x,y}(\mathbf{g}_x) = 0$, then $y \in F_{\widetilde{\mathsf{B}}}(\mathbf{g}_x)$ (because $\widetilde{\mathsf{B}}$ is fine), contradicting the fact that $\mathbf{g}_x \in \mathcal{C}_x \setminus \mathcal{C}_y$. Thus, we must have $\widetilde{\mathbf{b}}^{x,y}(\mathbf{g}_x) > 0$.

Meanwhile, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{c}_y) \leq 0$, because $\mathbf{c}_y \in \mathcal{C}_y$. Thus, $\widetilde{\mathbf{b}}^{x,y}(\mathbf{q}) = \widetilde{\mathbf{b}}^{x,y}(\mathbf{g}_x) - \widetilde{\mathbf{b}}^{x,y}(\mathbf{c}_y) > 0$. ◇ `Claim 1`

Let $r \in \mathcal{R}_{x,y}$; then $r = \mathbf{b}^{x,y}(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle}$. Suppose $r > 0$; we must show that $\alpha(r) > 0$.

If $x \underset{B}{\approx} y$, then $\mathbb{Z}^{\langle\mathcal{V}\rangle} = \mathcal{O}_{x,y} \cup \mathcal{Q}_{x,y} \sqcup \mathcal{Q}_{y,x}$; thus, either $\mathbf{z} \in \mathcal{O}_{x,y}$ or $\mathbf{z} \in \mathcal{Q}_{x,y}$ or $\mathbf{z} \in \mathcal{Q}_{y,x}$. But if $\mathbf{z} \in \mathcal{O}_{x,y}$, then $r = 0$. Thus, if $r > 0$, then $\mathbf{z} \notin \mathcal{O}_{x,y}$. Also, if $\mathbf{z} \in \mathcal{Q}_{y,x}$, then $\mathbf{z} \in \mathcal{P}_{y,x}$, so $r \leq 0$. Thus, if $r > 0$, then $\mathbf{z} \notin \mathcal{Q}_{y,x}$, either. Thus, we must have $\mathbf{z} \in \mathcal{Q}_{x,y}$. But then Claim 1(a) says that $\widetilde{\mathbf{b}}^{x,y}(\mathbf{z}) > 0$. But $\widetilde{\mathbf{b}}^{x,y} = \alpha \circ \mathbf{b}^{x,y}$,so this means that $\alpha(\mathbf{b}^{x,y}(\mathbf{z})) > 0$, which means $\alpha(r) > 0$.

This holds for all $r \in \mathcal{R}_{x,y}$; we conclude that α is strictly order-preserving, which means $\widetilde{\mathbf{b}}^{x,y}$ is a strict rescaling of $\mathbf{b}^{x,y}$.

(c) follows immediately from part (b). □

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