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Abstract

In this paper, we prove the existence of a recursive competitive equilibrium (RCE) for an Aiyagari style economy with permanent income shocks and perpetual youth structure. We show that there exist equilibria where borrowing constraints are never binding. This allows us to establish a non-trivial lower bound on the equilibrium interest rate. To solve the individual's problem, we present a new approach that uses lattices of consumption functions to deal with the non-compact state space and the unbounded utility function. The approach uses only the first order conditions of the problem (*Euler equations*). The proof is constructive and it serves as a theoretical foundation for the convergence of a policy function iteration procedure.

keywords : Permanent income shocks, incomplete markets, dynamic general equilibrium, heterogeneous agents
JEL codes : D51, D52, E21

1 Introduction

Over the last two decades, a large literature has studied the effects of income uncertainty on individual behavior in heterogeneous agents incomplete markets economies, a model class that is widely known as Aiyagari style models.[1] While applied researchers have extensively studied this class of models numerically, theoretical results on the existence, characterization, and computation of equilibria are rare. This paper makes three theoretical contributions with important economic implications. We prove the existence of recursive competitive equilibria (RCE) for an Aiyagari style model where income shocks are permanent. The proof is constructive and contains a convergence proof for a popular computational algorithm based on the first-order conditions of the agent's problem (*policy function iteration*). Regarding the characterization, we prove the existence of equilibria with non-binding borrowing constraints and with a non-trivial

*Contact: Department of Economics, University of Mannheim, L7, 3-5, 68131 Mannheim, Germany, email: mokuhn@rumms.uni-mannheim.de, http://webrum.uni-mannheim.de/vwl/mokuhn. This paper is part of my Ph.D. dissertation at the Center for Doctoral Studies in Economics (CDSE) at the University of Mannheim. I thank Philip Jung, Dirk Krüger, Felix Kubler, Nicola Pavoni, Melanie Schienle for comments and remarks and in particular my advisor Tom Krebs for his guidance and support. Furthermore I am grateful for helpful comments and remarks from seminar participants at Bonn, Mannheim, and UCL and participants at the EEA in Barcelona and the ESWM in Budapest. All remaining errors are mine.

[1] See for example Aiyagari (1994), Huggett (1993), Telmer (1993) or the textbook by Ljungqvist and Sargent (2000)

lower bound on the equilibrium interest rate. The characterization of the equilibrium allocation allows us to derive further important implications for the optimal consumption-saving decision of agents in equilibrium.

Applied researchers studying Aiyagari style economies have focused on finding RCE numerically trusting on their existence. In line with these studies, Duffie et al. (1994) and Miao (2006) have provided existence proofs for RCE where the state space is a compact set. The elements of the equilibrium description, like the optimal policy function or the distribution over individuals on the state space, are then functions (distributions) on a compact domain (support). Although the assumption of a compact state space seems to be a rather technical issue, it imposes important economic restrictions on individual income processes. For example, it rules out the possibility that the income process contains a unit root. However, the unit-root specification has become quite popular in the empirical literature on income risk because various empirical studies have provided evidence that individual income risk contains transitory and permanent (unit root) components.[2] Therefore, the analysis of a model with a non-compact state space does not only address a theoretical gap but it also provides the foundation to study the implications of permanent income shocks on the consumption-saving decision in Aiyagari style economies.

The equilibrium existence proof comprises three steps. The first step is to show the existence of an optimal solution to the agents' problem. The seminal textbook by Stokey and Lucas (1989) established the value function approach, the contraction property of the Bellman equation, and the principle of optimality as the standard tools to prove the existence of a solution for this kind of problem. In this paper, we depart from this approach by relying only on first order conditions of the agents' problem (*Euler equations*) to prove the existence of an optimal policy function.[3] Similar approaches have been taken in Deaton and Laroque (1992), Coleman (1991), and Rabault (2002). All three papers deal with functions on a metric space and in the case of Deaton and Laroque (1992) and Coleman (1991) apply only to problems with a compact state space and bounded utility.[4] Instead of dealing with functions in a metric space, we use a lattice of consumption functions and apply Tarski's fixed point theorem to prove the existence of a recursive policy function. This allows us to deal with the non-compactness of the state space and unboundedness of the utility function. Since the proof is constructive it establishes the convergence of the *policy function iteration* algorithm for consumption-saving problems, and thereby provides a theoretical justification for its widespread use. This proof has to our knowledge been missing from the literature.[5]

In the second step of the existence proof, we show that a unique stationary distribution exists, and in step three we derive the existence of a market clearing interest rate. As it turns out, the presence of prudence, i.e. strictly convex marginal utility, is crucial in order to get precautionary savings in an equilibrium with permanent income shocks. The reason is that borrowing constraints are potentially non-binding. This complements findings in Huggett and Ospina (2001), who have shown that in models with mean-reverting shocks, prudence of agents is not needed to get precautionary savings because borrowing constraints are always binding for some agents.

In fact, the existence of equilibria with non-binding borrowing constraints follows as a corollary to the existence proof. This result is of particular interest because it opposes the finding in

[2] For example Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Blundell, Preston, and Pistaferri (2008).

[3] Although the present paper focuses on the case of permanent income shocks, this step of the proof is presented for a general class of consumption-saving problems with Markovian income processes.

[4] Coleman (1991) analyzes a representative agent model. This changes the operator on the Euler equation.

[5] The approach in Deaton and Laroque (1992) and Coleman (1991) covers only the case of a compact state space. Furthermore, the operator in Coleman applies only to a representative agent economy. The approach by Rendahl (2007) assumes bounded utility and still relies on the convergence of the value function iteration.

standard incomplete markets models, where there is an intimate link between the existence of equilibria and binding borrowing constraints. Hence, it shows that the non-existence result for RCE with non-binding borrowing constraints on a compact state space (Krebs (2004)) does not extend to the case of a non-compact state space. The two sources of market incompleteness, namely missing insurance markets for idiosyncratic risk and borrowing constraints, can now be disentangled. This suggests that the existence of precautionary savings in Huggett and Ospina (2001) is indeed driven by the market imperfection induced by the borrowing constraint rather than by incomplete insurance markets, although the two sources of market incompleteness are intimately linked in models with mean-reverting shocks.

The present paper is not the first to study the implications of permanent income shocks. Constantinides and Duffie (1996) and Krebs (2007) are two examples that do this in a general equilibrium setup. The prediction for the consumption-saving decision from these papers is, however, highly stylized. The structure of the endowment process in these models allows it to construct no trade equilibria where all agents consume their endowment of the current period.[6] In contrast to these models, we consider a production economy. The consumption-investment good is produced using capital and labor as inputs to a neoclassical production function. Consequently, in equilibrium some agents have to hold positive assets, for which they receive a deterministic income in return. This rules out autarkic equilibria as they are constructed in the earlier papers.

Turning to our last result, we show that non-binding borrowing constraints imply a non-trivial lower bound on the equilibrium interest rate. This lower bound coincides with the equilibrium interest rate in no trade economies as in Krebs (2007). The reason for the higher interest rate in our model stems from the fact that in a production economy agents must hold on average assets in positive net supply.[7] The lower bound allows us to relate our results to existing partial equilibrium studies that examine consumption-saving decisions with permanent income shocks, like Deaton (1991) and Carroll (2004). In these studies, the authors restrict the interest rates to values that are below the lower bound that we establish. This provides an explanation for why they find borrowing constraints to be always binding.[8] These models predict, therefore, long-run consumption dynamics that are similar to those of models with autarkic equilibria like in Constantinides and Duffie (1996) and Krebs (2007), where consumption tracks income one-to-one.[9] In contrast, the model in this paper features asset trade in equilibrium, so that income shocks will not affect consumption one-to-one.

The rest of the paper is structured as follows: Section 2 presents the model. The existence of an optimal solution to the individual's problem is established in section 3. This section is more general and applies to a large class of Markovian income processes. In section 4, we prove the existence of a stationary distribution, and in section 5, we prove that a RCE exists. The discussion on borrowing constraints and the implications for the consumption-saving decision follows in section 6. Section 7 concludes. All proofs can be found in the appendix.

[6]Heathcote et al. (2009) build on this model setup to sustain analytic tractability in a model with permanent shocks but they allow for insurance of a certain fraction of income shocks.

[7]In Krebs (2007), the bond is in zero net supply.

[8]Carroll (2004) allows for zero income shocks and for transitory shocks. These additional shocks induce savings in his model. If we drop these additional shocks, the model reduces to the Deaton (1991) case, and we will find again that borrowing constraints are always binding.

[9]However, the model by Carroll (2004) generates a reaction that is less than one-to-one if all sources of income risk (transitory and zero income shock) as specified in the model are employed.

2 The model

We take time to be discrete and the periods are labeled by an index $t \in \mathbb{N}$. The economy is populated by a continuum of mass 1 of ex ante identical agents.[10] Every agent has an infinite planning horizon, but faces a constant probability of death in every period. An agent who dies is replaced by a newborn agent. The initial endowment in assets and labor productivity $\{a_0, z_0\}$ is drawn from a possibly degenerate distribution $\lambda(a, z, r)$. At the beginning of her life every agent chooses a recursive policy function that determines her behavior over time. We normalize the time endowment of every agent in every period to unity and assume an inelasitc labor supply of this unit of time. The only choice the agent has to make in the model is a consumption-saving decision. We assume that the preferences of agents over recursively generated consumption plans can be represented by the expected discounted sum of constant relative risk aversion (CRRA) utility functions.

Assumption 1. *The period utility function is of the CRRA type*

$$u(c) = \begin{cases} \log(c) & \gamma = 1 \\ \frac{c^{1-\gamma}}{1-\gamma} & otherwise \end{cases} \tag{1}$$

We denote the productivity state in period t by z_t.[11] The shocks to labor productivity are permanent, and we allow for a wide range of distributions for the innovation term. To capture the fact that an agent who died is replaced by a newborn agent, we use the following augmented labor productivity process

$$z_{t+1} = \begin{cases} z_t \varepsilon_{t+1} & \eta_{t+1} = 1 \\ z_0 & otherwise \end{cases} \tag{2}$$

ε_{t+1} denotes the shock to labor productivity that is realized at the beginning of period $t+1$, and η_{t+1} denotes a survival shock. For simplicity we assume that η_{t+1} has a binomial distribution. A realization $\eta_{t+1} = 1$ means that an agent survives the transition from period t to $t+1$. We also allow for transitory i.i.d. income shocks. We denote the transitory income shock in period t by ζ_t. We make the following assumptions on the random variables

Assumption 2. *The distributions of ε, ζ and η satisfy*

$$\begin{array}{llll} (i) & \nexists e \in supp(\varepsilon) : Prob(e) = 1 & (vi) & \mathbb{E}[\zeta] = 1 \\ (ii) & Prob(\varepsilon > 0) = 1 & (vii) & Prob(\zeta > 0) = 1 \\ (iii) & Prob(\eta = 0) = \theta > 0 & (viii) & \mathbb{E}[\zeta_t \varepsilon_s] = \mathbb{E}[\zeta_t]\,\mathbb{E}[\varepsilon_s] \qquad \forall s, t \geq 0 \\ (iv) & \mathbb{E}[\varepsilon] = 1 & (ix) & \mathbb{E}\left[\zeta^{1-\gamma}\right] = M < \infty \\ (v) & \beta \mathbb{E}\left[\varepsilon^{1-\gamma}\right] < 1 & & \end{array}$$

2.1 Agent's problem

We assume that the objective of the agent is to maximize her expected discounted lifetime utility from consumption. The objective function is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \left((1-\theta)\tilde{\beta}\right)^t u(c_t) \middle| \mathcal{F}_0 \right] \tag{3}$$

[10] We are aware of the technical issues regarding the measurability problem for models with a continuum of agents and i.i.d. income shocks. But we refer the interested reader to Green (1994) for detailed discussion of the appropriate construction of the set of agents to preserve measurability for all subset of agents. From now on we apply the law of large numbers in this paper without further discussion.

[11] Throughout, we do not use subscripts for individuals because they only increase the notational burden and are not necessary for the proofs.

where $\tilde{\beta}$ is the time discount factor and $(1-\theta)$ is the probability of surviving from period t to $t+1$. Hence, the expectations are only taken with respect to the realization of the stochastic productivity process $\{\varepsilon_{t+1}\}_{t=0}^{\infty}$ and the sequence of transitory income shocks $\{\zeta_{t+1}\}_{t=0}^{\infty}$. By $\mathcal{F}_t$ we denote the information set of the agent in period t. The set of admissible consumption choices is restricted by the fact that every plan must satisfy the intertemporal budget constraint

$$c_t + a_{t+1} = (1+r)a_t + w_t z_t \zeta_t \tag{4}$$

together with a no Ponzi condition. The condition we impose to rule out Ponzi schemes is an ad hoc debt constraint $a_{t+1} \geq 0$ for all periods $t > 0$. We discuss the impact of this borrowing constraint in section 6.

The state space S for this problem is the Cartesian product of possible asset holdings and productivity states. The information set $\mathcal{F}_t$ for every period contains the current state of the agent $\{a_t, z_t\}$ and all prices.

When we collect all ingredients to the agent's decision problem, we can write it as an optimal control problem under uncertainty

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} ((1-\theta)\tilde{\beta})^t \, u(c_t) \middle| \mathcal{F}_0 \right] \\
s.t. \quad & c_t + a_{t+1} = (1+r)a_t + w_t z_t \zeta_t \qquad \forall t \\
& z_{t+1} = \ z_t \varepsilon_{t+1} \qquad \forall t \\
& \{a_{t+1}, c_t\} \in [0, \infty) \times \mathbb{R}_+ \ \ \forall t \\
& \{a_0, z_0\} \subset \mathcal{F}_0
\end{aligned} \tag{5}$$

To simplify notation, we replace $(1-\theta)\tilde{\beta}$ by an implicit discount rate β

$$\beta := (1-\theta)\tilde{\beta}$$

Assumption 3. *θ and $\tilde{\beta}$ are such that $\beta < 1$.*

2.2 Firm's problem

Production in the model takes place in a perfectly competetive production sector. We model the production side of the economy as a representative firm producing at marginal costs. We assume that production takes place using a standard neoclassical production function.

Assumption 4.

$$Y_t = F(K_t, L_t) = L_t f(k_t) \tag{6}$$

$$F(0, L_t) = F(K_t, 0) = 0$$

and $f'(k_t) > 0$,$f''(k_t) < 0$.

where L_t denotes labor in productivity units, i.e. labor supply times productivity aggregated over all individuals. We construct the productivity process below such that aggregate effective labor supply is $L_t \equiv 1$ in all periods. From the first order conditions there exists a one-to-one mapping from wages to interest rates

$$w = f(f^{'-1}(r+\delta)) - (r+\delta)f^{'-1}(r+\delta) \tag{7}$$

We make the following assumption for the depreciation rate and the discount factor.

Assumption 5. *At $\bar{k}$ defined by*

$$\delta\bar{k} = f(\bar{k})$$

it holds that

$$\left(\beta(1 + f'(\bar{k}) - \delta)^{1-\gamma}\right)^{\frac{1}{\gamma}} < 1$$

The assumption imposes joint restrictions on the preferences of individuals and the production technology. This technical assumption is only needed to make sure that for every possible aggregate capital stock there exists a strictly positive lower bound to the consumption function. It can be easily verified that for a risk aversion parameter $\gamma \leq 1$, which includes the important case of log utility, the assumption does not impose any additional restrictions on the choice for model parameters.

2.3 Bequests and the probability of death

The reason to assume a constant probability of death is to guarantee the existence of a stationary distribution. To make the bequest scheme resource feasible, we require that in equilibrium bequests must be equal to asset holdings of agents who die.

Assumption 6. *The initial endowments $\{a_0, z_0\}$ of agents are drawn from some distribution $\lambda(a, z, r)$ that is continuous in r and satisfies*

$$\begin{aligned} \int z\lambda(da, dz, r) &= 1 \\ \int a\lambda(da, dz, r) &= f'^{-1}(r + \delta) \end{aligned}$$

The assumptions on the means ensure that the average labor productivity in the population is always one and that the assets allocated to the newborn generation equal on average the bequests of the old generation in equilibrium.

2.4 Equilibrium

We define a *recursive competitive equilibrium* (RCE) for this economy as a set of recursively generated asset choices $\{a^*_{t+1}\}$ and consumption choices $\{c^*_t\}$, a capital and labor demand K^d and L^d of the production sector together with equilibrium prices r^* and w^* and a stationary equilibrium distribution $\mu(a, z)$ over asset and productivity levels of agents such that

1. For every agent there is the sequence of recursively generated asset choices $\{a^*_{t+1}\}_{t=0}^{\infty}$ and consumption choices $\{c^*_t\}_{t=0}^{\infty}$ that solve the agent's optimization problem in (5) given equilibrium prices w^* and r^*.

2. The firm's demand for capital K^d and labor L^d maximizes firm's profits given equilibrium prices w^* and r^*.

3. Equilibrium prices are such that

$$\begin{aligned} \int a^*_t\mu(da, dz) = \quad K^* &= K^d \qquad \forall t \\ \int z_t\mu(da, dz) = \quad L^* &= L^d \qquad \forall t \end{aligned}$$

3 Individual problem

In this section, we consider a more general consumption-saving problem where we allow for a larger class of Markovian labor productivity processes and looser ad hoc debt constraints. However, we still require that

$$Prob(wz_t\zeta_t - rD > 0) = 1$$

The generalized consumption-saving problem is

$$\begin{aligned}
\max_{c_t, a_{t+1}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_0\right] \\
s.t. \quad & c_t + a_{t+1} = (1+r)a_t + wz_t\zeta_t \\
& z_{t+1} = f(z_t, \varepsilon_{t+1}) \\
& a_{t+1} \geq -D \\
& c_t \geq 0 \\
& \{a_0, z_0\} \subset \mathcal{F}_0
\end{aligned} \tag{8}$$

where $f(z_t, \varepsilon_{t+1})$ is the (Markovian) law of motion for $\{z_t\}_{t=0}^{\infty}$. We reformulate the problem using cash-at-hand. We define

$$x_t := (1+r)a_t + wz_t\zeta_t + D$$

and get

$$\begin{aligned}
\max_{c_t} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_0\right] \\
s.t. \quad & x_{t+1} = (1+r)(x_t - c_t) + wz_{t+1}\zeta_{t+1} - rD \\
& z_{t+1} = f(z_t, \varepsilon_{t+1}) \\
& x_t \geq c_t \\
& c_t \geq 0 \\
& \{x_0, z_0\} \subset \mathcal{F}_0
\end{aligned} \tag{9}$$

3.1 Characterization of the optimal solution

We know that every optimal solution to (9) must satisfy the first order conditions.

$$c_t^{-\gamma} + \kappa_t = \beta(1+r)\mathbb{E}\left[c_{t+1}^{-\gamma} \middle| \mathcal{F}_t\right] \qquad \forall t \tag{10}$$

$$\kappa_t(x_t - c_t) = 0 \qquad \forall t \tag{11}$$

where κ_t denotes the Lagrange multiplier on the debt constraint. In a RCE the optimal consumption plan must obey a recursive structure, therefore, we restrict attention to optimal solutions that have a recursive structure of the form

$$c_t = c(x_t, z_t)$$

where the dependence on z_t is necessary if the conditional distribution of income next period depends on the current state[12].

Once we have restricted the optimal solution to obey a recursive structure, the problem of finding a solution to the first order conditions can be formulated as finding a fixed point to the following equation

$$c(x,z) = \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}} (\mathbb{E}\left[(c(x',z'))^{-\gamma}\right])^{-\frac{1}{\gamma}}\right\} \tag{12}$$

where the min-operator captures the complementary slackness condition in (11). This approach has been proposed by Deaton and Laroque (1992) and has been applied to consumption-saving problems in Deaton (1991) and Rabault (2002)[13]. In the following, we establish the existence of a fixed point $c(x,z)$ to the modified Euler equation in (12). To establish the existence of a fixed point, we restrict the interest rate to a set $\left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$. As we show below, this is sufficient to establish the existence of a RCE.

3.2 Existence of an optimal solution

We have formulated the search for an optimal solution to the agents' problem as a fix point problem of the modified Euler equation. To prove the existence of a fixed point to this equation, we construct a lattice of consumption functions and an operator that is a selfmap on this set of functions. We then apply a version of Tarski's fixed point theorem to establish the existence of a fixed point to this operator in a constructive way. All definitions can be found in the appendix.

In the first step, we construct a set of candidate consumption functions for the optimal solution to the consumption-saving problem. We restrict attention to the following set of consumption functions

$$\begin{aligned} C_0 := \{&c : X \times Z \to \mathbb{R}_+ | \\ &\forall x_1, x_2 \in X : x_1 > x_2 \Rightarrow c(x_1,z) \geq c(x_2,z) \wedge x_1 - x_2 \geq c(x_1,z) - c(x_2,z)\} \end{aligned}$$

Hence, we only consider consumption functions that are increasing and Lipschitz continuous (with Lipschitz constant $L = 1$) in their first argument. For this class of functions, we apply the usual pointwise ordering

$$c_1(x,z) \geq c_2(x,z) \qquad \forall (x,z) \in X \times Z \Rightarrow c_1 \geq c_2$$

In the appendix, we show (lemma 10) that we can restrict the set of candidate solutions further by imposing an upper and a lower bound (c^u and c^l) on the set of consumption functions. The reason is that the operator that we will construct below is inward pointing[14] at the bounds. The restricted set of candidate solutions in which we are looking for a solution is the set C

$$C := \{c \in C_0 : c^l \leq c \leq c^u\}$$

The next step is to show that this set C together with the ordering just defined forms a complete lattice. To this end, we need to show that the supremum and the infimum for arbitrary sets

[12] It has been shown for example in Deaton (1991) that this dependence can be removed in the case of permanent income shocks.

[13] Both authors iterate on the optimal marginal utility function whereas we iterate on the optimal consumption policy directly.

[14] We call the operator T inward pointing if for the upper bound $\bar{x}$ it holds that $T\bar{x} \leq \bar{x}$ and respectively for the lower bound $\underline{x}$ it holds that $T\underline{x} \geq \underline{x}$.

always exist. In the appendix, we prove that we get the supremum (infimum) of two consumption functions as the upper (lower) envelope. Hence, we obtain the supremum $\bar{c}$ (infimum $\underline{c}$) by taking the pointwise maximum (minimum).

$$\begin{aligned}
\bar{c}(x,z) &= \max\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z \\
\underline{c}(x,z) &= \min\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z
\end{aligned}$$

Equivalently, we get the supremum $\bar{c}^{\infty}$ (infimum $\underline{c}^{\infty}$) of a possibly infinite subset of consumption functions $C' \subset C$ as the upper (lower) envelope.

$$\begin{aligned}
\bar{c}^{\infty}(x,z) &= \sup_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \\
\underline{c}^{\infty}(x,z) &= \inf_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z
\end{aligned}$$

Since the set C has an upper bound c^u and a lower bound c^l the supremum and the infimum always exist, and it holds that $\bar{c}^{\infty} \leq c^u$ and $\underline{c}^{\infty} \geq c^l$. It follows that $(C, \leq)$ is a complete lattice. In the next step, we go on an construct an operator on this set of functions. The operator T maps an element $c_i \in C$ to an element c_{i+1}

$$c_{i+1} = Tc_i$$

by the following operation

$$\begin{aligned}
&\forall (x,z) \; : \; c_{i+1}(x,z) = \lambda \;\text{ where } \lambda \text{ solves} \\
&\lambda = \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

and we define the following function

$$G_i(x,z,\lambda) := \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} - \lambda \qquad (13)$$

such that we can represent the operator as $c_{i+1} = Tc_i$ with $c_{i+1}(x,z) = \lambda$ iff $G(x,z,\lambda) = 0$ for all (x,z).
In the appendix, we prove that the function $G(x,z,\lambda)$ is (i) increasing and continuous in x, (ii) strictly decreasing and continuous in λ, and (iii) for fixed (x,z) there is a unique solution λ^* that solves $G(x,z,\lambda^*) = 0$. It follows, that the operator maps every element $c_i \in C$ to a unique element c_{i+1}. We prove that the operator has the properties of being (i) monotone increasing and (ii) a selfmap, i.e. $T : C \to C$. Furthermore, we prove that imposing an upper bound and a lower bound on the possible set of consumption functions is valid because the operator is inward pointing at these bounds. Thus, we have constructed a monotone increasing operator that is a selfmap on a complete lattice. This is already sufficient to prove the existence of a fixed point to the modifed Euler equation in (12) using the fixed point theorem by Tarski (1955).

Tarski 1. *Every monotone increasing mapping $T : X \to X$ on a complete lattice X has a smallest and a greatest fixed point.*

The theorem does not require a contraction property of the operator but lacks therefore also the uniqueness result of a contracting operator. The proof is not constructive and establishes only the existence of a fixed point. However, constructiveness would be desirable because it would provide an approach how the fix point can be attained. A constructive version of Tarski's theorem exists for continuous operators. The continuity of the operator T can be proven by exploiting the properties of the lattice of consumption fucntions. This fact allows us to apply the constructive version of Tarski's fixed point theorem[15].

Tarski 2. *For $x^u := \sup(X)$, $x^l := \inf(X)$ and a continuous increasing mapping $T : X \to X$ on a complete lattice X we get that $\lim_{n\to\infty} T^n x^u$ and $\lim_{n\to\infty} T^n x^l$ converge to the largest resp. lowest fixed point $\bar{x}$ resp. $\underline{x}$ of $T : X \to X$.*

This constructive version of the iteration procedure proves the convergence of the standard numerically approach of policy function iteration. The policy function iteration algorithm starts with an initial guess for the policy function and applies the operator T repeatedly to this guess. If c^u is taken as initial guess, then iterating on the operator T will attain a fixed point to the modified Euler equation.
Since the first order conditions are only necessary for an optimal solution, we still have to check if the transversality condition is satisfied at our candidate solution. In the appendix, we show that under the maintained assumptions the transversality condition for the case of permanent income shocks is satisfied. We also state additional conditions for the case of general Markovian income processes and borrowing constraints with $D > 0$. We can summarize the results of this section in the following proposition.

Proposition 1. *Under the maintained assumptions there exists for every $r \in [f'(\bar{k}) - \delta, \beta^{-1} - 1]$ an optimal recursive policy function to the agents' problem. It can be found as $\lim_{n\to\infty} T^n c^u$.*

4 Stationary distribution

For the existence of a stationary distribution, we restrict attention again to the case of permanent income shocks with a constant probability of death.[16]
The joint stochastic process for asset holdings and productivity is

$$\begin{bmatrix} a_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} \eta_{t+1}((1+r)a_t + wz_t - c^*(x_t, z_t)) + (1-\eta_{t+1})a_0 \\ \eta_{t+1} z_t \varepsilon_{t+1} + (1-\eta_{t+1}) z_0 \end{bmatrix}$$

where $c^*(x_t, z_t)$ denotes the optimal policy given r and w, and a_0 and z_0 are draws from $\lambda(a, z, r)$.
In the appendix, we prove that a unique stationary probability distribution for the process always exists. The idea of the proof is to exploit the renewal structure induced by the constant probability of death. With a positive probability of death the expected life-time of an agent is finite. Every time an agent dies there is a draw from a fixed distribution λ and the process starts from the support of λ. This implies that all sets with positive λ-mass must also have positive μ-mass. These two features of the stochastic process imply that the process is *recurrent* and *irreducible* such that a unique stationary distribution exists.[17]
We also establish the continuity in the interest rate of the stationary distribution on the interval $[f'(\bar{k}) - \delta, \beta^{-1} - 1]$. The proof relies on a result by Le Van and Stachurski (2007).
We summarize the results of the current section in the follwoing proposition

Proposition 2. *Under the maintained assumptions there exists for every $r \in [f'(\bar{k}) - \delta, \beta^{-1} - 1]$ a unique stationary distribution μ_r that is continuous in r on $[f'(\bar{k}) - \delta, \beta^{-1} - 1]$.*

[15] The constructive version of the theorem results from Kleene's (1952) first recursion theorem. See Cousot and Cousot (1979) for discussion and further references.

[16] All proofs also apply to the more general case of a Markovian process, if there is a positive probability of death, and an optimal recursive consumption policy exists.

[17] Further details and an extensive study of stability of Markovian processes can be found in the textbook by Meyn and Tweedie (1993).

Indeed, the stationary distribution in this model is a mixture over distributions of agents of different *'age cohorts'*, where an *age cohort* at time T contains all agents that have survived for t periods from $T-t$ to T. If we introduce an operator P that maps the distribution of agents' asset holdings and productivity levels of one cohort to their next period's distribution conditional on survival, then the stationary distribution can be shown to be an infinite mixture over initial distributions

$$\mu_r = \sum_{t=0}^{\infty}(1-\theta)^t P^t \lambda(a,z,r)$$

Remark 1. *The operator P maps asset holdings and productivity from the current period's distribution to next periods distribution conditional on survival, it depends therefore on the optimal consumption policy because the consumption policy affects the transition of assets.*

5 Equilibrium

In the previous sections, we have established the existence of an optimal recursive solution to the agents' problem and the existence of a stationary distribution for a wide range of interest rates. To statisfy the equilibrium conditions of a RCE in section 2.4, we have to find a stationary distribution μ_{r^*} such that all markets clear. The labor market is cleared by construction, and in the appendix, we show that the goods market clears for at least one interest rate in the set of interest rates for which an optimal solution to the agents' problem and a stationary distribution exist. The idea of the proof is to show that there is an interest rate low enough such that asset demand exceeds asset supply and an interest rate high enough such that the reverse is true. Since the asset demand and the asset supply are continuous in the interest rate, there must be at least one interest rate in between where asset markets clear. This proves the existence of a RCE for this model.
We summarize the results of this section again in a proposition.

Proposition 3. *Under the maintained assumptions a recursive competitive equilibrium always exists.*

When we establish the existence of an interest rate for which there is aggregate excess supply of capital, we find that for sufficiently high interest rates and only permanent income shocks borrowing constraints are not binding. For this case, we need that consumers are prudent, i.e. have a positive third derivative of the utility function, to rule out equilibria without positive precautionary savings. This case provides an example where the argument by Huggett and Ospina (2001) for the existence of precautionary savings does not apply. Their result of the irrelevance of prudence relies on the fact that borrowing constraints must be binding in equilibrium. However, as we show below, there are equilibria with incomplete markets and idiosyncratic income risk where borrowing constraints are non-binding and precautionary savings arise only due to prudence of consumers.[18]

6 Borrowing constraints

We have established the existence of a RCE in a model with permanent and transitory income shocks. In this section, we remove transitory income risk. This allows us to prove some interest-

[18] The same bound for the interest rate at which borrowing constraints would be non-binding has been established in Rabault (2002) who studies the consumption-saving decision in a partial equilibrium framework. However, he puts it as an open question whether non-binding borrowing constraints can be sustained indefinitely if marginal utility at the optimal solution is bounded.

ing properties of the equilibrium in this model. Especially, we prove that borrowing constraints *must* be non-binding. The following proposition summarizes this result

Proposition 4. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then borrowing constraints must be non-binding.*

To establish this result, it is important to recognize that the state space can be reduced to a single ratio variable[19]: *cash-at-hand to permanent labor income.* This variable is defined as follows

$$\tilde{x}_t := \frac{x_t}{wz_t} = (1+r)\frac{a_t}{wz_t} + 1$$

The reduction of the state space implies that the decision whether to save or not becomes independent of the current income level, however, the amount saved will still depend on the level. This characteristic property allows us to develop an intuitive understanding why borrowing constraints are non-binding.

Consider the case where asset holdings are zero ($\tilde{x}_t = 1$). At this point, the decision whether to save or not is the same for *all* agents. If agents with no asset holdings decided not to save, this would imply that agents with higher *cash-at-hand to permanent labor income* ratios save to sustain a positive aggregate capital stock in equilibrium. As we prove in the appendix, this can not be an optimal solution to the agent's problem. Hence, an optimal policy that is compatible with an equilibrium must be a policy where agents with zero assets do save, and hence, borrowing constraints are non-binding.

Exploiting the same property also provides a good starting point to develop an intuitive understanding for the optimal consumption-saving decision. First recall the case of mean-reverting shocks, there agents save income when they expect a future decline in income, and they spend additional funds - if available - when they expect a future growth in income. Hence, in situations with low income and low assets the borrowing constraint will be binding. The decision depends therefore crucially on the level of the current income state relative to the long-run mean of income. Intuitively, in a situation with mean-reverting shocks agents smooth income around the long-run mean by accumulating and decumulating assets. This behavior is generally known as *buffer-stock saving.* The intuition for the optimal behavior with permanent income shocks must differ from this case because a long-run mean no longer exists. The current income is now the best predictor for future income, and a policy that aims at smoothing income around this income level can not optimal if shocks are permanent and neither Ponzi schemes nor accumulating an infinite amount of assets is optimal. We think therefore that the optimal behavior in a situation with permanent income shocks should be rather described as balancing the risk exposure of total income by adjusting capital, i.e. the *cash-at-hand to permanent labor income* ratio. Now, agents buffer some of the shock by adjusting the stock of assets towards their old *cash-at-hand to permanent labor income* ratio from which the shock has put them apart rather than smoothing income around a long-run trend. This interpretation also provides an alternative explanation for the existence of non-binding borrowing constraints. With permanent shocks agents do neither expect a future increase nor decline in income but if they are in equilibrium willing to save at low cash-at-hand to permanent labor income ratios to balance their risk exposure, then they will be not constrained by the borrowing limit. This intuitive explanation leads us to associate the result of non-binding borrowing constraints rather with the existence of permanent income shocks than with the non-compactness of the state space although the two properties are inherently related.

[19] This result is well-known and can be found in Deaton (1991). We establish the result in the appendix (lemma 16).

From the existence of equilibria with non-binding borrowing constraints, we can derive further implications for the consumption-saving decision. One is the existence of a unique *target insurance ratio*.[20]

Corollary 1. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then there is a unique $\bar{\bar{x}}$ (target insurance ratio) such that the optimal policy is $a_t = a_{t+1}$.*

The *target insurance ratio* is characterized as the state in the reduced state space where the optimal decision of the agent is to keep assets constant between periods.[21] The uniqueness of the *target insurance ratio* implies that the dynamics induced by the optimal consumption saving decision drive - apart from stochastic fluctuations - the agents' cash-at-hand ratio towards the *target insurance ratio*. This aligns nicely with the intuition provided above that agents aim at balancing their risk exposure rather than sustaining a constant income level.
As a further corollary to the result of non-binding borrowing constraints, we can establish a non-trivial interval for the equilibrium interest rate[22].

Corollary 2. *If a RCE with non-binding borrowing constraints exists, then the equilibrium interest rate r lies in the interval* $[\underline{r}, \bar{r}] := \left((\beta\mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1; \beta^{-1} - 1\right)$

The lower bound interest rate $\underline{r}$ separates three ranges for the interest rate that have all been independently studied in different strands of the literature with quite different implications for the consumption-saving decision.
One strand of the literature has studied economies where the interest rate is exactly at the lower bound $\underline{r}$. These are the endowment economies as studied for example in Krebs (2007). In this model, assets are in zero net supply and the interest rate is chosen to balance the desire to accumulate and decumulate assets for all agents and there will be no trade in equilibrium. In this situation, the *target insurance ratio* is exactly at one ($\bar{\bar{x}} = 1$). A situation that is not compatible with an equilibrium in a production economy where capital is an essential input in the production technology. Intuitively, the higher interest rate in the production economy can then be explained by the fact that agents need an additional incentive to accumulate assets.
The interest rates below the lower bound, i.e. $r < \underline{r}$, have been extensively studied in papers by Deaton (1991) and Carroll (1997, 2004). In his paper, Deaton (1991) conjectures that agents always run down assets to zero, become borrowing constrained, and stay borrowing constrained forever. We prove that his interest rate is never an equilibrium interest rate, once we impose equilibrium restrictions on prices. The bound on the interest rate in the models by Deaton and Carroll arises naturally in the proof for the existence of an optimal policy function. It can, however, be shown that this condition can be slightly relaxed without loosing existence of the optimal solution if the lower bound on the optimal consumption function $\underline{c}$ (lemma 10) is taken into account. We exploit this property to prove that the transversality condition is always satisfied.

[20] The proof can be found in the appendix.

[21] It is important to notice, that this does not coincide with the target insurance rate as defined in Carroll (2004) which is

$$\mathbb{E}[\tilde{x}_{t+1}|\mathcal{F}_t] = \tilde{x}_t$$

To see this, plug $\tilde{c}_t = \frac{r}{1+r}\tilde{x}_t + \frac{1}{1+r}$ in the law of motion for the ratio variable, this yields

$$\mathbb{E}[\tilde{x}_{t+1}|\mathcal{F}_t] = \mathbb{E}[\varepsilon^{-1}](\tilde{x}_t - 1) + 1 \neq \tilde{x}_t$$

[22] The proof can be found in the appendix.

7 Conclusions

In this paper, we prove the existence of a recursive competitive equilibrium (RCE) for an Aiyagari style economy with permanent income shocks and perpetual youth structure. The available proofs for the existence of an equilibrium do not apply to this economy because they require a compact state space. To prove that there exists an optimal recursive solution to the agent's problem in our economy, we present an approach based only on first order conditions (*Euler equation*) and use lattices of consumption functions together with Tarski's fixed point theorem. This allows us to deal with the non-compact state space and an unbounded utility function. We present the approach for a general setting of Markovian income processes and show that it can be applied for a large class of consumption-saving problems. The fact that the proof is constructive serves as a theoretical foundation for the convergence of an *policy function iteration* algorithm that is popular in the quantitative literature.

In the second part of the paper, we prove that if there exists an equilibrium where only permanent income shocks are present, then borrowing constraints must always be non-binding. This shows that the non-existence result of equilibria with non-binding borrowing constraints on compact state spaces by Krebs (2004) does not extend to the case of a non-compact state space. However, it is important to notice that the result in our paper seems to be driven by the fact that income shocks are permanent rather than by the fact that the state space is non-compact. From this result, we can establish the existence of a unique target insurance ratio and a non-trivial lower bound on the equilibrium interest rate. If we compare this lower bound to the interest rates in existing studies, we find that the interest rates in these studies are not compatible with the equilibrium interest rates in our model.

A Proofs and definitions for the existence of an optimal solution

A.1 Mathematical preliminaries

The definitions are taken mostly from Zeidler (1986).

Definition 1. *1. A set M is called ordered iff M is nonempty and for certain pairs $(x, y) \in M \times M$ there is a relation $x \leq y$ which satisfies*

(a) $x \leq x$ for all $x \in M$

(b) if $x \leq y$ and $y \leq x$ then $x = y$

(c) if $x \leq y$ and $y \leq z$ then $x \leq z$

The notation $x < y$ means that $x \leq y$ and $x \neq y$

2. Let $N \subseteq M$ and let M be ordered. The set N is called a chain (of M) iff N is nonempty and for all $x, y \in N$, one of the two conditions $x \leq y$ and $y \leq x$ holds.

3. Let $N \subseteq M$ again. The element $x \in N$ is called greatest or smallest in N iff $y \leq x$ or $x \leq y$, respectively, for all $y \in N$. The element $x \in N$ is called a maximal element of N iff there is no $y \in N$ such that $x < y$.

4. The ordered set M is called well ordered iff every nonempty subset of M has a smallest element.

Definition 2. *Let $y \in M$ and $N \subseteq M$. Then y is called the supremum (smallest upper bound) of N iff y is an upper bound of N, i.e. $x \leq y$ for all $x \in N$, and $y \leq u$ for all upper bounds u of N. We write $y = \sup(N)$. Similarly, $\inf(N)$ is defined to be the greatest lower bound.*

Definition 3. *By a lattice we mean an ordered set M with the property that $\inf(\{x, y\})$ and $\sup(\{x, y\})$ exist for all $x, y \in M$. A lattice is called complete iff $\inf(N)$ and $\sup(N)$ exist for all nonempty subsets N of M.*

Definition 4. *An operator T is called continuous iff for every chain S*

$$\sup T(S) = T(\sup(S))$$

and

$$\inf T(S) = T(\inf(S))$$

Definition 5. *An operator T is called monotone increasing if for $x \geq y$ it holds that $Tx \geq Ty$.*

A.2 Set of consumption functions as complete lattice

Define

$$\begin{aligned} \bar{c}(x, z) &:= \max\{c_1(x, z), c_2(x, z)\} \qquad \forall (x, z) \in X \times Z \\ \underline{c}(x, z) &:= \min\{c_1(x, z), c_2(x, z)\} \qquad \forall (x, z) \in X \times Z \end{aligned}$$

Lemma 1. *For every two consumption functions $c_1, c_2 \in C$, it holds that $\underline{c} = \inf\{c_1, c_2\}$ and $\bar{c} = \sup\{c_1, c_2\}$. Furthermore, it holds that $\underline{c}, \bar{c} \in C$.*

Proof. Suppose not. Suppose there is a $\hat{c}$ such that $\hat{c} \geq c_1$ and $\hat{c} \geq c_2$ but $\hat{c} < \bar{c}$. This yields immediately a contradiction because $\bar{c}(x,z) = \max\{c_1(x,z), c_2(x,z)\}$ and it holds that either $\hat{c} \ngeq c_1$ or $\hat{c} \ngeq c_2$ or $\hat{c} \leq c_1$ or $\hat{c} \leq c_2$. The argument for $\underline{c}$ is equivalent.
We have $c_1, c_2 \in C$, and therefore, it holds that $\bar{c} \in C$ because $\bar{c}$ is the piecewise continuous composition of parts of c_1 and c_2. $\square$

Define

$$\begin{aligned} \bar{c}^{\infty}(x,z) &:= \sup_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \\ \underline{c}^{\infty}(x,z) &:= \inf_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \end{aligned}$$

Lemma 2. *For every subset of consumption functions $C' \subset C$, it holds that $\underline{c}^{\infty} = \inf(C')$ and $\bar{c}^{\infty} = \sup(C')$. Furthermore, it holds that $\underline{c}^{\infty}, \bar{c}^{\infty} \in C$.*

Proof. Suppose not. Suppose there exists a $\hat{c} < \bar{c}^{\infty}$ such that $c \leq \hat{c}$ for all $c \in C'$. This implies that there exist (x,s) such that $\hat{c}(x,z) < \bar{c}^{\infty}(x,z)$. By definition, it holds that $\bar{c}^{\infty}(x,z) = \sup_{c \in C'}\{c(x,z)\}$, hence, $\hat{c}(x,z) \geq c(x,s)$ implies that $\hat{c}(x,z) \geq \sup_{c \in C'}\{c(x,z)\}$ which yields a contradiction because

$$\sup_{c \in C'}\{c(x,z)\} = \bar{c}^{\infty}(x,z) > \hat{c}(x,z) \geq \sup_{c \in C'}\{c(x,z)\}$$

It follows immediately from the fact that all $c \in C'$ are Lipschitz continuous that $\bar{c}^{\infty}(x,z)$ is also Lipschitz continuous such that $\bar{c}^{\infty} \in C$ holds. An equivalent argument applies for the infimum. $\square$

Remark 2. *The fact that $\bar{c}^{\infty} \in C$ holds follows directly from the Lipschitz property because for all (x_1, z) and (x_2, z) with $x_1 \leq x_2$ it holds that*

$$\begin{aligned} \bar{c}^{\infty}(x_2,z) &= \sup_{c \in C'}\{c(x_2,z)\} \\ &\leq \sup_{c \in C'}\{c(x_1,z) + x_2 - x_1\} \\ &= \sup_{c \in C'}\{c(x_1,z)\} + x_2 - x_1 \\ &= \bar{c}^{\infty}(x_1,z) + x_2 - x_1 \end{aligned}$$

and the same argument applies to the infimum.

Lemma 3. *$(C, \geq)$ is a complete lattice.*

Proof. From lemma 1 it follows that $(C, \geq)$ is a lattice, and from lemma 2 follows that it is complete. $\square$

A.3 Properties of $G(x, z, \lambda)$

Lemma 4. *$G_i(x, z, \lambda)$ is*

(a) increasing and continuous in x

(b) strictly decreasing and continuous in λ

Proof. We consider the two arguments of the min-operator first separately

1. Suppose $G_i(x,z,\lambda) = x - \lambda$, (a) and (b) are obviously satisfied.

2. Suppose

$$G_i(x,z,\lambda) = \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda)+wz'\zeta'-rD,s'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} - \lambda \tag{14}$$

   Since $u'(\cdot)$ is a strictly decreasing function, its inverse is strictly decreasing as well. By assumption, $c_i(\cdot,z)$ is increasing and continuous in x. It follows that (14) must be increasing in x. The continuity of $c_i(\cdot,z)$ together with the continuity of $u'(\cdot)$ and its inverse imply that (14) satisfies (a) because $c_i \geq c^l > 0$. We apply the same arguments for (b) and $\lambda \leq x$, and we get that (14) satisfies (b).

Finally, we have to show that the min-operator preserves the properties of $G_i(\cdot,z,\cdot)$. The min-operator forms the lower envelope of two continuous and increasing respectively strictly decreasing functions in x and λ. It preserves, therefore, the monotonicity and continuity of these functions. Hence, $G_i(\cdot,z,\cdot)$ satisfies (a) and (b). $\square$

Lemma 5. *For every (x,z), $G(x,z,\lambda)=0$ has a unique solution λ.*

Proof. It follows from the properties of $u'(\cdot)$ that for $\lambda = 0$, $G(x,z,\lambda) \geq 0$ and for $\lambda \to x$, it follows from lemma 4 that $G(x,z,\lambda)$ is strictly decreasing with $G(x,z,\lambda) \leq 0$ if $\lambda = x$. Hence, the solution $G(x,z,\lambda) = 0$ must be unique. $\square$

A.4 Properties of T

Lemma 6. *The operator T is monotone increasing.*

Proof. Take $c_i^1 > c_i^2$. It follows from the fact that $u'(\cdot)$ and its inverse are strictly decreasing functions that

$$\min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^1((1+r)(x-\lambda)+wz'\zeta'-rD,z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \geq$$
$$\min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^2((1+r)(x-\lambda)+wz'\zeta'-rD,z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}$$

From lemma 4, we know that $G_i(x,z,\cdot)$ is decreasing in λ. Since it holds that $G_i^1(x,z,\cdot) \geq G_i^2(x,z,\cdot)$, it follows that for all (x,z) we get that $\lambda^1 \geq \lambda^2$. $\square$

Lemma 7. *The operator T maps elements of C to continuous and increasing functions.*

Proof. Again, we proceed in two steps. First, we show that if $c_i(\cdot,z)$ is continuous and increasing, then $c_{i+1}(\cdot,z)$ will be increasing, and in a second step, we show that it is also continuous.

1. (*increasing*)

   (a) If $\lambda = x$, this is obvious.

   (b) If $\lambda = \left(\beta(1+r)\mathbb{E}\left[\left(c_i((1+r)(x-\lambda)+wz'\zeta'-rD,z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}$ pick $x_1 > x_2$. Lemma 4 implies that $G_i(x_1,z,\lambda) \geq G_i(x_2,z,\lambda)$ and it follows that $\lambda_1 \geq \lambda_2$ because $G_i(x_1,z,\cdot)$ is strictly decreasing.

   From steps (1a) and (1b) it follows that $c_{i+1}(\cdot,z)$ must be an increasing function.

2. (*continuous*) The continuity of the optimal solution follows directly from the implicit function theorem (Kumagai (1980))[23]. To see this, note that $G_i(\cdot, z, \cdot)$ is a continuous map $G_i : X \subset \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. From lemma 5, we know that for all (x_0, z) there exists a unique solution $G_i(x_0, z, \lambda_0) = 0$, and from Kumagai (1980), it follows that $c_{i+1}(\cdot, z)$ is continuous in a neighborhood of x_0 if and only if there are open neighborhoods $B \subset X$ and $A \subset \mathbb{R}_+$ of x_0 and λ_0, respectively, and

$$\forall x_0 \in B : G_i(x_0, z, \cdot) : A \to \mathbb{R}$$

is locally one-to-one (injective). From lemma 4, we know that $G(x, z, \cdot)$ is strictly decreasing, and therefore, it is locally one-to-one. Hence, $c_{i+1}(x, z)$ will be continuous in x.

□

Lemma 8. *If $x_1 > x_2$ and $G(x_2, z, \lambda_2) = 0$ with $x_2 > \lambda_2$, then for $G(x_1, z, \lambda_1) = 0$ it holds that $x_1 > \lambda_1$.*

Proof. Suppose not. It follows from lemma 4 that

$$\begin{aligned}
\lambda_1 &= x_1 \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i(wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_2 - \lambda_2) + wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&= \lambda_2 \\
&< x_2
\end{aligned}$$

This yields a contradiction, and hence, it holds that if $x_1 > x_2$ and $x_2 > \lambda_2$, then also $x_1 > \lambda_1$. □

Lemma 9. *The operator T is a self-map. It maps Lipschitz continuous, increasing functions $c_i(\cdot, z)$ to Lipschitz continuous, increasing functions $c_{i+1}(\cdot, z)$ with Lipschitz constant $L = 1$, i.e.*

$$c_i(x_1, z) - c_i(x_2, z) \leq x_1 - x_2 \qquad \forall x_1, x_2 \in X$$

Proof. From lemma 7, we know that T maps continuous and increasing functions to continuous and increasing functions. Consider the case where $x_1 > x_2$. We know from lemma 7 that $\lambda_1 \geq \lambda_2$. We consider now all possible combinations

I. $\lambda_1 = x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 = \lambda_1 - \lambda_2$.

II. $\lambda_1 < x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 > \lambda_1 - \lambda_2$.

III. $\lambda_1 = x_1$ and $\lambda_2 < x_2$. Not possible, see lemma 8.

IV. $\lambda_1 < x_1$ and $\lambda_2 < x_2$.

   (a) $\lambda_1 = \lambda_2 \Rightarrow x_1 - x_2 > \lambda_1 - \lambda_2$

[23] Kumagai proves a theorem for the case of non-differentiable function.

(b) $\lambda_1 > \lambda_2$: (Proof by contradiction) Suppose that $x_1 - x_2 < \lambda_1 - \lambda_2$. This implies $x_1 - \lambda_1 < x_2 - \lambda_2$.

$$\begin{aligned}
\lambda_1 &= \left(\beta(1+r)\mathbb{E}\left[\left(c_i((1+r)(x_1-\lambda_1)+wz'\zeta'-rD,z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&\leq \left(\beta(1+r)\mathbb{E}\left[\left(c_i((1+r)(x_2-\lambda_2)+wz'\zeta'-rD,z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&= \lambda_2
\end{aligned}$$

but $\lambda_1 \leq \lambda_2$ yields a contradiction, because we started with the assumption that $\lambda_1 > \lambda_2$.

Hence, it must be true that

$$x_1 - \lambda_1 \geq x_2 - \lambda_2 \qquad \iff \qquad x_1 - x_2 \geq \lambda_1 - \lambda_2$$

and the proof is complete. □

Lemma 10. *For every r such that $\beta(1+r) \leq 1$ and $1 - (\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}} > 0$ there exists a supersolution c^u and a subsolution c^l to the operator T.*

1. *For $c^u(x,s) = x$, it holds that $Tc^u \leq c^u$.*

2. *For $c^l(x,s) = \iota x$ with $\iota := 1 - (\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}}$, it holds that $Tc^l > c^l$.*

Proof. 1. By construction, we get that $c_1 = Tc^u \leq x$. Since $c_1(x,s) = \lambda \leq x$ where λ solves

$$\lambda = \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c^u\left((1+r)(x-\lambda)+wz'\zeta',z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}$$

and it follows that $Tc^u \leq c^u$

2. Take $c^l(x,z) = \iota x$ and suppose that $G^l(x,z,\lambda) = 0$ for $\lambda \leq \iota x$ for some x. This implies that

$$\begin{aligned}
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}} \\
&\quad \left(\mathbb{E}\left[\left(c^l\left((1+r)(x-\iota x)+wz'\zeta'-rD,z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left(\iota\left((1+r)(1-\iota)x+wz'\zeta'-rD,z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
x &> (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[((1+r)(1-\iota)x)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
1 &> (\beta(1+r))^{-\frac{1}{\gamma}}(1+r)(1-\iota) \\
(1-\iota) &> (1-\iota)
\end{aligned}$$

which yields a contradiction. Hence, it must be true that $\lambda > \iota x$ for all (x,z), and therefore, it holds that $Tc^l > c^l$.

□

Lemma 11. *The operator $T : C \to C$ is continuous.*

Proof. For finite chains the proof is obvious. For infinite chains, take a chain $C^S \subset C$. Define $\bar{c}^\infty = \sup(C^S)$. Denote the image set of C^S by $C^{S'} = \left\{c' \in C : c' = Tc \,\forall c \in C^S\right\}$ and $\bar{c}' = \sup(C^{S'})$. For all $(x,z) \in X \times Z$, we have $c_i'(x,z) = \lambda_i^*$ where λ_i^* solves $G_i(x,z,\lambda) = 0$. Again, $\bar{c}'$ is defined pointwise as $\bar{c}'(x,z) = \sup \lambda^* =: \bar{\lambda}^*$. Since T is monotone increasing and C^S is a chain, it holds that $\lambda_i^* \geq \lambda_j^*$ if $c_i \geq c_j$. It follows from the definition of a chain that for all $c_i, c_j \in C^S$ we either have $c_i \geq c_j$ or $c_i \leq c_j$. Now fix $(x,z,\bar{\lambda}^\infty)$ where $\bar{\lambda}^\infty = T\bar{c}^\infty(x,z)$. Put $c_i \in C^S$ in increasing order and define $\Delta_i := G_i(x,z,\bar{\lambda}^\infty)$. The $\{\Delta_i\}$ sequence is increasing and bounded because $\bar{\lambda}^\infty$ solve $G(x,z,\bar{\lambda}^\infty) = 0$ for $\bar{c}^\infty$. Since we have $\bar{c}^\infty = \sup(C^S)$, it follows from the proof of lemma 2 that for every c_i there exists a $c_{i+1} \in C^S$ such that $\bar{c}^\infty \geq c_{i+1} \geq c_i$ because otherwise $\bar{c}^\infty$ can not be the supremum of C^S. It follows that $\sup(\Delta_i) = 0$. Hence, $G_i(x,z,\bar{\lambda}^\infty) \to 0$ holds, and this implies that $\lambda_i^* \to \bar{\lambda}^\infty$ because λ_i^* solves $G_i(x,z,\lambda) = 0$ and $G_i(x,z,\cdot)$ is continuous in λ. Hence, we get $\bar{\lambda}^* = \bar{\lambda}^\infty$ for all (x,z) such that $T\bar{c}^\infty = \sup(Tc)$ holds. The equivalent argument applies to the infimum and the elements of the chain put in decreasing order. It follows that according to definition 4, $T : C \to C$ is a continuous operator. □

A.5 Transversality condition

The transversality condition reads

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] = 0 \tag{15}$$

In the following, we need the definition for cash-at-hand $x_t = (1+r)a_t + wz_t\zeta_t + D$ and the result from lemma 10 that $c^*(x_t,z_t) > \iota x_t$ for all (x_t,z_t).

$$\begin{aligned}
\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] &= \lim_{t\to\infty} \beta^t \mathbb{E}\left[\left(\frac{c_t}{x_t}x_t\right)^{-\gamma}((1+r)a_t + wz_t\zeta_t + D - wz_t\zeta_t - D)\right] \\
&= \lim_{t\to\infty} \beta^t \mathbb{E}\left[\left(\frac{c_t}{x_t}\right)^{-\gamma} x_t^{-\gamma}(x_t - wz_t\zeta_t - D)\right] \\
&\leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma} - x_t^{-\gamma}wz_t\zeta_t - x_t^{-\gamma}D)\right] \\
&\leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma})\right]
\end{aligned}$$

Consider first the case of log utility ($\gamma = 1$)

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-1}x_t^0\right] = \lim_{t\to\infty} \beta^t \iota^{-1} = 0$$

For the $\gamma > 1$ case, we get

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}x_t^{1-\gamma}\right] \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(wz_t\zeta_t - rD)^{1-\gamma}\right]$$

We make the following additional assumption for the general case

Assumption 7. *If $\gamma \geq 1$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(wz_t\zeta_t - rD)^{1-\gamma}\right] = 0$$

From assumption 7, it follows that

$$\lim_{t\to\infty}\beta^t\mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right]\leq 0$$

For the case $D=0$, assumption 7 condition simplifies to

$$\lim_{t\to\infty}\beta^t\mathbb{E}\left[(wz_t\zeta_t)^{1-\gamma}\right]=0$$

and we get for the case of permanent income shocks the sufficient condition

$$\beta\mathbb{E}\left[\varepsilon^{1-\gamma}\right]<1$$

This condition is satisfied by assumption 2.
Finally, consider the $\gamma<1$ case

$$\begin{aligned}\lim_{t\to\infty}\beta^t\mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma})\right] &\leq \lim_{t\to\infty}\beta^t\mathbb{E}\left[\iota^{-\gamma}(1+(1-\gamma)(x_t-1))\right]\\ &\leq \lim_{t\to\infty}\left(\beta^t(\iota^{-\gamma}-(1-\gamma))+\beta^t\mathbb{E}\left[\iota^{-\gamma}x_t\right]\right)\end{aligned}$$

We can determine an upper bound on $\mathbb{E}[x_t]$

$$\begin{aligned}\mathbb{E}[x_t] &= \mathbb{E}\left[(1+r)a_t+wz_t\zeta_t+D\right]\\ &= \mathbb{E}\left[(1+r)a_t\right]+\mathbb{E}\left[wz_t\zeta_t\right]+D\\ &\leq \mathbb{E}\left[(1+r)\bar{a}_t\right]+\mathbb{E}\left[wz_t\zeta_t\right]+D\end{aligned}$$

where $\bar{a}_t$ is defined as follows

$$\begin{aligned}\bar{a}_1 &= (1+r)a_0+wz_0\zeta_0-\iota((1+r)a_0+wz_0\zeta_0)\\ \bar{a}_1 &= (1-\iota)((1+r)a_0+wz_0\zeta_0)\\ \bar{a}_2 &= ((1-\iota)(1+r))^2a_0+(1-\iota)^2(1+r)wz_0\zeta_0+(1-\iota)wz_1\zeta_1\\ \bar{a}_3 &= ((1-\iota)(1+r))^3a_0+(1-\iota)^3(1+r)^2wz_0\zeta_0+(1-\iota)^2(1+r)wz_1\zeta_1+(1-\iota)wz_2\zeta_2\\ &\vdots\\ \bar{a}_t &= ((1-\iota)(1+r))^ta_0+(1-\iota)\sum_{s=0}^{t-1}((1-\iota)(1+r))^swz_{t-1-s}\zeta_{t-1-s}\end{aligned}$$

We have $\beta(1+r)\leq 1$, and therefore, we get

$$\bar{a}_t\leq a_0+\frac{1}{1+r}\sum_{s=0}^{t-1}wz_{t-1-s}\zeta_{t-1-s}$$

and

$$\begin{aligned}\mathbb{E}[x_t] &\leq \mathbb{E}\left[\sum_{s=0}^{t}wz_{t-s}\zeta_{t-s}\right]+D+a_0(1+r)\\ &= x_0+\mathbb{E}\left[\sum_{s=0}^{t-1}wz_{t-s}\zeta_{t-s}\right]\\ &= x_0+\mathbb{E}\left[\sum_{s=0}^{t-1}wz_{t-s}\right]\end{aligned}$$

where the last equality holds because of assumption 2.
For the general case we have to make an additional assumption

Assumption 8. *If $\gamma < 1$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\sum_{s=0}^{t-1} wz_{t-s}\right] = 0$$

For the case of permanent income shocks, the expression simplifies to

$$\lim_{t\to\infty} \beta^t wz_0 \sum_{s=0}^{t-1} (\mathbb{E}[\varepsilon])^{t-s} = 0$$

and is satisfied because of assumption 2.
Hence, if for the general case 7 resp. 8 holds, then there exists an upper bound for the transversality condition

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0$$

For the case of permanent shocks assumption 2 is sufficient for the existence of the upper bound. To establish a lower bound, note that if $D = 0$, then the lower bound is trivially at zero. For the general case of $D > 0$ we need an additional assumption.

Assumption 9. *If $D > 0$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(wz_t\zeta_t - rD)^{-\gamma}\right] = 0$$

We have established an upper bound and an lower bound for the transversality condition

$$0 \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0 \qquad \Longrightarrow \qquad \lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] = 0$$

and we can conclude that the transversality condition is satisfied. Hence, the fixed point to the modified Euler equation is an optimal solution to the agents' problem in (9).

B Proofs and definitions for the existence of a stationary distribution

B.1 Mathematical preliminaries

The definitions are taken mostly from Meyn and Tweedie (1993). Let the state space for the stochastic process of labor productivity and asset holdings be S and the Borel σ-algebra on S be $\mathcal{B}(S)$. The stochastic process $\{a_t, z_t\}_{t=0}^{\infty}$ is denoted by Φ and the state in period t by $\Phi_t = \{a_t, z_t\}$.

Definition 6. *The return time probability from state Φ_0 to a set $A \in \mathcal{B}(S)$ is defined as*

$$L(\{a_0, z_0\}, A) := Prob(\Phi_t \text{ ever enters } A | \{a_0, z_0\})$$

Definition 7. *We call a Markov chain φ-irreducible if there exists a measure φ on $\mathcal{B}(S)$ such that, whenever $\varphi(A) > 0$, we have $L(\{a, z\}, A) > 0$ for all $\{a, z\} \in S$*

Definition 8. *The Markov chain is called ψ-irreducible if it is φ-irreducible for some φ and the measure ψ is a maximal irreducibility measure ($\psi \succ \varphi$).*

From the definitions and proposition 4.2.2 in Meyn and Tweedie (1993) we get immediately that if the Markov chain is φ-irreducible, it is also ψ-irreducible. Next, we introduce the concepts of *recurrence* and *transience*.

Definition 9. *The set A is called recurrent if $\mathbb{E}\left[\mathbb{1}(\Phi_t \in A)|(a,z)\right] = \infty$ for all $(a,z) \in A$. The set A is called uniformly transient if there exists a $M < \infty$ such that $\mathbb{E}\left[\mathbb{1}(\Phi_t \in A)|(a,z)\right] \leq M$ for all $(a,z) \in A$.*

These concepts can be extended to chains in the following way

Definition 10. *If every state is recurrent, the chain is recurrent, and if every state is transient, the chain is transient.*

Theorem 1. *Under the maintained assumptions there exists for every r with $\beta(1+r) \leq 1$ a unique stationary probability distribution μ_r.*

Proof. By construction, Φ is λ-irreducible, and every set in the support of λ is recurrent, hence, Φ is a recurrent chain (cf. theorem 8.1.2 Meyn and Tweedie (1993)). It follows from theorem 10.0.1 in Meyn and Tweedie (1993) that Φ has a unique stationary measure. It holds furthermore that the expected hitting time for every set in the support of λ is finite, and therefore, the stationary measure can be normalized to be a probability measure. □

It is important to notice that the intial endowments of agents are only resource feasible in equilibrium. If goods markets do not clear, then also the mean over assets of the exogenously fixed distribution does not coincide with the mean asset holdings of the agents' that died.

Remark 3. *The proof for the existence and uniqueness of a stationary distribution does not require that initial endowments $\{a_0, z_0\}$ are uncorrelated with $\{a_t, z_t\}$. It only requires that the conditional distribution for $\{a_0, z_0\}$ has the same support as $\lambda(a,z,r)$ and that the unconditional distribution over $\{a_0, z_0\}$ is $\lambda(a,z,r)$. Hence, we can allow for correlation in assets and productivity levels of agents that leave and their successors.*

Lemma 12. *The stationary distribution is continuous in the interest rate on the interval $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$.*

Proof. See proof of theorem 1 in Le Van and Stachurski (2007). The assumptions can be easily verified. Assumption 1 holds because the optimal consumption choice is continuous in the interest rate, the individual choice is independent from the cross-sectional distribution, and the initial distribution is continuous in the interest rate. Assumption 2 is satisfied[24] because we have for every r in $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$ a unique stationary distribution (theorem 1) such that we can directly evaluate at the limit. The bound for the stationary moments follow immediately from the positive probability of death (our assumption 2) and the lower bound on consumption (lemma 10). Finally, assumption 3 follows by a similar argument using that a highest sustainable capital stock exists (our assumption 5) and that the variance of productivity is bounded. We have already shown that the stationary distribution is unique (theorem 1), and hence, the stationary distribution is continuous in the interest rate (see remark 1 in Le Van and Stachurski). □

C Proof for the existence of a RCE

In this section, we establish the existence of an equilibrium interest rate in the interval $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$ such that all markets clear. We need the following lemmata.

Lemma 13. *If only permanent shocks are present, $D = 0$, and r is such that $\beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \geq 1$, then borrowing constraints are non-binding.*

[24] Using as Lyapunov function $V(a,z) = a + (z - \mathbb{E}[z])^2 = a + (z-1)^2$.

Proof. The borrowing constraints are non-binding if for all (x, z) it holds that $G(x, z, x) < 0$. If only permanent income shocks are present, then it can be easily checked that the inequality always holds if

$$1 > \beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right]$$

Hence, we get that for all r that satisfy this inequality, borrowing constraints must be non-binding. ☐

Lemma 14. *For $\beta(1+r) = 1$ aggregate asset supply is larger than aggregate asset demand.*

Proof. It follows from theorem 1 that a stationary distribution exists. Aggregate asset supply K^s is the sum of asset supply of newborn agents K^{new} and the asset holdings of agents that survived from the last period K^{old}, we get

$$K^s = \theta K^{new} + (1-\theta)K^{old}$$

The asset supply of the newborn generation K^{new} is determined by the initial distribution $\lambda(a, z, r)$. The asset supply of the surviving generation K^{old} has been determined by a sequence of optimal consumption choices. The consumption choice is characterized by the first order conditions of the agent's problem. We have to distinguish two cases.

(1) If borrowing constraints are binding for some agents, it follows from the first-order conditions (see Huggett and Ospina (2001)) that for $\beta(1+r) = 1$ there is expected consumption grwoth in the cross-section conditional on survival

$$1 > \mathbb{E}_\mu\left[\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma}\right] \Rightarrow \mathbb{E}_\mu\left[c^*_t\right] < \mathbb{E}_\mu\left[c^*_{t+1}\right]$$

where the μ subscript denotes the fact that the expectations are taken with respect to the stationary distribution μ.

(2) If lemma 13 applies, then borrowing constraints are non-binding. The Euler equation holds as an equality, and the argument by Huggett and Ospina (2001) does not apply.

$$1 = \mathbb{E}\left[\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma}\right]$$

There is only one riskless asset. Hence, $c_{t+1} = c_t$ is not an optimal choice for all realizations of ε_{t+1}. Hence, Jensen's inequality for strictly convex functions[25] applies, we get

$$1 = \mathbb{E}\left[\left(\frac{c^*_{t+1}}{c^*_t}\right)^{-\gamma}\right] > \left(\mathbb{E}\left[\frac{c^*_{t+1}}{c^*_t}\right]\right)^{-\gamma} \Rightarrow 1 < \mathbb{E}\left[\frac{c^*_{t+1}}{c^*_t}\right] \Rightarrow \mathbb{E}_\mu\left[c^*_t\right] < \mathbb{E}_\mu\left[c^*_{t+1}\right]$$

and again we get conditional on survival consumption growth in the cross-section.[26]

Since expected labor income is constant, consumption growth can only be financed by accumulating on average higher assets. If assets grow for all surviving agents between periods, it follows that $K^{old} > K^{new}$ because the average capital of all generations at the beginning of the life has been K^{new}. As a consequence, we get $K^s > K^{new} = K^d$. ☐

[25]Note that marginal utility is strictly convex if and only if $\frac{\partial^3 u(x)}{\partial x^3} > 0$.

[26]The same argument applies, if borrowing constraints were binding. The argument by Huggett and Ospina (2001) could therefore be replaced by this argument but to highlight the importance of prudence in the model with permanent shocks we decided to present the proof in two steps.

Lemma 15. *There exists an interest rate low enough such that aggregate asset demand is larger than aggregate asset supply.*

Proof. Suppose not. First determine the highest sustainable capital stock given zero consumption

$$\bar{k} = (1-\delta)\bar{k} + f(\bar{k})$$

Fix the interest rate at the implied interest rate

$$\underline{r} = f'(\bar{k}) - \delta$$

and allocate $\bar{k}$ arbitrarily in the population. Draw initial productivity levels from the stationary marginal distribution of productivity levels. To sustain the capital stock, all agents must consume $c_t = 0$ but this is never optimal. Hence, aggregate consumption must be positive and capital supply must be smaller than capital demand, but this yields a contradiction. $\square$

Theorem 2. *Under the maintained assumptions a recursive competitive equilibrium (RCE) exists.*

Proof. We have already shown that an optimal solution to the agents optimization problem and a stationary distribution exist. The stationary distribution is continuous in the interest rate. Lemmata 14 and 15 together with the fact that asset demand is downward sloped[27] imply that there must exist at least one interest rate such that the goods market clears. The labor market clears by construction. Hence, a recursive competitive equilibrium exists. $\square$

D Proof of non-binding borrowing constraints

Lemma 16. *If all income shocks are permanent or transitory and i.i.d., then the optimal policy only depends on a single variable.*

Proof. (i) Start with $c_0(x,z) = c^u(x,z) = x$.

$$\begin{aligned}
\lambda &= \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[((1+r)(x-\lambda)+wz'\eta)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\frac{\lambda}{wz} &= \min\left\{\frac{x}{wz}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left((1+r)\frac{(x-\lambda)}{wz}+\varepsilon\eta\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\tilde{\lambda} &= \min\left\{\tilde{x}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\varepsilon^{-\gamma}\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

where we define for all variables $\tilde{x} := \frac{x}{wz}$. It follows that $\tilde{c}_0(\tilde{x}) = \tilde{x}$, because $\tilde{x}' = \frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta$ and $\tilde{c}_1(\tilde{x}) = \tilde{\lambda}$ for all $\tilde{x}$.

(ii) Suppose $c_i(x,z) = wz\tilde{c}_i(\tilde{x})$, it follows that

$$\begin{aligned}
\lambda &= \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left(\tilde{c}_i\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\frac{\lambda}{wz})+\eta\right)wz\varepsilon\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\tilde{\lambda} &= \min\left\{\tilde{x}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left(\tilde{c}_i\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta\right)\varepsilon\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

[27] This follows immediately from assumption 4.

it follows that $\tilde{c}_{i+1}(\tilde{x}) = \tilde{\lambda}$ will also only be a function of $\tilde{x}$.

$\square$

For this policy we use the result from Carroll and Kimball (1996) that the optimal consumption function $c(\tilde{x})$ is concave[28]. Using this result, we prove that for the case where only permanent shocks are present borrowing constraints must be non-binding.

Theorem 3. *Assume only permanent income shocks are present. If a stationary recursive equilibrium exists, then borrowing constraints must be non-binding.*

Proof. The optimal recursive policy function of a RCE satisfies $c^* > c^l$ (Lemma 10). From Carroll and Kimball (1996) and Carroll (2004) it follows that $\tilde{c}(\tilde{x})$ is concave. This implies that ι as defined in Lemma 10 is also a lower bound to the slope of the optimal policy function in ratio form $\tilde{c}(\tilde{x})$. If an equilibrium exists, there must exist states where agents spend less than there current income, and states where they spend more than their current income. Current income in the reduced state space is

$$\frac{r}{1+r}\tilde{x} + \frac{1}{1+r}$$

and it can be easily shown that $\frac{r}{1+r} \leq \underline{\iota}$ in equilibrium because $\beta(1+r) \leq 1$

$$\underline{\iota} = 1 - (1+r)^{-1}\left(\beta(1+r)\right)^{\frac{1}{\gamma}} \geq 1 - \frac{1}{1+r} = \frac{r}{1+r}$$

If borrowing constraints are binding, then it holds for some $\tilde{x}$ that $\tilde{c}(\tilde{x}) = \tilde{x}$ and the continuity and the slope restriction for $\tilde{c}(\tilde{x})$ imply $\tilde{c}(\tilde{x}) > \frac{r}{1+r}\tilde{x} + \frac{1}{1+r}$ for all $\tilde{x}$. However, a situation where agents always spend more than their current income is not compatible with the existence of an equilibrium. This contradiction proves that borrowing constraints must always be non-binding in a RCE of this model. $\square$

Corollary 1. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then there is a unique $\bar{\tilde{x}}$ (*target insurance rate*) exists such that the optimal policy yields $a_t = a_{t+1}$.*

Proof. In equilibrium the optimal policy of the agent must such that optimal consumption is for some state smaller and for some states larger than current income. It follows directly from the continuity and concavity of the optimal policy function together with the lower bound c^l on the optimal policy that there must be a unique intersection of the optimal policy with current income. This intersection characterizes $\bar{\tilde{x}}$. $\square$

Corollary 2. *Given the assumptions of theorem 3, the equilibrium interest rate r lies in the interval $\left((\beta\mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1; \beta^{-1} - 1\right)$*

Proof. The upper bound follows from lemma 14. The lower bound can be derived from the fact that borrowing constraints are always non-binding. The Euler equation for the reduced state space variables and zero assets implies that if borrowing constraints are non-binding, then

$$\begin{aligned} & 1 < \beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \\ \Longleftrightarrow \quad & r > \left(\beta\mathbb{E}[\varepsilon^{-\gamma}]\right)^{-1} - 1 \end{aligned}$$

$\square$

[28] The result can also be used on the reduced state space as it is shown in Carroll (2004). The argument by Carroll and Kimball (1996) involves iteration on the Bellman equation but applies here as well because the sequences of consumption functions of the two approaches are equivalent. This can be easily verified because $G_i(x, z, \lambda) = 0$ is the necessary condition for updating the value function using the Bellman equation.

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