Liberal approaches to ranking infinite utility streams: When can we avoid interferences?

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12 July 2011

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Abstract

In this work we analyse social welfare relations on sets of infinite utility streams that verify various types of liberal non-interference principles. Earlier contributions have established that (finitely) anonymous and strongly Paretian quasiorderings exist that agree with axioms of that kind together with weak preference continuity and further consistency. Nevertheless Mariotti and Veneziani [12] prove that a fully liberal non-interfering view of a finite society leads to dictatorship if weak Pareto optimality is imposed. We first prove that extending the horizon to infinity produces a reversal of such impossibility result. Then we investigate a related problem: namely, the possibility of combining "standard" semicontinuity with efficiency in the presence of non-interference. We provide several impossibility results that prove that there is a generalised incompatibility between continuity and non-interference principles, both under ordinal and cardinal views of the problem. Our analysis ends with some insights on the property of representability in the presence of non-interference assumptions. In particular we prove that all social welfare functions that verify a very mild efficiency property must exert some interference (penalising both adverse and favorable changes) on the affairs of particular generations

JEL classification: D63; D71; D90.

Key words: Pareto axiom, Intergenerational justice, Social welfare relation, Non-interference, Continuity

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[1] I thank Roberto Veneziani for very detailed comments on a previous version of this paper. Errors are my own. This work has been supported by the Spanish Ministerio de Ciencia e Innovación under Project ECO2009-07682, and by Junta de Castilla y León under GR-99 Funding.

1 Introduction and motivation

In relation with the analysis of criteria for comparing allocations to a finite society, Hammond's [8] characterization of the leximin ordering is based on anonymity, the strong Pareto axiom, and a principle now called Hammond Equity. It has been recently proven that in the presence of anonymity and strong Pareto optimality, Hammond Equity is equivalent to a liberal non-interference property called *Harm Principle* (cf., Mariotti and Veneziani [14]). This alternative characterization of the leximin social ranking seems fairly surprising since the Harm Principle does not embody any egalitarian consideration while Hammond Equity is a strongly egalitarian property. Extensions of the analysis to the case of the leximax criterion and also to the case of infinitely-lived societies appear in Mariotti and Veneziani [12] and Lombardi and Veneziani [10,11]. They appeal to a 'dual' of the finite- or infinite-dimensional versions of the Harm Principle, namely, the *Individual Benefit Principle*. In particular, preference continuities permit to characterize infinite extensions of the leximin criterion both on the basis of Hammond Equity (cf., Asheim and Tungodden [3], Bossert et al. [6]) and of adapted versions of the Harm Principle. Nevertheless, [11] shows that in the evaluation of infinitely long streams by orderings, anonymity, the strong Pareto axiom, and preference continuity properties are incompatible with full non-interference. Restricting ourselves to a finite economy, Mariotti and Veneziani [12] prove that a fully liberal non-interfering view of the society –incorporating both the Harm Principle and the Individual Benefit Principle– leads to dictatorship if weak Pareto optimality is imposed.

In this paper we first prove that extending the horizon to infinity produces a reversal of the latter impossibility result (cf., Section 3). Afterwards we explore the consequences of adding *standard* continuity properties to non-interference (cf., Section 4). Our main interest lies on the case of infinite utility streams but we also state some paralel implications for the case of finitely-lived societies. The results above inform us of trivial incompatibilities that derive from lack of continuity of the leximin/leximax criteria. We investigate more accurate reasons for the conflict among non-interference, optimality, the equal treatment of the generations and continuity in the evaluation of infinity utility streams. Then we elaborate on less demanding views of non-interference that scarcely provide some routes of escape to the generalised impossibilities that arise.

In Section 5 we complement the analysis with some insights on the property of representability in the presence of non-interference assumptions. Prior constructions of weakly dominant and anonymous social welfare functions implement a cardinal view of non-interference. Despite that, we prove a rather extreme impossibility result: Representable social welfare relations that verify a relaxed version of the weak dominance axiom must violate *both* the Harm

Principle *and* the Individual Benefit Principle. We end up our analysis of social welfare functions by proving that like in the case of social welfare orderings, sufficient efficiency can be combined with *either* the Harm Principle and equity *or* the Individual Benefit Principle by avoiding any dictatorship (by the present and by the future).

2 Notation and axioms

A *social welfare relation* (SWR) is a binary relation $\succcurlyeq$ on $\mathbf{X} \subseteq \mathbb{R}^n$ with $n \in \mathbb{N} \cup \{+\infty\}$. Unless we state otherwise, it is assumed that it is reflexive. Its asymmetric factor is denoted by $\succ$ (i.e., $\mathbf{x} \succ \mathbf{y}$ iff $\mathbf{x} \succcurlyeq \mathbf{y}$ but not $\mathbf{y} \succcurlyeq \mathbf{x}$), and its symmetric factor is denoted by $\sim$ (i.e., $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{x} \succcurlyeq \mathbf{y}$ and $\mathbf{y} \succcurlyeq \mathbf{x}$). If $\succcurlyeq$ is an ordering (i.e., complete and transitive) then we call it a social welfare ordering or SWO.

When $\mathbf{X}$ denotes a subset of $\mathbb{R}^{\mathbb{N}}$, it represents a domain of utility sequences or infinite-horizon utility streams and we adopt the usual notation for such context: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. Besides, by $(y)_{con}$ we mean the constant sequence $(y, y, ...)$, and $(x_1, ..., x_k, (y)_{con}) = (x_1, ..., x_k, y, y, ...)$ denotes an eventually constant sequence. Denote by ${}_1\mathbf{x}_{H-1} = (x_1, ..., x_{H-1})$ the H-head of $\mathbf{x} \in \mathbf{X}$, and denote by ${}_T\mathbf{x} = (x_T, x_{T+1}, ...)$ its T-tail, thus $\mathbf{x} = {}_1\mathbf{x} = ({}_1\mathbf{x}_{n-1}, {}_n\mathbf{x})$ for each $n \in \mathbb{N}$. When the intergenerational terminology is adopted, the first component or generation is often called the present.

We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

We are concerned with axioms of different nature for SWRs. We state them for $\mathbf{X} = [0, 1]^{\mathbb{N}}$ but they can be easily regarded as axioms on $\mathbf{X} = [0, 1]^n$ with $n \in \mathbb{N}$ too, as is dutifully clarified along the exposition when needed. These are the settings that we examine in the following Sections.

Firstly we introduce equity axioms of two different classes for a SWR $\succcurlyeq$ on $\mathbf{X} = [0, 1]^{\mathbb{N}}$. Anonymity is the usual "equal treatment of all generations" postulate à-la-Sidgwick and Diamond.

Axiom AN (*Anonymity*). Any finite permutation of a utility stream produces a socially indifferent utility stream.

We now recall a consequentialist equity axiom that implements preference for egalitarian allocations of utilities among generations in various senses. Axiom HE below states that in case of a conflict between two generations, every other generation being as well off, the stream where the least favoured generation is better off must be weakly preferred.

Axiom HE (*Hammond Equity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{y} \succcurlyeq \mathbf{x}$.

Further we are concerned with the following axiom that was introduced in Asheim and Tungodden [3].

Axiom HEF (*Hammond Equity for the Future*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$, $\mathbf{y} = (y_1, (y)_{con})$ and $x_1 > y_1 > y > x$, then $\mathbf{y} \succcurlyeq \mathbf{x}$.

HEF states the following ethical restriction on the ranking of streams where the level of utility is constant from the second period on and the present generation is better-off than the future: If the sacrifice by the present conveys a higher utility for all future generations, then such trade off is weakly preferred.

In a different vein, Mariotti and Veneziani [12,14] introduce non-interference conditions in the context of a finite society. Under additional requirements they are intimately related to HE (cf., Mariotti and Veneziani [14, p. 127]). We proceed to recall their infinite counterparts, which are extensively analyzed in Lombardi and Veneziani [10,11]. Their respective versions for finite-length streams are the same except in that the restriction of the thesis to eventually coincident vectors does not apply.

Axiom HP (*Harm Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}' \in \mathbf{X}$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i < x_i$ and $y'_i < y_i$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

In case that only $\mathbf{x}' \succcurlyeq \mathbf{y}'$ is ensured in the definition above, we speak of Weak Harm Principle.

Axiom IBP (*Individual Benefit Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}' \in \mathbf{X}$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i > x_i$ and $y'_i > y_i$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

In case that only $\mathbf{x}' \succcurlyeq \mathbf{y}'$ is ensured in the definition above, we speak of Weak Individual Benefit Principle.

Quoting from [12], the core of these non-interference principles is the following idea: changes in one generation's welfare that leave all other generations unaffected should not be a motive for penalising that generation in the social judgement, whether the change involves a damage (HP) or a benefit (IBP) for it. A penalisation means a switch against the interest of that generation in society's strict ranking on distributions (with respect to the ranking of the original distributions).

We intend to account for some kind of efficiency too. Various axioms capture the general principle that with respect to a given infinite utility stream, adequate changes must produce socially better streams if every generation is at least as well off after the change. The *Weak Dominance* axiom captures the following spirit: Improving the welfare of *exactly one* generation suffices to produce a socially better stream. In turn, the *Weak Pareto* axiom requests that *all* generations increase their utility in order to obtain a socially better stream. The *Strong Pareto* axiom imposes that if *at least one* generation increases its utility then the resulting stream is socially better thus Strong Pareto and Weak Dominance coincide over sets of finite-length vectors. Formally:

Axiom WD (*Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{x} \succ \mathbf{y}$.

For weakly dominant SWOs on $\mathbf{X} = [0,1]^{\mathbb{N}}$, the Harm Principle (resp., the Individual Benefit Principle) and the Weak Harm Principle (resp., the Weak Individual Benefit Principle) coincide.

Axiom WP (*Weak Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom SP (*Strong Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

Another relaxed form of Strong Pareto that is unrelated to either WP or WD is the uncontroversial Monotonicity.

Axiom M (*Monotonicity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \geqslant \mathbf{y}$ then $\mathbf{x} \succcurlyeq \mathbf{y}$.

Observe that SWOs that verify M and WD are SP.

Finally, we list some semicontinuity properties. Below we discuss how they adapt to the case $\mathbf{X} \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$. For a reflexive binary relation $\succcurlyeq$ on $\mathbf{X} \subseteq \mathbb{R}^{\mathbb{N}}$, the following definitions apply:

Axiom RUSC (*Restricted upper semicontinuity with respect to the sup topology*). For each $\mathbf{x} \in \mathbf{X}$ eventually constant, $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succcurlyeq \mathbf{x}\}$ is closed with respect to the sup topology.

Axiom RLSC (*Restricted lower semicontinuity with respect to the sup topology*). For each $\mathbf{x} \in \mathbf{X}$ eventually constant, $\{\mathbf{y} \in \mathbf{X} : \mathbf{x} \succcurlyeq \mathbf{y}\}$ is closed with respect to the sup topology.

In general, the sup topology is finer than the product topology but when $\mathbf{X} \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$, both topologies coincide with the Euclidean topology. Also in this context, RUSC/RLSC are the ordinary USC/LSC (upper/lower semicontinuity with respect to the sup topology).

3 The possibility of non-interference with an infinite horizon

In the context of SWOs on allocations to a finite society, there are linear rankings that verify SP, HP and IBP (e.g., lexicographic orders) but all SWOs that verify WP, HP and IBP are dictatorial by a generation (cf., Mariotti and Veneziani [12, Theorem 1]). For example, lexicographic orders are dictatorial by the first generation. In this Section we prove that extending the horizon to infinity reverses the situation. We appeal to the following non-dictatorship axioms (cf., Chichilnisky [7], also Sakai [15], Asheim et al. [2]) that impose that the comparisons between pairs of streams do not depend on the welfare levels of present (resp., future) generations only.

Axiom NDP (*Non-Dictatorship of the Present*). The following is not true: If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} \succ \mathbf{y}$, there is $i \in \mathbb{N}$ for which $j \geqslant i$ and $\mathbf{z}, \mathbf{w} \in \mathbf{X}$ imply $({}_1x_j,\, {}_{j+1}z) \succ ({}_1y_j, {}_{j+1}\, w)$.

Axiom NDF (*Non-Dictatorship of the Future*). The following is not true: If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} \succ \mathbf{y}$, there is $i \in \mathbb{N}$ for which $j \geqslant i$ and $\mathbf{z}, \mathbf{w} \in \mathbf{X}$ imply $({}_1z_j,\, {}_{j+1}x) \succ ({}_1w_j,\, {}_{j+1}y)$.

Theorem 1 *There are SWOs on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ that verify M, WP, HEF, HP, IBP, NDP, and NDF.*

Proof: We define the following binary relation $\succcurlyeq$ on $\mathbf{X}$: $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if either $\liminf_n(x_n) > \liminf_n(y_n)$ or ($\liminf_n(x_n) = \liminf_n(y_n)$ and $x_1 \geqslant y_1$). This is a lexicographic composition of a long-run criterion and dictatorship of the present. Thus it is routine to check that $\succcurlyeq$ is a complete preorder. Its asymmetric part is defined by: $\mathbf{x} \succ \mathbf{y}$ if and only if either $\liminf_n(x_n) > \liminf_n(y_n)$ or ($\liminf_n(x_n) = \liminf_n(y_n)$ and $x_1 > y_1$).

In order to prove M, take $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $\mathbf{x} \geqslant \mathbf{y}$ thus $\liminf_n(x_n) \geqslant \liminf_n(y_n)$. If $\liminf_n(x_n) > \liminf_n(y_n)$ we obtain $\mathbf{x} \succ \mathbf{y}$. If $\liminf_n(x_n) = \liminf_n(y_n)$ then the fact that $x_1 \geqslant y_1$ yields $\mathbf{x} \succcurlyeq \mathbf{y}$.

In order to prove WP, take $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $\mathbf{x} \gg \mathbf{y}$ thus $\liminf_n(x_n) \geqslant \liminf_n(y_n)$. We proceed as above to check $\mathbf{x} \succ \mathbf{y}$.

The proof that $\succcurlyeq$ verifies a reinforced version of HEF is direct: If $\mathbf{x} =$

$(x_1,(x)_{con})$, $\mathbf{y}=(y_1,(y)_{con})$ and $x_1>y_1>y>x$ then $\mathbf{y}\succ\mathbf{x}$ because $\liminf_n(y_n)=y>x=\liminf_n(x_n)$.

Let us now prove HP. Suppose $\mathbf{x},\mathbf{y}\in\mathbf{X}$ are eventually coincident with $\mathbf{x}\succ\mathbf{y}$, and consider $\mathbf{x}',\mathbf{y}'\in\mathbf{X}$ such that: for some $i\in\mathbb{N}$, $j\neq i$ implies $x'_j=x_j$ and $y'_j=y_j$. Suppose further $x'_i<x_i$, $y'_i<y_i$ and $y'_i<x'_i$. Because $\liminf_n(x'_n)=\liminf_n(x_n)$ and $\liminf_n(y'_n)=\liminf_n(y_n)$, in case that $\liminf_n(x_n)>\liminf_n(y_n)$ we directly derive $\mathbf{x}'\succ\mathbf{y}'$. In case that $\liminf_n(x_n)=\liminf_n(y_n)$ and $x_1>y_1$, both when $i=1$ and when $i\neq 1$ we also deduce $\mathbf{x}'\succ\mathbf{y}'$.

The proof that $\succcurlyeq$ verifies IBP is analogous to the argument for HP.

In order to check that there is no dictatorship of the present, observe that if $\mathbf{x}=(1_{con}),\mathbf{y}=(0_{con})$ it is true that $\mathbf{x}\succ\mathbf{y}$ but for each $i\in\mathbb{N}$ and $j\geqslant i$, if $\mathbf{z}=(0_{con}),\mathbf{w}=(1_{con})$ one has $({}_1y_{j},{}_{j+1}w)\succ({}_1x_{j},\ {}_{j+1}z)$.

Finally, in order to check that there is no dictatorship of the future observe that if $\mathbf{x}=(1_{con}),\mathbf{y}=(0,1_{con})$ it is true that $\mathbf{x}\succ\mathbf{y}$ but for each $i>1$ and $j\geqslant i$, if $\mathbf{z}=(0_{con}),\mathbf{w}=(1_{con})$ one has $({}_1w_{j},\ {}_{j+1}y)\succ({}_1z_{j},\ {}_{j+1}x)$. $\square$

Remark 1 *We have fixed an exact expression in order to simplify the proof of Theorem 1. The reader can check that the generation that is looked upon when the first criterion is not decisive can be chosen in a random manner. To be precise: take any map* $\nu:[0,1]\longrightarrow\mathbb{N}$ *and define* $u_\nu:\boldsymbol{X}\longrightarrow[0,1]^2$ *according to* $u_\nu(\boldsymbol{x})=(\liminf_n(x_n),x_{\nu(\liminf_n(x_n))})$. *If we now define a binary relation* $\succcurlyeq_\nu$ *on* $\boldsymbol{X}$ *by the expression:* $\boldsymbol{x}\succcurlyeq_\nu\boldsymbol{y}$ *if and only if* $u_\nu(\boldsymbol{x})$ *lexicographically beats* $u_\nu(\boldsymbol{y})$, *then a straightforward modification of the argument proves that* $\succcurlyeq_\nu$ *verifies the thesis of Theorem 1. Obviously, in the proof above* ν *is constantly* 1.

Remark 2 *Let us define Restricted Weak Pareto (RWP) as* $\boldsymbol{x}\succ\boldsymbol{y}$ *when* $\boldsymbol{x}\gg\boldsymbol{y}$ *and* both $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are eventually constant (cf., Asheim et al. [2]). The limit inferior proves that with respect to the assumptions of Theorem 1, possibility remains when AN and representability are imposed at the cost of NDF and of relaxing WP to RWP. That is to say: There are SWFs that verify M, RWP, HEF, HP, IBP, NDP, and AN.*

4 Impossibility results for semicontinuous relations

In this Section we are interested in topological semicontinuity. We first produce various impossibility results for SWRs with non-interference properties on $\mathbf{X} = [0,1]^n$, a setting where SP and WD coincide. Afterwards we show that they naturally translate into results on $\mathbf{X} = [0,1]^{\mathbb{N}}$. A cardinal variant of the analysis completes this Section.

4.1 The case of a finite society

In this context it is known that AN, SP, and IBP (resp., either HE or HP) characterize the extensions of the leximax (resp., leximin): cf., Mariotti and Veneziani [12, Proposition 2, 3]. Let us recall the definitions of these orderings. For any $x \in \mathbb{R}^n$, let $\bar{x}$ denote the permutation of x whose components $\bar{x}_1, .., \bar{x}_n$ are ranked in ascending order. The leximin ordering $\succsim^{LM}$ is defined by: $x \succ^{LM} y$ if and only if either $\bar{x}_1 > \bar{y}_1$ or there exists $l > 1$ such that $\bar{x}_1 = \bar{y}_1$, ... , $\bar{x}_{l-1} = \bar{y}_{l-1}$, $\bar{x}_l > \bar{y}_l$. The leximax ordering $\succsim^{LX}$ is defined by: $x \succ^{LX} y$ if and only if either $\bar{x}_n > \bar{y}_n$ or there exists $l < n$ such that $\bar{x}_n = \bar{y}_n$, ... , $\bar{x}_{l+1} = \bar{y}_{l+1}$, $\bar{x}_l > \bar{y}_l$.

It is now trivial that AN, SP, HP, and IBP are incompatible properties for a SWO on $\mathbf{X} = [0,1]^n$ when $n > 1$. Relaxing SP to WP produces incompatibility too (WP, HP and IBP together entail dictatorship by an agent, which violates AN), but dropping either HP or IBP instead produces compatibility. Any dictatorship by an agent proves that the incompatibility among AN, SP, HP, and IBP is avoided if AN is dropped *and* SP is relaxed to M plus WP.

Since the extensions of the leximax (resp., leximin) do not verify lower (resp., upper) semicontinuity with respect to the sup topology, trivial impossibility consequences follow. [2] To be precise: No SWO on $\mathbf{X} = [0,1]^n$ verifies AN, SP, HP (res., IBP), and USC (resp., LSC). In this Subsection we clarify the extent of the conflict among non-interference principles, Pareto optimality, and semicontinuity by proving that (a) AN plays no role in such incompatibilities, and (b) if an extremely mild technical condition replaces WD/SP then we still obtain conflicting axiomatics.

[2] Consider the case of the leximax. For each $i \in \mathbb{N}$ let $\mathbf{y}^{(i)} = (1 - \frac{1}{i}, \frac{1}{2})$. With respect to the sup topology, $\mathbf{y}^{(i)}$ converges to $\mathbf{y} = (1, \frac{1}{2})$. However $(1,0) \succ^{LX} \mathbf{x}^{(i)}$ and $\mathbf{y} \succ^{LX} (1,0)$.
Now consider the case of the leximin. For each $i \in \mathbb{N}$ let $\mathbf{y}^{(i)} = (\frac{1}{i}, \frac{1}{2})$. With respect to the sup topology, $\mathbf{y}^{(i)}$ converges to $\mathbf{y} = (0, \frac{1}{2})$. However $\mathbf{y}^{(i)} \succ^{LM} (0,1)$ and $(0,1) \succ^{LM} \mathbf{x}$.

Regarding our first purpose, the following Propositions 1 and 2 are in order:

Proposition 1 *There is no complete SWR* $\succcurlyeq$ *on* $\boldsymbol{X} = [0,1]^n$, $n \in \{2,3,....\}$, *that verifies IBP, WD, and LSC.*

Proof: We prove that the combination of properties in the statement conveys an absurd conclusion. Let us first show $(0,1,0,...,0) \succcurlyeq (\frac{1}{2}, 1-\frac{1}{i}, 0,...,0)$ for each $i = 2,3,....$ Suppose the opposite, thus $(\frac{1}{2}, 1-\frac{1}{i_0}, 0,...,0) \succ (0,1,0,...,0)$ for some $i_0 \in \{2,3,...\}$. An appeal to IBP yields $(1, 1-\frac{1}{i_0}, 0,...,0) \succ (1-\frac{1}{m}, 1, 0,...,0)$ for each $m = 2,3,....$ Now LSC entails $(1, 1-\frac{1}{i_0}, 0,...,0) \succcurlyeq (1,1,0,...,0)$, contradicting WD.

With respect to the sup topology, $\{(\frac{1}{2}, 1-\frac{1}{i}, 0,...,0)\}_i$ converges to $(\frac{1}{2},1,0,...,0)$ thus LSC entails $(0,1,0,...,0) \succcurlyeq (\frac{1}{2},1,0,...,0)$, contradicting WD. $\square$

Proposition 2 *There is no complete SWR* $\succcurlyeq$ *on* $\boldsymbol{X} = [0,1]^n$, $n \in \{2,3,....\}$, *that verifies HP, WD, and USC.*

Proof: We prove that the combination of properties in the statement conveys an absurd conclusion. Let us first show $(\frac{1}{i}, \frac{1}{2}, 0,...,0) \succcurlyeq (0,1,0,...,0)$ for each $i = 2,3,....$ Suppose the opposite, thus $(0,1,0,...,0) \succ (\frac{1}{i_0}, \frac{1}{2}, 0,...,0)$ for some $i_0 \in \{2,3,...\}$. An appeal to HP yields $(0, \frac{1}{m}, 0,...,0) \succ (\frac{1}{i_0}, 0, 0,...,0)$ for each $m = 2,3,....$ Now USC entails $(0,0,0,...,0) \succcurlyeq (\frac{1}{i_0}, 0,0,...,0)$, contradicting WD.

With respect to the sup topology, $\{(\frac{1}{i}, \frac{1}{2}, 0,...,0)\}_i$ converges to $(0,\frac{1}{2},0,...,0)$ thus USC entails $(0,\frac{1}{2},0,...,0) \succcurlyeq (0,1,0,...,0)$, contradicting WD. $\square$

Regarding objective (b), we preliminarily explore the intimate relationship between the Harm Principle and Hammond Equity. This reveals another conflict between HP and USC, which bears comparison with the conclusion in Proposition 2.

Mariotti and Veneziani [13, Prop. 3] proved that when $\mathbf{X} = \mathbb{R}^2$, WD/SP and HE imply HP. These authors also proved that when $\mathbf{X} = [0,1]^n$, $n > 1$, HP and HE are equivalent in the presence of AN and WD/SP (cf., [14, p. 127]). Proposition 3 below shows that it is possible to deduce the egalitarian HE from HP if the generations are treated equally. The argument is exported to the case of infinitely-lived societies in subsection 4.2 below.

Proposition 3 *Let* $\succcurlyeq$ *be a SWO on* $\boldsymbol{X} = [0,1]^n$ *for some* $n \in \{2,3,....\}$. *If* $\succcurlyeq$ *verifies AN and HP then* $\succcurlyeq$ *verifies HE.* [3]

[3] This result is due to F. Maniquet, as has been communicated to the author by R. Veneziani.

Proof: Suppose the SWO $\succcurlyeq$ verifies AN and HP but not HE. Rejecting HE in the presence of AN ensures that there exist $x_2 > y_2 > y_1 > x_1$ such that $\mathbf{x} = (x_1, x_2, x_3, ..., x_n) \succ \mathbf{y} = (y_1, y_2, x_3, ..., x_n)$, $\mathbf{x}, \mathbf{y} \in [0,1]^n$. By reflexivity and AN, $(x_1, x_2, x_3, ..., x_n) \sim (x_2, x_1, x_3, ..., x_n)$. By transitivity, $(x_2, x_1, x_3, ..., x_n) \succ (y_1, y_2, x_3, ..., x_n)$ but now the HP assures $(y_2, x_1, x_3, ..., x_n) \succ (x_1, y_2, x_3, ..., x_n)$, violating AN. $\square$

Given Proposition 3, it is possible to use the following Proposition 4 to derive the subsequent Corollary 1 below:

Proposition 4 *Let $\succcurlyeq$ be a SWR on $\boldsymbol{X} = [0,1]^n$ for some $n \in \{2, 3, ...\}$. Suppose*

$$\exists\, \mathbf{x} \in \boldsymbol{X} \text{ such that } \mathbf{x} \not\succcurlyeq \mathbf{y} = (y_1, x_2, ..., x_n) \text{ and } y_1 > x_1 > x_2 \qquad (1)$$

Then $\succcurlyeq$ does not verify HE and RUSC simultaneously.[4]

Proof: By contradiction. Define $\mathbf{y}^{(k)}$ according to: $y_i^{(k)} = x_i$ if $i = 1, 3, 4, ..., n$, $y_2^{(k)} = x_2 + \frac{1}{k}$. With respect to the sup topology, $\mathbf{y}^{(k)}$ converges to $\mathbf{x}$. For each $k > \frac{1}{x_1 - x_2}$, HE entails $\mathbf{y}^{(k)} \succcurlyeq \mathbf{y}$ because $y_1 > x_1 = y_1^{(k)} > x_2 + \frac{1}{k} = y_2^{(k)} > x_2$. This means $\mathbf{x} \succcurlyeq \mathbf{y}$ due to RUSC, contradicting the assumption. $\square$

Corollary 1 *There is no SWO $\succcurlyeq$ on $\boldsymbol{X} = [0,1]^n$, $n \in \{2, 3, ...\}$, that verifies AN, HP, RUSC and condition (1) above.*

Proof: By Proposition 3, $\succcurlyeq$ verifies HE. Now Proposition 4 applies. $\square$

Our last result in this regard replicates the incompatibility shown by Corollary 1 in terms of IBP.

Proposition 5 *There is no SWO $\succcurlyeq$ on $\boldsymbol{X} = [0,1]^n$, $n \in \{2, 3, ...\}$, that verifies AN, IBP, RLSC and the following condition (2):*

$$\exists\, \mathbf{x} \in \boldsymbol{X} \text{ such that there are } x_2 > x_1 > y_1 \text{ with } \mathbf{x} \succ \mathbf{y} = (y_1, x_2, ..., x_n) \qquad (2)$$

Proof: By contradiction. Define $\mathbf{x}^{(k)} \in \mathbf{X}$ according to: $x_i^{(k)} = x_i$ if $i = 1, 3, 4, ..., n$, $x_2^{(k)} = x_2 - \frac{1}{k}$. With respect to the sup topology, $\mathbf{x}^{(k)}$ converges to $\mathbf{x}$. Thus there is k_0 such that $\mathbf{x}^{(k)} \succ \mathbf{y}$ when $k > k_0$, due to RLSC. Select $m > k_0$ such that $x_2 - \frac{1}{m} > y_1$. Therefore $(x_1, x_2 - \frac{1}{m}, x_3, ..., x_n) \succ (y_1, x_2, x_3, ..., x_n)$.

By reflexivity and AN, $(x_2 - \frac{1}{m}, x_1, x_3, ..., x_n) \sim (x_1, x_2 - \frac{1}{m}, x_3, ..., x_n)$ therefore $(x_2 - \frac{1}{m}, x_1, x_3, ..., x_n) \succ (y_1, x_2, x_3, ..., x_n)$. Now an appeal to IBP yields

[4] Observe that condition (1) holds under e.g., WD/SP. Also, recall that RUSC and USC (resp., RLSC and LSC) coincide in this setting.

$(x_2, x_1, x_3, ..., x_n) \succ (x_1, x_2, x_3, ..., x_n)$, contradicting AN. $\square$

Remark 3 *The reader is invited to mimick the proof of Proposition 5 in order to give a direct argument for Corollary 1 that circumvents Propositions 3 and 4. And alternatively, it is possible to mimick the proof of Proposition 3 in order to prove that AN and IBP entail a 'dual' of HE.* [5] *From such implication, Proposition 5 follows easily too.*

4.2 The case of an infinitely-lived society

Most of the arguments in the preceding subsection carry forward to the case of infinite sequences of utilities. In this subsection we discuss the details.

Lombardi and Veneziani [11, Theorem 5] take advantage of their characterizations of the standard leximin/leximax relations that compare infinite streams, in order to prove that there is no weakly complete SWR that verifies AN, SP, a Strong Preference Continuity axiom, HP, and IBP. Here we complement the analysis by appealing to usual continuity instead.

Firstly we study if semicontinuity imposes restrictions to non-interference in the presence of efficiency. In order to convert Propositions 1 and 2 into statements for infinitely-lived societies, neither the completeness axiom nor WD are needed in full capacity. We just need to refer to relations that are able to deal with streams with the same tail thus it suffices to appeal to their following respective versions:

Axiom MC (*Minimal Completeness*). [6] If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, there is $T > 1$ such that $({}_1\mathbf{x}_{T}, {}_{T+1}\mathbf{y}) \neq \mathbf{y} \Rightarrow ({}_1\mathbf{x}_{T}, {}_{T+1}\mathbf{y}) \succcurlyeq \mathbf{y}$ or $\mathbf{y} \succcurlyeq ({}_1\mathbf{x}_{T}, {}_{T+1}\mathbf{y})$.

Axiom RWD (*Restricted Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually constant, and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{x} \succ \mathbf{y}$.

With respect to RWD and its reinforcements, a reduction to the case of Propositions 1 and 2 yields Proposition 6 below:

Proposition 6 *There is no reflexive SWR on* $\boldsymbol{X} = [0, 1]^{\mathbb{N}}$ *that verifies MC, RWD, IBP (res., HP), and RLSC (resp., RUSC).*

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[5] Such dual property is sometimes used to characterise the leximax. See, e.g., d'Aspremont [4, pp. 56-57].

[6] See Lombardi and Veneziani [10, Section 4.1] for a prior use of this axiom.

Proof: Suppose $\succcurlyeq$ verifies MC, RWD, IBP (res., HP), and RLSC (resp., RUSC). Define a binary relation R on $[0,1]^2$ according to: $(x,y)R(x',y')$ iff $(x,y,0_{con}) \succcurlyeq (x',y',0_{con})$. It is straightforward to check that it is complete because $\succcurlyeq$ is reflexive and MC, and that it verifies IBP (res., HP), WD, and RLSC (resp., RUSC). This contradicts Proposition 1 (resp., Proposition 2). $\square$

As has been said, the formal incompatibility between HP-IBP and efficiency stronger than RWD is not shared by WP, since $\mathbf{W}(\mathbf{x}) = x_i$ produces a M, WP, HEF (if $i > 1$), HP, IBP, representable and continuous w.r.t. the sup topology (but dictatorial) evaluation. Dictatorship might be avoided by imposing the equal treatment of the generations. This leads us to the question if non-interference is possible under AN. We proceed to check that the answer is negative: The reader can easily borrow the arguments from Proposition 3 to Proposition 5 in order to produce the following twin statements for infinitely-lived societies. [7]

Proposition 7 *Let $\succcurlyeq$ be a SWO on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$. If $\succcurlyeq$ verifies AN and HP then $\succcurlyeq$ verifies HE.*

Proposition 8 *There is no SWO $\succcurlyeq$ on $\boldsymbol{X}$ satisfying either of the following sets of conditions:*

(a) AN, HP, RUSC, and condition $(1')$ below:

$\exists\, \mathbf{x} \in \boldsymbol{X}$ eventually constant such that there are $y_1 > x_1 > x_2$ for which $\mathbf{y} = (y_1,\, {}_2x) \succ \mathbf{x}$ $(1')$, or

(b) AN, IBP, RLSC, and condition $(2')$ below:

$\exists\, \mathbf{x} \in \boldsymbol{X}$ eventually constant such that there are $x_2 > x_1 > y_1$ for which $\mathbf{x} \succ \mathbf{y} = (y_1,\, {}_2x)$ $(2')$

With respect to Proposition 6, Proposition 8 brings to light an incompatibility under a technical condition (a very mild germ of RWD) when AN and further consistency are imposed.

4.3 Revisiting the analysis under a cardinal perspective

In order to explore some routes of escape to the generalized impossibilities that stem from semicontinuity, we now check for possible changes in the analysis

[7] Proposition 8 is reexplored in Subsection 4.3 below.

above when well-beings are universally comparable and cardinally measurable. We consider the following cardinal forms of the non-interference principles whose implications we have inspected thus far:

Axiom IEHP (*Individual Equal Harm Principle*). Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i = x_i - \varepsilon$ and $y'_i = y_i - \varepsilon$ for some $\varepsilon > 0$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

This axiom is a direct descendant of the Harm Principle thus it captures a related liberal spirit. A similar defense holds for the other side of the coin:

Axiom IEBP (*Individual Equal Benefit Principle*). Antecedent as in IEHP, thesis as follows: If $x'_i = x_i + \varepsilon$ and $y'_i = y_i + \varepsilon$ for some $\varepsilon > 0$ then $x'_i > y'_i$ implies $\mathbf{x}' \succ \mathbf{y}'$.

The respective versions for finite-length streams are the same except that the restriction of the conclusion to eventually coincident vectors does not apply.

Remark 4 *We do not need to explore the context of a finite society in depth because to the effect of comparing the ordinal and cardinal positions, summing up the components is a WD/SP, AN, IEHP, IEBP, continuous with respect to the sup topology evaluation. In fact Mariotti and Veneziani [12] state a property in line with the conjunction of adapted versions of IEHP and IEBP, namely, Uniform Additive Non-Interference. Then they prove that SWOs that verify SP, AN, and Uniform Additive Non-Interference only deviate from the utilitarian ordering in comparisons between indifferent elements for the utilitarian rule.*

Let us therefore focus on infinitely-lived societies.

1) Proposition 9 below proves that if a SWO is AN and IEHP then it verifies the following Weak Pigou-Dalton transfer principle (cf., Hara et al. [9, p. 185]).

Axiom WPDT (*Weak Pigou-Dalton transfer principle*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that there is $\varepsilon > 0$ with $x_j = y_j + \varepsilon > y_j \geqslant y_k > x_k = y_k - \varepsilon$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{y} \succcurlyeq \mathbf{x}$.

Proposition 9 *Let $\succcurlyeq$ be a SWO on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$. If $\succcurlyeq$ verifies AN and IEHP then $\succcurlyeq$ verifies WPDT.*

Proof: Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ be such that there is $\varepsilon > 0$ with $x_j = y_j + \varepsilon > y_j \geqslant y_k > x_k = y_k - \varepsilon$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. By contradiction, assume $\mathbf{x} \succ \mathbf{y}$. Due to AN we can fix $j = 1$, $k = 2$ thus ${}_3x = {}_3y$, and we also get $\mathbf{x} \succ (y_2, y_1, {}_3x)$.

Consider the vectors $\mathbf{x}' = (x_1 - \varepsilon,\, {}_2x)$ and $\mathbf{y}' = (y_2 - \varepsilon,\, y_1,\, {}_3x)$. They are obtained from $\mathbf{x}$ and $(y_2,\, y_1,\, {}_3x)$ by reducing the endowment of their respective presents by ε. Since $x_1 > y_2$ by assumption, we obtain $x_1 - \varepsilon > y_2 - \varepsilon$ thus IEHP yields $\mathbf{x}' = (x_1 - \varepsilon,\, {}_2x) = (y_1,\, {}_2x) \succ \mathbf{y}' = (y_2 - \varepsilon,\, y_1,\, {}_3x) = (x_2,\, y_1,\, {}_3x) \sim (y_1,\, x_2,\, {}_3x)$, an absurd. □

It is remarkable that in the presence of a procedural equity axiom like AN and consistency of the comparisons, cardinal non-interference implies a behavior that embodies a preference for egalitarian distribution of utilities among generations. If we further add the uncontroversial M then we also obtain HEF by Asheim et al. [2, Proposition 3].

2) A possibility result emerges from Proposition 6 by replacing HP/IBP with their cardinal variants above. This reduces to checking that discounted utilitarianism agrees with both IEHP and IEBP, as well as being SP, RUSC and RLSC, and representable. Besides, the utilitarian overtaking and catching up criteria also satisfy IEHP and IEBP, AN, and SP, although they are incomplete. [8] Thus by contrast with the case of general non-interference, utilitarianism can be reconciled with a cardinal approach to these principles.

3) It is less obvious that the conclusion in Proposition 8 does not vary if IEHP replaces HP in case (*a*), and IEBP replaces IBP in case (*b*). We prove this fact by showing that for infinitely-lived societies, the equal treatment of all generations is a cause for incompatibilities with cardinal non-interference principles under standard semicontinuity and very mild efficiency. [9]

Proposition 10 *There is no SWO on* $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ *satisfying either of:*

(a) AN, IEHP, RUSC, and condition $(1'')$ *below:*

$$\exists\, y_1 > x_1 > x_2 \text{ for which } \mathbf{y} = (y_1,\, (x_2)_{con}) \succ \mathbf{x} = (x_1, (x_2)_{con}) \qquad (1''), \quad \text{or}$$

(b) AN, IEBP, RLSC, and condition $(2'')$ *below:*

$$\exists\, x_2 > x_1 > y_1 \text{ for which } \mathbf{x} = (x_1, (x_2)_{con}) \succ \mathbf{y} = (y_1,\, (x_2)_{con}) \qquad (2'').$$

[8] In fact Asheim and Tungodden [3, Section 5] prove that a property with a formal similarity to the conjunction of IEHP and IEBP, namely 2-Generation Unit Comparability (or 2UC), permits to characterize the overtaking and catching up criteria. However that invariance property incorporates a behavior that has a strongly utilitarian component and can not be justified from a liberal perspective alone.

[9] Conditions $(1'')$ and $(2'')$ in Proposition 10 can be rephrased to resemble more $(1')$ and $(2')$. We believe that the technical effort does not compensate the benefit of doing so. In addition, observe that all $(1')$, $(2')$, $(1'')$ and $(2'')$ are derived from RWD.

Proof: We prove case (b) by contradiction. Suppose $\succcurlyeq$ is a SWO that verifies AN, IEBP, RLSC, and condition $(2'')$. Thus we have $x_2 > x_1 > y_1$ such that $\mathbf{x} = (x_1, (x_2)_{con}) \succ \mathbf{y} = (y_1, (x_2)_{con})$. Let us denote $m = x_1 - y_1$.

We now define the following sequence of streams: for each n sufficiently large,

$$\mathbf{x}^{(n)} = (x_1 - \frac{m}{n}, \overset{n}{\ldots\ldots}, x_n - \frac{m}{n}, {}_{n+1}x) \in \mathbf{X}$$

(one only needs $x_1 - \frac{m}{n}, x_2 - \frac{m}{n} \in [0,1]$). With respect to the sup topology, $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$. Thus there is $n' \in \mathbb{N}$ such that $\mathbf{x}^{(n)} \succ \mathbf{y}$ when $n \geqslant n'$, due to RLSC.

We proceed to use a recursive argument on $(x_1 - \frac{m}{n'}, \overset{n'}{\ldots\ldots}, x_{n'} - \frac{m}{n'}, {}_{n'+1}x) \succ (y_1, {}_2x)$. By appealing to IBP we compare the result of increasing their endowments to the present by $\frac{m}{n'}$ (which utilises $x_1 > y_1 + \frac{m}{n'}$ or $x_1 - y_1 = m > \frac{m}{n'}$). We then obtain

$$(x_1, x_2 - \frac{m}{n'}, \overset{n'-1}{\ldots\ldots}, x_{n'} - \frac{m}{n'}, {}_{n'+1}x) \succ (y_1 + \frac{m}{n'}, {}_2x)$$

and due to AN

$$(x_2 - \frac{m}{n'}, x_1, x_3 - \frac{m}{n'}, \overset{n'-2}{\ldots\ldots}, x_{n'} - \frac{m}{n'}, {}_{n'+1}x) \succ (y_1 + \frac{m}{n'}, {}_2x)$$

Again we appeal to IBP in order to compare the result of increasing their endowments to the present by $\frac{m}{n'}$ (which now utilises $x_2 > y_1 + \frac{2m}{n'}$ or $x_2 - y_1 > m \geqslant \frac{2m}{n'}$ because by the recursive assumption $2 \leqslant n'$). We then obtain

$$(x_2, x_1, x_3 - \frac{m}{n'}, \overset{n'-2}{\ldots\ldots}, x_{n'} - \frac{m}{n'}, {}_{n'+1}x) \succ (y_1 + \frac{2m}{n'}, {}_2x)$$

and due to AN

$$(x_3 - \frac{m}{n'}, x_1, x_2, x_4 - \frac{m}{n'}, \overset{n'-3}{\ldots\ldots}, x_{n'} - \frac{m}{n'}, {}_{n'+1}x) \succ (y_1 + \frac{2m}{n'}, {}_2x)$$

Now we repeat the argument until we reach the following conclusion after n' steps, which ends the proof:

$$\mathbf{x} = (x_1, x_2, \ldots\ldots, x_{n'}, {}_{n'+1}x) \succ (y_1 + \frac{n'm}{n'}, {}_2x) = \mathbf{x}$$

The other instance of the statement is proven by mimicking the proof above. $\square$

Remark 5 *Continuing our discussion in point* 2) *above in this Section, Asheim and Tungodden [3, Propositions 4 and 5] in particular prove the existence of reflexive and transitive relations with SP, AN, 2UC and two respective forms* *of preference continuity. As a matter of fact catching up (resp., overtaking) not only verifies SP, AN, 2UC, and Strong (resp., Weak) Preference Continuity, but also IEHP and IEBP. This speaks for the strong restrictions that weak semicontinuity axioms –in the usual sense– impose to anonymous equal harm/benefit behaviors.*

5 Results for representable relations

In this Section we are concerned with SWRs on $\mathbf{X} = [0,1]^{\mathbb{N}}$ that are representable by $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$, a social welfare function (SWF). Thus when $\succcurlyeq$ is represented by $\mathbf{W}$ one has: $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$ and one can proceed with $\mathbf{W}$ instead of $\succcurlyeq$ throughout. We investigate if the representability assumption is compatible with non-interference properties under efficiency and equity. Firstly we consider the case of restricted weakly dominant representations and then proceed to discuss the case of monotonic and weakly Paretian representations. Since M is necessary for efficiency and in conjunction with WD entails SP, our inspection is fairly exhaustive in that respect.

Alcantud and García-Sanz [1, Subsect. 5.1] proves that no SWF on $\mathbf{X}$ verifies WD and HE. From this and Proposition 7 we obtain a proof that WD, AN, and HP are incompatible for representable SWRs, and a twin argument proves the 'dual' assertion for IBP. We proceed to prove that even if we dispense with AN, a SWF that verifies RWD must contradict both HP and IBP. [10]

Proposition 11 *There is no SWF on $\boldsymbol{X} = [0,1]^{\mathbb{N}}$ that verifies RWD and HP (resp., IBP).*

Proof: We prove the statement for HP and leave the dual proof for IBP to the reader.

Step 1. If $\mathbf{W}$ on $\mathbf{X}$ verifies RWD, then there are $a, b, c \in (\frac{1}{8}, \frac{1}{2})$ such that $a < b < c$ and $\mathbf{W}(a, c, 0_{con}) < \mathbf{W}(b, b, 0_{con})$.

Suppose the contrary. For each $x \in (\frac{1}{4}, \frac{1}{2})$ we let $l(x) = \mathbf{W}(\frac{x}{2}, x, 0_{con})$, $r(x) = \mathbf{W}(x, x, 0_{con})$. The open interval $i(x) = (l(x), r(x))$ is nontrivial due to RWD. Now for each $x < y$, $x, y \in (\frac{1}{4}, \frac{1}{2})$ we observe $r(x) \leqslant l(y)$ because $\frac{1}{8} < a = \frac{y}{2} < \frac{1}{4} < b = x < c = y < \frac{1}{2}$ entails $\mathbf{W}(x, x, 0_{con}) \leqslant \mathbf{W}(\frac{y}{2}, y, 0_{con})$ by the assumption. But this is impossible because with each $x \in (\frac{1}{4}, \frac{1}{2})$ we would associate a different rational number.

[10] This is in contrast with the fact that the Basu-Mitra [5] construction of a WD and AN social welfare function on $[0,1]^{\mathbb{N}}$ verifies both IEHP and IEBP, as is easily checked. These authors acknowledge that a handicap of their construction is that it can not be monotonic.

Step 2. If $\mathbf{W}$ on $\mathbf{X}$ verifies RWD and IBP, then let us fix $a, b, c \in (\frac{1}{8}, \frac{1}{2})$ such that $a < b < c$ and $\mathbf{W}(a, c, 0_{con}) < \mathbf{W}(b, b, 0_{con})$ by Step 1. We proceed to obtain a contradiction.

For each $x \in [0, \frac{1}{8}]$ we let $L(x) = \mathbf{W}(x, b, 0_{con})$, $R(x) = \mathbf{W}(x, c, 0_{con})$. The open interval $I(x) = (L(x), R(x))$ is nontrivial due to RWD. Now we claim $R(x) < L(y)$ if $0 \leqslant x < y \leqslant \frac{1}{8}$. Observe that: $\mathbf{W}(b, b, 0_{con}) > \mathbf{W}(a, c, 0_{con})$, $b > \frac{1}{8} \geqslant y$, $a > \frac{1}{8} \geqslant y > x$ thus HP applies to prove the claim. But this is impossible since with each $x \in [0, \frac{1}{8}]$ we associate a different rational number. $\square$

Since non-interference properties do not assure an egalitarian behavior, criteria that only combine efficiency with non-interference may fail to verify unavoidable consistency requirements. For example, dictatorship by any generation is M, WP, HP, IBP, and representable. We now wonder: Can we preserve this list of good features without adhering to a dictatorship? If we dispense with representability then Theorem 1 says that the answer is positive even when HEF is requested. However, Mariotti and Veneziani [12, Theorem 1] prove that the answer is negative when the number of generations is finite: As has been said, WP, HP and IBP together entail dictatorship by a generation even if representability is dispensed with. If we want to use SWFs then imposing AN together with WP is impossible too (v., Basu and Mitra [5, Theorem 4]). In view of such restrictions, we now proceed to check that nice efficiency can be combined with adequate non-dictatorship axioms and each of those non-interference postulates.

Theorem 2 *There are SWFs on* $\boldsymbol{X}$ *that verify M, WP, HP, HEF, NDP, and NDF.*

Proof: Let us fix $\varepsilon > 0$, and select a surjective, strictly decreasing function $\phi : [2, \infty) \longrightarrow (-\varepsilon, 1]$. For each $\mathbf{x} \in \mathbf{X}$, there is a unique $m(\mathbf{x}) \in \mathbb{N}$ such that $\phi(m(\mathbf{x})+1) < lim\,inf\,(x_n) \leqslant \phi(m(\mathbf{x}))$. Clearly, when $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident one has $m(\mathbf{x}) = m(\mathbf{y})$. Define $\mathbf{V}(\mathbf{x}) = min\{x_k : k = 1, 2, ..., m(\mathbf{x})\}$.

(a) In order to prove that $\mathbf{V}$ is M and WP, suppose $\mathbf{x} \geqslant \mathbf{y}$. Because $m(\mathbf{x}) \leqslant m(\mathbf{y})$ one obtains

$$\mathbf{V}(\mathbf{x}) = min\{x_k : k = 1, 2, ..., m(\mathbf{x})\} \geqslant min\{x_k : k = 1, 2, ..., m(\mathbf{y})\} \geqslant$$
$$\geqslant min\{y_k : k = 1, 2, ..., m(\mathbf{y})\} = \mathbf{V}(\mathbf{y})$$

This proves M. If in fact $\mathbf{x} \gg \mathbf{y}$ then the last inequality above is strict, and WP follows.

(b) $\mathbf{V}$ is HP. Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i < x_i$ and $y'_i < y_i$ then then we need to prove $\mathbf{V}(\mathbf{x}') > \mathbf{V}(\mathbf{y}')$

under the assumption $x'_i > y'_i$. Observe $m(\mathbf{x}) = m(\mathbf{y}) = m(\mathbf{x}') = m(\mathbf{y}')$ thus we abbreviate this figure as m.

By assumption, $min\{x_k : k = 1, 2, ..., m\} > min\{y_k : k = 1, 2, ..., m\}$. If $i > m$ then nothing must be proven thus assume $m \geqslant i$. In case $min\{x_k : k = 1, 2, ..., m\} = min\{x'_k : k = 1, 2, ..., m\}$ we are done, otherwise

$$\mathbf{V}(\mathbf{x}') = min\{x'_k : k = 1, ..., m\} = x'_i > y'_i \geqslant min\{y'_k : k = 1, ..., m\} = \mathbf{V}(\mathbf{y}')$$

In order to prove (c) and (d) we observe that $m(\mathbf{x}) = 2$ when $lim\,inf\,(x_n) = 1$ ($\phi(3) < 1 \leqslant \phi(2) = 1$), and if N is the largest natural number with $\phi(N) \geqslant 0$ then $m(\mathbf{x}) = N$ when $lim\,inf\,(x_n) = 0$ ($\phi(N+1) < 0 \leqslant \phi(N)$). By suitable choosing ϕ, N can be made arbitrarily large (but fixed).

(c) $\mathbf{V}$ is NDP. Let $\mathbf{x} = (\frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}, ...)$, $\mathbf{y} = (1, 1, (0_{con}))$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. For $\mathbf{z} = (1_{con}) = \mathbf{w}$ and each $j > 2$, denote $\mathbf{u} = ({}_1x_j,\, {}_{j+1}z) = (\frac{1}{2}, ..., \frac{1}{2^j}, (1_{con}))$, $\mathbf{v} = ({}_1y_j, {}_{j+1}\, w) = (1, 1, 0, \overset{j-2}{\ldots\ldots}, 0, (1_{con}))$. Then $\mathbf{W}(\mathbf{u}) > \mathbf{W}(\mathbf{v})$ is false because $m(\mathbf{u}) = 2 = m(\mathbf{v})$.

(d) $\mathbf{V}$ is NDF. Let $\mathbf{x} = (1_{con})$, $\mathbf{y} = (0, 0, (1_{con}))$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. For $\mathbf{z} = (0_{con})$, $\mathbf{w} = (1_{con})$ and each $j > 2$, denote $\mathbf{u} = ({}_1z_j,\, {}_{j+1}x) = (0, \overset{j}{\ldots\ldots}, 0, (1_{con}))$, $\mathbf{v} = ({}_1w_j,\, {}_{j+1}y) = (1_{con})$. Then $\mathbf{W}(\mathbf{u}) > \mathbf{W}(\mathbf{v})$ is false because $m(\mathbf{u}) = 2 = m(\mathbf{v})$.

(e) $\mathbf{V}$ is HEF. If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$, $\mathbf{y} = (y_1, (y)_{con})$ and $x_1 > y_1 > y > x$, because $m(\mathbf{y}) \geqslant 2$ one has

$$\mathbf{V}(\mathbf{y}) = min(y_1, y) = y > x = min(x_1, x) = \mathbf{V}(\mathbf{x}) \qquad \square$$

Theorem 3 *There are SWFs on* $\boldsymbol{X}$ *that verify M, WP, IBP, NDP, and NDF.*

Proof: Select a surjective, strictly increasing function $\psi : [0, \infty) \longrightarrow [0, 1)$, e.g., $\psi(x) = \frac{x}{1+x}$ for each $x \in [0, \infty)$. For each $\mathbf{x} \in \mathbf{X}$, there is a unique $n(\mathbf{x}) \in \mathbb{N}$ such that $\psi(n(\mathbf{x})) > \frac{lim\,inf\,(x_n)}{2} \geqslant \psi(n(\mathbf{x}) - 1)$. Clearly, when $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident one has $n(\mathbf{x}) = n(\mathbf{y})$. Define $\mathbf{W}(\mathbf{x}) = max\{x_k : k = 1, 2, ..., n(\mathbf{x})\}$.

(a) In order to prove that $\mathbf{W}$ is M and WP, suppose $\mathbf{x} \geqslant \mathbf{y}$. Because $n(\mathbf{x}) \geqslant n(\mathbf{y})$ one obtains

$$\mathbf{W}(\mathbf{x}) = max\{x_k : k = 1, 2, ..., n(\mathbf{x})\} \geqslant max\{x_k : k = 1, 2, ..., n(\mathbf{y})\} \geqslant$$
$$\geqslant max\{y_k : k = 1, 2, ..., n(\mathbf{y})\} = \mathbf{W}(\mathbf{y})$$

This proves M. If in fact $\mathbf{x} \gg \mathbf{y}$ then the last inequality above is strict, and WP follows.

(b) $\mathbf{W}$ is IBP. Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are eventually coincident and $\mathbf{x} \succ \mathbf{y}$. Consider two streams $\mathbf{x}', \mathbf{y}'$ such that: for some $i \in \mathbb{N}$, $j \neq i$ implies $x'_j = x_j$ and $y'_j = y_j$. If $x'_i > x_i$ and $y'_i > y_i$ then we need to prove $\mathbf{W}(\mathbf{x}') > \mathbf{W}(\mathbf{y}')$ under the assumption $x'_i > y'_i$. Observe $n(\mathbf{x}) = n(\mathbf{y}) = n(\mathbf{x}') = n(\mathbf{y}')$ thus we abbreviate this figure as n.

By assumption, $max\{x_k : k = 1, 2, ..., n\} > max\{y_k : k = 1, 2, ..., n\}$. If $i > n$ then nothing must be proven thus assume $n \geqslant i$. In case $max\{y_k : k = 1, 2, ..., n\} = max\{y'_k : k = 1, 2, ..., n\}$ we are done, otherwise

$$\mathbf{W}(\mathbf{y}') = max\{y'_k : k = 1, ..., n\} = y'_i < x'_i \leqslant max\{x'_k : k = 1, ..., n\} = \mathbf{W}(\mathbf{x}')$$

(c) $\mathbf{W}$ is NDP. Let $\mathbf{x} = (0, (1_{con}))$, $\mathbf{y} = (\frac{1}{2}, (0_{con}))$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. For $\mathbf{z} = (0_{con}) = \mathbf{w}$ and each $j > 1$, denote $\mathbf{u} = ({}_1x_j,\ {}_{j+1}z) = (0, 1, \overset{j-1}{\ldots\ldots}, 1, (0_{con}))$, $\mathbf{v} = ({}_1y_j, {}_{j+1}\,w) = \mathbf{y}$. Then $\mathbf{W}(\mathbf{u}) > \mathbf{W}(\mathbf{v})$ is false because $n(\mathbf{u}) = 1 = n(\mathbf{v})$.

(d) $\mathbf{W}$ is NDF. Let $\mathbf{x} = (1, (0_{con}))$, $\mathbf{y} = (0_{con})$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$. For $\mathbf{z} = (0_{con})$, $\mathbf{w} = (1_{con})$ and each $j > 1$, denote $\mathbf{u} = ({}_1z_j,\ {}_{j+1}x) = (0_{con})$, $\mathbf{v} = ({}_1w_j,\ {}_{j+1}y) = (1, \overset{j}{\ldots\ldots}, 1, (0_{con}))$. Then $\mathbf{W}(\mathbf{u}) > \mathbf{W}(\mathbf{v})$ is false because $n(\mathbf{u}) = 1 = n(\mathbf{v})$. □

Remark 6 *The construction in Theorem 4 does not ensure any egalitarian behavior. Besides it suffers from certain insensitivity to the interests of the future, in the following sense. Let N be the first natural number such that $\psi(N) > \frac{1}{2}$, then $(z_1, ..., z_N, x_1, x_2, ...) \sim (z_1, ..., z_N, y_1, y_2, ...)$ for each $\mathbf{x}, \mathbf{y} \in \boldsymbol{X}$ such that $lim\,inf\, x_k = lim\,inf\, y_k = 1$. Likewise, the individual interest of a generation after the N threshold is not respected, thus WD fails to hold. It is nevertheless true that N can be made arbitrarily large by manipulating ψ and/or the way the $n(\mathbf{x})$ numbers are taken.*

6 Concluding remarks

We have investigated if separate non-interference properties (Harm Principle and Individual Benefit Principle, as well as their cardinal variants) are compatible with Paretian orderings both in finitely- and infinitely-lived societies. Our analysis added to prior studies when no further properties are presumed and we also extended the inspection in order to consider topological semi-continuities and representability. The results in this paper are summarized as follows:

1. In the case of finite societies, full non-interference leads to dictatorship when Weak Pareto is guaranteed. We have proved that the situation is quite the opposite when the horizon is infinite (cf., Section 3): Explicit SWOs can be designed that implement enough efficiency (in the form of M plus WP), minimal equity (in the form of HEF), as well as HP plus IBP and non-dictatorships by the present and by the future.

2. If we are interested in imposing standard semicontinuity then we have clarified the extent of the conflict among non-interference principles and Pareto optimality (cf., Section 4). We proved that renouncing anonymity is not a escape to the incompatibility that arises from the characterizations of the leximin/leximax in the finite context. In the same finite context, if we keep the equal treatment of the generations then a single suitable Paretian comparison yields a conflict. This confirms that for orderings of finite streams, anonymity and HP/IBP are almost universally incompatible in the presence of standard semicontinuity. The analysis translates faithfully to infinitely-long streams of utilities. In this case we define cardinal variants of HP/IBP and prove that the conclusion as to keeping AN and separate non-interference principles remains negative for semicontinuous SWOs.

3. Under the assumption of numerical representability, SWOs that respect HP and HEF, resp. IBP, are compatible with non-dictatorial rankings (by the present, by the future) in case that M and WP is requested: cf., Theorems 2 and 3. However if WD is needed then only a cardinal implementation of full non-interference can be made, under which the generations can be treated equally at the cost that monotonicity must be violated (cf., Footnote 10). From a purely ordinal position, RWD evaluations must exert some interference (penalising both adverse and favorable changes) on the affairs of particular generations: cf., Theorem 1.

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