Macro-economy in models for default probability.

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Abstract

We inspect the question how to adapt to macro-economical variables those probability of default (PD) estimates where Merton's model assumptions cannot be used. The need for this is to obtain trustworthy estimates of PD from a given economical situation. The structure of a known market-credit risk model is adapted. The key concept in this adaptation is the assumption of a different probabilistic situation for a firm before and at (first) default. If a corporate firm defaults we use a different probabilistic relation between macro-economical and market risk than in a firm's normal not default operation. We found a remarkable resemblance between relativity of physical space-time and the economical framework of variables. This means a solution of the calibration problem without using a Gaussian distribution estimates of the default probability.

Keywords: Credit worthiness index, statistical methodology

1 Introduction

In the present day risk literature there is more and more interest in the way macro-economic factors may enter risk models [1]. For instance studying the loss given default (LGD) there are in response to the Basel regulation already researchers that include macro-economical variables in their models [2].

In a well-established integrated model a credit worthiness index [3] [4] is defined as a sum of two factors related to

- an idiosyncratic component that determine the default of a company
- a systematic market risk component

If in the model the market risk variable is changed to allow macro-economic influences, then the structure of the index

$$Y_t = \beta\zeta_t + \alpha\epsilon_t \tag{1}$$

remains the same. Here, the ζ_t represents the systematic (summed) component of risk at time t. However, the usual market risk at time t is incorporated in the ζ_t and macro-economic factors enter the model 'together with' market risk. I.e. $\zeta_t = w_{mac}\zeta_{mac,t} + w_{mrk}\zeta_{mrk,t}$, where w_{mac} weights the macro-economic influence at t and w_{mrk} weights the (indirect) market factors. The latter is because the to be modeled company has to operate in a market and there is still a risk that competitors do it better in that market under the same macro-economical conditions.

The default time in this model is the (first) time that the index Y_t reaches a value below a limit $b_t = b(t)$.

$$\tau = \inf\{t|Y_t < b_t\} \tag{2}$$

The probability of default is the probability of the event $Y_t < b_t$ i.e. $P\{Y_t < b_t\}$.

2 Model considerations

The question is how to adapt the statistical description of default still having within probability the same defaulting event.

2.1 Expectation and covariance of variables

For the ease of the presentation let us write: $\zeta_{i,t}$ and $w_{i,t}$ with $i = 1, 2$ refering to macro ($i = 1$) and market ($i = 2$) components. Now if the distribution functions of both $\zeta_{i,t}$ are $N(0, 1)$, then the sum $\zeta_t = w_1\zeta_{1,t} + w_2\zeta_{2,t}$ will have a

$$E(\zeta_t) = w_1 E(\zeta_{1,t}) + w_2 E(\zeta_{2,t}) = 0 \tag{3}$$

The variance of ζ_t however is most likely different from unity because

$$E(\zeta_t^2) = E\left(w_1^2\zeta_{1,t}^2 + w_2^2\zeta_{2,t}^2 + 2w_1w_2\zeta_{1,t}\zeta_{2,t}\right) \tag{4}$$

The variance is 'contaminated' with the covariance between the macro and the market component. E.g. one firm can perform better under certain macro conditions than another despite the fact that idiosyncratic (i.e. firm internal) factors are comparable. Market and macro-economy influences have a correlated influence on the variance of the index Y_t. We find, using $E(\epsilon_t^2) = 1$ that

$$E(Y_t^2) = \beta^2 . \left[w_1^2 E(\zeta_{1,t}^2) + w_2^2 E(\zeta_{2,t}^2) + 2 w_1 w_2 E(\zeta_{1,t} \zeta_{2,t}) \right] + \alpha^2 \tag{5}$$

where it was assumed that $E(\epsilon_t \zeta_{it}) = 0$ for $i = 1, 2$. Note that despite the fact that in equation(5) there is no longer a unity variance of the index Y_t, as it is with the usual model. We will inspect the matter of the covariance shortly. One can follow the first steps of the calibration to statistical measures as long as one mathematically derives things back to the idiosyncratic component ϵ_t.

2.2 Reformulated index

To continue, please note that an alternative index, Y'_t can be defined by

$$Y'_t = \frac{Y_t}{\sqrt{\left[w_1^2 E(\zeta_{1,t}^2) + w_2^2 E(\zeta_{2,t}^2) + 2 w_1 w_2 E(\zeta_{1,t} \zeta_{2,t}) \right] \beta^2 + \alpha^2}} \tag{6}$$

Y'_t is $N(0, 1)$ distributed and we only need to adjust the $b(t)$ to

$$b'_t = \frac{b_t}{\sqrt{\left[w_1^2 E(\zeta_{1,t}^2) + w_2^2 E(\zeta_{2,t}^2) + 2 w_1 w_2 E(\zeta_{1,t} \zeta_{2,t}) \right] \beta^2 + \alpha^2}} \tag{7}$$

to obtain the default event $Y'_t < b'_t$. Note that the default event $Y_t < b_t$ as such is in probability equal to the $Y'_t < b'_t$. Only a weighted measure has been added on left and right hand side of the comparison. The practice of this method will make it necessary to monitor marco-economical factors like the price of crude oil or the level of public debt and public funding together with (meso?) market factors of the firm one considers. The α, β and w_i are model parameters.

2.3 The market - marco variables covariance

In the first place, let us suppose $t < \tau$. So we are before the first default time. In the second place, let us suppose that somehow we know the joint distribution $F_{1,2}(\zeta_{1,t}, \zeta_{2,t})$ of the macro and the market variables for $t < \tau$. The covariance then equals generally speaking

$$E(\zeta_{1,t} \zeta_{2,t}) = \int_{(\zeta_{1,t}, \zeta_{2,t}) \in Z} \zeta_{1,t} \, \zeta_{2,t} \, dF_{1,2}(\zeta_{1,t}, \zeta_{2,t}) \tag{8}$$

Here, Z is the domain of the macro and market variables. Note that if we in the first instance accept $\zeta_{i,t} \sim N(0, 1)$ before and at default, and if $Z = R^2$

then the covariance at default will vanish when $F_{1,2}(\zeta_1, \zeta_2) = \Phi(\zeta_1)\Phi(\zeta_2)$. This follows because

$$E(\zeta_1\zeta_2) = \int_{-\infty}^{+\infty} \zeta_1 d\Phi(\zeta_1) \int_{-\infty}^{+\infty} \zeta_2 d\Phi(\zeta_2) = 0 \tag{9}$$

Note that the t index, to alliviate the notation, is often suppressed for default times $t = \tau$.

In a less straightforward, but still relatively crude approach, we might take notice of the fact that market risks become important when a macro-economical variable such as e.g. public debt, consumers confidence or currency ratio in the economy at or close to a firm's default has dropped below level $\zeta_{0,1}$. However, at the same time the political economic situation will sometimes in such a situation not allow the firm's market risk to rise above level $\zeta_{0,2}$. Then if ζ_1 is the macro-economic variable and ζ_2 its market risk, at or close to default $t = \tau$, we might see a joint density like for instance,

$$f_{1,2}(\zeta_1, \zeta_2) = \varphi(\zeta_1)\varphi(\zeta_2)\theta(\zeta_{0,1} - \zeta_1)\theta(\zeta_{0,2} - \zeta_2). \tag{10}$$

Here, $\varphi(x)$ is the standard normal density such that $\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$. Moreover, $\theta(x)$ is the Heaviside function, $\theta(x) = 1$ when $x \geq 0$ and $\theta(x) = 0$ when $x < 0$. The covariance in the latter case equals $E(\zeta_1\, \zeta_2) = \frac{1}{2\pi} \exp\left[-(\zeta_{0,1}^2 + \zeta_{0,2}^2)/2\right]$. We see that macro variable and market risk covary positively in this model case with a maximum value of $1/2\pi$.

2.4 Sample estimate of covariance

Subsequently, we want to estimate the covariance at default, $t = \tau$ from macro-economical and market risk data before the default, $t < \tau$. If a bank has assembled figures (e.g. indices) that indicate macro variables together with figures (indices) related to market risk then the sampling of data can be seen in the time as a series of moments t_n for $n = 1, 2, ..., N$ for which the indices or figures are avialable. Now if a first default of a firm occurs at $t_{N+1} = \tau$ we may *estimate* the covariance $E(\zeta_1\zeta_2)$ from equations (6) and (7) from the previous sampling points in time like

$$\hat{E}(\zeta_1\zeta_2) = \frac{1}{N} \sum_{n=1}^{N} \left(\frac{Z_{1,t_n} - M_1}{S_1} \right) \left(\frac{Z_{2,t_n} - M_2}{S_2} \right). \tag{11}$$

Here, M_i and S_i for $i = 1, 2$ are the sampling mean and standard deviation. Hence, at $t = t_{N+1} = \tau$ we take $E(\zeta_1\zeta_2) = E(\zeta_{1,t}\zeta_{2,t})|_{t=\tau} = \hat{E}(\zeta_{1,t}\zeta_{2,t})|_{t<\tau}$ from equation (11).

2.5 Remark concerning the expectations

Note that in the evaluation of equation (4) in $E(\zeta_{i,t}^2)$, we implicitly have taken $\Phi(\zeta_1)\Phi(\zeta_2)$ for the joint distribution of $\zeta_{1,t}$ and $\zeta_{2,t}$ at $t=\tau$. This implies $E(\zeta_{i,t}^2)=1$ at $t=\tau$. If we consider what has been said in the previous paragraphs it is obvious that with this model assumptions we would have $E(\zeta_1\zeta_2)=E(\zeta_{1,t}\zeta_{2,t})|_{t=\tau}=0$ at default. However, this would ignore the statistically to be expected relation between macro-economical and market risk variables. Hence, we have taken in the model the perspective of $t<\tau$ estimations of covariance between macro and market variables leading to default.

In fact the expectation in equation (5) would under N(0,1) conditions for $t=\tau$ read like $E(Y_t^2)=[w_1^2+w_2^2]\,\beta^2+\alpha^2$. However, the model differentiates between 'before'and 'at default'. Hence, we take

$$E(Y_t^2)|_{t=\tau}=\left[w_1^2+w_2^2+2w_1w_2\hat{E}(\zeta_{1,t}\zeta_{2,t})|_{t<\tau}\right]\beta^2+\alpha^2 \qquad (12)$$

At default the index's variance is corrected with pre-default covariance. For brevity it is, $E(Y^2)=\left[w_1^2+w_2^2+2w_1w_2\hat{E}\,\right]\beta^2+\alpha^2$, where $\hat{E}=\hat{E}(\zeta_{1,t}\zeta_{2,t})|_{t<\tau}$ refers to the estimate in equation (11). Similarly for the index (6) and the boundary (7) at $t=\tau$ we have for the index at default

$$Y'=\frac{Y_{t=\tau}}{\sqrt{\left[w_1^2+w_2^2+2w_1w_2\hat{E}\,\right]\beta^2+\alpha^2}} \qquad (13)$$

and for the boundary

$$b'=\frac{b_{t=\tau}}{\sqrt{\left[w_1^2+w_2^2+2w_1w_2\hat{E}\,\right]\beta^2+\alpha^2}}. \qquad (14)$$

Because of default, we have $Y'<b'$ which follows from $Y<b$ in the usual model i.e. the one without weighted introduction of macro-economical variables.

3 Calibration through a linear wave equation

3.1 Preliminaries

Let us take for defaulting index $Y'=\alpha'\epsilon'+\beta'\zeta'$ and subsequently relate it to Y variables at $t=\tau$ given in equation (1). In the first place, we define $\lambda=\sqrt{\alpha^2+\beta^2\hat{E}(\zeta^2)}$. Here, as was discussed in the previous sections, $\hat{E}(\zeta^2)=$

$w_1^2 + w_2^2 + 2w_1 w_2 \hat{E} = \hat{\sigma}_\zeta^2$. In the second place let us take $\alpha' = (\alpha/\lambda)$ next to $\epsilon' = \epsilon$ and ensure that

$$\beta'\zeta' = \frac{\beta\zeta}{\lambda} \tag{15}$$

together with the requirement $E(\zeta'^2) = 1$. Hence, $\beta'^2 = \left(\frac{\beta^2}{\lambda^2}\right)\hat{E}(\zeta^2)$ and because of the definition of λ we may, noting $E(\epsilon\zeta) = 0$, conclude that $\alpha'^2 + \beta'^2 = \frac{\alpha^2}{\lambda^2} + \frac{\beta^2}{\lambda^2}\hat{E}(\zeta^2) = 1$ such as is necessary for consistency.

With the parametrization, $\alpha'^2 + \beta'^2 = 1$, for the a transformed credit worthiness index, $Y' = \alpha'\epsilon' + \beta'\zeta'$, as provided above, in a way we return to the usual model of one idiosyncratic and one systematic factor. Both ϵ' and ζ' are also $N(0,1)$ distributed. However, in the present case the systematic factor ζ' is driven by predefault macro-economical and by market risk factors plus their possible interaction.

3.2 Unconditional default probability

The best way to show the pre-default macro influences is to notice that in the literature [4] the unconditional probability of default is $p = \Phi(b')$. As we can see from the expression for the boundary value b' in equation (14) and notice that the $\hat{E}$ is obtained from covariance between market risk and macro-economical variables, it follows that via the boundary value, the unconditional PD is in the present model driven by pre-default covariance between market risk and macro-economical variables.

Moreover, because $p = \Phi(b')$ is cumulative, we can qualitatively make some predictions from the model to check its value. If we in an extreme case have taken w_1 and w_2 such that $\hat{E} = -\left(\frac{w_1^2+w_2^2}{2w_1w_2}\right)$ then it is easy to affirm that the smaller the α i.e. when a corporate firm has little influence on its credit worthiness index, the larger its unconditional probability of default, at or near the default, will be. This can be obtained looking at b' in equation (14). This qualitative result seems to make sense for corporate firms under certain macro-economical conditions. Of course idiosyncratic ϵ will contain chance processes but the author believes this is an indication for allowing firms close to default their own 'salvage' operations to increase their α coefficient in the present model. The macro-economic adaptation of the model does not come with counterintuitive statistics in this case.

3.3 Conditional default probability

In order to study the influences of the variables on the default event, we will have to study the conditional PD. The conditional probability of default is given by $Pr\{Y' < b'|\zeta'\} = P(\zeta')$. Because this can be reformulated as $\epsilon' < (b' - \beta'\zeta')/\alpha'$ and $\epsilon' = \epsilon \sim N(0,1)$ we find

$$P(\zeta') = \Phi\left(\frac{b' - \beta'\zeta'}{\alpha'}\right). \tag{16}$$

If we are looking for the distribution of the conditional probability of default a random variable $u = P(\zeta')$ is introduced such that for a boundary value $v \in [0,1]$ the probability $Pr\{u < v\}$ equals

$$Pr\{u < v\} = Pr\left\{\Phi\left(\frac{b' - \beta'\zeta'}{\alpha'}\right) < v\right\} = Pr\left\{\zeta' > \frac{b' - \alpha'\Phi^{-1}(v)}{\beta'}\right\} \tag{17}$$

As was demonstrated previously, $\zeta' \sim N(0,1)$, it follows that the distribution function for the PD can be written as

$$F(v) = 1 - \Phi\left(\frac{b' - \alpha'\Phi^{-1}(v)}{\beta'}\right). \tag{18}$$

3.4 Usual calibration

Usually the calibration of $u(w_1, w_2, b')$ is given in terms of an expectation problem

$$E\left[u(w_1, w_2, b', \beta')\right] = p \tag{19}$$

and

$$E\left[u^2(w_1, w_2, b', \beta')\right] = p^2 + \sigma^2. \tag{20}$$

In our paper we will attempt to find a different route that is closely related to modeling of small vibrations in physics.

3.5 d'Alembertian wave equation

Basing oneself on the expression for $\zeta = w_1\zeta_1 + w_2\zeta_2$ we can derive a law, also known in the physics for small vibrations [5], governing the development of the probability of default given in equation (16). In this section we will use the model parameters ζ_1, ζ_2, b'. For completeness, $\hat{\sigma}_\zeta = \sqrt{w_1^2 + w_2^2 + 2w_1w_2\hat{E}}$ and

$$x_1 = \sqrt{2}\left(\frac{\beta'}{\alpha'}\right)\left(\frac{w_1}{\hat{\sigma}_\zeta}\right)\zeta_1 \tag{21}$$

together with

$$x_2 = \sqrt{2}\left(\frac{\beta'}{\alpha'}\right)\left(\frac{w_2}{\hat{\sigma}_\zeta}\right)\zeta_2 \tag{22}$$

and $z = b'/\alpha'$. If we subsequently note that $\frac{\partial \Phi(x)}{\partial x} = \varphi(x)$ and $\frac{\partial \varphi(x)}{\partial x} = -x\varphi(x)$ then it obviously follows that: $\frac{\partial^2 \Phi(x)}{\partial x^2} = -x\varphi(x)$.

Taking the operator $\frac{\partial}{\partial \zeta_1}$ of $P(\zeta')$ we can observe the following

$$\frac{\partial}{\partial \zeta_1} P(\zeta') = -\left(\frac{\beta'}{\alpha'}\right)\left(\frac{w_1}{\hat{\sigma}_\zeta}\right)\Phi\left(\frac{b' - \beta'\zeta'}{\alpha'}\right) \tag{23}$$

Repeating the operation $\frac{\partial}{\partial \zeta_1}$ on equation (23) and employing $\frac{\partial^2 \Phi(x)}{\partial x^2} = -x\varphi(x)$ together with the definitions in (21) and (22) noting a similar transformation for $\frac{\partial}{\partial b'}$ we find

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial z^2}\right) u(x_1, x_2, z) = 0 \tag{24}$$

with, as defined in the calibration problem, $u(x_1, x_2, z) = P(\zeta')$. For completeness, the wave equation in the 'crude' variables is equal to

$$\left[\frac{1}{2\left(\frac{\beta'}{\alpha'}\right)^2\left(\frac{w_1}{\hat{\sigma}_\zeta}\right)^2}\frac{\partial^2}{\partial \zeta_1^2} + \frac{1}{2\left(\frac{\beta'}{\alpha'}\right)^2\left(\frac{w_2}{\hat{\sigma}_\zeta}\right)^2}\frac{\partial^2}{\partial \zeta_2^2} - \alpha'^2\frac{\partial^2}{\partial b'^2}\right] u(\zeta_1, \zeta_2, b') = 0 \tag{25}$$

From theoretical physics we know that equation (24) is invariant under Lorentz transformations denoting, in physics, the relativity of space and time. Let us inspect one single Lorentz transformation in the two 'spatial' variables x_1 and x_2 and one 'temporal' variable z. Lorentz transformed variables are denoted by $(\tilde{x}_1, \tilde{x}_2, \tilde{z})$. Suppose, for the sake of the argument, that $\tilde{x}_2 = x_2$. Hence, no coordinate transformations along the x_2 axis related to market risk. If a transformation factor $v \in [0, 1]$ is defined similar to a 'velocity' of movement along the coordinate plane (x_1, z) we have a Lorentz contraction factor $1/\sqrt{1 - v^2}$. The coordinate transformation then can be defined as

$$\tilde{x}_1 = \frac{x_1 - vz}{\sqrt{1 - v^2}} \tag{26}$$

and

$$\tilde{z} = \frac{z - vx_1}{\sqrt{1 - v^2}}. \tag{27}$$

We observe that

$$\frac{\partial}{\partial x_1} = \frac{1}{\sqrt{1 - v^2}}\frac{\partial}{\partial \tilde{x}_1} - \frac{v}{\sqrt{1 - v^2}}\frac{\partial}{\partial \tilde{z}} \tag{28}$$

and

$$\frac{\partial}{\partial z} = -\frac{v}{\sqrt{1 - v^2}}\frac{\partial}{\partial \tilde{x}_1} + \frac{1}{\sqrt{1 - v^2}}\frac{\partial}{\partial \tilde{z}}. \tag{29}$$

From the previous equations (28) and (29) it easily follows that the probability of default is invariant under Lorentz type of coordinate transformations i.e.

$$\left(\frac{\partial^2}{\partial \tilde{x}_1^2} + \frac{\partial^2}{\partial \tilde{x}_2^2} - \frac{\partial^2}{\partial \tilde{z}^2}\right) u(\tilde{x}_1, \tilde{x}_2, \tilde{z}) = 0 \tag{30}$$

This means that if x_1 is interpreted as, weighted (21), macro-economical influence on default, x_2 interpreted as, weighted (22), market risk and z as (weigthed) boundary defining the default that mixing of the coordinates, i.e. mixing of e.g. macro variables and boundary selection, at least in a Lorentz type coordinate transformation, leaves the probability of default invariant. This makes a calibration using solutions of the wave equation robust against probabilistic, $v \in [0, 1]$ Lorentz transformation 'contaminations' of the economical variables with each other. This raises the interesting possiblity to research into economical processes and variables with and without Lorentz (affinine geometry) invariance.

3.6 Mathematical solution of the wave equation

The wave equation calibration for the probability of default also enables a general form for a solution akin to the solution for e.g. a vibrating string equation. If, $\omega = \sqrt{k_1^2 + k_2^2}$ with k_i and ω form constants it follows that $\exp\left[-k_1 x_1 - k_2 x_2 - \omega z\right]$ is a so-called standing wave solution of equation (30).

4 Conclusion and Discussion

In the present paper we have introduced macro-economical variables in addition to risk variables. Used was made of the general model framework for probability of default. We found an alternative road to the calibration of the model when macro-economical variables are introduced next to the market risk factor. The key concept of the paper is that the pre-default situation and the 'at default' situation can be approached with different probabilistic modeling. The covariance of macro-economic and risk variables is estimated in the pre-default situation and is used as a correction factor for the variance in the default probabilistic model.

Contrary to the usual calibration methodology we managed to derive a wave equation for the probability of default. We found that the probability of default is invariant under the Lorentz transformations of the coordinate framework. Lorentz transformations are part of relativity theory and are among the affine geometric coordinate transformations. The invariance under Lorentz transformation implies a test for the 'robustness' of the employed economical variables and the boundary value determining the default. If those variables can be Lorentz transformed without affecting the (experimentally obtained) probability of default, the researcher knows that use was made of proper economic concepts and the calibration of the PD will lead to correct estimations of the default event. This test for orthogonality of the variables derives from the framework of theoretical physics where the orthogonal space-time coordinates can be Lorentz transformed without affecting the law that describe the physical object. In other words: the phenomenon does not change despite of a different view or perspective of it. In this case the economic 'law' is related to the probability of default and the set of coordinates arises from the economical variables that like space-time actually describe a relevant economical framework. Simply said, the economical variables employed then have the 'same reality' as e.g. length, breadth, depth and time. It should be noted that, because of a transformation like in equations (26) and (27), for a usual two parameter model such as in [4], the same Lorentz invariance test on the economical concepts applies.

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