Core stable bidding rings in independent private value auctions with externalities

Biran, Omer

Paris Dauphine University

March 2011

Core stable bidding rings in independent private value auctions with externalities

Omer Biran*

March 2011 (First version: June 2010)

Abstract

We consider a second price auction between bidders with independently and identically distributed valuations, where a losing bidder suffers a negative direct externality. Considering ex-ante commitments to form bidding rings we study the question of core stability of the grand coalition, namely: is there a subset of bidders that prefers forming a small bidding ring rather than participating in the grand cartel? We show that in the presence of direct externalities between bidders the grand coalition is not necessarily core stable, as opposed to the zero externality case, where the stability of the grand coalition is a known result. Finally, we study collusion in auctions as a mechanism design problem, insisting on the difficulty to compare ex-ante and interim commitments. In particular, we show that there are situations in which bidders prefer colluding before privately learning their types.

Keywords: Auctions, collusion, externalities, Bayesian games, core, partition function game, mechanism design.

JEL classification: C71, C72, D44.

1 Introduction

The question of collusion in auctions receives great attention in the auction theory literature. From an empirical point of view, there is clear evidence

*CEREMADE, université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris cedex 16.

of collusion in real life auctions and auction-like situations, although strictly prohibited in many countries. From a theoretical point of view, collusion in auctions provides a great challenge as the formation of bidding rings violates the symmetry between bidders, which is a common assumption in auction theory.

We address the question of collusion in auctions in the presence of direct externalities between bidders, focusing on the grand coalition, i.e., a bidding ring that includes all bidders. We examine the core stability of the grand coalition, as early as in the ex-ante stage, namely, before bidders learn their types. Our main goal is to answer the question: Is the grand coalition plausible in the presence of externalities, in the sense that no group of bidders prefers seceding?

1.1 A 3-player art auction example

The following example demonstrates the motivation to our work. Consider three art collectors who wish to acquire a valuable piece of art in a future auction. We assume that the object receives a restricted attention in the market, and that the three art collectors are the only possible potential bidders, a fact which is common knowledge. For the time being the auction house publishes a catalog with some details about the art object, however, the object itself is not yet available for close examination by the interested potential bidders or experts on their behalf. As such an examination is necessary in order to determine the true value of such a valuable object (e.g., to determine how well preserved it is), the three collectors can only have a rough estimation regarding their valuations in the planned auction. We, therefore, interpret this stage as the ex-ante stage.

As art collectors extremely vary in their personal preferences (e.g., personal taste), the value that they assign to the good (once it is finally accessible for a close inspection) is considered private.

A losing bidder in this example suffers a negative utility (i.e., direct externality). For instance, if the auctioned object is part of a collection, losing the auction may result in a decrease of value of other works of art that the losing collector already owns from the same collection.[1]

The formation of a bidding ring prior to the auction can be profitable. For instance, full cooperation between the three collectors will eliminate com-

[1] As opposed to art dealers, art collectors do not buy in order to resell.

pletely the competition in the auction (bid rigging), allowing them to win the object for a low price. Note, however, that in such a case the collector who will eventually own the good will most probably need to compensate the other ring members for their expected loss (i.e., externality). The presence of externalities may therefore interfere with cooperation.

We wish to see to which extent, in the presence of externalities, an ex-ante commitment of the grand coalition is plausible, in the sense that no group of players wishes to secede. In example 5.1 we demonstrate a market where a player, anticipating that the two other players will continue cooperating if he secedes, finds it profitable to deviate in the presence of externalities.[2] In such a setup, the grand coalition is said to be "instable".

1.2 Related Literature

Empirical evidence of collusion, and in particular of partial collusion, i.e., bidding rings which do not include all bidders, can be found in the work of Porter and Zona (1999) who find proof of bid rigging in school milk procurement in Ohio, as well as in Porter and Zona (1993) who study Long-Island highway construction contracts. Bajari and Ye (2003) study collusion in Midwest seal coats contracts. In particular, these studies provide some interesting links between observed bidders' behavior and theoretical notions. For example, Porter and Zona (1993) describe bidding rings that participate in several auctions, which can be interpreted as an ex-ante commitment, as ring members commit to cooperate in several auctions before learning the considered auctions' details, and in particular, before learning their private types.

With the help of mechanism design tools, the question of collusion in private value auctions without externalities was studied in, e.g., McAfee and McMillan (1992) who consider first price auctions, Mailath and Zemsky (1991), and Graham and Marshall (1987) who study collusion in second price auctions, and Marshall and Marx (2007) and Robinson (1985) who compare between first and second price auctions in order to find which is more vulnerable to bidder collusion.

We follow the model of markets with direct externalities between players, which was studied by Caillaud and Jehiel (1998) and Jehiel, Moldovanu

[2] For the sake of simplicity we demonstrate the secession of a single player rather than of a group of players. In order to demonstrate the secession of a group of players one needs to consider a larger market which significantly complicates the computations.

and Stacchetti (1999) in incomplete information setups, as well as by Jehiel and Moldovanu (1996, 1999) and Jehiel, Moldovanu and Stacchetti (1996) in complete information. Caillaud and Jehiel (1998) study collusion in identically distributed independent private value second price auctions with direct externalities, however, they do not consider the seceding of coalitions but of individuals only. The question of partial collusion, i.e., the formation of bidding rings smaller than the grand coalition, in complete information auctions with direct externalities was studied by Biran (2009).

The property of core-stability, to which we refer in this paper, was presented by Biran and Forges (2011), who develop tools to study the plausibility of coalitions in Bayesian games. They propose an application of core-stability of bidding rings in independent private value (first and second price) auctions. They also consider examples of instability in the presence of direct externalities with complete information. One of the key notions in their work, which naturally plays an important role in the current paper as well, is coalitional equilibrium in games with incomplete information, extending the work of Ray and Vohra (1997) and Ray (2007). When discussing core stability we rely on two basic core notions, the core with merging expectations introduced by Maskin (2003), and the core with singleton expectations introduced by Hafalir (2007).

Following Biran and Forges (2011) we focus on core-stability in the ex-ante stage. The literature which deals with ring formation in the ex-ante stage includes, e.g., Bajari (2001), Waehrer (1999), and Marshall et al. (1994). The question of ex-ante commitment to an incentive compatible (I.C.) mechanism was addressed by, e.g., Forges, Mertens and Vohra (2002) and Forges and Minelli (2001).

1.3 Core-stability

As in the model of Caillaud and Jehiel (1998), we consider a second price auction with a reserve price organized between a set of bidders with independent private values which are identically distributed. A losing bidder suffers a deterministic negative externality which is not identity dependent and hence assumed to be of common knowledge.

As we wish to study bidder collusion, given a partition of bidders, we define an auxiliary auction game in which the players are bidding rings, namely, cells in the considered partition.

Consider the existence of a Nash equilibrium in every auxiliary game, with respect to ex-ante expected utilities. In other words, for every bidder partition assume the existence of a *coalitional equilibrium* of a second price auction held between the bidding rings in this partition. Such a mapping of bidder partitions to coalitional equilibria, defines a partition form game (see, e.g., Lucas and Thrall (1963)), in which the value of a coalition (given a partition in which it is a cell) is its ex-ante expected utility in a Nash equilibrium of the corresponding auxiliary game.

Biran and Forges (2011) prove that with appropriate transfer payments between coalition members a coalitional equilibrium can be made I.C. More precisely, the latter results from their proposition 1 where they show that for any coalition S, given a strategy of the others and a best response for S, there exists a mechanism for S, composed of an exactly balanced transfer scheme between the members of S and the best response strategy of S, such that this mechanism is I.C. We therefore assume, thus WLOG, that types are common knowledge inside a coalition. In particular, the type of a bidding ring in the auxiliary game is the highest valuation of its members (net of their externalities).

Consider a mapping which determines the conjecture of every coalition on the behavior (i.e., partitioning) of the others if it decides to deviate and secede from the grand coalition. Namely, to every coalition we assign a partition in which it is a cell. With respect to this mapping the previously described partition form game reduces to a characteristic form game. The grand coalition is said to be *core-stable* if the core of this characteristic form game is not empty. The interpretation is that the grand coalition is plausible as no subset of bidders has an interest to deviate.

In order to examine the question of the stability of the grand coalition in auctions with externalities, we start by proving the existence of a coalitional equilibrium for any given bidder partition. Note that as opposed to the non-collusive auction game in which bidders are symmetric, in an auxiliary auction game the "bidders" (i.e., bidding rings) are asymmetric. Symmetry between bidders is violated in two senses in the auxiliary auction game. First, while bidders' valuations are distributed identically in the non-collusive auction, in the auxiliary auction each bidding ring has a different distribution which depends on its size. That is as the type of a bidding ring is the maximal valuation of its members.

Second, in the non-collusive auction externalities on a losing bidder are not identity dependent. A losing bidder suffers the same externality regardless of his (or the winner's) identity. In an auxiliary auction it is no longer true. The externality of a losing ring is cumulative and is a function of its size, as every member of the ring suffers a personal externality due to a loss. These two aspects of asymmetry introduce significant complications in the equilibrium analysis.

In this model Caillaud and Jehiel (1998) already identified an equilibrium in symmetric auctions with externalities where all bidders act individually (i.e., no bidding rings are allowed). They proved that bidding the difference between one's type and the externality term whenever the type exceeds the reservation price (and making an irrelevant bid otherwise) is in equilibrium. The reservation price therefore serves as a *participation threshold* for bidders. This equilibrium, however, depends crucially on the fact that bidders have identical distribution functions, and does not extend to asymmetric setups, where bidding rings (with nonidentical distribution functions) are considered.

In order to prove the existence of equilibrium in asymmetric collusive auctions (i.e., the auxiliary auction game), where each participating bidding ring has a different distribution function, we borrow the idea of participation thresholds. We prove that for **any** collusion scheme (i.e., bidder partition) there exist in the corresponding (asymmetric) auxiliary auction game participation thresholds which constitute an equilibrium. Specifically, given a strategy of the others, a ring has a threshold type between the reservation price and the sum of the reservation price and the ring's negative externality, such that if the ring's type is higher than its threshold type its best response is to bid the difference between its type and its externality, and whenever its type is lower than the threshold type its best response is to make an irrelevant bid.

We then rely on two specific mappings defining the conjecture of a seceding coalition regarding the partitioning of the others, starting with Maskin's (2003) *merging expectations* (see also, Hafalir (2007)). With merging expectations a seceding coalition expects the others to form the complementary coalition. Such an assumption yields an auxiliary auction game with two (usually asymmetric) bidders. Afterwards we consider the case where due to the secession of a coalition cooperation breaks down completely, as a result of which bidders outside the seceding coalition act individually. Hafalir (2007) refers to such a scenario as *singleton expectations*. It yields an auction game in which a "strong" bidder competes with "weak" bidders (see also, e.g., Maskin and Riley (2000)). We compute explicitly the participation thresholds of the coalitions in these two cases.

Once the equilibrium analysis is completed and the question of stability is to be addressed, for the sake of simplicity, we restrict our attention to an auction with three bidders.[3] As opposed to the case without externalities in which the grand coalition is stable (see, e.g., Mailath and Zemsky (1991)), we demonstrate that in the presence of externalities a bidder with merging expectations may prefer acting individually rather than participating in the all-bidders cartel. We therefore conclude that externalities may lead to the instability of the grand coalition.

As a last part of the paper we study collusion in auctions with externalities as a mechanism design problem, insisting on the differences between ex-ante and interim collusion. More specifically, we look at collusive ex-post efficient mechanisms which are budget-balanced and incentive compatible and compare interim individual rationality with ex-ante group participation constraints.

As a benchmark we refer to the results of Mailath and Zemsky (1991) in second price auctions without direct externalities. They prove the existence of such collusive mechanisms with ex-ante group participation constraints. In the presence of direct externalities Caillaud and Jehiel (1998) identify a necessary and sufficient condition for the existence of an ex-post efficient mechanism for the grand coalition which is budget-balanced, incentive compatible and satisfies interim individual rationality. Translating our results as described above, (obtained in a model similar to the one of Caillaud and Jehiel (1998)), to the language of mechanism design, the grand coalition being core-stable means the existence of an ex-post efficient mechanism for the grand coalition which is budget-balanced, incentive compatible and satisfies ex-ante group participation constraints.

In order to show that the ex-ante and interim approaches are not logically comparable we give two examples. In the first the grand coalition has an ex-post efficient mechanism, which is budget balanced, incentive compatible and satisfies interim individual participation constraints but not ex-ante group participation constraints. In the second example the grand coalition has an ex-post efficient mechanism which is budget-balanced, incentive compatible and satisfies ex-ante group participation constraints but is not interim individually rational.

We insist that although interim individual rationality is stronger than

[3]This is the smallest auction in which the instability of the grand coalition can be demonstrated. Examples exist for the general n-bidder case, $n > 3$. However, the grand coalition is always stable in a 2-bidder auction due to symmetry and the super-additivity of the grand coalition (see, section 5).

ex-ante individual rationality, comparing ex-ante and interim commitments is difficult. Biran and Forges (2011) explain this difficulty by saying that the decision of a coalition to block in the interim stage is a function of its conditional expected utility given the types of its members, and therefore one cannot use a straight forward transfer scheme for the grand coalition in order to ensure that no coalition would block interim (as opposed to ex-ante blocking). Finally, we conclude from this discussion that there are situations in which the grand coalition would try to collude in the ex-ante stage (given that such a stage can be identified), rather than letting the players learn their types before asking them to commit.

Our paper takes the following form: Section 2 presents the model. In section 3 we prove the existence of coalitional equilibrium in auctions with asymmetric bidders for any given bidder partition. In section 4 we compute the participation thresholds of bidding rings with merging and singleton expectation, which are used in section 5 to demonstrate the instability of the grand coalition in the presence of externalities. In section 6 we study collusion in auctions with externalities as a mechanism design problem, and section 7 concludes. In appendix A we bring the proof of coalitional equilibrium with singleton expectations, and appendix B presents simulations of core stability of the grand coalition with singleton expectations.

2 Model

2.1 Second price auction with externalities

As in Caillaud and Jehiel (1998) we consider a single indivisible object second price auction Γ with a reserve price $R > 0$, held in a market with a set of bidders $N = \{1, 2, ..., n\}$. Bidder i assigns a valuation t_i to the object, which is the utility he derives from the object if winning it. The valuation, or type, of i is private, and is identically and independently distributed with respect to a common continuous density $f > 0$ in $[\underline{t}, \overline{t}]$, with distribution F.

Additionally, symmetric direct external effects are considered. Specifically, a losing bidder suffers a negative externality $e < 0$. The externality term is not identity dependent. Namely, every losing bidder gets a utility equals to e regardless of his or the winner's identity. The externality is, therefore, assumed to be of common knowledge. For the sake of simplicity we assume that the parameters maintain $\underline{t} < R + e$ as well as $R < \overline{t}$. The interpretation is that a highest type bidder is expected to participate in the auction, whereas a lowest type is not.[4]

Let $b \in \mathbb{R}^n_+$ be a bidding vector. If no bidder makes a relevant bid, i.e., $\max_{j \in N} b_j < R$ then the seller keeps the good and all bidders get a utility normalized to zero. Otherwise, the good is allocated to the bidder who makes the highest bid, i, for the second highest relevant price: $p = \max\{R, \max_{j \neq i}\{b_j\}\}$. Considering quasi-linear utilities, i gets $t_i - p$, while every other bidder suffers the externality e. In case of a tie we assume that each of the bidders who placed the highest bid wins with equal probability.

2.2 The auxiliary collusion game

We extend now the auction game to consider bidding rings. Note P a partition of N, where the interpretation is that $S \in P$ is a bidding ring. We note t_S the valuation of such a ring, defined by $t_S = \max_{i \in S} t_i + (s-1)e$, where $s = |S|$, as a bidding ring wishes to maximize its profit.

Clearly, when defining the valuation of a coalition in this way we assume full revelation of information between coalition members. This assumption is WLOG as Biran and Forges (2011) show that a given coalitional equilibrium can be made I.C. with appropriate transfers within each coalition.[5] Types are therefore common knowledge inside a coalition. Hence, the distribution of a coalition's type in terms of the original distribution F is:

$$F_S(t) = (F(t - (s-1)e))^s \tag{2.1}$$

for all $t \in [\underline{t}+(s-1)e, \overline{t}+(s-1)e]$. Additionally, if S does not win the auction it suffers an externality $e_S = se$. Note, that the formation of coalitions clearly violates the symmetry in the market in two different senses. First, valuations of "bidders", i.e., bidding rings, in a collusive auction are not identically distributed, as F_S depends on the size of S. Second, the externality of a losing ring depends on its identity as e_S is a function of $|S|$.

Given a partition P we can therefore consider the auxiliary auction game $\Gamma(P)$ where players are coalitions in P. Let $b \in \mathbb{R}^{|P|}_+$ be a bidding vector in $\Gamma(P)$, where the interpretation is that in every coalition the highest valuation ring member makes a relevant bid, while the other members of the coalition make irrelevant bids (below the reservation price) which can be conveniently

[4] Our results may be recovered without this assumption.

[5] See Biran and Forges (2011)'s corollary of proposition 1.

ignored. Once again, WLOG we may assume full revelation of information within a coalition, hence, such a definition of a bidding vector in $\Gamma(P)$ is justified.

As before, if there is no relevant bid, i.e., $\max_{S\in P} b_S < R$ then the seller keeps the good and all coalitions get a null utility. Otherwise, the good is allocated to the coalition S that made the highest bid, for the second highest relevant price: $p = \max\{R, \max_{T\neq S}\{b_T\}\}$. S derives therefore a utility of $t_S - p$, while every other coalition suffers the (identity dependent) externality e_S. In case of a tie we assume that each of the coalitions that placed the highest bid wins with equal probability.

2.3 Core-stability of the grand coalition

Following Biran and Forges (2011) we address the question of the stability of the grand coalition, applied to the considered second price auction. Roughly speaking, they say that the grand coalition is core-stable if the core of the underlying cooperative game is non-empty.

As a first step in defining the underlying characteristic function, they consider for every coalition $S \subset N$, a partition $B(S)$ such that $S \in B(S)$. $B(S)$ is interpreted as the conjecture of S on the partitioning of the rest of the players, $N \setminus S$, if S secedes.

Assume the existence of a coalitional equilibrium mapping σ (see, e.g., Ray (2007)). Namely, for every partition P of N, $\sigma(P)$ is a Nash equilibrium of $\Gamma(P)$. With respect to σ, Biran and Forges (2011) derive a characteristic function, w_σ^B, which assigns to every coalition S its (ex-ante) expected utility in the equilibrium $\sigma(B(S))$ of the auxiliary game $\Gamma(B(S))$.

Finally, as mentioned above, they say that with respect to the mappings σ and B the grand coalition is core-stable if the core of w_σ^B is non-empty.

In order to examine the stability of the grand coalition in auctions with externalities, we start by proving the existence of a coalitional equilibrium mapping σ (see, section 3). We then compute the coalitional equilibria given two specific examples of (symmetric) mappings B (see, section 4).[6] Finally, we compute the corresponding characteristic functions w_σ^B (see, sections 5).

As the grand coalition may win the auction by offering the reserve price, or alternatively get a zero utility if not participating (no trade), it participates

[6]For instance, singleton expectations yield a symmetric mapping, as every seceding coalition conjectures that the others will partition themselves in the same way, to singletons. Merging expectations is another example for a symmetric mapping.

in equilibrium if and only if the highest valuation of its members net of the reserve price is greater than the externalities of the other two members. The value of the grand coalition is therefore:

$$w_\sigma^B(N) = w(N) = E((\max_i \tilde{t}_i + (n-1)e - R)I(\max_i \tilde{t}_i + (n-1)e > R)) \quad (2.2)$$

where, $\tilde{t}_i$ is a random variable with distribution F, and I is the indicator function. Obviously, the value of the grand coalition does not depend on the mappings σ and B (see also, e.g., Biran and Forges (2011)). If the core of the underlying cooperative game is not empty, then by symmetry, the allocation in which every agent gets an equal share of the value of the grand coalition is in the core. More precisely, the core is not empty if and only if it contains the following payment vector:

$$\left(\frac{w_\sigma^B(N)}{n}\right)_{i=1}^n \quad (2.3)$$

Hence, in order to prove the instability of the grand coalition, it suffices to identify a coalition S, such that:

$$\frac{w_\sigma^B(N)}{n} < \frac{w_\sigma^B(S)}{|S|} \quad (2.4)$$

Example 5.1 demonstrates a case where (2.4) holds for a singleton (i.e., $|S| = 1$) in a 3-bidder auction, illustrating the instability of the grand coalition in the presence of externalities.[7]

3 Coalitional equilibrium

Caillaud and Jehiel (1998) prove that in a symmetric market, if no bidding rings are considered, then for every distribution F there exists an equilibrium in the auction, where a bidder bids his valuation augmented by the externality term if his valuation is greater than the reserve price, and 0 otherwise. We refer to such a strategy as a *participation threshold bidding strategy* with a threshold R:

$$\sigma_i(t_i) = \begin{cases} t_i - e & \text{If } t_i > R \\ 0 & \text{Otherwise} \end{cases} \quad (3.1)$$

[7]This is for the sake of simplicity as demonstrating the secession of a group of players requires more bidders which significantly complicates the computations. See also, footnote 2.

We revise their analysis to establish the existence of equilibrium in threshold strategies in collusive auctions. Fix a partition P of N and a coalition $S \in P$. For all types t_S of S and bids $b \in \mathbb{R}_+^{|P|}$ of the participants in the auxiliary game $\Gamma(P)$, we denote $u_S(t_S; b)$ the utility of S of type t_S in $\Gamma(P)$ when b is played, as defined in section 2.2. Conveniently, we denote b_{-S} the bidding vector of all the participants but S. Consider the following lemmas:

Lemma 3.1.

$$\forall b_{-S}, \forall t_S < R + se, \forall b_S \geq R \quad u_S(t_S; (0, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.2.

$$\forall b_{-S}, \forall t_S \geq R + se, \forall b_S \geq R \quad u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.3.

$$\forall b_{-S}, \forall t_S > R, \forall b_S < R \quad u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.4. $\forall b_{-S}, \forall (t'_S, t_S), t'_S > t_S \geq R + se$, *if* $u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$ *for any* $b_S < R$, *then*

$$u_S(t'_S; (t'_S - e_S, b_{-S})) \geq u_S(t'_S; (b_S, b_{-S})) \text{ for any } b_S < R$$

The proofs of the lemmas are analogous to Caillaud and Jehiel (1998) and are therefore omitted. Loosely speaking, we conclude from the lemmas that in a collusive auction every bidding ring S has a "dominant" strategy (except for the interval $[R + se, R]$, where a "monotonous" behavior maintains).

Specifically, if the type of S is lower than $R + se$ its "best response" is not to participate regardless of the actions of the others as by winning it gets a utility lower than its externality (lemma 3.1). If its type is higher than the reserve price it should participate by bidding its type augmented by its externality (lemmas 3.2 and 3.3). Finally, if an intermediate type in the interval $[R + se, R]$ prefers participating then any higher type prefers participating as well (lemma 3.4). It follows that given any strategy of the others there exist a participation threshold strategy which is a best response. The existence of the following tractable equilibrium in threshold strategies follows.

Proposition 3.5. *Let P be a partition of N. Then for all $S \in P$ there exists $t^*_S \in [R + e_S, R]$ such that the following $(\sigma_S)_{S \in P}$ is an equilibrium of $\Gamma(P)$:*

$$\sigma_S(t_S) = \begin{cases} t_S - e_S & \text{If } t_S > t^*_S \\ 0 & \text{Otherwise} \end{cases} \tag{3.2}$$

We refer to t^*_S as the *participation threshold* of S in the equilibrium σ. An immediate implication of the proposition is that for a given valuation, the bigger a coalition is, the higher it is likely to bid if participating. The intuition is quite clear, as a big coalition suffers a greater externality if losing.

Finally, proposition 3.5 provides the coalitional equilibrium mapping σ which will be used to demonstrate the instability of the grand coalition in the presence of externalities (see, example 5.1).

4 Participation threshold

We wish to compute the coalitional equilibrium (namely, the participation threshold of a coalition) identified in the previous section in two specific cases. We start with the case where the complementary of a seceding coalition forms due to a secession. Then, we go on with the case where cooperation breaks down completely due to the secession of a coalition, namely, all players outside the seceding coalition act individually. The first case corresponds to the core with merging expectation (m-core) introduced by Maskin (2003), while the second case corresponds to the core with singleton expectations (s-core) introduced by Hafalir (2007).

4.1 Merging expectations - The two cartels case

Let us start with an analysis of the participation threshold of a coalition which expects its complementary to act cooperatively if seceding, i.e., $\forall S \subset N$, $B(S) = \{S, N \setminus S\}$. Suppose, thus WLOG, that $|S| > |N \setminus S|$.[8] We first claim (proposition 4.1) that there exists an equilibrium in threshold strategies where the participation threshold of the smaller coalition is simply the reserve price, while that of the bigger one is in the interval $[R - e_{N \setminus S} + e_S, R]$. We then derive an explicit expression of the latter.

[8]In case of an equality the collusive auction is symmetric and the equilibrium reduces to the one identified by Caillaud and Jehiel (1998). Namely, both coalitions have a participation threshold equals to the reservation price R (see, section 3).

Proposition 4.1. *Let $P = \{S, N \setminus S\}$ where $|S| > |N \setminus S|$. Then there exists $t^*_S \in [R - e_{N\setminus S} + e_S, R]$, such that t^*_S and $t^*_{N\setminus S} = R$ constitute an equilibrium of $\Gamma(P)$ in threshold strategies.*

Proof. Suppose first that $N\setminus S$ follows a bidding strategy with a participation threshold equals to R. From the preceding analysis we conclude that there exists a best response strategy for S with a threshold $t^*_S \in [R + e_S, R]$. Let us compute the interim utility of S of type $t_S \geq R + e_S$ if it chooses to participate in the auction and if it chooses not to participate. If S makes an irrelevant bid, then it gets 0 if $N \setminus S$ does not participate as well, and e_S otherwise, which yields:

$$seP(\tilde{t}_{N\setminus S} > R) \tag{4.1}$$

where, $|S| = s$ and $\tilde{t}_{N\setminus S}$ denotes a random variable with distribution $F_{N\setminus S}$.

If on the other hand S chooses to participate, we distinguish between 3 cases. Either, $N \setminus S$ does not participate in which case S wins the good for the reserve price. Or, $N \setminus S$ participates and wins, in which case S gets its externality. Or, finally, both coalitions participate and S wins, paying the bid of $N \setminus S$. Note that as we consider threshold strategies, if a coalition chooses to participate it bids the difference between its type and its externality (see, (3.2)). The interim utility of S if participating is therefore:

$$\begin{aligned}&(t_S - R)P(\tilde{t}_{N\setminus S} \leq R) + seP(\tilde{t}_{N\setminus S} > R \text{ and } \tilde{t}_{N\setminus S} - (n-s)e > t_S - se)\\&+ E((t_S - (\tilde{t}_{N\setminus S} - (n-s)e))I(\tilde{t}_{N\setminus S} > R \text{ and } \tilde{t}_{N\setminus S} - (n-s)e < t_S - se))\end{aligned} \tag{4.2}$$

where, $|N \setminus S| = n - s$ and I is the indicator function. As we consider continuous distribution, ties occur with zero probability with respect to threshold strategies.

As the interim utility functions of S if participating, (4.2), or not participating, (4.1), are continuous with respect to the type of S, we conclude that in equilibrium, S of threshold type t^*_S is indifferent between participating or not. Namely, replacing t_S with t^*_S in (4.1) and (4.2) yields an equality:

$$\begin{aligned}seP(\tilde{t}_{N\setminus S} > R) = {}&(t^*_S - R)P(\tilde{t}_{N\setminus S} \leq R) + seP(\tilde{t}_{N\setminus S} > \max\{R, t^*_S\\&+ (n-2s)e\}) + E((t^*_S - (\tilde{t}_{N\setminus S} - (n-s)e))I(\tilde{t}_{N\setminus S} > R\\&\text{and } \tilde{t}_{N\setminus S} - (n-s)e < t^*_S - se))\end{aligned} \tag{4.3}$$

Suppose by way of contradiction that $t_S^* < R - e_{N\setminus S} + e_S$. Then $R = \max\{R, t_S^* + (n-2s)e\}$ and $I(\tilde{t}_{N\setminus S} > R \texttt{ and } \tilde{t}_{N\setminus S} - (n-s)e < t_S^* - se) = 0, \forall \tilde{t}_{N\setminus S}$. Hence, (4.3) reduces to:

$$0 = (t_S^* - R)P(\tilde{t}_{N\setminus S} \leq R) \tag{4.4}$$

or equivalently, as $f > 0$, $t_S^* = R$, which is a contradiction.

Suppose now that S is following a bidding strategy with a participation threshold $t_S^* \in [R - e_{N\setminus S} + e_S, R]$. There exists a best response strategy for $N \setminus S$ with thresholds $t_{N\setminus S}^* \in [R + e_{N\setminus S}, R]$. Following the same analysis we conclude that in equilibrium:

$$\begin{aligned}(n-s)eP(\tilde{t}_S > t_S^*) = (t_{N\setminus S}^* - R)P(\tilde{t}_S \leq t_S^*) + (n-s)eP(\tilde{t}_S > \max\{t_S^*,\\ t_{N\setminus S}^* - (n-2s)e\}) + E((t_{N\setminus S}^* - (\tilde{t}_S - se))I(\tilde{t}_S > t_S^*\\ \texttt{and } \tilde{t}_S - se < t_{N\setminus S}^* - (n-s)e))\end{aligned} \tag{4.5}$$

where, $\tilde{t}_S$ is a random variable with distribution F_S. As $t_{N\setminus S}^* \leq R \leq t_S^* + e_{N\setminus S} - e_S$ it holds that $t_S^* = \max\{t_S^*, t_{N\setminus S}^* - (n-2s)e\}$ and $I(\tilde{t}_S > t_S^* \texttt{ and } \tilde{t}_S - se < t_{N\setminus S}^* - (n-s)e) = 0$, $\forall \tilde{t}_S$. It follows that $t_{N\setminus S}^* = R$, which concludes the proof. $\square$

In order to compute the participation threshold of the bigger coalition, we derive from (4.3):[9]

$$\begin{aligned}0 = -se(1 - (F(R-(n-s-1)e))^{n-s}) + (t_S^* - R)(F(R-(n-s-1)e))^{n-s}\\ + se(1 - (F(t_S^* - (s-1)e))^{n-s}) + \int_R^{t_S^*+(n-2s)e} (t_S^* - t + (n-s)e)(n-s)\\ (F(t-(n-s-1)e))^{n-s-1} f(t-(n-s-1)e)\,dt\end{aligned} \tag{4.6}$$

which, by integration in parts, gives a characterization of t_S^* in the considered case:

$$0 = -(n-2s)e(F(R-(n-s-1)e))^{n-s} + \int_{R-(n-s-1)e}^{t_S^*-(s-1)e} (F(t))^{n-s}\,dt \tag{4.7}$$

[9]For the sake of simplicity we assume here $\bar{t} + (n-2)e > R$, otherwise S might have a participation threshold which never allows it to participate in the auction. The analysis can be repeated in the complementary case.

Note, that there exists a unique threshold satisfying (4.7) as the function $h(\tau) = -(n-2s)e(F(R-(n-s-1)e))^{n-s} + \int_{R-(n-s-1)e}^{\tau-(s-1)e} (F(t))^{n-s}\,dt$, $\tau \in [R-(n-2s)e, R]$, is strictly increasing and maintains $h(R-(n-2s)e) \leq 0$ and $h(R) \geq 0$.

4.2 Singleton expectations - The cartel vs. individuals case

We repeat the previous analysis in a market where due to the secession of a coalition, cooperation breaks down completely. As a result of which, all players outside the seceding coalition act individually, namely, $\forall S \subset N$, $B(S) = \{S, \{i\}_{i \notin S}\}$. We prove in a similar way (see, appendix A) that each individual participates if his type is greater than the reserve price while the coalition participates if its valuation exceeds some $t_S^* \in [R+(s-1)e, R]$, characterized in (4.8) below.

Proposition 4.2. *Let $P = \{S, \{i\}_{i \notin S}\}$. Then there exists $t_S^* \in [R+(s-1)e, R]$, such that t_S^* and $t_i^* = R$ for all $i \notin S$ constitute an equilibrium of $\Gamma(P)$ in threshold strategies.*

Finally, as in the previous section we derive the following characterization of t_S^* in the considered case:

$$0 = (s-1)e(F(R))^{n-s} + \int_R^{t_S^*-(s-1)e} (F(t))^{n-s}\,dt \tag{4.8}$$

As before, there exists a unique threshold satisfying (4.8).

5 Stability in a 3-player market with externalities

We consider a symmetric market with 3 players, as it is the smallest market in which the instability of the grand coalition can be demonstrated.[10] Mailath and Zemsky (1991) prove that in a standard second price auction, without

[10]Due to symmetry and grand coalition superadditivity (see, e.g., Biran and Forges (2011)) the grand coalition is always stable in a 2-player market with externalities. The instability of the grand coalition in the presence of externalities can be demonstrated also in the general n-player case, $n \geq 3$.

direct externalities, the grand coalition is core-stable, namely, no subset of bidders has a profitable deviation. We wish to go further and examine the stability of the grand coalition in the presence of externalities.

5.1 Coalitions' values

In order to verify (2.4), namely, to compare the per-capita utility of a seceding coalition with the per-capita utility in the grand coalition, we compute the values of the different coalitions. Let us start with the value of the grand coalition. Consider, therefore, the auxiliary game $\Gamma(\{1,2,3\})$. Recalling (2.2), the value (or, ex-ante utility) of the grand coalition is,

$$v = E((\max_i \tilde{t}_i + 2e - R)I(\max_i \tilde{t}_i + 2e > R)) \tag{5.1}$$

Consider now a seceding coalition of two, denoted S. Note, that its value, denoted y, does not depend on the considered mapping B. Consider, therefore, the auxiliary game $\Gamma(\{\{i\},S\})$. We refer to the threshold equilibrium constructed in proposition 4.1, namely, the individual i participates if and only if his valuation is greater than the reserve price, offering his valuation augmented by the externality term, while the coalition S acts similarly with respect to the participation threshold t^* given by (4.7).[11]

In a similar way to the analysis in the previous section, as a first step in computing y we compute the interim utility function of S. If the type of S is lower than its participation threshold, $t_S \le t^*$, it gets: e_S if the individual participates (i.e., if $t_i > R$), and 0 otherwise. Its interim utility in this case is therefore,

$$e_S P(\tilde{t}_i > R) \tag{5.2}$$

If, alternatively, $t_S > t^*$, i.e., S participates, we distinguish between 3 cases. If the individual does not participate, S wins the good for the reserve price R. If the individual participates (offering his valuation augmented by the externality term) and the coalition overbids him (offering its valuation augmented by its externality) the coalition gets the good paying the offer of the individual. Finally, if the individual overbids the coalition, then the latter gets its externality. Hence, if participating, the coalition gets the following

[11] As in the computation of (4.7) we assume here $\bar{t} + e > R$. See also footnote 9.

interim utility:

$$(t_S - R)P(\tilde{t}_i \le R) + E[(t_S - (\tilde{t}_i - e))I(\tilde{t}_i > R \texttt{ and } t_S - e_S > \tilde{t}_i - e)] \\ + e_S P(\tilde{t}_i > R \texttt{ and } t_S - e_S < \tilde{t}_i - e) \quad (5.3)$$

By taking expectations on (5.2) and (5.3), recalling (2.1), and by integrating in parts, we get the value (or, ex-ante utility) of a coalition of two:

$$y = 2e(1 - F(R)) + \int_{t^*-e}^{\bar{t}} (F(t) - F(t)^3)\, dt \quad (5.4)$$

with a (unique) participation threshold t^* (see, (4.7)) given by:

$$0 = eF(R) + \int_{R}^{t^*-e} F(t)\, dt \quad (5.5)$$

Finally, we need to compute the value of a seceding single player. We need to distinguish between two cases. With merging expectations the value of a seceding individual is his ex-ante utility with respect to the equilibrium $\sigma(\{\{i\}, S\})$, denoted z. With singleton expectations his value is his ex-ante utility w.r.t $\sigma(\{\{1\}, \{2\}, \{3\}\})$, denoted x.

In a similar way to the computation of the term y above we get:

$$z = F(t^* - e)^2(t^* - 2e - R) + e - F(t^* - e)^2 \int_{R}^{t^*-e} F(t)\, dt \\ + \int_{t^*-e}^{\bar{t}} (F(t)^2 - F(t)^3)\, dt \quad (5.6)$$

In order to compute x, recall that the participation threshold of any individual in this case is the reserve price R. If $t_1 \le R$, 1 gets e if any of the others participates and 0 otherwise, which yields his interim utility if not participating:

$$eP(\max\{\tilde{t}_2, \tilde{t}_3\} > R) \quad (5.7)$$

If $t_1 > R$, the interim utility of 1 is:

$$(t_1 - R)P(\max\{\tilde{t}_2, \tilde{t}_3\} \le R) + E[(t_1 - (\max\{\tilde{t}_2, \tilde{t}_3\} - e))I(\max\{\tilde{t}_2, \tilde{t}_3\} > R \texttt{ and } t_1 - e > \max\{\tilde{t}_2, \tilde{t}_3\} - e)] + eP(\max\{\tilde{t}_2, \tilde{t}_3\} > R \texttt{ and } t_1 - e < \max\{\tilde{t}_2, \tilde{t}_3\} - e) \quad (5.8)$$

The distribution of $\max\{\tilde{t}_2, \tilde{t}_3\}$ being F^2, we conclude by taking expectations on the interim utilities and integrating in parts that the ex-ante utility of an individual facing two other individuals is:

$$x = e(1 - F(R)^2) + \int_R^{\bar{t}} (F(t)^2 - F(t)^3)\, dt \tag{5.9}$$

5.2 Emptiness of the core

As stated above in order to examine the stability of the grand coalition we need to verify whether a coalition of two players or an individual can gain more by not cooperating. By symmetry, a coalition of two players secedes if $\frac{y}{2} > \frac{v}{3}$. However, if an individual secedes, there are two possible partitions for the remaining two agents. To demonstrate the instability of the grand coalition in the presence of externalities we focus on merging expectations (m-core), namely, a seceding individual expects the others to form the complementary cartel. Therefore, by symmetry, an individual profitably secedes if $z > \frac{v}{3}$.[12]

We conclude that the grand coalition is m-core-stable, (i.e., the m-core is non-empty), if and only if:[13]

$$z \leq \frac{v}{3} \text{ and } \frac{y}{2} \leq \frac{v}{3} \tag{5.10}$$

Consider first, as a benchmark, a market without externalities. We know that in such a market offering one's valuation, as long as it is higher than the reserve price, constitutes an equilibrium in weakly dominant strategies. Recalling that the valuation of a coalition is equal to the maximal type of its members, Biran and Forges (2011) prove that the partition function reduces in this case to a characteristic function. In other words, in the absence of externalities the value of a coalition does not depend on the mapping B. In particular, the s-core and the m-core coincide, and $x = z$. Furthermore, they

[12]Note that an individual with singleton expectations will not deviate as by grand coalition superadditivity $x \leq \frac{v}{3}$.

[13]As for the core with singleton expectations (s-core), it is non-empty if and only if:

$$\frac{y}{2} \leq \frac{v}{3} \tag{5.11}$$

It follows that the s-core contains the m-core in a 3-player market.

prove that without externalities all rings are core-stable (see also Mailath and Zemsky (1991) and Barbar and Forges (2007)).[14]

Introducing direct externalities between agents interferes severely with cooperation. While as collusion reduces competition which leads to potential greater profits due to price reduction, agents find it more difficult to collude in the presence of externalities as in large coalitions the cumulating effect of negative externalities decreases dramatically the coalition's net profit.

In the following example an individual facing a cartel of two gains more than if he chooses to join the cartel in order to form the grand coalition. We therefore conclude that the grand coalition is not stable with merging expectations.

Example 5.1. Consider a 3-agent symmetric market where the valuation each agent assigns to the good is distributed uniformly in the unit interval, $F \sim U[0,1]$. A second price auction is held in this market with a reserve price $R = \frac{9}{10}$. The externality on a non-winning agent is $e = -\frac{1}{4}$. Note that as $1 + 2e < R$, if the grand coalition forms it profitably chooses not to participate in the auction. Hence, $v = 0$. By (5.6) it is readily verified that $z > 0$, hence an individual expecting the others to act cooperatively in case he secedes from the grand coalition, will profitably act independently. The grand coalition is therefore instable with merging expectation. □

Note, that considering "merging expectations" in the example is easily justified. Once a player secedes from the grand coalition the remaining two players can either cooperate and get $\frac{y}{2}$ each, or further split up gaining x each. From (5.4) and (5.9) we learn that $\frac{y}{2} > x$ in the example, hence, the remaining two players will profitably cooperate.

The individual's behavior in the example can also be interpreted as "free-riding". If he forms a coalition with the two other agents he gets a null payoff. By letting the two others form a coalition he gains a higher payoff, $z > 0$.

Let us note that stability is violated in the presence of externalities also when externalities are close to zero. One should not expect some sort of continuity, in the sense that for a small enough externality term all coalitions remain stable. This discontinuity may be explained, for instance, by the fact that for any negative externality the utility of a coalition depends on the

[14] The stability of a coalition of two players may also be recovered from Waehrer (1999). He proves that in the absence of externalities the per capita share in a second price auction increases with the size of the coalition. Specifically, $\frac{y}{2} \geq z = x$.

partition of the others. In particular, we can consider an externality term as small as we wish, for a large enough reserve price, the instability with merging expectations demonstrated in the example maintains.

As a concluding remark for this section we recall a property proved by Waehrer (1999): the per capita share in a second price auction increases with the size of the coalition. As the example shows, the property does not extend to markets with direct externalities. Specifically, we obtain in the example $\frac{y}{2} < z$.

6 Collusive mechanisms

The question of collusion in auctions is frequently addressed in the literature as a mechanism design problem. Loosely speaking, a collusive mechanism for the grand coalition determines the bidding profile of the players, as well as transfer payments between them in order to share the coalition's gain. When studying collusion in auctions with the means of mechanism design, we try to establish which properties the collusive mechanism may possess.

Consider a second price auction with independent private value (IPV) without direct externalities between bidders. Mailath and Zemsky (1991) construct a collusive mechanism for the grand coalition (with heterogeneous bidders) which is ex-post efficient, incentive compatible (IC), interim individually rational (IR) and budget-balanced (BB).[15] As is well known, IPV second price auctions have an equilibrium in weakly dominant strategies, where every bidder bids his type, or valuation. Full revelation within bidding rings yields an analogous result in collusive auctions, namely, a ring submits in equilibrium a bid equals to the highest valuation of its members (see, e.g., Biran and Forges (2011)). In particular a ring conveys no strategic externality upon players outside the ring, since the ring's highest type member would make the same bid if the ring did not collude. This allows Mailath and Zemsky (1991) to prove the "coalitional stability" of the grand coalition, namely, the existence of a mechanism for the grand coalition which is ex-post efficient, IC, BB and satisfies ex-ante group participation constraints.[16]

[15] Allowing agreements which involve the seller as well, Myerson and Satterthwaite (1983) prove an impossibility theorem regarding the existence of an ex-post efficient, BB, IC, interim IR mechanism. This indicates that in this result, the seller takes an important part with respect to the (in)stability of collusive agreements.

[16] Mailath and Zemsky (1991) prove the existence of such a mechanism using the famous

Caillaud and Jehiel (1998) study collusion in auctions with direct externalities where individual bidders decide whether to participate in the grand coalition at the interim stage. They mainly assume that bidders have veto power, namely, the refusal of a bidder to participate in the grand coalition leads to an auction where all bidders act individually.[17] They prove that the following condition is necessary and sufficient for the existence of a mechanism for the grand coalition which is ex-post efficient, BB, IC and interim IR:

$$e + \int_{R+e}^{\bar{t}} F(t)^{n-1}\, dt \leq \frac{1}{n} v + \int_{\underline{t}}^{\bar{t}} \left(\int_{\max\{\tau, R-(n-1)e\}}^{\bar{t}} F(t)^{n-1}\, dt \right) f(\tau)\, d\tau \quad (6.1)$$

whenever $R \leq \bar{t} + (n-1)e$. In the complementary case, namely, when the grand coalition has absolutely no interest in participating in the auction, whatever its valuation is, the necessary and sufficient condition for the existence of a mechanism for the grand coalition which is ex-post efficient, BB, IC and interim IR is:

$$e + \int_{R+e}^{\bar{t}} F(t)^{n-1}\, dt \leq 0 \quad (6.2)$$

It can be verified that in the case without direct externalities ($e = 0$) the necessary and sufficient condition identified by Caillaud and Jehiel (1998) holds, which in turn revalidates the result of Mailath and Zemsky (1991) regarding the existence of an ex-post efficient mechanism for the grand coalition which is IC, BB and interim IR. Specifically, as $e = 0$ it holds that $R \leq \bar{t} + (n-1)e$ and we should therefore look at condition (6.1). Changing the order of integration in the RHS of (6.1) replacing $v = E((\max_i \tilde{t}_i + (n-1)e - R) I(\max_i \tilde{t}_i + (n-1)e > R))$ (see, (2.2)) and setting $e = 0$ yields,

Bondareva-Shapley theorem (see, e.g., Shapley (1967)). In particular, no constructive proof is available, as opposed to the case with interim individual participation constraints, as mentioned above.

[17] Caillaud and Jehiel (1998) state that one can assume veto power to bidders without loss of generality. Since they define the individual rationality level of i as his minmax payoff, it indeed does not depend on the partitioning of $N \setminus \{i\}$. Nevertheless, in our model individual rationality levels are calculated in equilibrium and do depend on the partitioning of the others, as demonstrated by example 5.1.

$$\int_R^{\bar{t}} F(t)^{n-1}\,dt \le \int_R^{\bar{t}} (t-R)F(t)^{n-1}f(t)\,dt + \int_R^{\bar{t}} F(t)^n\,dt \tag{6.3}$$

Integrating in parts the first integral in the RHS of (6.3) yields the following equivalent inequality:

$$\int_R^{\bar{t}} F(t)^{n-1}\,dt \le \int_R^{\bar{t}} \frac{1}{n}(1+(n-1)F(t)^n)\,dt \tag{6.4}$$

which can be verified by looking at the difference between the integrand in the RHS of (6.4) and the integrand in the LHS of (6.4) as a function of the type in the interval $[\underline{t}, \bar{t}]$. This function is non-increasing and is equal to 0 in $\bar{t}$.

With respect to Caillaud and Jehiel (1998)'s condition, consider example 5.1. In terms of mechanism design, the coalitional equilibrium considered in the example yields a mechanism for the grand coalition which is ex-post efficient, BB and IC.[18] The fact that a single player (with merging expectations) prefers seceding at the ex-ante stage in this example means that the mechanism fails to be ex-ante IR.

As ex-ante IR is a weaker property than interim IR, such a mechanism would also fail to be interim IR given that an individual has merging expectations, in particular, without veto power. Nevertheless, it can be readily verified that condition (6.2) does hold in this example, which means that the grand coalition has an ex-post efficient, BB, IC and interim IR mechanism if individuals have veto power. We conclude that the veto power assumption is strong.

To complete the discussion we propose another example, where the grand coalition is core-stable while as the necessary and sufficient condition of Caillaud and Jehiel (1998) does not hold. The interpretation is that there exists an ex-post efficient mechanism for the grand coalition which is BB, IC and satisfies ex-ante group participation constraints (for groups with either merging or singleton expectations, i.e., veto power or not), yet a mechanism for the grand coalition which is ex-post efficient, BB and IC, fails to be interim IR.

[18]While we follow an ex-ante approach, we achieve ex-ante incentive efficiency in the grand coalition by constructing an ex-post efficient mechanism which is also incentive compatible (see also, e.g., Holmstrom and Myerson (1983), Mas-Colell et al. (1995)).

Example 6.1. Consider a 3-player symmetric market where the valuation that each player assigns to the good is distributed uniformly in the unit interval, $F \sim U[0,1]$. A second price auction is held in this market with a reserve price $R = \frac{3}{5}$. The externality on a non-winning agent is $e = -\frac{1}{4}$. Note that as $1 + 2e < R$ if the grand coalition forms it profitably chooses not to participate in the auction.[19] It is readily verified that the LHS of (6.2) is positive, therefore, condition (6.2) does not hold.

Once again, as $1 + 2e < R$, the grand coalition is not expected to participate in the auction, namely $v = 0$. It can be readily verified that $y, z < 0$ (see, (5.4) and (5.6)), hence, the m-core of this game is non-empty (see, (5.10)), which in turn yields that the s-core is not empty as well (see, footnote 13). Hence the grand coalition is ex-ante core stable with both merging or singleton expectations. $\square$

The latter example demonstrates the existence of an ex-post efficient, IC and BB mechanism which satisfies ex-ante group participation constraints while interim IR fails to be maintained. We find this result important for two reasons. First, the context of the mechanism design problem might be so that agents have to commit ex-ante, the interim stage being too late for collusion. Consider, for instance, the example of an art auction previously mentioned (see, e.g., section 1.1) where waiting for the art object to be available at the auction house for close examination by experts on behalf of the art collectors (i.e., interim stage) might be too adjacent to the auction, leaving no time for coalition formation negotiations. Another example, Porter and Zona (1999) study collusion in school milk procurements in Ohio where local diaries formed a bidding ring which repeatedly participated in the yearly school milk procurement. Forming a ring with the intention to participate in future auctions explicitly assumes the ability and willing of players to commit ex-ante. That is since the auction details are not yet published while collusion takes place, and therefore the players cannot calculate their cost if winning, and in particular they cannot calculate their valuations or types.

A second reason relates to the point of view of the colluding ring. As proved by Biran and Forges (2011) a coalition can strive to achieve a "first best" solution at the ex-ante stage, namely, a collusive mechanism which is ex-post efficient, and satisfies group participation constraints, which can be made IC by implementing an appropriate balanced transfer scheme, as

[19] The same example can be revisited with the parameters $R = \frac{1}{2}$ and $e = -\frac{1}{5}$ for which the grand coalition does participate in the auction.

introduced, for instance, by Groves (1973). It is well known that a transfer scheme of that kind usually fails from being IR at the interim stage, leaving the coalition with a "second best" solution.

Moreover, as already explained by Biran and Forges (2011), defining group participation constraints in the interim stage is difficult. The decision of a coalition to secede in the interim stage is a function of its "type". But what is the "type" of a coalition? How, and to what extent, do coalition members share their information, if at all?[20] Moreover, formulating interim blocking as a function of the conditional expected utility of a coalition given its "type" raises another difficulty. As opposed to the ex-ante blocking case, one can no longer use straight forward transfer payments between the members of the grand coalition in order to avoid all coalitions from blocking. Nevertheless, in example 6.1 an individual is blocking in the interim stage. Therefore, even if we could define group participation constraints in the interim stage, the grand coalition would be interim blocked, although, as demonstrated above, it would not be blocked in the ex-ante stage by both individuals and groups. We conclude that in this example the grand coalition would prefer colluding in the ex-ante stage.

7 Conclusion

We considered a second price auction in the presence of externalities, studying the question of the core stability of the grand coalition. We derived an auxiliary auction game between coalitions, or bidding rings, from the original non-collusive auction game, defining the valuation of a coalition as the highest valuation of its members net of the externalities of the coalition's members. We then proved the existence of a Nash equilibrium in the auxiliary game for any given partition of the bidders, computing it specifically in two concrete cases. The merging expectations case, where a seceding coalition conjectures that the others will form a complementary coalition. And the singleton expectations case, where a seceding coalition conjectures that the others will act individually.

Given these results, we showed that as opposed to the case without externalities where the grand coalition is always stable, the presence of external-

[20]Biran and Forges (2011) propose looking at the incentive compatible coarse-core (see, e.g., Vohra (1999)), where coalitions block on the basis of common knowledge events, i.e., communication between coalition members is reduced to a minimal level.

ities makes cooperation harder. In particular, we demonstrated an auction with three bidders, where a singleton with merging expectations prefers acting individually than joining the grand coalition, as early as in the ex-ante stage. It follows that the grand coalition is not necessarily stable in the presence of externalities.

Finally, we studied collusion in auctions using mechanism design tools. We showed that ex-ante and interim commitments are not logically dependent, insisting on the difficulty to define interim group participation constraints, and demonstrating an auction where the grand coalition would prefer to collude as early as in the ex-ante stage.

A Appendix: Participation threshold of a coalition facing individuals

Proof of proposition 4.2. Suppose first that all $i \notin S$ follow a bidding strategy with a participation threshold equals to R. Then S has a best response strategy with a threshold $t_S^* \in [R+e_S, R]$. Let us compute the interim utility of S of type t_S^* if it chooses to participate in the auction and if it chooses not to participate. If S makes an irrelevant bid, then it gets 0 if all $i \notin S$ do not participate as well, and e_S otherwise, which yields:

$$seP(\max_{i \notin S} \tilde{t}_i > R) \tag{A.1}$$

where $\tilde{t}_i$ denotes a random variable with distribution F.

If on the other hand S chooses to participate, we distinguish between three cases. Either all $i \notin S$ do not participate, in which case S wins the good for the reserve price. Either there is some $i \notin S$ who participates and wins, in which case S gets its externality. Or finally there is some $i \notin S$ who participates, however S wins paying the second highest bid $\max\limits_{i \notin S} \tilde{t}_i - e$. The interim utility of S is therefore:

$$\begin{aligned}(t_S^* - R)P(\max_{i \notin S} \tilde{t}_i \leq R) + seP(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e > t_S^* - se) \\ + E((t_S^* - (\max_{i \notin S} \tilde{t}_i - e))I(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e < t_S^* - se))\end{aligned} \tag{A.2}$$

In equilibrium S of type t_S^* is indifferent between participating or not. Suppose by way of contradiction that $t_S^* < R + (s-1)e$. Then $R =$

$\max\{R, t_S^* - (s-1)e\}$ and $I(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e < t_S^* - se) = 0, \forall (\tilde{t}_i)_{i \notin S}$. Hence,

$$seP(\max_{i \notin S} \tilde{t}_i > R) = (t_S^* - R)P(\max_{i \notin S} \tilde{t}_i \leq R) + seP(\max_{i \notin S} \tilde{t}_i > R) \tag{A.3}$$

or equivalently, $t_S^* = R$, which is a contradiction.

Suppose now WLOG that $1 \notin S$ and that S is following a bidding strategy with a participation threshold $t_S^* \in [R+(s-1)e, R]$, while all the individuals but 1 follow a strategy with a threshold R. There exists a best response strategy for 1 with threshold $t_1^* \in [R+e, R]$. Following the same analysis we conclude that in equilibrium:

$$\begin{aligned} eP(\max_{i \notin S \cup \{1\}} \tilde{t}_i > R \texttt{ or } \tilde{t}_S > t_S^*) = {} & (t_1^* - R)P(\max_{i \notin S \cup \{1\}} \tilde{t}_i \leq R \texttt{ and } \tilde{t}_S \leq t_S^*) \\ & + eP(\max_{i \notin S \cup \{1\}} \tilde{t}_i > \max\{R, t_1^*\} \\ & \quad \texttt{or } \tilde{t}_S > \max\{t_S^*, t_1^* + (s-1)e\}) \\ & + E((t_1^* - \max\{\max_{i \notin S \cup \{1\}} \tilde{t}_i - e, \tilde{t}_S - se\}) \\ & \quad I(\max_{i \notin S \cup \{1\}} \tilde{t}_i \in (R, t_1^*] \\ & \quad \texttt{or } \tilde{t}_S \in (t_S^*, t_1^* + (s-1)e]) \end{aligned} \tag{A.4}$$

It follows that $t_1^* = R$. $\square$

B Appendix: Simulations of core stability of the grand coalition with singleton expectations

We consider in this appendix singleton expectations, namely, a deviating coalition expects the others to act individually. We present several simulations of markets with three agents, showing that the grand coalition is core stable with singleton expectations, i.e., the s-core is non-empty. We note, however, that given example 5.1 where the grand coalition was proved to be unstable with merging expectation, the question of the stability of the grand coalition in a given market depends strongly on the conjecture of a seceding coalition on the partitioning of the remaining bidders.

We first recall, with respect to the notations in section 5, that the s-core is non-empty if and only if $\frac{y}{2} \leq \frac{v}{3}$. Namely, the per capita utility in a coalition of two if seceding is lower than the per capita utility in the grand coalition (see, (5.11)).

We consider symmetric markets where the valuation that each player assigns to the good is distributed uniformly in the unit interval, $F \sim U[0, 1]$. According to the parameters of the market, (i.e., the reservation price, R, and the externality term, e), we distinguish between two market types . The first market type maintains $R < 1 + 2e$, namely, a maximal type grand coalition participates in the auction as it can gain a positive utility by bidding the reserve price.

The second type maintains $R > 1 + 2e$, which means that for any type realization, the grand coalition, if it forms, does not participate in the auction. Nevertheless, we also demand $R < 1 + e$ to avoid a degenerated case, where a coalition of two players does not participate as well.[21]

We executed the following MATLAB simulation using the Symbolic Math Toolbox in order to calculate: v, y, z, x. (Correspondingly, the payoff of: The grand coalition; Coalition of two; Singleton facing a coalition of two; Singleton facing two individuals.)[22]

```
syms t;
v = @(e,R) int((t+2*e-R)*3*t^2,R-2*e,1);
% tstar is the participation threshold of a coalition of two
tstar = @(e,R) e+sqrt(R^2-2*e*R);
y = @(e,R) 2*e*(1-R)+int(t-t^3,tstar(e,R)-e,1);
z = @(e,R) (tstar(e,R)-e)^2*(tstar(e,R)-2*e-R)+e-
           (tstar(e,R)-e)^2*int(t,R,tstar(e,R)-e)+
           int(t^2-t^3,tstar(e,R)-e,1);
x = @(e,R) e*(1-R^2)+int(t^2-t^3,R,1);
```

The following table presents the simulation results in markets of the first type (i.e., the grand coalition may participate):

[21] Simulations ran in the complementary case, $1 + 2e > R > 1 + e$, also find the grand coalition core stable with singleton expectations.

[22] In the second market type, as the grand coalition does not participate, we simply set: v = 0;

R	e	v	y	z	x
0.01	-0.005	0.73	0.24	0.078	0.078
0.99	-0.001	$9.5 \cdot 10^{-5}$	$6 \cdot 10^{-5}$	$3.1 \cdot 10^{-5}$	$2.9 \cdot 10^{-5}$
0.33	-0.32	0.001	-0.31	-0.13	-0.21
0.5	-0.2	0.014	-0.124	-0.045	-0.093

It can be readily verified that all simulations maintain $\frac{y}{2} \leq \frac{v}{3}$, namely, the grand coalition is core stable with singleton expectations.

Simulation results of markets of the second type, i.e., the grand coalition does not participate, are presented in the following table:

R	e	v	y	z	x
0.34	-0.33	0	-0.33	-0.13	-0.22
0.99	-0.007	0	$-1.3 \cdot 10^{-4}$	$7.4 \cdot 10^{-6}$	$-9 \cdot 10^{-5}$
0.5	-0.49	0	-0.47	-0.03	-0.31
0.9	-0.25	0	-0.03	0.07	-0.04

Here also, all simulations maintain $\frac{y}{2} \leq \frac{v}{3}$. Note, that the last simulation corresponds to the market from example 5.1, where the grand coalition was not core stable with merging expectations.

References

[1] Bajari, P. (2001), "Comparing competition and collusion: a numerical approach", Economic Theory 18, 187-205.

[2] Bajari, P. and L. Ye (2003), "Deciding between competition and collusion", Review of Economics and Statistics 85, 971-989.

[3] Barbar, R. and F. Forges (2007), "Collusion dans les enchères: quelques apports des jeux coopératifs", Revue Economique 58, 965-984.

[4] Biran, O. (2009), "Strategic partial collusion in auctions with externalities", mimeo, Université Paris Dauphine.

[5] Biran, O. and F. Forges (2011), "Core-stable rings in auctions with independent private values", Games and Economic Behavior.

[6] Caillaud, B. and P. Jehiel (1998), "Collusion in auctions with externalities", RAND Journal of Economics 29, 680-702.

[7] Forges, F., J., F. Mertens and R. Vohra (2002), "The ex ante incentive compatible core in the absence of wealth effects", Econometrica 70, 1865-1892.

[8] Forges, F. and E. Minelli (2001), "A note on the incentive compatible core", Journal of Economic Theory 98, 179-188.

[9] Graham, D. A. and R., C. Marshall (1987), "Collusive bidder behavior at single-object second-price and English auctions", The Journal of Political Economy 95, 1217-1239.

[10] Groves, T. (1973), "Incentives in teams", Econometrica 41, 617631.

[11] Hafalir, I. (2007), "Efficiency in coalition games with externalities", Games and Economic Behavior 61, 242-258.

[12] Holmstrom, B. and R., B. Myerson (1983), "Efficient and durable decision rules with incomplete information", Econometrica 51, 1799-1819.

[13] Jehiel, P. and B. Moldovanu (1996), "Strategic nonparticipation", RAND Journal of Economics 27, 84-98.

[14] Jehiel, P. and B. Moldovanu (1999), "Resale markets and the assignment of property rights", Review of Economic Studies 66, 971-991.

[15] Jehiel, P., B. Moldovanu and E. Stacchetti (1996), "How (not) to sell nuclear weapons", American Economic Review 86, 814-829.

[16] Jehiel, P., B. Moldovanu and E. Stacchetti (1999), "Multidimensional mechanism design for auctions with externalities", Journal of Economic Theory 85, 258-293.

[17] Lucas, W. and R. Thrall (1963), "n-person games in partition function form", Naval Research Logistics Quarterly 10, 281-298.

[18] Mailath, G. J. and P. Zemsky (1991), "Collusion in second price auctions with heterogeneous bidders", Games and Economic Behavior 3, 467-486.

[19] Mas-Colell A., M., D. Whinston and J., R. Green (1995), "Microeconomic theory", New York, Oxford University press.

[20] Marshall, R. C. and L., M. Marx (2007), "Bidder collusion", Journal of Economic Theory 133, 374-402.

[21] Marshall, R., M. Meurer, J., F. Richard and W. Stromquist (1994), "Numerical analysis of asymmetric first price auctions", Games and Economic Behavior 7, 193-220.

[22] Maskin, E. (2003), "Bargaining, coalitions and externalities", Presidential Address to the Econometric Society, Institute for Advanced Study, Princeton University.

[23] Maskin, E. and J. Riley (2000), "Asymmetric auctions", Review of Economic Studies 67, 413-438.

[24] Myerson, R., B. and M., A. Satterthwaite (1983), "Efficient mechanisms for bilateral trading", Journal of Economic Theory 29, 265-281.

[25] McAfee, R., P. and J. McMillan (1992), "Bidding rings", The American Economic Review 82, 579-599.

[26] Porter, R. and D. Zona (1993), "Detection of bid rigging in procurement auctions", Journal of Political Economy 101, 518-538.

[27] Porter, R. and D. Zona (1999), "Ohio school milk markets: an analysis of bidding", RAND Journal of Economics 30, 263-288.

[28] Ray, D. (2007), "A game-theoretic perspective on coalition formation", Oxford University Press.

[29] Ray, D. and R. Vohra (1997), "Equilibrium binding agreements", Journal of Economic Theory 73, 30-78.

[30] Robinson, M., S. (1985), "Collusion and the choice of auction", The RAND Journal of Economics 16, 141-145.

[31] Shapley, L., S. (1967), "On balanced sets and cores", Naval Research Logistics Quarterly 14, 453-460.

[32] Vohra, R., (1999), "Incomplete information, incentive compatibility and the core", Journal of Economic Theory 86, 123-147.

[33] Waehrer, K. (1999), "Asymmetric auctions with application to joint bidding and mergers", International Journal of Industrial Organization 17, 437-452.