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Patent Licensing in Spatial Competition – Does Pre-innovation Cost Asymmetry Matter ?

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Abstract

We consider the optimal licensing strategy of an insider patentee in a circular city of Salop's model and in a linear city of Hotelling's model when firms have asymmetric pre-innovation marginal costs of production and compete in prices. We completely characterize the optimal licensing policies using a fixed fee and per-unit royalty under the drastic and non-drastic innovations. We find that when the innovative firm is efficient compared to the licensee at the pre-innovation stage then the results regarding optimal licensing policy coincide with the results described in the literature with symmetric firms. However, this is not true when the innovative firm is inefficient in the pre-innovation stage compared to the licensee. To that end, we show that even a drastic innovation can be licensed using a royalty scheme when the patentee is highly inefficient compared to licensee in the pre-innovation stage and the size of the innovation is intermediate. We also show that in this set-up, fixed fee licensing is never optimal.

Keywords : Innovation, Technology transfer, Salop model, Hotelling model, Patent licensing, symmetric and asymmetric costs

JEL codes : D43, D45, L13

1 Introduction

Patent licensing is a fairly common practice that takes place in almost all industries. It is a source of profit for the innovator (also called licensor or patentee) who earns rent from the licensee by transferring a new technology. The theoretical literature has mainly considered the following three modes of patent licensing: a royalty on per unit of output produced with the patented technology, a fixed fee that is independent of the quantity produced with the patented technology, or an auction of a certain number of licenses, that is, offering a fixed number of licenses to the highest bidders. As for the patentee is concerned, two types are studied closely, namely the outsider and insider patentee. When the patentee is an independent R&D organization and not a competitor of the licensee in the product market, it is an outsider patentee; whereas when it competes with the licensee it becomes an insider patentee. In the literature on insider patentee, which is also the focus of this paper, the transfer of new technology is essentially studied in a framework where the competing firms are symmetric in terms of costs of production in the pre-innovation stage. We depart from this standard framework to an environment where the competing firms are asymmetric in terms of costs of production at the pre-innovation stage. The patentee can be inefficient or efficient compared to the licensee in our framework. There we ask the question does the pre-innovation cost asymmetry matter when it is compared with the symmetric case as for the optimal licensing is concerned? We find the answer to this question can be yes and no. It does not matter when the innovative firm is efficient relative to the licensee and there the results regarding optimal licensing policy coincide with the case when the firms are symmetric. However, it does matter when the innovative firm is inefficient compared to the licensee; and there the decision to license (using appropriate policy) or not very much depends on the size of the cost-reducing innovation and the degree of pre-innovation cost asymmetry. Thus doing the analysis of insider patent licensing in a more general framework like this gives us some valuable insight related to the pre-innovation cost structures of the competing firms and optimal licensing. Also realistically speaking in any market, no two firms are exactly symmetric; they are asymmetric in nature in almost all the time. Thus, an analysis of this nature is vital to realistically understand the licensing pattern in various industries.¹ We aim to do that in the paper.²

¹ See Rostoker (1983), Caves et al. (1983), Macho-Stadler et al. (1996), Jensen and Thursby (2001) and Vishwasrao (2007), among others for empirical findings on the nature licensing contracts in various industries.

² Poddar and Sinha (2010) in a recent paper, first studied optimal patent licensing in a situation where firms are

Now to study optimal patent licensing in a spatial competition, we use the framework of the two most celebrated spatial models in economics, namely the circular model of Salop (1979) and linear model of Hotelling (1929). There is a vast literature (see Kamien (1992) for a survey on patent licensing, and Sen and Tauman (2007) for general licensing schemes), which focuses on the optimal licensing arrangement by the patentee in a wide variety of situations.³ Interestingly, all these studies are done in a standard framework of price and/or quantity competition (i.e. the representative consumer approach of product differentiation) but very few studies are done in spatial framework.⁴ We believe that the spatial models, like Salop and Hotelling, are an appropriate place to study the licensing behaviour of firms in the industries where markets are already developed and not growing over time; while the differentiation over the brands is well established and is not changing rapidly. In a typical location model, the full market is always served and the demand does not change. We believe this particular feature in a location model is important, when one compares across equilibrium outcomes (equilibrium prices, profits of the firms) under different licensing regimes as the market size (or aggregate demand) remains constant across the regimes. Also, this feature in the location model has some advantage over standard models (e.g. Singh and Vives, 1984) of product differentiation. In the Singh and Vives (1984) model, the demand is elastic. Hence, comparisons between equilibrium values across different licensing regimes could be misleading because of varying aggregate demand across regimes. Thus, we find that Salop or Hotelling type spatial models are more appropriate place to study optimal licensing policies from different licensing regimes in

asymmetric and the patentee is relatively cost-inefficient compared to licensee under a Cournot duopoly; and obtained various interesting results. In this paper, among other things, we aim to pursue a similar story when firms are Bertrand competitors instead of Cournot.

³ In the literature, the results for optimal licensing policies under complete information frameworks are: if the patentee is an outsider, upfront fixed-fee licensing (or auctioning off a certain number of licenses) is optimal for the patentee (see Kamien & Tauman (1986); Katz & Shapiro (1986); also see Sen (2005), for an exception), whereas royalty licensing is optimal for the patentee when the patentee is an insider (see Rockett (1990); Wang (1998); Kamien & Tauman (2002); Erkal (2005)). There is also a small literature which has started to focus on the strategic trade policy in the presence of technology licensing (e.g. Kabiraj & Marjit (2003); Mukherjee & Pennings (2006); Mukherjee (2007); Ghosh & Saha (2008)) using various licensing schemes.

⁴ Patent licensing in a spatial model were initially studied by Caballero et. al. (2002); and Poddar & Sinha (2004). Caballero et. al. used Salop's circular city model to analyze optimal licensing with one outsider patentee and two potential licensees when the licensees have the same marginal costs of production at the pre-licensing stage. On the other hand, Poddar and Sinha used Hotelling's linear city model to analyze the case of an insider (as well as an outsider) patentee when competing firms have the same marginal costs of production at the pre-innovation stage (and pre-licensing stage for the outsider patentee case). Our analysis here in the Hotelling's framework (see section 5) is an extension of Poddar and Sinha (2004), when firms are asymmetric in the pre-innovation stage.

developed and matured markets. It is also a study of optimal licensing policies in markets where products are horizontally differentiated.

In our spatial framework here, we consider a market with two firms, one patentee and the other is potential licensee. The patentee comes up with an innovation which brings down its unit cost of production. This is the starting point of our analysis. In the circular Salop model, firms are situated symmetrically on the circumference and in the linear Hotelling model firms located at the end points. Consumers are uniformly distributed over the circumference in the Salop model and over the linear segment in the Hotelling model. In each case, the locations of the competing firms are fixed. The good produced by both firms is identical in nature; however, due to the presence of transport cost incurred by the consumer to buy the good from either of the firms, the goods are horizontally differentiated in the eyes of the consumers. We consider two possible types of cost-reducing innovations from the patentee that are generally described in the literature, namely, the drastic and non-drastic innovations. In our analysis, we completely characterize the equilibrium licensing outcomes under the fixed fee and per-unit royalty, and find the optimal strategy of the patentee in offering (or not offering) the license to its rival.⁵

As mentioned earlier, our major point of departure here from the existing literature is the asymmetry in initial costs of production in the pre-innovation stage. We show that when the patentee is efficient compared to the licensee; more specifically, when the marginal cost of production of the patentee is lower than that of the licensee before innovation, the results related to optimal licensing coincide with the traditional situation when both firms are symmetric with respect to the marginal costs at the pre-innovation stage. This result is true in Salop's model as well as in Hotelling's model. However, when the patentee is inefficient compared to the licensee in the pre-innovation stage, both in the Salop's and Hotelling's model, we find new results that were not obtained in the literature before. There we show that the initial cost asymmetry along with the size of the innovation can play a crucial role in determining the optimal licensing policy for the patentee. To that end, we find a drastic innovation can be optimally licensed using a royalty scheme when the patentee is highly inefficient compared to the licensee in the pre-innovation stage but the size of the drastic innovation is not too large. In our study, we also completely characterize the optimal licensing policies of the patentee using a fixed fee and per-

⁵ Note that with one rival firm auctioning is not a choice and we also show that a two-part tariff licensing can never be optimal here. Thus considering only fixed fee and per-unit royalty scheme is sufficient to find optimal licensing strategy of the patentee under this set up.

unit royalty in all possible regimes of drastic and non-drastring innovations under both types of pre-innovation cost asymmetry i.e. when the patentee is inefficient as well as efficient. We find that fixed fee licencing alone is never optimal and is dominated either by royalty or no-licencing. We show that it is dominated by royalty when the patent holder is inefficient in the pre-innovation stage and is dominated by no-licensing when the patent holder is efficient or firms are symmetric in the pre-innovation stage.

In the literature Poddar and Sinha (2010) also analyze the situation of optimal patent licensing when the patentee is inefficient compared to the licensee in the pre-innovation stage. However, they considered a quantity competition in the product market and obtained a two-part tariff licensing (i.e. the licencing contract consists of a fixed fee component and a royalty component) as optimal in various situations.⁶ In contrast to that study, here we consider price competition between the firms in the product market and showed that a two-part tariff licensing is never optimal. The contrasting results in these two scenarios arises due to the fact that under quantity competition, a fixed fee licencing alone is shown to be optimal (when compared with royalty and no-licencing) in certain situations, whereas in our study with price competition, a fixed fee licensing alone is shown to be never optimal in any situation. Thus a licensing strategy (like a two-part tariff contract) that consists a fixed fee component is never optimal in our framework. On the other hand, our results of optimal licensing under symmetric pre-innovations costs in Salop model or when the patentee is the efficient firm in Salop and Hotelling model, coincide with the results of Wang (1998, 2002) where the firms (i.e. the patentee and the licensee) are symmetric but compete in quantities.

The rest of the paper is organized as follows. In the next section, we set-up the Salop model for price competition. In section 3, we analyze optimal licensing under symmetric costs in Salop model. In section 4, we do our main analysis of optimal licensing under pre-innovation asymmetric costs in Salop's model. In section 5, a similar analysis is done in the Hotelling's model. Section 6, concludes.

2. Salop Model – Price Competition

Consider a circular city with unit circumference and two firms A and B producing a

homogeneous good, and located symmetrically on the city. Suppose firm A is located at 0 and firm B is located at $1/2$. Consumers are uniformly distributed over the circular rim. Each buys exactly one unit of the good either from firm A or B .

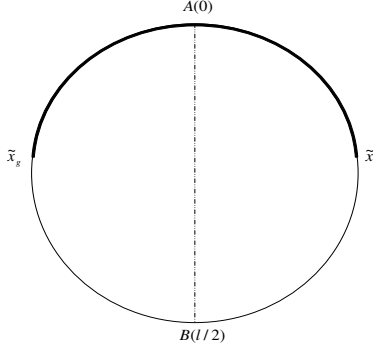


Figure 1

The utility function of a consumer located at x and buying from firm A is :

$$U_A = \begin{cases} -p_1 - x & \text{if } 0 < x < \frac{1}{2} \\ -p_1 - (1-x) & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

The utility function of a consumer located at x and buying from firm B is :

$$U_B = \begin{cases} -p_2 - \left(\frac{1}{2} - x\right) & \text{if } 0 < x < \frac{1}{2} \\ -p_2 - \left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

We derive the demand for firms A and B by equating the utility of the person who is indifferent between buying from A or B . We distinguish here between two marginal consumers: the first one located at \tilde{x}_d and the second one at \tilde{x}_g . Assume the marginal costs of production of A and B are c_1 and c_2 .

Demand function of firm A is :

⁶ The optimality of a two-part tariff licensing contract under complete information is also analysed by Fauli-Oller and Sandonis (2002) in the context of differentiated goods; by Sen and Tauman (2007) in a Cournot oligopoly and by Saracho (2002) in the context of strategic delegation.

$$D_A = \begin{cases} 1 & \text{if } p_1 < p_2 - \frac{1}{2} \\ \tilde{x}_d + (1 - \tilde{x}_g) & \text{if } p_2 - \frac{1}{2} \leq p_1 \leq p_2 + \frac{1}{2} \\ 0 & \text{if } p_1 > p_2 + \frac{1}{2} \end{cases} \Leftrightarrow D_A = \begin{cases} 1 & \text{if } p_1 \in \text{int}_1^A \\ \frac{1}{2} + (p_2 - p_1) & \text{if } p_1 \in \text{int}_2^A \\ 0 & \text{if } p_1 \in \text{int}_3^A \end{cases}$$

$$\text{Where } \text{int}_1^A = [c_1, p_2 - \frac{1}{2}), \text{int}_2^A = [p_2 - \frac{1}{2}, p_2 + \frac{1}{2}], \text{int}_3^A = (p_2 + \frac{1}{2}, \infty)$$

Demand function of firm B is :

$$D_B = \begin{cases} 0 & \text{if } p_2 > p_1 + \frac{1}{2} \\ \tilde{x}_g - \tilde{x}_d & \text{if } p_1 - \frac{1}{2} \leq p_2 \leq p_1 + \frac{1}{2} \\ 1 & \text{if } p_2 < p_1 - \frac{1}{2} \end{cases} \Leftrightarrow D_B = \begin{cases} 0 & \text{if } p_2 \in \text{int}_1^B \\ \frac{1}{2} + (p_1 - p_2) & \text{if } p_2 \in \text{int}_2^B \\ 1 & \text{if } p_2 \in \text{int}_3^B \end{cases}$$

$$\text{Where } \text{int}_1^B = (p_1 + \frac{1}{2}, \infty), \text{int}_2^B = [p_1 - \frac{1}{2}, p_1 + \frac{1}{2}], \text{int}_3^B = [c_2, p_1 - \frac{1}{2})$$

The profit function of firm A is:

$$\pi_A = \begin{cases} (p_1 - c_1) & \text{if } p_1 \in \text{int}_1^A \\ (p_1 - c_1)(\frac{1}{2} + (p_1 - p_2)) & \text{if } p_1 \in \text{int}_2^A \\ 0 & \text{if } p_1 \in \text{int}_3^A \end{cases}$$

The profit function of firm B is:

$$\pi_B = \begin{cases} 0 & \text{if } p_2 \in \text{Int}_1^B \\ (p_2 - c_2)(\frac{1}{2} + (p_1 - p_2)) & \text{if } p_2 \in \text{Int}_2^B \\ (p_2 - c_2) & \text{if } p_2 \in \text{Int}_3^B \end{cases}$$

To find a interior Nash equilibrium, the prices p_1 and p_2 must satisfy: $|p_1 - p_2| \leq \frac{1}{2}$.

Profits maximization with respect to prices gives the following reaction functions:

$$p_1 = \frac{1}{4}(2p_2 + 1 + 2c_1)$$

$$p_2 = \frac{1}{4}(2p_1 + 1 + 2c_2)$$

$$\text{Equilibrium prices are: } p_1^* = \frac{1}{6}(3+4c_1+2c_2), p_2^* = \frac{1}{6}(3+2c_1+4c_2) \quad (1)$$

$$\text{Equilibrium profits are: } \pi_A^* = \frac{1}{36}(3-2(c_1-c_2))^2 \text{ and } \pi_B^* = \frac{1}{36}(3+2(c_1-c_2))^2 \quad (2)$$

$$\text{Equilibrium demands are: } D_A = \frac{1}{2} + \frac{1}{3}(c_2 - c_1), D_B = \frac{1}{2} + \frac{1}{3}(c_1 - c_2) \quad (3)$$

3. Licensing under Symmetric Costs

Assume the pre-innovation marginal costs of firm A and B are $c_1 = c_2 = c$. Suppose the innovative firm A (also called patentee) comes up with a cost reducing innovation allowing to reduce the marginal cost of production by ε which also measures the size of the innovation. Assume firm B is the potential licensee.

A licensing game consists of three stages. In the first stage, the patent holding firm A decides whether to provide license or not to firm B for the new technology, and if decides to provide then it either sets a fixed licensing fee or a royalty rate. In the second stage, the rival firm B decides whether to accept or reject the offer from firm A . In the last stage, both firms compete in prices.

Firm A sets a fixed fee or royalty rate in order to maximize the sum of the profit from its own production and the licensing revenue. In a fixed fee licensing scheme firm B must make an upfront fixed payment and in a royalty licensing scheme firm B must pay a fixed amount on each unit quantity produced using the new technology.

We will start with the case where firm A does not offer any license to firm B . We call this as no-licensing (NL) regime.

3.1 No Licensing

In this regime, firm A profits alone from its innovation while firm B uses the old technology, Thus, $c_1 = c - \varepsilon$ and $c_2 = c$. Using (1) we get:

$$p_1^* = \frac{1}{2} + c - \frac{2}{3}\varepsilon \text{ and } p_2^* = \frac{1}{2} + c - \frac{1}{3}\varepsilon$$

Firm B using the old technology can only make a non negative profit when $D_B > 0$ or

$$p_2^* > c \Leftrightarrow \varepsilon < \frac{3}{2}.$$

Thus the innovation is non drastic if $\varepsilon < \frac{3}{2}$ and it is drastic otherwise.

When innovation is not drastic $\left(\varepsilon < \frac{3}{2}\right)$, the equilibrium profits are:

$$\pi_A^{NL} = \frac{1}{36}(3+2\varepsilon)^2 \text{ and } \pi_B^{NL} = \frac{1}{36}(3-2\varepsilon)^2$$

$$\text{Equilibrium demands are: } D_A = \frac{1}{2} + \frac{\varepsilon}{3}, D_B = \frac{1}{2} - \frac{\varepsilon}{3}$$

When innovation is drastic $\left(\varepsilon \geq \frac{3}{2}\right)$, the monopoly price of the patent holder is $p_1 = c - \frac{1}{2}$ and its

$$\text{profit is } \pi_A^{NL} = \varepsilon - \frac{1}{2}.$$

3.2 Fixed Fee Licensing

In this regime, firm B can use the new technology by paying the fixed fee denoted by F to the patent holding firm. The maximum amount that firm A can choose is equal to the increase of firm B 's profit when using the new technology. Thus, $F = \pi_B^F - \pi_B^{NL}$ will ensure that firm B accepts to buy the license.

Thus, firm A and firm B will produce under unit costs $c_1 = c_2 = c - \varepsilon$. Hence, from (2) we find :

$$\pi_A = \frac{1}{4} \text{ and } \pi_B = \frac{1}{4}.$$

Equilibrium demands for firm A and firm B are : $D_A = \frac{1}{2}$ and $D_B = \frac{1}{2}$

For a non drastic innovation, fixed fee equals: $F = \frac{1}{9}\varepsilon(3-\varepsilon)$

Thus, total profit of the patent holding firm is:

$$\Pi_A^F = \pi_A + F = \frac{1}{4} + \frac{1}{3}\varepsilon - \frac{1}{9}\varepsilon^2$$

For a drastic innovation, fixed fee is: $F = \frac{1}{4}$

Thus, total profit of the patent holding firm is:

$$\Pi_A^F = \pi_A + F = \frac{1}{2}$$

Lemma 1

In Salop's circular model with symmetric firms, no licensing is always better than fixed fee for the patentee when innovation is drastic and non drastic.⁷

Proof

$$\text{For } \varepsilon < \frac{3}{2}, \Pi_A^F - \pi_A^{NL} = -\frac{2}{9}\varepsilon^2 < 0$$

$$\text{For } \varepsilon \geq \frac{3}{2}, \Pi_A^F - \pi_A^{NL} = -\varepsilon < 0$$

3.3 Royalty Licensing

In the royalty regime, the cost-reducing innovation is sold to the firm B using the royalty scheme. The maximum royalty that firm A can charge is ε (given $0 \leq r \leq \varepsilon$).

After licensing, the marginal costs of firms A and B becomes respectively $c_1 = c - \varepsilon$ and $c_2 = c - \varepsilon + r$. From (1) we find equilibrium prices: $p_1 = \frac{1}{2} + c - \varepsilon + \frac{1}{3}r$ and $p_2 = \frac{1}{2} + c - \varepsilon + \frac{2}{3}r$

$$\text{Equilibrium profits are: } \pi_A = \frac{1}{36}(3+2r)^2 \text{ and } \pi_B = \frac{1}{36}(3-2r)^2$$

$$\text{Equilibrium demands are: } D_A = \frac{1}{2} + \frac{r}{3} \text{ and } D_B = \frac{1}{2} - \frac{r}{3}$$

Thus, total profit of firm A under royalty licenising is :

$$\Pi_A^r = \frac{1}{36}(3+2r)^2 + r\left(\frac{1}{2} - \frac{r}{3}\right)$$

Note for firm B 's demand to be positive: $D_B = \frac{1}{2} - \frac{r}{3} \geq 0$ which means that $r \leq \frac{3}{2}$.

When innovation is non drastic, we have $r \leq \varepsilon < \frac{3}{2}$. The derivative of the patent holding firm

A 's total revenue under a per unit royalty with respect to r is: $\frac{\partial \Pi_A^r}{\partial r} = \frac{5}{6} - \frac{4}{9}r > 0$ since

⁷ Same result is showed in Hotelling's model of linear city by Poddar and Sinha (2004) where no licensing is always

$r \leq \varepsilon < \frac{3}{2}$, we have optimal royalty as $r^* = \varepsilon$.

For drastic case : $\left(\varepsilon \geq \frac{3}{2}\right)$, $r^* = \frac{3}{2}$

Summarizing the two cases we get:

	$\varepsilon < \frac{3}{2}$	$\varepsilon > \frac{3}{2}$
Π_A^r	$-\frac{2}{9}\varepsilon^2 + \frac{5}{6}\varepsilon + \frac{1}{4}$	1

Since fixed fee licensing is always dominated by no licensing as we have seen in Lemma 1, the optimal licenising policy here is essentially a comparison between royalty and no licensing.

Proposition 1

*In Salop circular's model with symmetric firms, the patentee licenses its innovation using royalty when innovation is non drastic. For a drastic innovation, licensing does not occur and the patentee becomes a monopoly.*⁸

Proof

For $\varepsilon < \frac{3}{2}$, $\Pi_A^r - \pi_A^{NL} = \frac{\varepsilon}{3} \left(\frac{3}{2} - \varepsilon \right) > 0$. Here two firms are active since $|p_1 - p_2| \leq \frac{1}{2} \Leftrightarrow r \leq \frac{3}{2}$

which is verified.

For $\varepsilon > \frac{3}{2}$, $\Pi_A^r - \pi_A^{NL} = \frac{1}{2} - \varepsilon < 0$

4. Licesning under Asymmetric Costs

Now, we suppose that the unit production costs of the two firms A and B are not equal initially and let us first suppose that the patent holder is the inefficient firm i.e. $c_1 > c_2$ (we will denote by $\delta = c_1 - c_2$ the difference between the costs). Later we do the case where the patent holder is

found to be better than fixed fee licensing.

⁸ Same results are obtained in Wang (1998) under a Cournot duopoly and in Hotelling's model of linear city by Poddar and Sinha (2004) under price competition.

efficient (i.e. $c_1 < c_2$).

4.1 Inefficient Patentee

4.1.1 Fixed Fee Licensing

The two firms here use the new technology. Production unit costs of firms A and B are respectively $c_1' = c_1 - \varepsilon$ and $c_2' = c_2 - \varepsilon$. Using (1), (2) and (3) we can find equilibrium prices, profits and demands.

For a non-drastic innovation, ($\varepsilon < \frac{3}{2} + \delta$), the fixed fee is $F = \pi_B^F - \pi_B^{NL} = \frac{1}{3}\varepsilon - \frac{1}{9}\varepsilon^2 + \frac{2}{9}\delta\varepsilon$.

Total revenue of the patent holder becomes

$$\Pi_A^F = \pi_A^F + F = \left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{1}{3} + \frac{2}{9}\delta\right)\varepsilon + \left(\frac{1}{9}\delta^2 - \frac{1}{3}\delta + \frac{1}{4}\right)$$

For a drastic innovation ($\varepsilon > \frac{3}{2} + \delta$), the fixed fee is $F = \pi_B^F - \pi_B^{NL} = \frac{1}{36}(3 + 2\delta)^2$

The total revenue for this case is $\Pi_A^F = \pi_A^F + F = \frac{2}{9}\delta^2 + \frac{1}{2}$

Now to have both the firms active in the market under fixed fee licensing, we need to have equilibrium prices satisfy $|p_2 - p_1| < \frac{1}{2}$.

We see that $|p_2 - p_1| < \frac{1}{2} \Leftrightarrow \delta < \frac{3}{2} = 1.5$ (using equilibrium prices). So this condition needs to be satisfied for a valid fixed fee licencing regime.

Summarizing the equilibrium profits for the patentee under fixed fee:

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
Π_A^F	$\left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{1}{3} + \frac{2}{9}\delta\right)\varepsilon + \left(\frac{1}{9}\delta^2 - \frac{1}{3}\delta + \frac{1}{4}\right)$	$\frac{2}{9}\delta^2 + \frac{1}{2}$

4.1.2 Royalty Licensing

Similar to the symmetric case, the optimal royalty rate must be $0 \leq r \leq \varepsilon$ to ensure that firm B will buy the license. Production unit costs are $c_1' = c_1 - \varepsilon$ and $c_2' = c_2 - \varepsilon + r$.

Using (1) we find equilibrium prices as:

$$p_1^* = \frac{1}{6}(3 + 4c_1 + 2c_2 - 6\varepsilon + 2r); \quad p_2^* = \frac{1}{6}(3 + 2c_1 + 4c_2 - 6\varepsilon + 4r)$$

Using (3): Equilibrium demands are : $D_A = \frac{1}{2} + \frac{1}{3}(c_2 - c_1 + r)$, $D_B = \frac{1}{2} + \frac{1}{3}(c_1 - c_2 - r)$

Using (2) : Equilibrium profits are : $\pi_A^r = \frac{1}{36}(3 - 2(c_1 - c_2 - r))^2$ and $\pi_B^r = \frac{1}{36}(3 + 2(c_1 - c_2 - r))^2$

Total revenue of the patent holder is : $\Pi_A^r = \pi_A^r + rD_B = \frac{1}{36}(3 - 2(\delta - r))^2 + r\left(\frac{1}{2} + \frac{1}{3}(\delta - r)\right)$

Maximizing this total revenue with respect to the royalty rate we find :

$$\left. \begin{array}{l} \frac{\partial \Pi_A^r}{\partial r} = \frac{1}{18}(2\delta - 8r + 15) \\ \frac{\partial^2 \Pi_A^r}{\partial r^2} = -\frac{4}{9} < 0 \end{array} \right\} \Rightarrow r^* = \frac{15}{8} + \frac{1}{4}\delta$$

However, we must check that the optimal value r^* is not higher than ε and that the demand of firm B is non-negative. In fact $D_B > 0 \Leftrightarrow r < \frac{3}{2} + \delta$.

We distinguish between two cases of non-drastic and drastic innovation.

For non drastic innovation ($\varepsilon < \frac{3}{2} + \delta$), the optimal royalty rates are :

$$\left\{ \begin{array}{l} r^* = \varepsilon \quad \text{if } \varepsilon < \frac{3}{2} + \delta \text{ and } \delta < \frac{1}{2} \\ r^* = \frac{15}{8} + \frac{1}{4}\delta \quad \text{if } \frac{15}{8} + \frac{1}{4}\delta < \varepsilon < \frac{3}{2} + \delta \text{ and } \delta > \frac{1}{2} \end{array} \right. \quad (4) \text{ and } (5)$$

For drastic innovation ($\varepsilon > \frac{3}{2} + \delta$), the optimal royalty rates are:

$$\left\{ \begin{array}{l} r^* = \frac{3}{2} + \delta \quad \text{if } \frac{3}{2} + \delta < \frac{15}{8} + \frac{1}{4}\delta \Leftrightarrow \delta < \frac{1}{2} \\ r^* = \frac{15}{8} + \frac{1}{4}\delta \quad \text{if } \frac{15}{8} + \frac{1}{4}\delta < \frac{3}{2} + \delta \Leftrightarrow \delta > \frac{1}{2} \end{array} \right. \quad (6) \text{ and } (7)$$

We also need to satisfy the condition $|p_2 - p_1| < \frac{1}{2}$ for interior price equilibrium, so that both firms are active under royalty scheme.

Using equilibrium prices, $|p_2 - p_1| < \frac{1}{2} \Leftrightarrow \delta - \frac{3}{2} < r^* < \delta + \frac{3}{2}$; and which is true if and only if $\delta < 4.5$ (proof see Appendix (i) where we also verify the legitimate values of all the optimal royalties described from (4) to (7)).

Summarizing the equilibrium profits for the patentee under royalty:

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$\delta < \frac{1}{2}$	$\Pi_A^r = \left(-\frac{2}{9}\right)\varepsilon^2 + \left(\frac{5}{6} + \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{9}\delta^2 - \frac{1}{3}\delta + \frac{1}{4}\right)$	$\Pi_A^r = 1$
$\delta > \frac{1}{2}$	$\Pi_A^r = \frac{33}{32} + \frac{1}{8}\delta^2 - \frac{1}{8}\delta$	$\Pi_A^r = \frac{33}{32} + \frac{1}{8}\delta^2 - \frac{1}{8}\delta$

4.1.3 Fixed Fee versus Royalty Licensing

We use the table below to compare between fixed fee and royalty licensing :

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$\delta < \frac{1}{2}$	$\Pi_A^r - \Pi_A^F = \frac{1}{18}\varepsilon(-2\delta - 2\varepsilon + 9)$	$\Pi_A^r - \Pi_A^F = \frac{1}{2} - \frac{2}{9}\delta^2$
$\delta > \frac{1}{2}$	$\Pi_A^r - \Pi_A^F = \frac{1}{9}\varepsilon^2 + \left(-\frac{1}{3} - \frac{2}{9}\delta\right)\varepsilon + \left(\frac{1}{72}\delta^2 + \frac{5}{24}\delta + \frac{25}{32}\right)$	$\Pi_A^r - \Pi_A^F = \frac{17}{32} - \frac{1}{8}\delta - \frac{7}{72}\delta^2$

Working out the signs of the expressions in each of these components in the above table (see Appendix (ii) for details), we get the following.⁹

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$\delta < \frac{1}{2}$	$\Pi_A^r - \Pi_A^F > 0$	$\Pi_A^r - \Pi_A^F > 0$
$\frac{1}{2} < \delta < 1.7815$	$\Pi_A^r - \Pi_A^F > 0$	$\Pi_A^r - \Pi_A^F > 0$

⁹ Note that under fixed fee licensing when both firms are active we must satisfy $\delta < 1.5$; hence considering the interval $\frac{1}{2} < \delta < 1.7815$ is sufficient.

Thus, we have the following result.

Lemma 2

In Salop's circular model with pre-innovation asymmetric costs, when the patentee is inefficient, royalty is always better than fixed fee irrespective of drastic or non drastic innovation.

4.1.4 No Licensing

Suppose now the patentee uses the new technology alone. Its production unit cost is $c_1' = c_1 - \varepsilon$ while the licensee firm B uses the old technology and its production unit cost is $c_2' = c_2$.

Under non-drastic innovation, the profits of the two firms in the non-drastic case are:

$$\pi_A^{NL} = \frac{1}{36}(3 - 2(\delta - \varepsilon))^2 \text{ and } \pi_B^{NL} = \frac{1}{36}(3 + 2(\delta - \varepsilon))^2$$

For a drastic innovation, firm B leaves the market and the patent holder becomes a monopoly. In

this case $p_1 = c_2 - \frac{1}{2}$, $\pi_A^{NL} = \varepsilon - \frac{1}{2} - \delta^{10}$ and $\pi_B^{NL} = 0$.

Thus we have :

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
Firm A	$\pi_A^{NL} = \frac{1}{36}(3 - 2(\delta - \varepsilon))^2$	$\pi_A^{NL} = \varepsilon - \frac{1}{2} - \delta$
Firm B	$\pi_B^{NL} = \frac{1}{36}(3 + 2(\delta - \varepsilon))^2$	$\pi_B^{NL} = 0$

4.1.5 Optimal Licensing

So to find optimal licensing we need to compare only between royalty and no licencing as we have seen fixed fee is always dominated by royalty in this case. Now consider the following table:

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$\delta < \frac{1}{2}$	$\Pi_A^r - \pi_A^{NL} = \frac{1}{6}\varepsilon(2\delta - 2\varepsilon + 3) > 0$	$\Pi_A^r - \pi_A^{NL} = \delta - \varepsilon + \frac{3}{2} < 0$

¹⁰ $\pi_A^{NL} = p_1 - c_1 + \varepsilon = c_2 - \frac{1}{2} - c_1 + \varepsilon = \varepsilon - \frac{1}{2} - \delta$

$\delta > \frac{1}{2}$	$\Pi_A^r - \pi_A^{NL} = \left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{2}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{72}\delta^2 + \frac{5}{24}\delta + \frac{25}{32}\right) > 0$	$\Pi_A^r - \pi_A^{NL} = \frac{1}{8}\delta^2 + \frac{7}{8}\delta - \varepsilon + \frac{49}{32}$ - to be signed
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Working out the signs of the expressions in each of these components in the above table (see Appendix (iii) for details), we get the following.

	$\varepsilon < \frac{3}{2} + \delta$ Non-Drastic Case	$\frac{3}{2} + \delta < \varepsilon < \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32}$ Drastic Case	$\varepsilon > \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32}$ Drastic Case
$\delta < \frac{1}{2}$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r < \pi_A^{NL}$	
$\delta > \frac{1}{2}$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r < \pi_A^{NL}$

Thus, we have our main result.

Proposition 2

In Salop's circular model with pre-innovation asymmetric costs, when the patentee is inefficient, royalty licensing is optimal for a non drastic innovation.

In the drastic case, royalty can be optimal only when pre-innovation cost asymmetry is large (i.e.

$\delta > \frac{1}{2}$) and the size of the innovation is intermediate (i.e. when $\frac{3}{2} + \delta < \varepsilon < \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32}$). For

all other drastic innovations, the patentee does not license its technology and becomes a monopoly.

Intuition: In the case of non-drastic innovation, royalty licensing is optimal since by charging an appropriate per unit royalty the patentee can hold its cost advantage when it competes and at the same time collects the extra revenue coming from royalty.¹¹ In this case, the pre-innovation costs asymmetry does not have any effect. However, in the case of drastic innovation, the pre-innovation costs asymmetry and the size of the drastic innovation could matter in the following. For a large enough initial cost difference and with a moderate innovation, the efficiency gain is

significant when the production is also shifted to the efficient licensee, and this gain then can be appropriated by charging a suitable royalty. However, when the drastic innovation is really large, then the initial costs asymmetry again does not matter, and in that case no-licensing i.e. staying as monopoly is more beneficial to the patentee.

Corollary

In Salop's circular model with pre-innovation asymmetric costs, given that fixed fee is never a part of optimal licensing, a two-part tariff licensing can never be optimal as well.

4.2 Efficient Patentee

Now let us consider the situation when the patentee is efficient in costs (i.e. $c_1 - c_2 = \delta < 0$)

4.2.1 No Licensing and Fixed Fee Licsning

In the case of no licensing (under non-drastic innovation) as well as for fixed fee licensing, the

demand of firm B is positive only when $\delta > -\frac{3}{2}$ (see ¹²)

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
Π_A^F	$\left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{1}{3} + \frac{2}{9}\delta\right)\varepsilon + \left(\frac{1}{9}\delta^2 - \frac{1}{3}\delta + \frac{1}{4}\right)$	$\frac{2}{9}\delta^2 + \frac{1}{2}$
π_A^{NL}	$\frac{1}{36}(3 - 2(\delta - \varepsilon))^2$	$\varepsilon - \frac{1}{2} - \delta$
$\Pi_A^F - \pi_A^{NL}$	$\frac{2}{9}\varepsilon(2\delta - \varepsilon)$	$\frac{2}{9}\delta^2 + \delta + 1 - \varepsilon$

Under non drastic innovation : $\varepsilon < \frac{3}{2} + \delta$ with $-\frac{3}{2} < \delta < 0$, we find : $\pi_A^{NL} > \Pi_A^F$

Under drastic innovation : $\varepsilon > \frac{3}{2} + \delta$ with $-\frac{3}{2} < \delta < 0$, we also find $\pi_A^{NL} > \Pi_A^F$.

Lemma 3

In Salop's circular model with pre-innovation asymmetric costs, when the patentee is efficient, offering no license is better than fixed fee irrespective of drastic or non drastic innovation.

¹¹ Note in the case of no licensing, the only advantage of the patentee is just the cost advantage.

¹² $D_B = \frac{1}{2} + \frac{1}{3}(c_1' - c_2')$ in a fixed fee licensing $c_1' = c_1 - \varepsilon$ and $c_2' = c_2 - \varepsilon$.

Note that this result is qualitatively same as Lemma 1 under symmetric pre-innovation costs.

4.2.2 Royalty Licensing

Optimal r maximizing the total revenue of the patent holder is $r^* = \frac{15}{8} + \frac{1}{4}\delta$

Demand of firm B must be positive : $D_B > 0 \Leftrightarrow r < \frac{3}{2} + \delta$

Since $0 < r$ then we must have $\frac{3}{2} + \delta > 0 \Leftrightarrow \delta > -\frac{3}{2}$ (in this case : $-\frac{3}{2} < \delta < 0$)

Also since $r \leq \varepsilon$ we have:

$$r^* = \varepsilon \text{ if } \varepsilon < \frac{3}{2} + \delta$$

$$r^* = \frac{3}{2} + \delta \text{ if } \varepsilon > \frac{3}{2} + \delta$$

The optimal royalty rate when the innovation is non drastic here is $r^* = \varepsilon$. To have a Nash equilibrium with two firms, we must have $|p_2 - p_1| < \frac{1}{2} \Leftrightarrow \delta - \frac{3}{2} < r < \delta + \frac{3}{2}$. We can check that

$\delta - \frac{3}{2} < \varepsilon < \delta + \frac{3}{2}$ since $\delta < 0 < \varepsilon + \frac{3}{2}$ and $\varepsilon < \delta + \frac{3}{2}$ (non drastic case).

Total revenue of the patentee under royalty are as follows :

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$-\frac{3}{2} < \delta < 0$	$\Pi_A^r = \left(-\frac{2}{9}\right)\varepsilon^2 + \left(\frac{5}{6} + \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{9}\delta^2 - \frac{1}{3}\delta + \frac{1}{4}\right)$	$\Pi_A^r = 1$

4.2.3 Optimal Licensing

It has to be a comparison between royalty licencing and no-licensing as no-licensing is always better than fixed fee.

Then $D_B = \frac{1}{2} + \frac{1}{3}\delta > 0 \Leftrightarrow \frac{1}{3}\delta > -\frac{1}{2} \Leftrightarrow \delta > -\frac{3}{2}$

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
$-\frac{3}{2} < \delta < 0$	$\Pi_A^r - \pi_A^{NL} = \frac{1}{6}\varepsilon(2\delta - 2\varepsilon + 3) > 0$	$\Pi_A^r - \pi_A^{NL} = \delta - \varepsilon + \frac{3}{2} < 0$

Proposition 3

In Salop's circular model with pre-innovation asymmetric costs, when the patentee is efficient, we find the same results as in a Salop model with symmetric costs (proposition 1) where royalty is optimal when innovation is non drastic and non licensing is optimal when the innovation is drastic.

5. Licencing under Hotelling's Model with Asymmetric Costs

This is an extension of Poddar and Sinha (2004) for the case of insider patentee. Poddar and Sinha assumed symmetric pre-innovation costs of the patentee and the licensee, and analysed optimal licensing policies. Here we find the optimal licensing policy for the patentee when the pre-innovation costs are not symmetric. Like Salop's model, first we will suppose that the patentee is not efficient in costs ($c_1 > c_2$) and later we deal with the case when it is efficient ($c_1 < c_2$). Like Poddar and Sinha (2004), we also assume that the patentee and the potential licensee are located at the end points of the linear city.

Thus, the location of the marginal consumer:

$$U_A = -p_1 - x$$

$$U_B = -p_2 - (1-x)$$

Demands are :

$$D_A = \begin{cases} 1 & \text{if } p_1 < p_2 - 1 \\ \frac{1}{2} - \frac{p_1 - p_2}{2} & \text{if } p_2 - 1 < p_1 < p_2 + 1 \\ 0 & \text{if } p_1 > p_2 + 1 \end{cases}$$

$$D_B = \begin{cases} 0 & \text{if } p_2 > p_1 + 1 \\ \frac{1}{2} + \frac{p_1 - p_2}{2} & \text{if } p_1 - 1 < p_2 < p_1 + 1 \\ 1 & \text{if } p_2 < p_1 - 1 \end{cases}$$

The profits are:

$$\pi_A = \begin{cases} p_1 - c_1 & \text{if } p_1 < p_2 - 1 \\ (p_1 - c_1) \left(\frac{1}{2} - \frac{p_1 - p_2}{2} \right) & \text{if } p_2 - 1 < p_1 < p_2 + 1 \\ 0 & \text{if } p_1 > p_2 + 1 \end{cases}$$

$$\pi_B = \begin{cases} 0 & \text{if } p_2 > p_1 + 1 \\ (p_2 - c_2) \left(\frac{1}{2} + \frac{p_1 - p_2}{2} \right) & \text{if } p_1 - 1 < p_2 < p_1 + 1 \\ p_2 - c_2 & \text{if } p_2 < p_1 - 1 \end{cases}$$

Thus when $|p_1 - p_2| < 1$ we have

$$\pi_A = (p_1 - c_1) \left(\frac{1}{2} - \frac{p_1 - p_2}{2} \right)$$

$$\pi_B = (p_2 - c_2) \left(\frac{1}{2} + \frac{p_1 - p_2}{2} \right)$$

Maximizing profits with respect to the prices we find equilibrium prices as:

$$p_1 = 1 + \frac{2}{3}c_1 + \frac{1}{3}c_2$$

$$p_2 = 1 + \frac{1}{3}c_1 + \frac{2}{3}c_2$$

Equilibrium profits are :

$$\pi_A = \frac{1}{18} (3 - c_1 + c_2)^2$$

$$\pi_B = \frac{1}{18} (3 + c_1 - c_2)^2$$

Equilibrium demands are :

$$D_A = \frac{1}{2} - \frac{1}{6} (c_1 - c_2)$$

$$D_B = \frac{1}{2} + \frac{1}{6} (c_1 - c_2)$$

5.1 Inefficient Patentee

The patentee is inefficient compared to the licensee and thus ($\delta = c_1 - c_2 > 0$).

5.1.1 Fixed Fee Licensing

In a fixed fee licensing, the marginal costs of firm A and firm B respectively are $c_1' = (c_1 - \varepsilon)$ and $c_2' = (c_2 - \varepsilon)$. The equilibrium prices are :

$$p_1 = 1 + \frac{2}{3}c_1' + \frac{1}{3}c_2' = 1 + \frac{2}{3}(c_1 - \varepsilon) + \frac{1}{3}(c_2 - \varepsilon) = \frac{2}{3}c_1 - \varepsilon + \frac{1}{3}c_2 + 1$$

$$p_2 = 1 + \frac{1}{3}c_1' + \frac{2}{3}c_2' = 1 + \frac{1}{3}(c_1 - \varepsilon) + \frac{2}{3}(c_2 - \varepsilon) = \frac{1}{3}c_1 - \varepsilon + \frac{2}{3}c_2 + 1$$

Equilibrium profits are:

$$\pi_A^F = \frac{1}{18}(3 - c_1' + c_2')^2 = \frac{1}{18}(3 - (c_1 - \varepsilon) + (c_2 - \varepsilon))^2 = \frac{1}{18}(c_2 - c_1 + 3)^2 = \frac{1}{18}(3 - \delta)^2$$

$$\pi_B^F = \frac{1}{18}(3 + c_1' - c_2')^2 = \frac{1}{18}(3 + (c_1 - \varepsilon) - (c_2 - \varepsilon))^2 = \frac{1}{18}(c_1 - c_2 + 3)^2 = \frac{1}{18}(3 + \delta)^2$$

Equilibrium Demands are:

$$D_A = \frac{1}{2} - \frac{1}{6}(c_1' - c_2') = \frac{1}{2} - \frac{1}{6}((c_1 - \varepsilon) - (c_2 - \varepsilon)) = \frac{1}{2} - \frac{1}{6}(c_1 - c_2) = \frac{1}{2} - \frac{\delta}{6}$$

$$D_B = \frac{1}{2} + \frac{1}{6}(c_1' - c_2') = \frac{1}{2} + \frac{1}{6}((c_1 - \varepsilon) - (c_2 - \varepsilon)) = \frac{1}{2} + \frac{1}{6}(c_1 - c_2) = \frac{1}{2} + \frac{\delta}{6}$$

The fixed fee that the patentee charges is $F = \pi_B^F - \pi_B^{NL}$

Below we summarize the relevant equilibrium expressions:

	Non Drastic innovation	Drastic innovation
	$\varepsilon < 3 + \delta$	$\varepsilon \geq 3 + \delta$
π_A^F	$\frac{1}{18}(3 - \delta)^2$	$\frac{1}{18}(3 - \delta)^2$
π_B^{NL}	$\frac{1}{18}(\varepsilon - \delta - 3)^2$	0
$F = \pi_B^F - \pi_B^{NL}$	$\frac{\varepsilon}{18}(2\delta - \varepsilon + 6)$	$\frac{1}{18}(3 + \delta)^2$

π_B^F	$\frac{1}{18}(3-\delta)^2$	$\frac{1}{18}(3-\delta)^2$
$\Pi_A^F = \pi_A^F + F$	$\left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{3} + \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{18}\delta^2 - \frac{1}{3}\delta + \frac{1}{2}\right)$	$\frac{1}{9}\delta^2 + 1$

For an interior equilibrium prices we must have : $|p_2 - p_1| < 1$. Using equilibrium prices we get:

$$|p_2 - p_1| = \left(\frac{1}{3}c_1 - \varepsilon + \frac{2}{3}c_2 + 1\right) - \left(\frac{2}{3}c_1 - \varepsilon + \frac{1}{3}c_2 + 1\right) = \frac{1}{3}c_2 - \frac{1}{3}c_1 = \frac{1}{3}\delta$$

Hence for valid equilibrium prices under fixed fee licensing we satisfy the following condition :

$$|p_2 - p_1| < 1 \Leftrightarrow \frac{1}{3}\delta < 1 \Leftrightarrow \delta < 3$$

5.1.2 Royalties Licensing

Production unit costs here become : $c_1' = (c_1 - \varepsilon)$ and $c_2' = (c_2 - \varepsilon + r)$

$$p_1 = 1 + \frac{2}{3}c_1' + \frac{1}{3}c_2' = 1 + \frac{2}{3}(c_1 - \varepsilon) + \frac{1}{3}(c_2 - \varepsilon + r) = \frac{1}{3}r - \varepsilon + \frac{2}{3}c_1 + \frac{1}{3}c_2 + 1$$

$$p_2 = 1 + \frac{1}{3}c_1' + \frac{2}{3}c_2' = 1 + \frac{1}{3}(c_1 - \varepsilon) + \frac{2}{3}(c_2 - \varepsilon + r) = \frac{2}{3}r - \varepsilon + \frac{1}{3}c_1 + \frac{2}{3}c_2 + 1$$

Equilibrium profits are:

$$\pi_A^r = \frac{1}{18}(3 - c_1' + c_2')^2 = \frac{1}{18}(3 - (c_1 - \varepsilon) + (c_2 - \varepsilon + r))^2 = \frac{1}{18}(3 + r - \delta)^2$$

$$\pi_B^r = \frac{1}{18}(3 + c_1' - c_2')^2 = \frac{1}{18}(3 + (c_1 - \varepsilon) - (c_2 - \varepsilon + r))^2 = \frac{1}{18}(3 - r + \delta)^2$$

Equilibrium demands are:

$$D_A = \frac{1}{2} - \frac{1}{6}(c_1' - c_2') = \frac{1}{2} - \frac{1}{6}((c_1 - \varepsilon) - (c_2 - \varepsilon + r)) = \frac{1}{6}(3 + r - \delta)$$

$$D_B = \frac{1}{2} + \frac{1}{6}(c_1' - c_2') = \frac{1}{2} + \frac{1}{6}((c_1 - \varepsilon) - (c_2 - \varepsilon + r)) = \frac{1}{6}(3 - r + \delta)$$

Total revenue of firm A after licensing is :

$$\Pi_A^r = \pi_A^r + rD_B = \frac{1}{18}(3 + r - \delta)^2 + r\frac{1}{6}(3 - r + \delta)$$

$$\frac{\partial \Pi_A^r}{\partial r} = \frac{1}{18}(\delta - 4r + 15) \Leftrightarrow r^* = \frac{15}{4} + \frac{1}{4}\delta$$

Now $D_B > 0 \Leftrightarrow \frac{1}{6}(3-r+\delta) > 0 \Leftrightarrow r < 3+\delta$, we also must check that $r < \varepsilon$

Thus, we find :

For a non drastic innovation $\varepsilon < 3+\delta$

$$\begin{cases} r^* = \varepsilon & \text{if } \varepsilon < 3+\delta \text{ and } \delta < 1 \\ r^* = \frac{15}{4} + \frac{1}{4}\delta & \text{if } \frac{15}{4} + \frac{1}{4}\delta < \varepsilon < 3+\delta \text{ and } \delta > 1 \end{cases}$$

For a drastic innovation $\varepsilon > 3+\delta$

$$\begin{cases} r^* = 3+\delta & \text{if } 3+\delta < \frac{15}{4} + \frac{1}{4}\delta \Leftrightarrow \delta < 1 \\ r^* = \frac{15}{4} + \frac{1}{4}\delta & \text{if } \frac{15}{4} + \frac{1}{4}\delta < 3+\delta \Leftrightarrow \delta > 1 \end{cases}$$

Checking for the condition: $|p_2 - p_1| < 1$

$$|p_2 - p_1| = \left(\frac{2}{3}r - \varepsilon + \frac{1}{3}c_1 + \frac{2}{3}c_2 + 1 \right) - \left(\frac{1}{3}r - \varepsilon + \frac{2}{3}c_1 + \frac{1}{3}c_2 + 1 \right) = \frac{1}{3}r - \frac{1}{3}c_1 + \frac{1}{3}c_2 = \frac{1}{3}r - \frac{1}{3}\delta$$

Now, $|p_2 - p_1| < 1 \Leftrightarrow -1 < \frac{1}{3}r - \frac{1}{3}\delta < 1 \Leftrightarrow -1 + \frac{1}{3}\delta < \frac{1}{3}r < 1 + \frac{1}{3}\delta \Leftrightarrow -3 + \delta < r < 3 + \delta$

The royalty rates are valid when $-3 + \delta < r^* < 3 + \delta$ which is true if and only if $\delta < 9$ (using the

fact : $-3 + \delta < r^* = \left(\frac{15}{4} + \frac{1}{4}\delta \right)$)

Total revenue of the firm A is :

$\Pi_A^r = \frac{1}{18}(3+r-\delta)^2 + r\frac{1}{6}(3-r+\delta)$ and using $r^* = \left(\frac{15}{4} + \frac{1}{4}\delta \right)$, we get the following:

Total Revenue at equilibrium:

$$\frac{1}{18} \left(3 + \left(\frac{15}{4} + \frac{1}{4}\delta \right) - \delta \right)^2 + \left(\frac{15}{4} + \frac{1}{4}\delta \right) \frac{1}{6} \left(3 - \left(\frac{15}{4} + \frac{1}{4}\delta \right) + \delta \right) = \frac{1}{16}\delta^2 - \frac{1}{8}\delta + \frac{33}{16}$$

	$\varepsilon < 3+\delta$	$\varepsilon > 3+\delta$
$\delta < 1$	$\Pi_A^r = \left(-\frac{1}{9} \right) \varepsilon^2 + \left(\frac{5}{6} + \frac{1}{18}\delta \right) \varepsilon + \left(\frac{1}{18}\delta^2 - \frac{1}{3}\delta + \frac{1}{2} \right)$	$\Pi_A^r = 2$
$\delta > 1$	$\Pi_A^r = \frac{1}{16}\delta^2 - \frac{1}{8}\delta + \frac{33}{16}$	$\Pi_A^r = \frac{1}{16}\delta^2 - \frac{1}{8}\delta + \frac{33}{16}$

5.1.3 Comparison Between Fixed Fee and Royalty Licensing:

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$\delta < 1$	$\Pi_A^r - \Pi_A^F = \frac{\varepsilon}{18}(9 - \delta - \varepsilon) > 0$	$\Pi_A^r - \Pi_A^F = \frac{1}{9}\delta^2 + 1 > 0$
$\delta > 1$	$\Pi_A^r - \Pi_A^F = \frac{1}{18}\varepsilon^2 + \left(-\frac{1}{3} - \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right)$	$\Pi_A^r - \Pi_A^F = \frac{17}{16} - \frac{1}{8}\delta - \frac{7}{144}\delta^2$

Working out the signs of the expressions in each of these components in the above table (see Appendix (iv) for details), we get the following.

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$\delta < 1$	$\Pi_A^r > \Pi_A^F$	$\Pi_A^r > \Pi_A^F$
$1 < \delta < 3.563$	$\Pi_A^r > \Pi_A^F$	$\Pi_A^r > \Pi_A^F$

Lemma 4

In Hotelling's model with pre-innovation asymmetric costs, when the patentee is inefficient, royalty is always better than fixed fee irrespective of drastic or non drastic innovation.

Note that this result is qualitatively same as Lemma 2.

5.1.4 No Licensing

The production unit costs of respectively firm A and firm B are $c_1' = c_1 - \varepsilon$ and $c_2' = c_2$

The prices at the equilibrium are :

$$p_1 = 1 + \frac{2}{3}c_1' + \frac{1}{3}c_2' = 1 + \frac{2}{3}(c_1 - \varepsilon) + \frac{1}{3}c_2 = \frac{2}{3}c_1 - \frac{2}{3}\varepsilon + \frac{1}{3}c_2 + 1$$

$$p_2 = 1 + \frac{1}{3}c_1' + \frac{2}{3}c_2' = 1 + \frac{1}{3}(c_1 - \varepsilon) + \frac{2}{3}c_2 = \frac{1}{3}c_1 - \frac{1}{3}\varepsilon + \frac{2}{3}c_2 + 1$$

For a non drastic innovation ($\varepsilon < 3 + \delta$), the equilibrium profits are:

$$\pi_A = \frac{1}{18}(3 - c_1' + c_2')^2 = \frac{1}{18}(3 - (c_1 - \varepsilon) + c_2)^2 = \frac{1}{18}(\varepsilon - c_1 + c_2 + 3)^2 = \frac{1}{18}(\varepsilon - \delta + 3)^2$$

$$\pi_B = \frac{1}{18}(3 + c_1' - c_2')^2 = \frac{1}{18}(3 + (c_1 - \varepsilon) - c_2)^2 = \frac{1}{18}(\varepsilon - c_1 + c_2 - 3)^2 = \frac{1}{18}(\varepsilon - \delta - 3)^2$$

Equilibrium demands are :

$$D_A = \frac{1}{2} - \frac{1}{6}(c_1 - c_2) = \frac{1}{2} - \frac{1}{6}((c_1 - \varepsilon) - c_2) = \frac{1}{2} + \frac{1}{6}(\varepsilon - c_1 + c_2) = \frac{1}{2} + \frac{1}{6}(\varepsilon - \delta)$$

$$D_B = \frac{1}{2} + \frac{1}{6}(c_1 - c_2) = \frac{1}{2} + \frac{1}{6}((c_1 - \varepsilon) - c_2) = \frac{1}{2} - \frac{1}{6}(\varepsilon - c_1 + c_2) = \frac{1}{2} - \frac{1}{6}(\varepsilon - \delta)$$

For a drastic innovation ($\varepsilon \geq 3 + \delta$) we have:

$$p_A = c_2 - 1 \text{ and } \pi_A = (p_1 - c_1 + \varepsilon) = (c_2 - 1 - c_1 + \varepsilon) = (\varepsilon - 1 - \delta)$$

$$\pi_B = 0$$

Summarizing profits under no licensing :

	$\varepsilon < 3 + \delta$	$\varepsilon \geq 3 + \delta$
π_A^{NL}	$\frac{1}{18}(\varepsilon - \delta + 3)^2$	$(\varepsilon - 1 - \delta)$
π_B^{NL}	$\frac{1}{18}(\varepsilon - \delta - 3)^2$	0

5.1.5 Optimal Licising

We need to compare between royalty licening and no licensing only as fixed fee is always dominated by royalty.

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$\delta < 1$	$\Pi_A^r - \pi_A^{NL} = \frac{\varepsilon}{6}(\delta - \varepsilon + 3) > 0$	$\Pi_A^r - \pi_A^{NL} = (3 + \delta - \varepsilon) < 0$
$\delta > 1$	$\Pi_A^r - \pi_A^{NL} = \left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right)$	$\Pi_A^r - \pi_A^{NL} = \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16} - \varepsilon$

Working out the signs of the expressions in each of these components in the above table (see Appendix (iv) for details), we get the following.

	$\varepsilon < 3 + \delta$ Non Drastic Case	$3 + \delta < \varepsilon < \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}$ Drastic Case	$\varepsilon > \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}$ Drastic Case
$\delta < 1$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r < \pi_A^{NL}$	
$\delta > 1$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r > \pi_A^{NL}$	$\Pi_A^r < \pi_A^{NL}$

Proposition 4

In Hotelling's linear model with pre-innovation asymmetric costs, when the patentee is inefficient, royalty licensing is optimal for a non drastic innovation.

In the drastic case, royalty can be optimal only when pre-innovation cost asymmetry is large (i.e.

$\delta > 1$) and the size of the innovation is intermediate (i.e. when $3 + \delta < \varepsilon < \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}$).

For all other drastic innovations, the patentee does not license its technology and becomes a monopoly.¹³

5.2 Efficient Patentee

Let's now suppose an efficient patentee, thus ($\delta = c_1 - c_2 < 0$)

5.2.1 No Licensing and Fixed Fee Licesning

In the case of no licensing (under non-drastric innovation) as well as for fixed fee licensing the demand of firm B is positive only when $\delta > -3$ (see ¹⁴)

	$\varepsilon < \frac{3}{2} + \delta$	$\varepsilon > \frac{3}{2} + \delta$
Π_A^F	$\left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{3} + \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{18}\delta^2 - \frac{1}{3}\delta + \frac{1}{2}\right)$	$\frac{1}{9}\delta^2 + 1$
π_A^{NL}	$\frac{1}{18}(\varepsilon - \delta + 3)^2$	$\varepsilon - 1 - \delta$
$\Pi_A^F - \pi_A^{NL}$	$\frac{\varepsilon}{9}(2\delta - \varepsilon)$	$\frac{1}{9}\delta^2 + \delta + 2 - \varepsilon$

Now for the non-drastric case : $\varepsilon < 3 + \delta$ with $-3 < \delta < 0$:

It can be shown that $\Pi_A^F - \pi_A^{NL} = \frac{\varepsilon}{9}(2\delta - \varepsilon) < 0$

For drastric case : $\varepsilon \geq 3 + \delta$ with $-3 < \delta < 0$:

It can be shown that $\Pi_A^F - \pi_A^{NL} = \frac{1}{9}\delta^2 + \delta - \varepsilon + 2 = \left(\frac{1}{9}\delta^2 + \delta + 2 - \varepsilon\right)$

Now for $-3 < \delta < 0$ we have $\varepsilon \geq 3 + \delta > \frac{1}{9}\delta^2 + \delta + 2$, hence $\Pi_A^F < \pi_A^{NL}$

¹³ Here the intuition is same as described after Proposition 2.

Lemma 5

In Hotelling's linear model with pre-innovation asymmetric costs, when the patentee is efficient, offering no license is better than fixed fee irrespective of drastic or non drastic innovation.

Note that this result is qualitatively same as Lemma 3.

5.2.2 Royalty Licensing

Optimal royalty rates when $\delta < 0$ are : $r^* = \varepsilon$ for a non drastic innovation and $r^* = 3 + \delta$ for a drastic innovation.

Since $r > 0$ then we must have $3 + \delta > 0 \Leftrightarrow \delta > -3$ (in this case : $-3 < \delta < 0$)

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$-3 < \delta < 0$	$r^* = \varepsilon$	$r^* = 3 + \delta$

We also verify that $|p_1 - p_2| < 1 \Leftrightarrow -3 + \delta < r^* < 3 + \delta$.

The total revenues of the patent holder under royalty licensing are:

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$-3 < \delta < 0$	$\Pi_A^r = \left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{5}{6} + \frac{1}{18}\delta\right)\varepsilon + \left(\frac{1}{18}\delta^2 - \frac{1}{3}\delta + \frac{1}{2}\right)$	$\Pi_A^r = 2$

Comparing total revenue of the patentee under royalty licensing and non licensing we get:

	$\varepsilon < 3 + \delta$	$\varepsilon > 3 + \delta$
$-3 < \delta < 0$	$\Pi_A^r - \pi_A^{NL} = \frac{\varepsilon}{6}(\delta - \varepsilon + 3) > 0$	$\Pi_A^r - \pi_A^{NL} = (3 + \delta - \varepsilon) < 0$

Then we have the following result:

Proposition 5

In Hotelling's linear model with pre-innovation asymmetric costs, when the patentee is efficient, we find the same results as in Salop's or Hotelling's model with symmetric costs. Royalty is optimal when innovation is non drastic and non licensing is optimal when the innovation is

¹⁴ $D_B = \frac{1}{2} + \frac{1}{6}(c_1' - c_2') = \frac{1}{2} + \frac{1}{6}(c_1 - \varepsilon - c_2 + \varepsilon) = \frac{1}{6}\delta + \frac{1}{2} > 0$ iff $\frac{1}{6}\delta > -\frac{1}{2} \Leftrightarrow \delta > -3$

drastic.

6. Conclusion

In this paper, we considered the problem of optimal licensing strategy of an insider patentee to a potential licensee in a circular city of Salop's model and in a linear city of Hotelling's model. Firms have asymmetric pre-innovation marginal costs of production and compete in prices in the product market. The patentee comes up with an innovation which brings down its unit cost of production. In the circular Salop model, firms are situated symmetrically on the circumference and in the linear Hotelling model firms located at the end points. Consumers are uniformly distributed over the circumference in the Salop model and over the linear segment in the Hotelling model. The good produced by both firms is identical in nature; however, due to the presence of transport cost incurred by the consumer to buy the good from either of the firms, the goods are horizontally differentiated in the eyes of the consumers. We consider two possible types of cost-reducing innovations from the patentee that are generally described in the literature, namely, the drastic and non-drastic innovations and completely characterize the equilibrium licensing outcomes under the fixed fee and per-unit royalty in order to find the optimal strategy of the patentee in offering (or not offering) the license to its rival. We find that when the innovative firm is efficient compared to the licensee at the pre-innovation stage then the results regarding optimal licensing policy coincide with the results described in the literature when firms are symmetric in the pre-innovation stage. This result is true in Salop's model as well as in Hotelling's model. However, this is not true when the innovative firm is inefficient in the pre-innovation stage compared to the licensee. To that end, we show that both in Salop's and Hotelling's model, a drastic innovation can be licensed using a royalty scheme when the patentee is highly inefficient compared to licensee in the pre-innovation stage and the size of the innovation is intermediate. We also show that in this set-up, fixed fee licensing alone is never optimal.

Note that in this paper, we only dealt with the case of an insider patentee. In our future work, we plan to consider the case of an outsider patentee in a similar framework of spatial competition.

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Appendix (i)

From (4) we get: when $\varepsilon < \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$, we find $r^* = \varepsilon$

Check two firms are always active here

$\delta - \frac{3}{2} < r^* < \delta + \frac{3}{2} \Leftrightarrow \delta - \frac{3}{2} < \varepsilon < \delta + \frac{3}{2}$, verified since $\varepsilon < \delta + \frac{3}{2}$ (non drastic case); and

$$\delta < \frac{1}{2} \Leftrightarrow \delta - \frac{3}{2} < 0 \text{ then } \delta - \frac{3}{2} < 0 < \varepsilon$$

From (5) we get : when $\varepsilon < \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$, we find $r^* = \frac{15}{8} + \frac{1}{4}\delta$

Check two firms are active here only when $\delta < 4.5$

$$\delta - \frac{3}{2} < r^* < \delta + \frac{3}{2} \Leftrightarrow \delta - \frac{3}{2} < \frac{15}{8} + \frac{1}{4}\delta < \delta + \frac{3}{2}, \text{ verified since } \frac{15}{8} + \frac{1}{4}\delta < \delta + \frac{3}{2} \Leftrightarrow \delta > \frac{1}{2}; \text{ and}$$

$$\delta - \frac{3}{2} < \frac{15}{8} + \frac{1}{4}\delta \Leftrightarrow \delta < 4.5$$

From (6) we get : when $\varepsilon > \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$, we find $r^* = \frac{3}{2} + \delta$

Check two firms are always active here

$$\delta - \frac{3}{2} < r^* < \delta + \frac{3}{2} \Leftrightarrow \delta - \frac{3}{2} < \frac{3}{2} + \delta < \delta + \frac{3}{2} \text{ verified}$$

From (7) we get : when $\varepsilon > \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$, we find $r^* = \frac{15}{8} + \frac{1}{4}\delta$

Check two firms are active here only when $\delta < 4.5$

$$\delta - \frac{3}{2} < r^* < \delta + \frac{3}{2} \Leftrightarrow \delta - \frac{3}{2} < \frac{15}{8} + \frac{1}{4}\delta < \delta + \frac{3}{2}, \text{ verified since } \frac{15}{8} + \frac{1}{4}\delta < \delta + \frac{3}{2} \Leftrightarrow \delta > \frac{1}{2}.$$

$$\text{And } \delta - \frac{3}{2} < \frac{15}{8} + \frac{1}{4}\delta \Leftrightarrow \delta < 4.5$$

Appendix (ii)

*when $\varepsilon < \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$ we have $\Pi'_A - \Pi^F_A = \frac{1}{18}\varepsilon(-2\delta - 2\varepsilon + 9) > 0$ since $\varepsilon < \frac{3}{2} + \delta < \frac{9}{2} - \delta$

for $\delta < \frac{1}{2}$

*when $\varepsilon > \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$ we have $\Pi'_A - \Pi^F_A = \frac{1}{2} - \frac{2}{9}\delta^2 = \frac{2}{9}\left(\frac{3}{2} - \delta\right)\left(\frac{3}{2} + \delta\right) > 0$ since $\delta < \frac{1}{2}$

*when $\varepsilon > \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$ we have $\Pi'_A - \Pi^F_A = \frac{17}{32} - \frac{1}{8}\delta - \frac{7}{72}\delta^2 = -\frac{7}{72}\delta^2 - \frac{1}{8}\delta + \frac{17}{32}$

the roots of the polynomial $-\frac{7}{72}\delta^2 - \frac{1}{8}\delta + \frac{17}{32}$ are $\delta' = -3.0672$ and $\delta'' = 1.7815$

if $\frac{1}{2} < \delta < 1.7815$ then $\Pi_A^r - \Pi_A^F = \frac{17}{32} - \frac{1}{8}\delta - \frac{7}{72}\delta^2 > 0$

*when $\varepsilon < \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$ we have $\Pi_A^r - \Pi_A^F = \frac{1}{9}\varepsilon^2 + \left(-\frac{1}{3} - \frac{2}{9}\delta\right)\varepsilon + \left(\frac{1}{72}\delta^2 + \frac{5}{24}\delta + \frac{25}{32}\right)$

This polynomial can be solved if the discriminant term $\Delta = 7\delta^2 + 9\delta - \frac{153}{4} > 0$

we can check that $\Delta < 0$ when $\frac{1}{2} < \delta < 1.7815 \Rightarrow$ no roots $\Rightarrow \Pi_A^r - \Pi_A^F > 0$

Appendix (iii)

For $\varepsilon < \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$ easy to see that $\Pi_A^r - \pi_A^{NL} = \frac{1}{6}\varepsilon(2\delta - 2\varepsilon + 3) > 0$

For $\varepsilon < \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$ we have $\Pi_A^r - \pi_A^{NL} = \left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{2}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{72}\delta^2 + \frac{5}{24}\delta + \frac{25}{32}\right)$

Studying the sign of this polynomial we find that roots are $\varepsilon' = \delta - \frac{3}{2} - \frac{3}{4}\sqrt{2}\sqrt{\delta^2 - \delta + \frac{33}{4}}$ and

$\varepsilon'' = \delta - \frac{3}{2} + \frac{3}{4}\sqrt{2}\sqrt{\delta^2 - \delta + \frac{33}{4}}$. Note that ε' and ε'' exists since $\sqrt{\delta^2 - \delta + \frac{33}{4}} > 0$. Next we

verify that $\varepsilon' < 0$ and $\varepsilon'' - (\frac{3}{2} + \delta) > 0$ ¹⁵.

Thus, we have the following : $\varepsilon' < 0 < \varepsilon < \frac{3}{2} + \delta < \varepsilon''$

Thus, we can say that if $\delta > \frac{1}{2}$ and $\varepsilon < \frac{3}{2} + \delta$ then :

$$\Pi_A^r - \pi_A^{NL} = \left(-\frac{1}{9}\right)\varepsilon^2 + \left(\frac{2}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{72}\delta^2 + \frac{5}{24}\delta + \frac{25}{32}\right) > 0 \Rightarrow \Pi_A^r > \pi_A^{NL}$$

For $\varepsilon > \frac{3}{2} + \delta$ and $\delta < \frac{1}{2}$ easy to see that $\Pi_A^r - \pi_A^{NL} = \delta - \varepsilon + \frac{3}{2} < 0$ since $\varepsilon > \frac{3}{2} + \delta \Rightarrow \Pi_A^r < \pi_A^{NL}$

For $\varepsilon > \frac{3}{2} + \delta$ and $\delta > \frac{1}{2}$ we have $\Pi_A^r - \pi_A^{NL} = \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32} - \varepsilon$

we distinguish here between two cases:

If $\frac{3}{2} + \delta < \varepsilon < \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32}$ then $\pi_A^r - \pi_A^{NL} > 0 \Rightarrow \Pi_A^r > \pi_A^{NL}$

If $\varepsilon > \frac{1}{8}\delta^2 + \frac{7}{8}\delta + \frac{49}{32}$ then $\pi_A^r - \pi_A^{NL} < 0 \Rightarrow \Pi_A^r < \pi_A^{NL}$

Appendix (iv)

For $\delta > 1$ and $\varepsilon > 3 + \delta$ we have $\Pi_A^r - \Pi_A^F = \frac{17}{16} - \frac{1}{8}\delta - \frac{7}{144}\delta^2$

For $\frac{17}{16} - \frac{1}{8}\delta - \frac{7}{144}\delta^2$, solution is:

$$-\frac{9}{7} - \frac{24}{7}\sqrt{2} = -6.1344$$

$$\frac{24}{7}\sqrt{2} - \frac{9}{7} = 3.563$$

If $1 < \delta < 3.563$ then $\Pi_A^r - \Pi_A^F > 0 \Leftrightarrow \Pi_A^r > \Pi_A^F$

For $\delta > 1$ and $\varepsilon < 3 + \delta$ we have $\Pi_A^r - \Pi_A^F = \frac{1}{18}\varepsilon^2 + \left(-\frac{1}{3} - \frac{1}{9}\delta\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right)$

The discriminant term $\Delta = -153 + 18\delta + 7\delta^2$

Solution is:

$$-\frac{9}{7} - \frac{24}{7}\sqrt{2} = -6.1344$$

$$\frac{24}{7}\sqrt{2} - \frac{9}{7} = 3.563$$

then

$\Delta < 0$ if $\delta < 3.563 \Rightarrow$ no solutions $\Rightarrow \Pi_A^r - \Pi_A^F > 0 \Leftrightarrow \Pi_A^r > \Pi_A^F$

Appendix (v)

* for $\varepsilon > 3 + \delta$ and $\delta > 1$

$$\Pi_A^r - \pi_A^{NL} = \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16} - \varepsilon = \left(\frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16} - \varepsilon\right)$$

¹⁵ Check that $\delta - \frac{3}{2} + \frac{3}{4}\sqrt{2}\sqrt{\delta^2 - \delta + \frac{33}{4}} - \left(\frac{3}{2} + \delta\right) = \frac{3}{4}\sqrt{2}\sqrt{\delta^2 - \delta + \frac{33}{4}} - 3 > 0$

we have: $\left(\frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}\right) > (3+\delta)$ since $\left(\frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}\right) - (3+\delta) = \frac{1}{16} - \frac{1}{8}\delta + \frac{1}{16}\delta^2 > 0$

Then if $3+\delta < \varepsilon < \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}$ we have $\Pi'_A - \pi_A^{NL} > 0 \Leftrightarrow \Pi'_A > \pi_A^{NL}$

while if $\varepsilon > \frac{1}{16}\delta^2 + \frac{7}{8}\delta + \frac{49}{16}$ we have $\Pi'_A - \pi_A^{NL} < 0 \Leftrightarrow \Pi'_A < \pi_A^{NL}$

* For $\varepsilon < 3+\delta$ and $\delta > 1$

$$\Pi'_A - \pi_A^{NL} = \left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right)$$

$$\left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right)$$

Solution is:

$$\varepsilon' = \delta - 3 - \frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2}$$

$$\varepsilon'' = \delta - 3 + \frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2}$$

we can check that $\Delta = 33 - 2\delta + \delta^2 > 0$ (we have two solutions)

$$\varepsilon' = \delta - 3 - \frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2} < 0$$

$$\varepsilon'' - (3+\delta) = \delta - 3 + \frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2} - (3+\delta) = \frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2} - 6$$

the difference between the squares is :

$$\left(\frac{3}{4}\sqrt{2}\sqrt{33-2\delta+\delta^2}\right)^2 - 36 = \frac{9}{8}\delta^2 - \frac{9}{4}\delta + \frac{9}{8} > 0$$

$$\Rightarrow \varepsilon'' > (3+\delta)$$

$$\varepsilon' < 0 < \varepsilon < (3+\delta) < \varepsilon'' \text{ then}$$

$$\Pi'_A - \pi_A^{NL} = \left(-\frac{1}{18}\right)\varepsilon^2 + \left(\frac{1}{9}\delta - \frac{1}{3}\right)\varepsilon + \left(\frac{1}{144}\delta^2 + \frac{5}{24}\delta + \frac{25}{16}\right) > 0 \Leftrightarrow \Pi'_A > \pi_A^{NL}$$