Equilibrium Market and Pricing Structures in Virtual Platform Duopoly

Behringer, Stefan

Goethe Universität Frankfurt

21 December 2005

Equilibrium Market and Pricing Structures in Virtual Platform Duopoly

Stefan Behringer
Universität Frankfurt

December 21, 2005

Abstract

We investigate the equilibrium market sturcture in virtual platform duopoly (auctions or other market forms) that are prevalent in internet settings. We take full account of the complexity of network effects in such markets and determine optimal pricing strategies. We invstigate the welfare implications of such strategies, look at the impact of non-exclusive services and at what happens in large markets.

1 Introduction

Virtual market platforms such as auction platforms often reveal very different price strategies despite the fact that such intermediaries offer homogenous products. Competition between Yahoo! vs. eBay in the US (see P.-L. Yin (2004)) or Mobile.de vs. Autoscout24 the German used-car platforms (see FAZ (2003) suggests that a cheaper platform is not able to attract all (or even more) customers. In the latter case, the unilateral introduction of seller charges has only led to very minor and temporary changes in market shares. An important explanation for this observation is the presence of network externalities.

Optimal pricing strategies in two-sided platform duopoly have been investigated recently by Rochet, J.-C. & Tirole, J. (2003) and M. Armstrong (2004). Both papers focus on intermediation services for two types of agents but refrain from modeling a stage in which transactions take place explicitly. Intermediation between heterogenous agents such as buyers and sellers who bargain about some good generates direct, 'congestion' externalities (from more agents of their own type) and indirect network externalities (from more agents of the other type). These more complex forms of network externalities often remain unmodelled an exception being the work of Ellison, G., Fudenberg, D., & Möbius, M. (EFM, 2004).

The role of 'congestion' externalities on the transaction price is very relevant in practice as has been argued in the business literature for sellers on eBay auctions (see for example McGrath, S. (2005)). Whereas the analysis in EFM shows that stable equilibria in such markets may be asymmetric, the consequences of asymmetry for optimal platform pricing strategies is not pursued (see Section 7 in their paper). Endogenous competitive pricing decisions are studied by Caillaud, B. & Jullien, B. (2001, 2003) albeit in a continuum model with homogenous agents on both sides of the platform focusing on indirect network externalities only.

In this paper we model price competition of virtual platforms extending the framework of EFM. We first look at the effects of introducing exogenous charges into the model and investigate the effects on the equilibrium market structure. We then determine charges endogenously and find that *optimal* pricing decisions of platforms with potentially asymmetric market shares of buyers and sellers exist and reveal strategic complementarities. Focusing on the role of prices in this setting also allows us to tighten the set of equilibria of the model compared to EFM and shows that we can often focus on a particularly simple sub-class of equilibria which allows us to investigate the dependence prices on the fundamentals of the model.

The model in EFM reveals a disconnected equilibrium set as corners (or "dominant firm" outcomes) are part of the equilibrium set of the game but there is a strictly positive critical mass of buyers necessary to make a second auction site viable (see Section 4 in their paper). Their finding that two competing platforms can coexists in equilibrium despite offering homogenous products is thus not robust to extreme differences in size and raises implications for the viability of new entrants in such markets and hence for regulatory concerns. We are able to show that optimal seller charges can reestablish the connectedness of the equilibrium set so that two competing platforms may coexist even for extreme size differences. We eventually investigate the behaviour of the model for platforms with many and possibly multihoming agents.

2   The Model

We model the duopolistic platform competition departing from a simple two-stage game presented in EFM (2004).

The timing of the game is as follows: In the first stage risk-neutral buyers ($B \in \mathcal{N}_0$) with uniformly and i.i.d. distributed valuation from the unit interval and risk-neutral sellers ($S \in \mathcal{N}_0$) simultaneously decide whether to attend platform 1 or platform 2 before knowing their valuation for the homogeneous good.

In the second stage they learn their valuations and bargaining for the object takes place. We model this bargain as a uniform price (multiobject if $S > 1$) auction on each platform. By the revenue equivalence theorem this choice of the bargaining process is quite general. Each buyer only demands one homogeneous good. In order to guarantee strictly positive prices we make the 'non-triviality assumption' that

$$B > S + 1 \tag{1}$$

for both being positive integers. Risk neutral sellers have zero reservation value and their expected utility is given by the expected price on their chosen platform. A buyer's utility on a platform with B buyers and S sellers is given by his expected net utility conditional on winning the good i.e.

$$u_B = E\left\{v - v^{S+1,B} \middle| v \geq v^{S,B}\right\} \Pr\left\{v \geq v^{S,B}\right\} \tag{2}$$

where $v^{k,n}$ gives the k highest order statistic of a draw of n values and thus in this auction format the uniform price is simply the $S + 1$ highest of the buyers valuations $v^{S+1,B}$ (i.e. the highest losing bid). This is the typical mathematical convention as long as we deal with a discrete model.

Larger markets are more efficient than smaller ones as they come closer to the ex-post efficient outcome to allocate a good to a buyer iff his valuation is high. The ex-post efficient outcome implies that the buyers with the S highest values obtain the good, so that the expectation of the maximum total ex-ante surplus (welfare) is

$$B \Pr\left\{v \geq v^{S,B}\right\} E\left\{v \middle| v \geq v^{S,B}\right\} = SE\left\{v \middle| v \geq v^{S,B}\right\} =$$
$$SE\left\{v \middle| v > v^{S+1,B}\right\} = S \int_0^1 \left(\int_x^1 v f(v \middle| v > x) dv\right) f^{S+1,B}(x) dx \tag{3}$$

where $f^{S+1,B}$ is the density function of $v^{S+1,B}$, the $S + 1$ highest order statistic of a draw of B values under the uniform distribution.

Lemma 0 (EFM):

Under the uniform distribution total welfare on one platform can be written as the sum of buyer and seller utilities

$$w(B,S) = S(1 - \frac{1}{2}\frac{1+S}{B+1}) = S\left(\frac{B-S}{B+1}\right) + B\left(\frac{S(1+S)}{2B(B+1)}\right)$$

Proof:
See Appendix.∎

The result is intuitive: The total value of a sale is $E\left\{v\,|\,v>v^{S+1,B}\right\}$, i.e. expected value of v *given* $v>p$. Under the uniform distribution this is $1-\frac{1}{2}\frac{1+S}{B+1}=p+\frac{1-p}{2}$. Clearly the second term is the value for one buyer $E\left\{v-v^{S+1,B}\,\middle|\,v>v^{S+1,B}\right\}=\frac{1-p}{2}$ with the remaining p (as calculated above) going to the seller and to obtain total welfare we multiply with the number of sales.

Note that

$$\frac{\partial w(1,\frac{S}{B}=\bar{x}<1)}{\partial B}=\frac{1}{2}\bar{x}\frac{(2-\bar{x})(B+2)B+1}{(B+1)^2}>0 \tag{4}$$

showing that for constant shares of sellers to buyers larger markets are more efficient than smaller ones. The efficiency deficit makes it more difficult for small markets to survive but the sequential structure of the game allows for equilibria with two active platforms whenever the impact of switching of buyer and/or seller on his expected surplus more than outweighs the efficiency advantage.

The game is solved by backward induction and the solution concept is Sub-game Perfect Nash equilibrium (SPNE). The transaction of the good in stage two yields ex-ante utility in stage one for a seller of

$$u_S(B,S)=p=\frac{B-S}{B+1} \tag{5}$$

and for a potential buyer of

$$u_B(B,S)=\frac{1-p}{2}\frac{S}{B}=\frac{S(1+S)}{2B(1+B)}. \tag{6}$$

Note that holding S/B (the relative advantages of buyers and sellers) constant, sellers prefer larger, *more liquid* markets (where the expected equilibrium price is higher) and buyers prefer small, less efficient markets as

$$\frac{\partial u_S(1,\frac{S}{B}=\bar{x}<1)}{\partial B}=\frac{\partial p(1,\frac{S}{B}=\bar{x}<1)}{\partial B}>0 \tag{7}$$

and

$$\frac{\partial u_B(1,\frac{S}{B}=\bar{x}<1)}{\partial B}<0 \tag{8}$$

4

Extending the setting of EFM, platforms may charge buyers and/or sellers some fee for participating. Without loss of generality we assume that such a fee takes a non-negative value.

As buyers and sellers simultaneously decide which platform to join in stage one, we can set up the relevant constraints that determine the set of *all* possible SPNE of the game subject to the qualification that the integer constraint holds otherwise we will speak of a *quasi-equilibrium*. The constraints to keep buyers in place in stage one given buyer charge difference $p_{2B} - p_{1B} \equiv \Delta_B \geq 0$ are (B1)

$$u_B(B_1, S_1) \geq u_B(B_2 + 1, S_2) - \Delta_B \tag{9}$$

and (B2)

$$u_B(B_2, S_2) - \Delta_B \geq u_B(B_1 + 1, S_1) \tag{10}$$

In words: A buyer on platform 1 needs to have an expected utility from the bargaining stage correcting for charges paid to the platform owner such that a change to the other platform and the implied effect on the equilibrium bargaining outcome there deters him from doing so.

To keep sellers in place in stage one given seller charge difference $\Delta_S \geq 0$ we need (S1)

$$u_S(B_1, S_1) \geq u_S(B_2, S_2 + 1) - \Delta_S \tag{11}$$

and (S2)

$$u_S(B_2, S_2) - \Delta_S \geq u_B(B_1, S_1 + 1) \tag{12}$$

to hold. The motivation for the constraints is analogous.

3 Exogenous buyer charges

We now look explicitly at the form of the constraints and thus at the set of possible SPNE with some exogenous charge differences $\Delta_B > 0$ to (winning) buyers in auction two. Note that this does not imply that charges are made only by one of the platforms but only that it is the difference between such charges that influence location incentives.
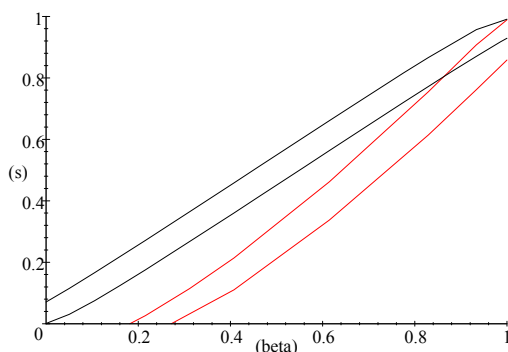
Denoting s as the *share* of sellers on platform one and β as the *share* of buyers at platform one the buyer constraint (9) becomes

$$\frac{sS(1+sS)}{2\beta B(1+\beta B)} \geq \frac{(1-s)S(1+(1-s)S)}{2((1-\beta)B+1)(1+(1-\beta)B+1)} - \Delta_B \qquad (13)$$

and (B2) is

$$\frac{(1-s)S(1+(1-s)S)}{2(1-\beta)B(1+(1-\beta)B)} - \Delta_B \geq \frac{sS(1+sS)}{2(\beta B+1)(1+\beta B+1)} \qquad (14)$$

A numerical example (with $B = 10, S = 5$) may make clear how the buyer constraints change. The two buyer constraints with $\Delta_B = 0$ and $\Delta_B = 0.2$ are:

where the share of sellers on platform one (s) is on the ordinate and the share of buyers on platform one (β) is on the abscissa.

The interpretation of this finding is as follows: The lower (B1) constraint gives the condition that buyers stay on platform one if the fraction of sellers s is large enough or, alternatively if β is low enough. The higher, (B2) constraint gives the condition under which buyers stay on platform 2, i.e. if s is small (and thus $(1-s)$ the fraction of seller on his own platform is large enough). Between the two curves is the candidate set of SPNE (we still need to check if the seller constraints hold).

Now with a charge of $\Delta_B > 0$ to buyers on the second platform both the (B1) and the (B2) constraint shift downwards, i.e. the set of SPNE allows for equilibria with a lower share of sellers on platform one for a given share of buyers. The (B2) constraint also shifts downwards. i.e. buyers move from the second platform at higher levels of s already, (and thus for a lower fraction of seller $(1-s)$ on his own platform) than before given the new charge.
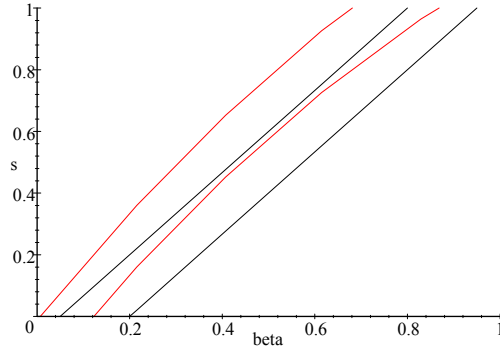
4 Exogenous seller charges

We now introduce an exogenous charge difference Δ_S for sellers of platform 2. Seller constraints are (S1)

$$\frac{\beta B - sS}{\beta B + 1} \geq \frac{(1-\beta)B - ((1-s)S + 1)}{(1-\beta)B + 1} - \Delta_S \tag{15}$$

and (S2)

$$\frac{(1-\beta)B - (1-s)S}{(1-\beta)B + 1} - \Delta_S \geq \frac{\beta B - (sS + 1)}{\beta B + 1} \tag{16}$$

With $\Delta_S = 0.3$ we find the picture with the seller constraints becomes:

The interpretation of this finding is as follows: For the upper linear (S1) constraint, a seller stays on platform 1 if s is not too high for a given share of β, otherwise he will go to platform 2. For the lower linear (S2) constraint, a seller stays at platform 2 if s is high (i.e. his own seller share $1-s$ is low) otherwise he will go to platform one. Between the two curves is the candidate set of SPNE (we need to check if the buyer constraint holds simultaneously).

Now that there is a charge of $\Delta_S > 0$ to the sellers on the second platform, the (S1) constraint in no longer linear and shifts upwards: Seller stay at platform 1 even if s is much higher than before for given β. Similarly the (S2) constraint is no longer linear and also shifts upwards: Seller will move from platform 2 even if s is much higher (hence their own seller share $1 - s$ much lower) than before.

The numerical example with $\Delta_S = 0.3$ yields both seller and buyer constraints as
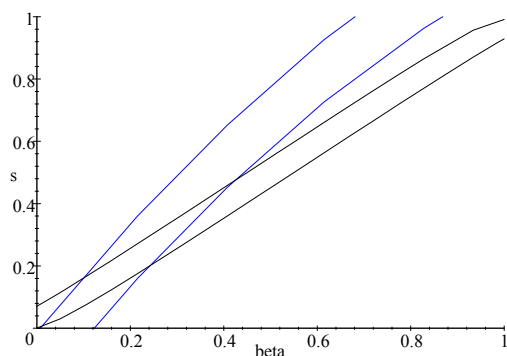
Only $\beta = 0.2, s = 0.2$ is a viable equilibrium here and the previous candidate $\beta = 0.4, s = 0.4$ is no longer viable.

The result reveals that charging sellers on platform 2 allows for higher s tolerance for given β on platform 1. Also, equally sized platforms are no longer viable. As sellers like larger, more liquid platforms where the uncertainty about the resulting final price is lower we find the following claim:

Claim 1: *A positive and exogenous relative seller charge difference of platform 2 can only be an equilibrium if platform 2 also has the larger share of sellers.*

5 Equilibrium properties

We now propose some more general results that allow us to characterize the set of SPNE more tightly than in EFM:

Definition 1 *We call a platform duopoly equilibrium "proportional" if the fraction of buyers and sellers on each platform is identical.*

Absent charges the set of subgame perfect quasi-equilibria as defined by the incentive constraints is disconnected from the cornered market outcomes. The cornered market outcomes ($\beta = 0, s = 0$ and $\beta = 1, s = 1$), i.e. the case where the market is "tipping" is always a true equilibrium and hence by definition part of the quasi-equilibrium set.

Lemma 2 *Given $\Delta_S = \Delta_B = 0$ the set of SPNE is not connected.*

Proof:

From looking at the numerical example for $\Delta_S = 0$ the seller constraint (S1) prevents corner outcomes (0,0) just as symmetrically (S2) prevents corner outcomes (1,1). The first constraint (15) reveals the intercept with the abscissa as

$$\beta = \frac{B - S - 1}{B(3 + S)} > 0 \tag{17}$$

which always holds from non-triviality.∎

The practical implication of this finding is that there may always be an outcome with only one active platform but given that there are two platforms operating there exists a *critical mass of buyers* necessary to render this second platform operational. (See EFM (2004), Proposition 4).

Most importantly we find that the set of SPNE can be characterized further than undertaken in EFM and we may often focus on the particular class of proportional equilibria.

Proposition 3 *Given $\Delta_B = 0$ the set of SPNE of the game does not contain non-proportional equilibria even if exogenously $\Delta_S > 0$.*

Proof:
See Appendix.∎

The fact that the buyer constraint is the 'stricter' one relative to the seller constraint around the proportional quasi-equilibrium set can be seen in the numerical example. That this result holds for any B, S is quite intriguing and can be rationalized by the fact that by the non-triviality assumption there are strictly more buyers than sellers and hence they are more averse to inequalities with regard to the buyer seller ratio (and hence their ex-ante probability to obtain the good in the auction) than sellers.

6   Welfare and Equilibrium refinements

Total welfare, given as the sum of welfare on each platform can be written as

$$W(\beta, s, B, S) = \frac{1}{2} S \frac{\begin{array}{c}(2S + SB)\, s^2 + (B - 2S - 2SB\beta - 2B\beta)\, s \\ -2B + B\beta + S + SB\beta - 2B^2\beta - 1 + 2B^2\beta^2\end{array}}{(B\beta + 1)\,(-B + B\beta - 1)}$$

subject to the constraints that $0 \leq s, \beta \leq 1$ for any $\Delta_B, \Delta_S \geq 0$ as those charges are only redistributed between buyers, sellers and the proprietors of the platform. We then find that

Proposition 4 *Constraint maximization of the welfare function yields corner outcomes $W(0, 0, B, S)$ and $W(1, 1, B, S)$ for all B, S.*

Proof:
See Appendix.∎

Lemma 5 *Welfare of proportional equilibria is strictly decreasing in β until $\beta = \frac{1}{2}$, the welfare worst proportional (quasi-)equilibrium.*

Proof:
See Appendix.∎

As seen above, holding the relative advantages of buyers and sellers constant, sellers prefer large markets (where the expected equilibrium price is higher) and buyers prefer small, less efficient markets. The previous Proposition shows that aggregating these welfare differentials from an overall welfare perspective, a single platform is optimal in the set of *all* SPNE. From the above Proposition in conjunction with the previous Lemma we can conclude that if $\Delta_B = 0$ then an exogenous charge difference $\Delta_S > 0$ will *always* be welfare improving.

As noted by EFM, the multiplicity of equilibria of the game cannot be disposed of by simple equilibrium refinement as outcomes cannot be Pareto-ranked. Hence, for example, a coalition proof Nash equilibrium has no bite here. A single Pareto-optimal equilibrium does not exit and thus we may not reduce the set of SPNE set to some focal point. This observation makes the implication of Proposition 3 even more valuable as it allows us to restrict the SPNE set without further refinements and we will make ample use of the result below.

7 Endogenous price competition

In order to discuss price formation in the above platform game we now introduce a *pricing game* in a stage prior to the two-stage game above. In order to tackle the issue of multiple equilibria in this game we make the assumption of "bad-expectation" *beliefs* (as in Caillaud, Jullien, 2003, p. 314) such that for any given equilibrium in market shares a price deviation of a platform that violates any of the incentive constraints (S1),(S2),(B1),(B2) will lead to a new market equilibrium allocation in which the profit of the deviating platform is strictly lower.

In other words: A platform charging sellers such that each one would be better off leaving the current equilibrium allocation for the other platform will also convey a signal to buyers whose beliefs are such that they will also decide to leave. By assumption then both will coordinate on a new equilibrium that yields strictly lower profits to the deviating platform. Given that optimal prices are fully anticipated by buyers and sellers when they decide about locations *a rational expectations equilibrium* can be derived.

We find the following result for the case where networks simultaneously decide about profit-maximizing seller charges:

Proposition 6 *If the model has a rational expectations equilibrium in stage two with S_1 sellers and B_1 buyers on platform 1 and S_2 sellers and B_2 buyers on platform 2 then the pricing game has an equilibrium where platform 1 charges sellers at*

$$p_{1s}^* = f_s + \frac{1}{3} \frac{S_1 B_2 + 2S_2 B_1 + 2S_2 + S_1 + 3 + 3B_1}{(B_1 + 1)(B_2 + 1)} \tag{18}$$

and symmetrically platform 2 charges sellers at

$$p_{2s}^* = f_s + \frac{1}{3} \frac{S_2 B_1 + 2S_1 B_2 + 2S_1 + S_2 + 3 + 3B_2}{(B_1 + 1)(B_2 + 1)} \tag{19}$$

Proof:
See Appendix.∎

Note that equilibrium prices are *strategic complements* here, i.e. the reaction functions are increasing in the price choice of the other platform. We can derive the following equilibrium comparative statics results:

Lemma 7 *The equilibrium seller charge (and profit) of platform 1 satisfies the following comparative statics*

$$\frac{\partial p_{1s}^*}{\partial S_1} = \frac{1}{3(B_1 + 1)} > 0$$

$$\frac{\partial p_{1s}^*}{\partial S_2} = \frac{2}{3(B_2 + 1)} > 0$$

$$\frac{\partial p_{1s}^*}{\partial B_1} = -\frac{1}{3}\frac{S_1}{(B_1 + 1)^2} < 0$$

$$\frac{\partial p_{1s}^*}{\partial B_2} = -\frac{1}{3}\frac{2S_2 + 3}{(B_2 + 1)^2} < 0$$

Proof:
By differentiation.∎

This result may surprise at first. Although the expected gains of a seller in the bargaining stage are decreasing in the number of other sellers on its platform the optimal seller charge paid to platform owners increases. Why is this the case? Note that the optimal seller charge is also increasing in the number of sellers on the *other* platform and at a larger magnitude if $2B_1 > B_2$. This implies that a larger own share of sellers also increases the price of the other platform which relaxes the participation constraint for sellers on platform 1 (S1) so that a platform can increase its own seller charge without violating the constraint.

The reasoning for the change in the seller charge in the number of own buyers is more involved. Having more buyers on any platform will reduce the optimal seller charge but the difference in magnitude depends on the number of buyers and sellers on both platforms. If there are relatively few sellers and many buyers on platform 1 again the cross-effect is stronger.

If we rewrite equilibrium allocations in terms of *market shares* we find

$$p_{1s}^* = f_s + \frac{1}{3}\frac{sS(1-\beta)B + 2(1-s)S\beta B + 2(1-s)S + sS + 3 + 3\beta B}{(\beta B + 1)((1-\beta)B + 1)} \tag{20}$$

and

$$p_{2s}^* = f_s + \frac{1}{3}\frac{S\beta B - 3sS\beta B + 2sSB + sS + S + 3 + 3B - 3\beta B}{(\beta B + 1)((1-\beta)B + 1)} \tag{21}$$

By Proposition 3 we can focus on proportional equilibria, thus

$$p_{1s}^*(s = \beta) = f_s + \frac{1}{3}\frac{(3\beta((1-\beta)S + 1))B + 3 + S(2-\beta)}{(\beta B + 1)((1-\beta)B + 1)} \tag{22}$$

Interestingly eBay now prevents seller-fee-shifting, i.e. prevents sellers from passing on sales- and listing-fees onto buyers which strengthens the case for looking at proportional equilibria when looking at actual competition of on-line auctions.

With a symmetric market allocation we find

$$p_{1s}^*(s = \beta = \frac{1}{2}) = f_s + \frac{S+2}{B+2} \tag{23}$$

which is increasing in the total number of sellers (S) and decreasing in the total number of buyers (B). More general equilibrium comparative statics results are:

Lemma 8 *The equilibrium seller charge (and profit) of platform 1 in terms of market shares satisfies the following comparative statics*

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial S} > 0$$

and for sufficiently many buyers B

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial B} < 0$$

and

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial \beta} > 0$$

Proof:
See Appendix.∎

Thus we find the result that optimal seller charges are always increasing if the total number of sellers on the market increases. With regard to an increase in the total number of buyers or the buyer market share of platform 1 there exists a critical total number of buyers such that the optimal seller charges of platform 1 is decreasing in the former and increasing in the latter. Clearly profits π_1^* of platform 1 reveals the same comparative statics.

We may also look at what happens to equilibrium seller utility. From above, platform 1 equilibrium seller utility will clearly be increasing in B_1 and decreasing in S_1 as the expected utility of the bargaining stage is reduced and optimal charges paid to the platform owner is increased. For B_2 and S_2 only the price effect matters so that the equilibrium seller utility increasing in the former and decreasing in the latter.

Rewriting equilibrium allocations in terms of *market shares* and using Proposition 3 we also find:

Lemma 9 *The equilibrium platform 1 customer utility satisfies the following comparative statics*

$$\frac{\partial u_{1s}^*(s = \beta)}{\partial S} < 0 \qquad (24)$$

and

$$\frac{\partial u_{1B}^*(s = \beta)}{\partial \beta} < 0 \qquad (25)$$

Also for sufficiently many buyers B

$$\frac{\partial u_{1s}^*(s = \beta)}{\partial B} > 0 \qquad (26)$$

and

$$\frac{\partial u_{1s}^*(s = \beta)}{\partial \beta} < 0 \qquad (27)$$

if $\beta > \frac{1}{2}$.

Proof:
See Appendix.■

We see that an increase in the total number of sellers unambiguously reduces net utility of sellers on platform 1 as it implies a higher optimal seller charge both because of more sellers on the own but also because of more sellers on the other platform. In addition the expected gains in the bargaining stage are reduced. As buyers are unaffected by seller price changes only the bargaining stage matters. Here in accordance with (8) buyers prefer the more illiquid market.

Given there are many buyers, (which guarantees u_{1s}^* to be strictly positive) optimal seller charges are reduced by a further increase as seen in the previous Lemma and expected gains in the bargaining stage are increased leading to a

higher expected seller utility. In contrast to (7) with endogenous seller charges now sellers on the larger platform would also prefer a more illiquid market as they face prices that are increasing in market scale which then outweigh their preference for more liquid markets (and higher expected gains in the bargaining stage).

Now we look at the implications of different prices on differences in the number of buyers and sellers.

Lemma 10 *Equilibrium prices satisfy* $p_{1s}^* > p_{2s}^*$ *iff*

$$\frac{S_2}{(B_2 + 1)} - \frac{S_1}{(B_1 + 1)} + \frac{3(B_1 - B_2)}{(B_1 + 1)(B_2 + 1)} > 0 \tag{28}$$

or in terms of market shares iff

$$\beta = s > \frac{1}{2}. \tag{29}$$

Proof:
See Appendix.∎

Note that the LHS in (28) is increasing in B_1 and S_2 and decreasing in B_2 and S_1. Looking at Lemma 7 we see that a larger S_2 will imply that p_{1s}^* is increasing and this effect is more pronounced if B_2 is small.

Note that Claim 1 above for exogenous charges has established a positive seller price differential as being a necessary condition for having a larger share of sellers. When endogenizing seller prices we find that the condition is also *sufficient*.

Claim 2: *A positive and endogenous relative seller charge difference of platform 2 is an equilibrium iff platform 2 also has the larger share of sellers.*

15

Given the optimal prices above are rationally anticipated by buyers and sellers when they decide about locations with charge differences being

$$\Delta_s^* = p_{2s}{}^* - p_{1s}{}^* = \frac{1}{3}\frac{(s-\beta)SB + (2s-1)S + 3(1-2\beta)B}{(\beta B+1)\left((1-\beta)B+1\right)} \tag{30}$$
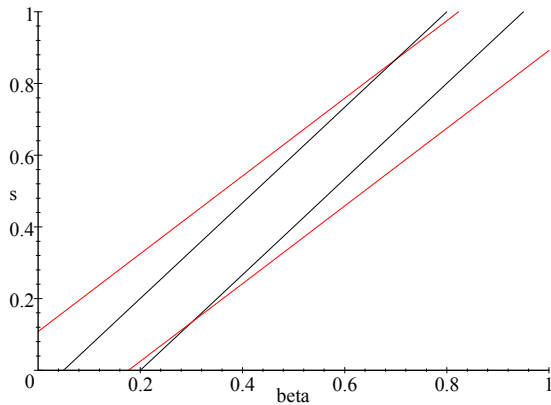
seller incentive constraints change accordingly.

Proposition 11 *Given $\Delta_s = \Delta_s^*$ the set of SPNE is connected.*

Proof:
See Appendix.∎

For the numerical example we find the following picture:

Corner equilibria where one network does not have any buyers and sellers are now connected to the quasi-equilibrium set. In other words: there no longer is a critical mass of buyers necessary to make *any* platform site viable.

8   Large platforms

The above analysis finds that equilibria of this game may have non-Bertrand outcomes where pricing differences between the two platforms may prevail in subgame perfect equilibrium. We now investigate the robustness of this property of the model for large platforms.

Proposition 12 *On large platforms any equilibrium is proportional and charges satisfy* $\Delta_B = \Delta_S = 0$.

Proof:
See Appendix.∎

The intuition for this limit result is straightforward: The possibility that the switching of either buyer or seller has a tangible impact on expectations decreases as the number of buyers and sellers increases so that in the limit the friction disappears from the model and we get a Bertrand type outcome with regard to the charge differences and proportional equilibria. This Proposition can be easily extended to an unspecified distribution of valuations and is thus highly robust.

We also have a result for welfare on large platforms: As total welfare of a platform goes out of bounds if the platform gets large we look at total welfare per buyer and seller respectively

$$\frac{w(B,S)}{B} = u_B(B,S) + \bar{x}u_S(B,S) = \bar{x}(1 - \frac{\bar{x}}{2}) \tag{31}$$

and

$$\frac{w(B,S)}{S} = \frac{1}{\bar{x}}u_B(B,S) + u_S(B,S) = 1 - \frac{\bar{x}}{2} \tag{32}$$

and by the non-triviality assumption the per capita welfare contribution of a buyer is always lower than that of a seller.

9 Multihoming buyers and sellers

So far the analysis has assumed that in the first stage of the game buyer and sellers exclusively choose which platform to locate on being sensitive to relative prices. In actual markets (especially in Internet marketplaces) buyers (and to a lesser degree sellers) may decide to be present on *both* platforms.

We denote the total number of multihoming buyers as M and that of multihoming sellers as N both of which also only buy and sell one good. We find the following result:

Proposition 13 *Any numerical example with $B = a > 0$, $S = b > 0$, $M = c > 0$, and $N = d > 0$ satisfying $B + M > S + N + 1$ and $a, b, c, d \in \mathcal{N}_0$ can be transformed into one with $B = a + c$, $S = b + d$, $M = 0$, and $N = 0$ such that the incentive analysis in the location stage remains unchanged.*

Proof:
See Appendix.∎

Whence all the above location results remain valid as having multihoming buyers and sellers affects seller and buyer incentives just as having more singlehoming buyers and sellers would do. In particular, the results about the prevalence of proportional equilibria remain intact as they are valid for *any B* and S. Having more buyers just tightens the buyer switching constraint further.

If we have $B = 10$, $S = 5$, $M = 10$ and $N = 0$ the constraints of the singlehoming buyers and sellers (S1,2) and (B1,2) can be represented by $B = 20$, $S = 5$, $M = 0$ and $N = 0$. We have to observe that the equilibrium set is constrained by the fact that $s_{\min} = \frac{S_1 + N/2}{S + N} = 0$ and $\beta_{\min} = \frac{B_1 + M/2}{B + M} = \frac{5}{20} = \frac{1}{4}$ now whereas previously $\beta_{\min} = 0$. The result is intuitive as multihomers benefits both platforms simultaneously and hence equilibrium market shares become 'more equal'. Note that if we have $M \to \infty$, (or $\beta = 0$ and $M > 0$) we find an equal market sharing outcome as the quasi equilibrium set degenerates to a point whereas $B \to \infty$ allows for a continuum of proportional equilibria.

10   Conclusion

If buyers cannot be charged for participating on a platform, (or getting the good, which is the case at platforms such as Mobile.de where the transaction cannot be observed or at eBay where seller-fee-shifting is not allowed) the strictness of the buyer switching constraints implies that independently of whether or not there are charges to the sellers, the equilibrium market structure of the two platform duopoly will imply proportional equilibria which in contrast to EFM restricts the set of equilibria of the game substantially.

Endogenizing the pricing choice of the two platforms we find further evidence for why it is that the platform with the larger share of sellers also has the higher seller charge which turns out to be profit maximizing. Contrary to this, having more buyers on ones platform will actually lead to a lower optimal equilibrium seller charge being asked for in equilibrium. What were necessary conditions for the link between the exogenous seller charge differentials and market shares in the equilibria of the simple location game turns out to be necessary and sufficient once a pricing game is introduced in which platforms set prices endogenously. We also show that given sufficiently liquid markets we can expect prices to rise in a duopoly with increasing total numbers of sellers and decreasing with total numbers of buyers, no matter on which platform they decide to locate. We further show that given optimal seller charges, sellers may prefer to be on *less liquid* platforms (as do buyers) contrary to the case without charges in EFM as charges on such liquid platforms will be higher, overcompensating the increased benefits from the auction stage.

Our results are in accordance with casual observations of the virtual auction platform competition between Yahoo! and eBay in the US. As eBay has gained market share against its competitors by increasing its advertising and word-of-mouth spreading it was able to raise its charges to sellers without loosing them to competitors on the ground that being the relatively more liquid platform increased the seller's willingness to pay for the service. Charges for buyers are not imposed at all so we can be confident that buyer switching guarantees proportional equilibrium outcomes and hence a situation where a seller charge differential (as is often the case in practice, see eBay vs. Yahoo or Mobile.de vs. Autoscout24) is welfare superior. If this seller charge differential is the result of optimal platform pricing choices we may also observe platforms with extremely different sizes to co-exist.

11 Appendix

Proof of Lemma 0:

Under the uniform distribution on [0,1] the ith *lowest* order statistic out of n draws is distributed $\text{Beta}(i, n - i + 1)$ with probability density function

$$f^{i,n-i+1}(x) = \begin{cases} \frac{x^{i-1}(1-x)^{n-i}}{\int_0^1 u^{i-1}(1-u)^{n-i}du}, & x \in (0,1) \\ 0 \text{ otherwise} \end{cases} \tag{33}$$

and expectation

$$\int_0^1 x f^{i,n-i+1}(x)dx = \frac{i}{n+1} \tag{34}$$

As the order statistic of the $S + 1$ highest of B draws is also that of the $B - S$ *lowest,* the expectation of the price given by the $S + 1$ highest buyer valuation can be rewritten as

$$\int_0^1 x f^{S+1,B}(x)dx = \frac{B - S}{B + 1} \tag{35}$$

which is also expected seller surplus due to the normalized reservation value. Thus the density of the order statistic $v^{S+1,B}$ is

$$f^{S+1,B}(x) = \begin{cases} \frac{x^{B-S-1}(1-x)^S}{\int_0^1 u^{B-S-1}(1-u)^S du}, & x \in (0,1) \\ 0 \text{ otherwise} \end{cases} \tag{36}$$

Total welfare $w(B, S)$ on one platform given uniformly distributed valuations can thus be written as

$$w(B,S) = S \int_0^1 \left(\int_x^1 v f(v \,|\, v > x)dv \right) f^{S+1,B}(x)dx =$$

$$\frac{S}{\int_0^1 u^{B-S-1}(1-u)^S du} \int_0^1 \left(\int_x^1 v \left(\frac{1}{1-x} \right) dv \right) (x^{B-S-1}(1-x)^S)dx =$$

$$\frac{S}{2 \int_0^1 u^{B-S-1}(1-u)^S du} \int_0^1 (x + 1)(x^{B-S-1}(1-x)^S)dx =$$

$$= S(1 - \frac{1}{2}\frac{1+S}{B+1}) = S\left(\frac{B-S}{B+1} \right) + B\left(\frac{S(1+S)}{2B(B+1)} \right). \blacksquare$$

Proof of Proposition 3:

We look at the seller constraint and the buyer constraint in turns and show whether it is possible to have them satisfied for non-proportional equilibria. Constraint (S1) is

$$\frac{B\beta - Ss}{B\beta + 1} = \frac{B(1-\beta) - (S(1-s)+1)}{B(1-\beta) + 1} - \Delta_S \tag{37}$$

or

$$s_{(S1)} = \frac{3B\beta - B + SB\beta + S + 1 + \Delta_S B^2\beta - \Delta_S B^2\beta^2 + \Delta_S(B+1)}{S(B+2)} \tag{38}$$

and (S2) is

$$\frac{B(1-\beta) - S(1-s)}{B(1-\beta) + 1} - \Delta_S = \frac{B\beta - (Ss+1)}{B\beta + 1} \tag{39}$$

or

$$s_{(S2)} = \frac{-2B + 3B\beta + SB\beta + S + \Delta_S B^2\beta - \Delta_S B^2\beta^2 + \Delta_S(B+1) - 1}{S(B+2)} \tag{40}$$

with vertical difference between the two seller constraints

$$s_{(S1)} - s_{(S2)} = \frac{1}{S} \tag{41}$$

for any Δ_S. Thus it may be possible to have the seller constraint strictly satisfied at a non-proportional equilibrium by 'squeezing in' non-proportional equilibrium candidate vertically.

For $\Delta_S = 0$ we have

$$\beta_{(S1)} = \frac{SsB + 2Ss + B - S - 1}{B(3+S)} \tag{42}$$

and

$$\beta_{(S2)} = \frac{2B - S + 2Ss + SsB + 1}{B(3+S)} \tag{43}$$

so that the horizontal difference between the two seller constraints is

$$\beta_{(S2)} - \beta_{(S1)} = \frac{B+2}{B(3+S)} > \frac{1}{B} \tag{44}$$

21

as $B > S + 1$. Thus it may be possible to have the seller constraint satisfied at a non-proportional equilibrium by 'squeezing in' non-proportional equilibrium candidate horizontally.

We now look at the critical buyer constraints: Now (B1) is

$$\frac{sS(1 + sS)}{2\beta B(1 + \beta B)} \geq \frac{(1 - s)S(1 + (1 - s)S)}{2((1 - \beta)B + 1)(1 + (1 - \beta)B + 1)} - \Delta_B \qquad (45)$$

and (B2) is

$$\frac{(1 - s)S(1 + (1 - s)S)}{2(1 - \beta)B(1 + (1 - \beta)B)} - \Delta_B \geq \frac{sS(1 + sS)}{2(\beta B + 1)(1 + \beta B + 1)} \qquad (46)$$

For $\Delta_B = 0$ the solution to (B1) is

$$\beta_{(B1)} = \frac{1}{2(1 + S)(2s - 1)} \times \qquad (47)$$
$$\frac{(4S + 2SB)s^2 + (-2S + 2 + 2B)s + 1 + S - \sqrt{\Psi}}{B}$$

and the one for (B2) is

$$\beta_{(B2)} = \frac{1}{2(1 + S)(2s - 1)} \times \qquad (48)$$
$$\frac{(4S + 2SB)s^2 + (2B - 2 - 6S)s + 3 + 3S - \sqrt{\Psi}}{B}$$

with

$$\Psi = 4S^2(B + 2)^2 s^4 - 8S^2(B + 2)^2 s^3 + \qquad (49)$$
$$\left(-16SB - 12 + 4S^2B^2 - 4SB^2 - 8S + 20S^2 - 16B + 16S^2B - 4B^2\right)s^2 -$$
$$4(1 + S)(S - 4B - B^2 - 3)s + 1 + S^2 + 2S$$

The horizontal difference is then

$$B(1) - B(2) = \frac{1}{B} \qquad (50)$$

Thus given that the two constraints with $\Delta_B = 0$ are always on opposite sides of the $\beta = s$ diagonal there will always be proportional equilibrium candidates and it is impossible to 'squeeze in' another non-proportional equilibrium candidate horizontally. Note that the result does not hold for $\Delta_B > 0$ although the horizontal difference remains the same.

22

The vertical difference between (B2) and (B1) is difficult to calculate directly. However we can use the fact that the distance between (B1) and the diagonal is monotone increasing in β and the mirror image, between (B2) and the diagonal is monotone decreasing in β which follows from buyers preference for the smaller platform. Hence if we can show that this distance for (B1) at $\beta = 1$ (or the distance for (B2) at $\beta = 0$ is smaller than $\frac{1}{S}$ again we can be sure that no non-proportional equilibrium can be 'squeezed in' next to the proportional equilibria on the diagonal.

Using (B1)

$$\beta_{(B1)} = \frac{1}{2(1+S)(2s-1)} \times \tag{51}$$
$$\frac{(4S+2SB)s^2 + (-2S+2+2B)s + 1 + S - \sqrt{\Psi}}{B}$$

we solve this equation for the relevant root and evaluate it at $\beta = 1$ to find

$$s_{(B1)} = -\frac{1}{2}\frac{\begin{array}{c}-2SB - 2SB^2 - B - B^2 - 2+\\ \sqrt{(4 + 4B + 5B^2 + B^4 + 2B^3 + 16SB + 8B^2S^2 + 8BS^2 + 16SB^2)}\end{array}}{S(B+B^2-2)} \tag{52}$$

The vertical distance to the diagonal is then given as

$$1 - s_{(B1)}(\beta = 1) = \tag{53}$$
$$\frac{1}{2}\frac{\begin{array}{c}-4S - B - B^2 - 2+\\ \sqrt{(4 + 4B + 5B^2 + B^4 + 2B^3 + 16SB + 8B^2S^2 + 8BS^2 + 16SB^2)}\end{array}}{S(B+B^2-2)}$$

Now

$$1 - s_{(B1)}(\beta = 1) < \frac{1}{S} \tag{54}$$

will hold if

$$B < -S \tag{55}$$

which cannot hold, or if

$$B > S - 1 \tag{56}$$

which holds by the non-triviality constraint that $B > S + 1.\blacksquare$

23

Proof of Proposition 4:

In order to obtain general results we need to maximize using the Kuhn-Tucker approach

$$Max_{s,\beta}W = \frac{1}{2}S\frac{\begin{array}{c}(2S+SB)\,s^2+(B-2S-2SB\beta-2B\beta)\,s\\-2B+B\beta+S+SB\beta-2B^2\beta-1+2B^2\beta^2\end{array}}{(B\beta+1)\,(-B+B\beta-1)} \quad (57)$$

subject to the constraints

$$\beta \geq 0, s \geq 0, \beta \leq 1, s \leq 1 \quad (58)$$

so that we can set up the Lagrangian

$$L(\beta, s, \lambda_1, \lambda_2, \lambda_3, \lambda_4,) = \quad (59)$$

$$\frac{1}{2}S\frac{\begin{array}{c}(2S+SB)\,s^2+(B-2S-2SB\beta-2B\beta)\,s\\-2B+B\beta+S+SB\beta-2B^2\beta-1+2B^2\beta^2\end{array}}{(B\beta+1)\,(-B+B\beta-1)}$$

$$-\lambda_1(-\beta)-\lambda_2(-s)-\lambda_3(\beta-1)-\lambda_4(s-1)$$

and the first order conditions can be written as

$$\frac{\partial L}{\partial \beta} = \frac{1}{2}SB\frac{\begin{array}{c}(2s-1)\left(B^2\beta^2(S+1)+S(2B\beta+1)+1\right)+\\(1-2\beta)\,sB\,(sS(B+2)+B)+(s-\beta)2B\end{array}}{(B\beta+1)^2\,(B(1-\beta)+1)^2}+\lambda_1-\lambda_3=0 \quad (60)$$

$$\frac{\partial L}{\partial s} = \frac{1}{2}S\frac{(1-2s)2S+(2\beta-1)B+(\beta-s)2SB}{B^2\beta(1-\beta)+B+1}+\lambda_2-\lambda_4=0 \quad (61)$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1\beta = 0 \quad (62)$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \quad (63)$$

$$\frac{\partial L}{\partial \lambda_3} = 1-\beta \geq 0, \lambda_3 \geq 0, \lambda_3(1-\beta) = 0 \quad (64)$$

$$\frac{\partial L}{\partial \lambda_4} = 1-s \geq 0, \lambda_4 \geq, \lambda_4(1-s) = 0 \quad (65)$$

Note that the first term in (60) given by

$$A \equiv \frac{1}{2}SB\frac{\begin{array}{c}(2s-1)\left(B^2\beta^2(S+1)+S(2B\beta+1)+1\right)+\\(1-2\beta)\,sB\,(sS(B+2)+B)+(s-\beta)2B\end{array}}{(B\beta+1)^2\,(B(1-\beta)+1)^2} \quad (66)$$

24

is strictly negative if $\beta \in \left(\frac{1}{2}, 1\right]$ and $s \in \left[0, \frac{1}{2}\right]$ (or if $\beta \in \left[\frac{1}{2}, 1\right]$ and $s \in \left[0, \frac{1}{2}\right)$) and positive otherwise.

Note that the first term B in (61) given by

$$B \equiv \frac{1}{2}S\frac{(1 - 2s)2S + (2\beta - 1)B + (\beta - s)2SB}{B^2\beta(1 - \beta) + B + 1} \tag{67}$$

is strictly negative if $\beta \in \left[0, \frac{1}{2}\right]$ and $s \in \left(\frac{1}{2}, 1\right]$ (or if $\beta \in \left(0, \frac{1}{2}\right]$ and $s \in \left[\frac{1}{2}, 1\right]$) and positive otherwise.

We need to consider *three cases* in turn:

a) Given

$$0 \leq \beta \leq \frac{1}{2} < s \leq 1$$

we may rewrite the constraints as:

$$\frac{\partial L}{\partial \beta} = (A > 0) + \lambda_1 - \lambda_3 = 0 \tag{68}$$

$$\frac{\partial L}{\partial s} = (B < 0) + \lambda_2 - \lambda_4 = 0 \tag{69}$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1\beta = 0 \tag{70}$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \tag{71}$$

$$\frac{\partial L}{\partial \lambda_3} = 1 - \beta \geq 0, \lambda_3 \geq 0, \lambda_3(1 - \beta) = 0 \tag{72}$$

$$\frac{\partial L}{\partial \lambda_4} = 1 - s \geq 0, \lambda_4 \geq 0, \lambda_4(1 - s) = 0 \tag{73}$$

As $\lambda_1 \geq 0$ we find that $\lambda_3 > 0$ and as $\lambda_4 \geq 0$ we find that $\lambda_2 > 0$. Then it follows that we need that $\beta = 1$ and $s = 0$ which cannot be the case. Contradiction.

b) Given

$$0 \leq s \leq \frac{1}{2} < \beta \leq 1$$

we may rewrite the constraints as:

$$\frac{\partial L}{\partial \beta} = (A < 0) + \lambda_1 - \lambda_3 = 0 \tag{74}$$

$$\frac{\partial L}{\partial s} = (B > 0) + \lambda_2 - \lambda_4 = 0 \tag{75}$$

$$\frac{\partial L}{\partial \lambda_1} = \beta \geq 0, \lambda_1 \geq 0, \lambda_1 \beta = 0 \tag{76}$$

$$\frac{\partial L}{\partial \lambda_2} = s \geq 0, \lambda_2 \geq 0, \lambda_2 s = 0 \tag{77}$$

$$\frac{\partial L}{\partial \lambda_3} = 1 - \beta \geq 0, \lambda_3 \geq 0, \lambda_3(1 - \beta) = 0 \tag{78}$$

$$\frac{\partial L}{\partial \lambda_4} = 1 - s \geq 0, \lambda_4 \geq 0, \lambda_4(1 - s) = 0 \tag{79}$$

As $\lambda_3 \geq 0$ we find that $\lambda_1 > 0$ and as $\lambda_2 \geq 0$ we find that $\lambda_4 > 0$. Then it follows that we need that $s = 1$ and $\beta = 0$ which cannot be the case. Contradiction.

$c)$ If we assume that

$$s = \beta$$

the Lagrangian reduces to

$$L(\beta, s, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = L(s, \lambda) \tag{80}$$

$$\frac{1}{2}S\frac{\begin{array}{c}2Ss^2 - SBs^2 + 2Bs - 2sS - 2Bs^2 - \\ 2B + S + SBs - 2B^2s - 1 + 2B^2s^2\end{array}}{(Bs + 1)(Bs - B - 1)} - \lambda_1(-s) - \lambda_2(s - 1)$$

$$\frac{\partial L}{\partial s} = \left(C = \frac{1}{2}S\frac{(2s - 1)(B + 2)(B - S)}{(Bs + 1)^2(B(1 - s) + 1)^2}\right) + \lambda_1 - \lambda_2 = 0 \tag{81}$$

$$\frac{\partial L}{\partial \lambda_1} = s \geq 0, \lambda_1 \geq 0, \lambda_1 s = 0 \tag{82}$$

26

$$\frac{\partial L}{\partial \lambda_2} = 1 - s \geq 0, \lambda_2 \geq 0, \lambda_2(1 - s) = 0 \tag{83}$$

Note that C negative if $s < \frac{1}{2}$ and positive if $s > \frac{1}{2}$.
Case $s > \frac{1}{2}$:

$$\frac{\partial L}{\partial s} = (C > 0) + \lambda_1 - \lambda_2 = 0 \tag{84}$$

$$\frac{\partial L}{\partial \lambda_1} = s \geq 0, \lambda_1 \geq 0, \lambda_1 s = 0 \tag{85}$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - s \geq 0, \lambda_2 \geq 0, \lambda_2(1 - s) = 0 \tag{86}$$

As $\lambda_1 \geq 0$ and $C > 0$ then $\lambda_2 > 0$ then $s = 1$, a corner solution.

Case $s < \frac{1}{2}$:

As $\lambda_2 \geq 0$ and $C < 0$ then $\lambda_1 > 0$ then $s = 0$, a corner solution.
All we need to show for the second order condition to hold is that

$$S(B + 2)(S - B) \frac{3B^2 s(1 - s) - B - 1 - B^2}{(Bs + 1)^3 (B(1 - s) + 1)^3} < 0 \tag{87}$$

At the first corner solution $s = 1$ we find

$$H_{11}(s = 1) = S(B + 2)(B - S) \frac{B + 1 + B^2}{(B + 1)^3} < 0 \tag{88}$$

Similarly at the second corner solution $s = 0$ we find

$$H_{11}(s = 0) = S(B + 2)(B - S) \frac{B + 1 + B^2}{(B + 1)^3} < 0 \tag{89}$$

and thus both solutions are indeed maximizing the welfare function.∎

Proof of Lemma 5:

Any proportional equilibrium implies that $\beta = s$ and so total welfare reduces to

$$W = \frac{1}{2} S \frac{\begin{array}{c}\left(2B + SB - 2B^2 - 2S\right)\beta^2 + \\ \left(2B^2 + 2S - 2B - SB\right)\beta + 2B - S + 1\end{array}}{(B\beta + 1)(B(1-\beta)+1)}$$

with first derivative

$$\frac{\partial W}{\partial \beta} \overset{!}{=} 0 = \frac{1}{2} S \frac{(2\beta - 1)(B+2)(B-S)}{(B\beta+1)^2 (B(1-\beta)+1)^2}$$

with solution $\beta^* = 1/2$. See that $\partial W / \partial \beta < 0$ for $\beta < 1/2$ and $\partial W / \partial \beta > 0$ for $\beta > 1/2$ and any B, S.

The second order condition is

$$\frac{\partial^2 W}{\partial \beta^2}\big|_{\beta=\beta^*} = 16 S \frac{B - S}{(B+2)^3} > 0$$

as $B > S + 1$ and hence β^* yields a minimum of the welfare function when we look at proportional equilibria, the welfare worst proportional equilibrium.∎

Proof of Proposition 6:

First note that constraint (S1) is strictly binding, otherwise platform 1 can increase its charge and profits. Then rearranging (S1) we find

$$S_1 = B_1 - (\frac{B_2 - (S_2 + 1)}{B_2 + 1} - p_{2s} + p_{1s})(B_1 + 1) \tag{90}$$

so that the profit maximization takes the form

$$\begin{aligned}\max_{p_{1s}} \pi &= (p_{1s} - f_s)S_1 = \\ &\quad (p_{1s} - f_s)\left(B_1 - \left(\frac{B_2 - (S_2 + 1)}{B_2 + 1} - p_{2s} + p_{1s}\right)(B_1 + 1)\right)\end{aligned}$$

with first order condition

$$\frac{\partial \pi}{\partial p_{1s}} = \left(B_1 - (\frac{B_2 - (S_2 + 1)}{B_2 + 1} - p_{2s} + p_{1s})(B_1 + 1)\right) + (p_{1s} - f_s)(-B_1 - 1) = 0 \tag{91}$$

giving the best response function

$$p_{1s} = \frac{1}{2} \frac{2B_1 - B_2 + S_2 B_1 + S_2 + 1}{(B_1 + 1)(B_2 + 1)} + \frac{1}{2} p_{2s} + \frac{1}{2} f_s \tag{92}$$

and second order condition

$$\frac{\partial^2 \pi}{\partial p_{1s}^2} = -B_1 - 1 < 0 \tag{93}$$

and

$$\frac{\partial^2 \pi}{\partial p_1 \partial p_2} = B_1 + 1 \tag{94}$$

revealing *strategic complementarities*. Using symmetry we find the other best response as

$$p_{2s} = \frac{1}{2}\frac{2B_2 - B_1 + S_1 B_2 + S_1 + 1}{(B_2 + 1)(B_1 + 1)} + \frac{1}{2}p_{1s} + \frac{1}{2}f_s \tag{95}$$

Solving simultaneously yields equilibrium prices as

$$p_{1s}^* = f_s + \frac{1}{3}\frac{S_1 B_2 + 2S_2 B_1 + 2S_2 + S_1 + 3 + 3B_1}{(B_1 + 1)(B_2 + 1)} \tag{96}$$

and

$$p_{2s}^* = f_s + \frac{1}{3}\frac{S_2 B_1 + 2S_1 B_2 + 2S_1 + S_2 + 3 + 3B_2}{(B_1 + 1)(B_2 + 1)} \tag{97}$$

as claimed.∎

Proof of Lemma 8:

By differentiation we find that

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial S} = \frac{1}{3}\frac{3\beta(\beta - 1)B - 2 + \beta}{(\beta B + 1)((\beta - 1)B - 1)} > 0 \tag{98}$$

and for sufficiently large B

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial B} = -\frac{1}{3}\frac{\begin{array}{c}\left(3\beta^2\left(\beta - 1\right)\left((\beta - 1)S - 1\right)\right)B^2 + \\ \left(2\beta\left(\beta - 1\right)\left(S(\beta - 2) - 3\right)\right)B + \\ 3 + 2S - 4\beta S + 3S\beta^2 - 3\beta\end{array}}{(\beta B + 1)^2\left((1 - \beta)B + 1\right)^2} < 0 \tag{99}$$

and

$$\frac{\partial p_{1s}^*(s = \beta)}{\partial \beta} = -\frac{1}{3}\frac{-3\beta^2 B^3 + \left(-6\beta - S + S\beta^2 + 2\beta S\right)B^2 + \left(-3 + 6\beta S - 2S\right)B + S}{(\beta B + 1)^2\left((1 - \beta)B + 1\right)^2} > 0 \tag{100}$$

.∎

Proof of Lemma 9:

Taking the derivative we find

$$\frac{\partial u_{1s}^*(s=\beta)}{\partial S} = -\frac{2}{3}\frac{3\beta B(1-\beta)+\beta+1}{(\beta B+1)((1-\beta)B+1)} < 0 \tag{101}$$

and

$$\frac{\partial u_{1B}^*(s=\beta)}{\partial \beta} = \frac{1}{2}S\frac{S-B}{B(\beta B+1)^2} < 0 \tag{102}$$

from non-triviality. Also

$$\frac{\partial u_{1s}^*(s=\beta)}{\partial B} = \frac{1}{3}\frac{\begin{array}{c}(3\beta(1-\beta)(2S\beta(1-\beta)+1))B^2+ \\ (4\beta(1-\beta)((1+\beta)S+3))B+ \\ 3+2S-4\beta S+6S\beta^2\end{array}}{(\beta B+1)^2((1-\beta)B+1)^2} \tag{103}$$

which is positive for sufficiently large B. Then

$$\frac{\partial u_{1s}^*(s=\beta)}{\partial \beta} = -\frac{1}{3}\frac{\begin{array}{c}(-3+6\beta)B^3+ \\ (12\beta+4S-6+2S\beta^2-8\beta S)B^2+ \\ (-12\beta S+8S)B+2S\end{array}}{(\beta B+1)^2((1-\beta)B+1)^2} \tag{104}$$

which is negative for $\beta > \frac{1}{2}$ and sufficiently large B.∎

Proof of Lemma 10:

Equilibrium prices are given by (18) and (19). Thus $p_{1s}^* > p_{2s}^*$ implies

$$\frac{S_2}{(B_2+1)} - \frac{S_1}{(B_1+1)} + \frac{3(B_1-B_2)}{(B_1+1)(B_2+1)} > 0 \tag{105}$$

which is increasing in B_1 and S_2 and decreasing in B_2 and S_1. Hence large S_2 at low B_2 (or low S_1 at large B_1) will make this inequality hold. Written in market shares we have

$$(1-s)S > \frac{(1-\beta)B+1}{\beta B+1}sS - 3\frac{\beta B-(1-\beta)B}{\beta B+1} \tag{106}$$

and using Proposition 3 again

$$\beta = s > \frac{1}{2}.∎ \tag{107}$$

Proof of Proposition 11:

The seller constraint (S1) becomes

$$\frac{B\beta - Ss}{B\beta + 1} \geq \frac{B(1-\beta) - (S(1-s) + 1)}{B(1-\beta) + 1} - \left(\frac{1}{3}\frac{(s-\beta)SB + (2s-1)S + 3(1-2\beta)B}{(\beta B + 1)((1-\beta)B + 1)}\right) \tag{108}$$

and (S2) is

$$\frac{B(1-\beta) - S(1-s)}{B(1-\beta) + 1} - \left(\frac{1}{3}\frac{(s-\beta)SB + (2s-1)S + 3(1-2\beta)B}{(\beta B + 1)((1-\beta)B + 1)}\right) \geq \frac{B\beta - (Ss+1)}{B\beta + 1} \tag{109}$$

(S1) can be solved for

$$s_{(S1)} = \frac{1}{2}\frac{3 + 2S}{S(B+2)}(\beta B + 1) \tag{110}$$

and the second constraint (S2) for

$$s_{(S2)} = \frac{1}{2}\frac{-3B + 3\beta B + 2S\beta B + 2S - 3}{S(B+2)} \tag{111}$$

Corner outcomes satisfy the buyer constraint if $\Delta_B = 0$ but not the seller constraints if $\Delta_S = 0$. Now with $\Delta_S = \Delta_S^*$ we need to show that for $\beta = 0$ we have $s_{(S1)} > 0$ and $s_{(S2)} < 0$ and for $\beta = 1$ we have $s_{(S1)} > 1$ and $s_{(S2)} < 1$. Thus

$$s_{(S1)}(\beta = 0) = \frac{1}{2}\frac{3 + 2S}{S(B+2)} > 0 \tag{112}$$

which always holds. Also

$$s_{(S2)}(\beta = 0) = \frac{1}{2}\frac{-3B + 2S - 3}{S(B+2)} < 0 \tag{113}$$

or $B > \frac{2}{3}S - 1$ which holds by non-triviality. Symmetrically the same conditions are found by

$$s_{(S1)}(\beta = 1) = \frac{1}{2}\frac{3 + 2S}{S(B+2)}(B+1) > 1 \tag{114}$$

or $B > \frac{2}{3}S - 1$ and

$$s_{(S2)}(\beta = 1) = \frac{1}{2}\frac{-3B + 3B + 2SB + 2S - 3}{S(B+2)} < 1 \tag{115}$$

which always holds. ∎

31

Proof of Proposition 12:

The buyer constraints are given above as (9) and (10). Letting the share of buyers to sellers on each platform be fixed at some $\bar{x}_i = S_i/B_i$ $i = 1, 2$ we find that the first constraint becomes

$$\frac{\bar{x}_1(\frac{1}{B_1} + \bar{x}_1)}{2(\frac{1}{B_1} + 1)} \geq \frac{\bar{x}_2(\frac{1}{B_2} + \bar{x}_2)}{2(1 + \frac{1}{B_2})(\frac{2}{B_2} + 1)} - \Delta_B \tag{116}$$

and on large platforms where $B_1, B_2 \to \infty$ we find that this reduces to

$$\frac{(\bar{x}_1)^2}{2} \geq \frac{(\bar{x}_2)^2}{2} - \Delta_B \tag{117}$$

for any share $\bar{x}_1$ as $u_B(1, \bar{x}_1) \to (\bar{x}_1)^2/2$. The second constraint can similarly be reduced to

$$\frac{(\bar{x}_2)^2}{2} - \Delta_B \geq \frac{(\bar{x}_1)^2}{2} \tag{118}$$

so that the only outcome that satisfies these constraints has $\Delta_B = 0$ and $\bar{x}_1 = \bar{x}_2$. Similarly for sellers we have from (15) that

$$1 - \bar{x}_1 \geq 1 - \bar{x}_2 - \Delta_S \tag{119}$$

and (16)

$$1 - \bar{x}_2 - \Delta_S \geq 1 - \bar{x}_1 \tag{120}$$

which again can only be satisfied for $\Delta_S = 0$ and $\bar{x}_1 = \bar{x}_2$ as $u_S(1, \bar{x}_1) \to 1 - \bar{x}_1$. The conclusion follows from noting that $\bar{x}_1 = \bar{x}_2 \Leftrightarrow \beta = s$. Note that endogenous seller charges also satisfy $\lim_{B \to \infty}(\Delta_s^*) = 0$.■

Proof of Proposition 13:

As in stage one of the game valuations are unknown each multihoming buyer and seller will transact on each platform with equal probability. Hence the buyer constraints (9) become

$$\frac{(sS + \frac{N}{2})(1 + sS + \frac{N}{2})}{2(\beta B + \frac{M}{2})(1 + \beta B + \frac{M}{2})} \geq \frac{((1-s)S + \frac{N}{2})(1 + (1-s)S + \frac{N}{2})}{2((1-\beta)B + \frac{M}{2} + 1)(1 + (1-\beta)B + \frac{M}{2} + 1)} - \Delta_B$$
(121)

and (10) is

$$\frac{((1-s)S + \frac{N}{2})(1 + (1-s)S + \frac{N}{2})}{2((1-\beta)B + \frac{M}{2})(1 + (1-\beta)B + \frac{M}{2})} - \Delta_B \geq \frac{(sS + \frac{N}{2})(1 + sS + \frac{N}{2})}{2(\beta B + \frac{M}{2} + 1)(1 + \beta B + \frac{M}{2} + 1)}$$
(122)

When writing these constraints in term of *total market shares* with $s = \frac{sS + N/2}{S+N}$ and $\beta = \frac{\beta B + M/2}{B+M}$ we find (B1)

$$\frac{s(S + N)(1 + s(S + N))}{2(\beta(B + M))(1 + \beta(B + M))} \geq \frac{((1-s)(S + N))(1 + (1-s)(S + N))}{2((1-\beta)(B + M) + 1)(1 + (1-\beta)(B + M) + 1)} - \Delta_B$$
(123)

and (B2)

$$\frac{(1-s)(S + N)(1 + (1-s)(S + N))}{2(1-\beta)(B + M)(1 + (1-\beta)(B + M))} - \Delta_B \geq \frac{(s(S + N))(1 + s(S + N))}{2(\beta(B + M) + 1)(1 + \beta(B + M) + 1)}$$
(124)

Similarly the seller constraints change to (S1)

$$\frac{\beta(B + M) - s(S + N)}{\beta(B + M) + 1} \geq \frac{(1-\beta)(B + M) - ((1-s)(S + N) + 1)}{(1-\beta)(B + M) + 1} - \Delta_S$$
(125)

and (S2)

$$\frac{(1-\beta)(B + M) - (1-s)(S + N)}{(1-\beta)(B + M) + 1} - \Delta_S \geq \frac{\beta(B + M) - (s(S + N) + 1)}{\beta(B + M) + 1}$$
(126)

so that the choice of M and N is a perfect substitute for that of B and S in any of the four incentive constraints.■

12 References

Armstrong, M. (2004): "Competition in Two-Sided Markets", University College London, (mimeo.).

Caillaud, B. & Jullien, B. (2001): " Competing Cybermediaries", *European Economic Review*, 45, 797-808.

Caillaud, B. & Jullien, B. (2003): "Chicken & Egg: Competition Among Intermediation Service Providers", *Rand Journal of Economics,* 34(2), 309-328.

FAZ (2003): "Mobile verliert Marktführerschaft im Online Autogeschäft", *Frankfurter Allgemeine Zeitung*, 10.11.2003, Nr. 261, p.19.

Ellison, G., Fudenberg, D., & Möbius, M. (2004): "Competing Auctions", *The Journal of the European Economic Association*, 2(1), p.30-66.

McGrath, S. (2005): "Understanding Supply and Demand on eBay", available at http://www.auctionbytes.com/cab/abu/y205/m03/abu0138/s03 (02.12.2005)

Rochet, J.-C. & Tirole, J. (2004): "Two-sided Markets: An Overview", IDEI, Toulouse, (mimeo.).

Yin, P.-L. (2004): "Online Auction Markets", *Harvard Business School Case Study*, 9-705-411, August 23, 2004.