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OPTIMAL MANAGEMENT AND INFLATION PROTECTION FOR DEFINED CONTRIBUTION PENSION PLANS

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Abstract

Due to the increasing risk of inflation and diminishing pension benefits, insurance companies have started selling inflation-linked products. Selling such products the insurance company takes over some or all of the inflation risk from their customers. On the other side financial derivatives which are linked to inflation such as inflation linked bonds are traded on financial markets and appear to be of increasing popularity. The insurance company can use these products to hedge its own inflation risk. In this article we study how to optimally manage a pension fund taking positions in a money market account, a stock and an inflation linked bond, while financing investments through a continuous stochastic income stream such as the plan member's contributions. We use the martingale method in order to compute an analytic expression for the optimal strategy and express it in terms of observable market variables.

Keywords: Pension mathematics, inflation, long-term investment, stochastic optimal control, martingale method

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1. Introduction

In a classical defined contribution pension plan, the plan member bears a considerable risk due to inflation. As investment into a pension plan is in general carried out over a long period of 30 years or more, the plan member may lose a considerable amount in real value of his pension benefit. In fact, many plan members may not be aware that the benefits they will obtain from a classical, non-inflation-linked pension plan may not be sufficient to carry their expenses in the future, as price levels may have increased due to inflation. A simple calculation shows that given an annual inflation rate of $1,5\%$ over 30 years will reduce the real value of 100.000 Euro then to 63.976 Euro today. It therefore makes sense to link pension products to inflation. Selling such products, the insurance company enters into a considerable risk itself. The financial management of the insurance company must therefore think how to invest the plan members' contributions optimally in the presence of inflation. One way to decrease the risk due to inflation is to trade in so called inflation-linked products. These products enjoy increasing popularity in the UK, Canada, Australia and some continental European states, while in Germany the state agency for Finance is still evaluating in how far interest rate related costs can be lowered by the introduction of inflation-linked bonds. Other inflation-linked products are for example inflation swaps, puts, calls, caps and floors. For an illustration of these products we refer to Korn and Kruse [12]. In this article, we mainly consider inflation-linked bonds of the following type.

Definition 1. An inflation-linked coupon bearing bond with non-inflation protected face value F is a bond paying coupons $C_i \frac{I(t_i)}{I(t_0)}$ at times t_i, $i = 1, ..., n$ and a final payment F at time $t_n = T$ where $I(t)$ denotes the value of some specific consumer price index at time t..

In theory individual investors could use these inflation-linked products to insure their pension benefits against inflation risk, but according to Korn and Kruse [12], the demand in inflation-linked products among private investors is rather low. On the other side, firms whose profits are strongly negatively correlated with inflation, and to which insurance companies clearly belong, have recognized the advantages of inflation-linked products and trade large quantities of them in their portfolios. In our study we consider the optimal asset allocation problem of an insurance company, that trades in a money market account, a stock and an inflation-linked bond while financing its investments by the pension plan's members contributions. We assume that the money market account and the stock are specified as in the classical Black-Scholes model, see [4], while the inflation index $I(t)$ will be specified as a geometric Brownian motion , whose drift rate is determined from the classical Fisher equation [9]. We will give a heuristic derivation of this in the following section. For simplicity, we fix a time horizon $[0, T]$ which corresponds to the time period spanning a specified plan member's entry into the pension plan until the point when he starts to receive the pension benefit. We furthermore assume that each plan member's contributions to the pension fund are defined as a percentage c of his salary $Y(t)$, which we assume is stochastic and follows a geometric Brownian motion. We allow a rather general correlation structure between stock $S(t)$, inflation index $I(t)$ and salary $Y(t)$. Obviously, the problem of optimal management of pension funds has been dealt with before, both in discrete and continuous time. Important contributions addressing the continuous time framework are due to Blake et al. [3], Cairns ([5], [6]) and Deelstra et al ([7],[8]). All of these authors, except Delstra et al in [8], use stochastic dynamic programming in order to solve the corresponding optimization problems. Being in line with [8], we use the so called martingale approach, which we describe in section 4. None of the articles above considers the feature of inflation and its consequences on pension fund management. Pension plan management in the presence of inflation has been considered by Battocchio and Menoncin [1] but these authors do not consider inflation linked bonds. Furthermore they use the stochastic dynamic programming approach. The main innovation of our framework is that we introduce inflation-linked bonds in our model and thereby give the insurance company effective means in order to hedge the risk due to inflation.

Our article is organized as follows. We give a heuristic derivation for the dynamics of the inflation index based on the Fisher equation in Section 2. In Section 3, we set up the mathematical framework of our model, while in section 4, we compute the optimal asset allocation problem of the insurance company by solving analytically the corresponding stochastic optimal control problem. In section 5, we consider some numerical examples and discuss the qualitative behavior of the optimal investment strategy. We summarize the main results in section 6.

2. Inflation as a stochastic process and the Fisher equation

We consider an inflation index such as the MUCPI (Monetary Union Consumer Price Index) which is a measure of inflation, different from the monthly or yearly inflation rate, which is often announced in the news, in particular at the end of the financial year or before critical votes. In this section we will specify the dynamics of the inflation index $I(t)$ based on the well known Fisher equation. Fisher [9] gives a derivation based on macroeconomic principles, which relates the nominal interest rate r_N, the real interest rate r_R and the expected rate of inflation over the specific planning horizon in the following formula

$$r_N - r_R = \tilde{\mathbb{E}}\,(i) \tag{1}$$

Fisher's original formulation does not include time dependency nor does it take into account any consideration under which measure the equation above is satisfied. The modern theory of arbitrage leads us to the assumption that the expectation in the Fisher equation has to be satisfied under a risk neutral measure. It is not clear whether this risk neutral measure is unique as the market is likely to be incomplete. We assume however that one risk neutral measure is chosen and fixed for the remaining of this article. We denote expectations under this measure with $\tilde{\mathbb{E}}$. On the other side Fisher's equation refers to a rather static setup. This can however easily be adapted to a dynamic, but discrete time framework. One way to do this is to define the relative inflation within the period

$[t, t+\Delta t]$ as measured by the inflation index via

$$i(t, t+\Delta t) = \frac{I(t+\Delta t) - I(t)}{I(t)} \tag{2}$$

The Fisher equation would then translate into

$$r_N(t) - r_R(t) = \frac{1}{\Delta t}\tilde{\mathbb{E}}\left(i(t, t+\Delta t) \middle| \mathcal{F}_t \right)$$

where $\mathcal{F}_t$ denotes the information available at time t. One way to obtain a continuous time inflation rate would now be to consider the limit $i(t) = \lim_{\Delta t \to 0} \frac{i(t,t+\Delta t)}{\Delta t}$. Assuming that $I(t)$ is a stochastic process itself, it is not clear whether this limit exists and if so, in what sense. In fact, if it would exist in the classical sense of standard calculus, an immediate consequence would be, that the inflation index could be written as $I(t) = I(0)\exp\left(\int_0^t i(s)ds\right)$. Such a model has been discussed in [12] where it was assumed that $i(t)$ follows a mean reverting Ornstein-Uhlenbeck process. Assuming that $i(t)$ is an Itô-process in this case however has the consequence that $I(t)$ is a finite first variation process. This would lead us to severe mathematical problems. Though a systematic quantitative analysis of the dynamic of the inflation index still needs to be carried out, we believe that the inflation index rather behaves like a stock which in general is assumed to display infinite first variation. One way to resolve this issue, is to assume that the process $i(t)$ is in fact the time derivative of an Itô process in the distributional sense which leads us into the theory of white-noise and Malliavin calculus. We avoid this rather technical and mathematical discussion and instead simply assume that the inflation index is an Itô process of the following type

$$\frac{dI(t)}{I(t)} = \mu(t)dt + \sigma_I(t)d\tilde{W}(t) \tag{3}$$

where $\tilde{W}$ is a Brownian motion under the chosen risk neutral measure. If we believe that a Brownian motion setup is accurate and that the inflation index remains positive, this is the most general form the inflation index may have. This equation suggests that the relative growth of the inflation index $\frac{dI(t)}{I(t)}$ consists of a drift part $\mu(t)dt$ and a part without trend $\sigma_I(t)dW(t)$, which specifies the level of volatility. For simplicity we assume here that the volatility $\sigma_I(t)$ is constant in time and deterministic, a similar assumption as in the Black-Scholes model. In the following we will relate heuristically the drift term $\mu(t)$ to the Fisher equation. Based on equation (3) the relative inflation over the period $[t, t+\Delta t]$ can be approximated as follows

$$i(t, t+\Delta t) = \frac{I(t+\Delta t) - I(t)}{I(t)} \approx \mu(t)\Delta t + \sigma_I \Delta \tilde{W}(t) \tag{4}$$

Taking conditional expectation leads to

$$\tilde{\mathbb{E}}\left(i(t, t+\Delta t)|\, \mathcal{F}_t\right) \approx \mu(t) \cdot \Delta t \tag{5}$$

and finally dividing by Δt and taking the limit for $\Delta t \to 0$ we obtain from the Fisher equation

$$r_N(t) - r_R(t) = \lim_{\Delta t \to 0} \mathbb{E}\left(i(t, t+\Delta t)|\mathcal{F}_t\right) = \mu(t) \tag{6}$$

Note that we obtained the limit only after taking expectation and that this argument could be generalised by taking distributional derivatives. Nevertheless, we obtain for the dynamic of the inflation index under the subjective measure, taking into account a market price of inflation risk θ_I,

$$\frac{dI(t)}{I(t)} = \left(r_N(t) - r_R(t) + \sigma_I \theta_I\right) dt + \sigma_I dW_I(t). \tag{7}$$

We will use in the following discussion. Note that this form of the dynamics of the inflation is also in line with the specification in Korn and Kruse [12] where the dynamics of the inflation were derived in analogy to the Garman and Kohlhagen model for exchange rate dynamics.

3. Management of pension funds with inflation-linked products

In this section we will setup the mathematical framework in which an insurance company is able to invest into a riskless money market account, a classical stock and an inflation-linked bond, financing its investments from the contributions of plan members.

For this we assume that the inflation index $I(t)$ follows the dynamic discussed in the previous section and which is displayed in equation (7). Let us consider the inflation-linked bond from Definition 1. Assuming that inflation is of the type discussed in the previous section and that r_N resp. r_R are constants, a fair price for the inflation-linked bond from above can be derived with a Black-Scholes like argument, see Korn and Kruse [12]. This price $B(t, I(t))$ satisfies

$$B(t, I(t)) = \sum_{i=1}^{n} C_i \frac{I(t)}{I(t_0)} e^{-r_R(t-t_i)} + F \frac{I(t)}{I(t_0)} e^{-r_R(T-t)} \tag{8}$$

An application of the Itô formula to equation (8) shows that the price of the inflation-linked bond and the inflation index are strongly related to each other

$$\begin{aligned} \frac{dB(t, I(t))}{B(t, I(t))} &= r_R dt + \frac{dI(t)}{I(t)} \\ &= (r_N + \sigma_I \theta_I) dt + \sigma_I dW_I(t) \end{aligned}$$

Under our assumption that the real interest rate is deterministic, we see that the inflation-linked bond and the inflation index, assuming it would represent a financial asset itself, are financially equivalent in the sense that they can perfectly replicate each other. In addition to the inflation linked bond, we assume that the insurance company has the opportunity to invest in a riskless money market account $S_0(t)$ offering a deterministic interest rate which coincides with the nominal interest rate, i.e.

$$\frac{dS_0(t)}{S_0(t)} = r_N dt \tag{9}$$

with $S_0(0) = 1$ and a stock $S(t)$, which we allow to be correlated to the inflation index and which follows the dynamic

$$\frac{dS(t)}{S(t)} = b dt + \sigma_S^1 dW_I(t) + \sigma_S^2 dW_S(t) \tag{10}$$

where b and $\sigma_S = (\sigma_S^1, \sigma_S^2)^\top$ are assumed to be a constants, while $\mathbb{W}(t) = (W_I(t), W_S(t))^\top$ is a two dimensional Brownian motion. Unlike in Battocchio/Menoncin [1], where the stock price is viewed as an inflation forecaster, we have considered the stock price as a variable following the inflation index. We will study this relationship in more detail later on in this discussion. Our insurance company therefore faces a market which consists of one riskless assets and two risky asset, all of whom are tradeable. We assume that $\sigma_I \neq 0 \neq \sigma_S^2$. Then the volatility matrix

$$\sigma := \begin{pmatrix} \sigma_I & 0 \\ \sigma_S^1 & \sigma_S^2 \end{pmatrix} \tag{11}$$

corresponding to the two risky assets satisfies $det(\sigma) = \sigma_I \cdot \sigma_S^2 \neq 0$. The market is therefore complete and there exists a unique market price of risk θ satisfying $\theta = \sigma^{-1}(\alpha - r_N \mathbf{1})$ where $\mathbf{1} = (1,1)^T$, $\alpha = (r_N + \sigma_I \theta_I, b)^T$. An elementary computation shows that

$$\theta := \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \theta_I \\ \frac{b - r_N - \theta_I \sigma_S^1}{\sigma_S^2} \end{pmatrix} \tag{12}$$

In a defined contribution (DC) pension plan, the contributions payable by both employee and employer, are defined, which in most cases corresponds to a fixed percentage of the salary. Here we assume the salary of a pension plan member follows the dynamics:

$$\frac{dY(t)}{Y(t)} = (r_N - r_R + \kappa)dt + \sigma_Y^1 dW_I(t) + \sigma_Y^2 dW_S(t) \tag{13}$$

with $Y(0) = y$, where κ and $\sigma_Y = (\sigma_Y^1,\ \sigma_Y^2)^\top$ are constants. The particular form of the drift term has been chosen so that the growth rate of the salary consists of two parts, the first one $r_N - r_R$ adjusting the workers' salary for inflation and the second one κ adjusting for economic growth and an increase in welfare.

In the presence of two independent Brownian motions W_I and W_S in our model and three stochastic key-variables $I(t), S(t)$ and $Y(t)$ it is possible to express any one of them by a combination of the other times a deterministic function. We will outline this thought in the following. Let us define the cross correlation matrices

$$\Sigma^{I,Y} = \begin{pmatrix} \sigma_I & \sigma_Y^1 \\ 0 & \sigma_Y^2 \end{pmatrix}, \quad \Sigma^{S,I} = \begin{pmatrix} \sigma_S^1 & \sigma_I \\ \sigma_S^2 & 0 \end{pmatrix}, \quad \Sigma^{S,Y} = \begin{pmatrix} \sigma_S^1 & \sigma_Y^1 \\ \sigma_S^2 & \sigma_Y^2 \end{pmatrix}$$

and the cross correlation cofactors

$$\begin{aligned} l_1 &= -\frac{\left|\Sigma^{I,Y}\right|}{\left|\Sigma^{S,I}\right|} = -\frac{\sigma_Y^2}{\sigma_S^2} \\ l_2 &= \frac{\left|\Sigma^{S,Y}\right|}{\left|\Sigma^{S,I}\right|} = \frac{\left|\Sigma^{S,Y}\right|}{\sigma_S^2 \cdot \sigma_I}. \end{aligned}$$

Furthermore define the constant l_0 as

$$\begin{aligned} l_0 &= \left(r_N - r_R + \kappa - \frac{1}{2}\left(\left(\sigma_Y^1\right)^2 + \left(\sigma_Y^2\right)^2\right)\right) \\ &\quad - \left(b - \frac{1}{2}\left(\left(\sigma_S^1\right)^2 + \left(\sigma_S^2\right)^2\right) l_1\right) \\ &\quad - \left(r_N - r_R + \theta_I \sigma_I - \frac{1}{2}\sigma_I^2\right) l_2 \end{aligned}$$

We then have the following proposition:

Proposition 1. *The normalized market variables $Y(t)/Y(0)$,$S(t)/S(0)$ and $I(t)/I(0)$ are related via the following equation*

$$\frac{Y(t)}{Y(0)} = e^{l_0 t} \left(\frac{S(t)}{S(0)}\right)^{l_1} \left(\frac{I(t)}{I(0)}\right)^{l_2}. \tag{14}$$

Equation (14) is obviously equivalent to the following equation

$$\frac{I(t)}{I(0)} = e^{-\frac{l_0}{l_2} t} \left(\frac{S(t)}{S(0)}\right)^{-\frac{l_1}{l_2}} \left(\frac{Y(t)}{Y(0)}\right)^{\frac{1}{l_2}}. \tag{15}$$

This equation represents inflation as a function of the stock price and the salary. One might as well think of a stock index as something which partly measures the state of production of the economy and the salary process as a measure for how much the economy spends for production. One may use this structural identity in order to calibrate the model and adapt the model parameters such that the constants l_0, l_1 and l_2 and Equation (15) become compatible with market data. We omit this statistical aspect here. Let us also note, that alternative to our derivation of the dynamic of the inflation index one may instead start with a structural relationship as expressed by equation (15) and then relate the parameters l_0,l_1 and l_2 to make the inflation index compatible with the Fisher equation.

Let us come back to the insurance companies investment problem. For simplicity we assume that there is only one member in the pension plan. Alternatively we may think of aggregate variables and a representative plan member. If the initial value of this member's pension account is $x > 0$, the contribution rate (i.e. the percentage of the member's salary) is $c > 0$, and $1 - \pi_1(t) - \pi_2(t)$, $\pi_1(t)$, $\pi_2(t)$ are the proportions of the pension fund invested in the riskless bond, the inflation-linked bond and the stock respectively. Then the corresponding portfolio process, which we denote by $X(t)$, is governed by the following equation

$$dX(t) = X(t)[r_N dt + \pi^T(t)\sigma(\theta dt + d\mathbb{W}(t))] + cY(t)dt \tag{16}$$

with $X(0) = x$ where σ and θ are given by equation (11), (12) respectively, while $\pi(t) = (\pi_1(t),\ \pi_2(t))^T$ is called the portfolio. Note that the contributions are assumed to be invested continuously over time. For the insurance company's decision how to invest the plan member's contributions optimally, the expectation of the plan member's future contribution plays an important part. We therefore define

Definition 2. The discounted expected future contribution process is defined as

$$D(t) = \mathbb{E}\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds \middle| \mathcal{F}_t\right] \tag{17}$$

where $\mathcal{F}_t$ represents the filtration generated by the Brownian motion $\mathbb{W}(t)$ and

$$H(t) := e^{-r_N t - \frac{1}{2}\|\theta\|^2 t - \theta^\top \mathbb{W}(t)} \tag{18}$$

is the discount factor which adjusts for nominal interest rate and market price of risk. We set $d = D(0)$

Definition 3. The pension fund value process is defined as

$$P(t) = X(t) + D(t) \tag{19}$$

where $X(t)$ and $D(t)$ satisfy equation (16), (17) respectively.

Equation (19) should be interpreted as follows : The value of the pension fund of a DC plan, at time t, is equal to:

- the value of the portfolio $X(t)$ plus
- the discounted expected value of future contributions to the plan.

The expected value of future contributions has the disadvantage that it is not directly observable and it is not clear whether the insurance company is likely to base its investment decision on the process $D(t)$. An alternative, which is observable, is the salary process $Y(t)$. The following proposition shows that the processes $D(t)$ and $Y(t)$ are strongly linked and differ merely by a deterministic function. It also characterizes the distribution of $D(t)$.

Proposition 2. *The expected future contributions process $D(t)$ and the salary process $Y(t)$ are related via the equation*

$$D(t) = \frac{1}{\beta}\left(e^{\beta(T-t)} - 1\right) c \cdot Y(t) \tag{20}$$

with $\beta = \kappa - r_R - < \sigma_Y, \theta >$. In particular the distribution of the expected future income $D(t)$ is log-normal, but expectation and variance change with time, both reaching 0 at the end of the planning horizon T

Proof. By definition we have

$$\begin{aligned} D(t) &= \mathbb{E}\left(\int_t^T \frac{H(s)}{H(t)} cY(s) ds \middle| \mathcal{F}_t \right) \\ &= c \cdot Y(t) \mathbb{E}\left(\int_t^T \frac{H(s)}{H(t)} \frac{Y(s)}{Y(t)} ds \middle| \mathcal{F}_t \right) \end{aligned}$$

Both processes $H(\cdot)$ and $Y(\cdot)$ are geometric Brownian motions and therefore it follows easily that $\frac{H(s)}{H(t)}\frac{Y(s)}{Y(t)}$ is independent of $\mathcal{F}_t$. The conditional expectation therefore collapses to an unconditional expectation and we obtain

$$D(t) = c \cdot Y(t) \cdot g(t, T)$$

with the deterministic function $g(t,T)$ which according to the Markovian setup of our model can be computed as

$$g(t,T) = \mathbb{E}\left(\int_0^{T-t} H(u)\frac{Y(u)}{Y(0)}du\right)$$

Noting

$$\begin{aligned} H(u)\frac{Y(u)}{Y(0)} &= e^{(\kappa-r_R)u}e^{(\sigma_Y-\theta)^\top \mathbb{W}(u)-\frac{1}{2}\left(\|\theta\|^2+\|\sigma_Y\|^2\right)u} \\ &= e^{\beta u}e^{(\sigma_Y-\theta)^\top \mathbb{W}(u)-\frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)u} \end{aligned} \tag{21}$$

we obtain

$$\mathbb{E}(H(u)\frac{Y(u)}{Y(0)}) = e^{\beta u}$$

Integration over time gives

$$\begin{aligned} g(t,T) &= \mathbb{E}\left(\int_0^{T-t} H(u)\cdot\frac{Y(u)}{Y(0)}du\right) = \int_0^{T-t}\mathbb{E}(H(u)\frac{Y(u)}{Y(0)})du \\ &= \int_0^{T-t} e^{\beta u}du = \frac{1}{\beta}\left(e^{\beta(T-t)}-1\right) \end{aligned}$$

which is the desired result. Given the relationship between $D(t)$ and $Y(t)$ the second statement now follows immediately from our assumption that the salary process is a geometric Brownian motion.

We will later make use of the martingale method to find the optimal investment strategy for the insurance company. In order to do this it will be essential that the pension fund value process is a martingale, when discounted with the state price process $H(\cdot)$.

Proposition 3. *The discounted pension fund process $H(\cdot)P(\cdot)$ is a martingale.*

Proof. By definition of $D(t)$ we find that

$$H(t)D(t) + \int_0^t H(s)cY(s)ds = \mathbb{E}\left[\int_0^T H(s)cY(s)ds\,\middle|\,\mathcal{F}_t\right] \tag{22}$$

which is a Martingale with respect to the Brownian filtration $\mathcal{F}_t$. It follows from the martingale representation theorem (see [11], page 71) that there exists a progressively measurable process $\psi(\cdot)$, with $\mathbb{E}\left[\int_0^T \|\psi(t)\|^2 dt\right] < \infty$ such that

$$d\left(H(t)D(t)\right) + H(t)cY(t)dt = \psi^T(t)dW(t) \ a.s. \tag{23}$$

which is obviously the same as

$$d\left(H(t)D(t)\right) = -H(t)cY(t)dt + \psi^T(t)dW(t) \ a.s. \tag{24}$$

Applying the Itô product rule to $H(t)X(t)$, we get

$$d\left(H(t)X(t)\right) = H(t)X(t)\left(\pi^T(t)\sigma - \theta^T\right)dW(t) + H(t)cY(t)dt$$

and therefore obtain

$$\begin{aligned} d\left(H(t)P(t)\right) &= d\left(H(t)X(t)\right) + d\left(H(t)D(t)\right) && (25) \\ &= \left[H(t)X(t)\left(\pi^T(t)\sigma - \theta^T\right) + \psi^T(t)\right]dW(t) && (26) \end{aligned}$$

which shows that $H(\cdot)P(\cdot)$ is a martingale.

The process $\psi(\cdot)$ obtained in the previous proposition as the integrand in the martingale representation theorem will play a significant role in the identification of the optimal investment strategy later and it is therefore necessary to compute it in a more explicit way.

Proposition 4. *The process $\psi(\cdot)$ from equation (23) takes the following form*

$$\psi(t) = cy\frac{1}{\beta}\left(e^{\beta T} - e^{\beta t}\right)M(t)(\sigma_Y - \theta) \tag{27}$$

with β as defined in Proposition 2 and $M(t) := e^{(\sigma_Y-\theta)^T W(t) - \frac{1}{2}\|\sigma_Y-\theta\|^2 t}$. Furthermore we have

$$H(t)D(t) = cy\frac{1}{\beta}\left(e^{\beta T} - e^{\beta t}\right)M(t) \tag{28}$$

Proof. Equations (20) and (21) imply

$$\begin{aligned} H(t)D(t) &= c\left(\int_0^{T-t} e^{\beta u}du\right)H(t)Y(t) && (29) \\ H(t)Y(t) &= ye^{\beta t}M(t) && (30) \end{aligned}$$

Substituting equation (30) in the right side of equation (29) leads to

$$
\begin{aligned}
H(t)D(t) &= cy\left(\int_0^{T-t} e^{\beta u}du\right)e^{\beta t}M(t)\\
&= cy\left(\int_t^{T} e^{\beta u}du\right)M(t) = cy\frac{1}{\beta}\left(e^{\beta T}-e^{\beta t}\right)M(t)
\end{aligned}
$$

which proves the second statement of the proposition. Now, taking differential in equation (29) leads to

$$
\begin{aligned}
d(H(t)D(t)) &= -cye^{\beta t}M(t)dt + cy\frac{1}{\beta}\left(e^{\beta T}-e^{\beta t}\right)dM(t)\\
&= cy\left(-e^{\beta t}M(t)dt + \frac{1}{\beta}\left(e^{\beta T}-e^{\beta t}\right)M(t)(\sigma_Y-\theta)^\top d\mathbb{W}(t)\right)\\
&= -H(t)cY(t)dt + cy\frac{1}{\beta}\left(e^{\beta T}-e^{\beta t}\right)M(t)(\sigma_Y-\theta)^\top d\mathbb{W}(t)
\end{aligned}
$$

where the last equation was obtained by substituting equation (30) again. The first statement now follows from comparing the last expression with (24) and the uniqueness of this representation.

4. Optimal management of the pension fund

Our objective is to maximize the expected utility of the pension fund at a member's retirement age T. We therefore have to solve the following optimization problem:

$$
\begin{aligned}
&\max_{\pi(\cdot)\in\mathcal{A}} \mathbb{E}[U(P(T))] \ \textit{subject to}\\
&\quad \mathbb{E}[H(T)P(T)] = x + d > 0
\end{aligned}
$$

where, $x^- = \max\{0,-x\}$, U is as utility function and $\mathcal{A}$ denotes the class of admissible portfolio strategies $\pi(\cdot)$, i.e. those satisfying that $\pi(t)$ *is* $\mathcal{F}_t$ measurable for all t, $\int_0^T \left\|\pi(t)^\top\sigma\right\|^2 dt < \infty \ P-a.s.$ and $\mathbb{E}\left[U^-(P(T))\right] < \infty$ (see for example [11], page 206). In this article we choose as U the constant relative risk aversion (CRRA) utility function

$$
U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma \in (-\infty, 1)\backslash\{0\}. \tag{31}
$$

This utility function is very popular as it often allows to derive closed form solutions of associated stochastic optimal control problems. Nevertheless the following discussion caries over to other utility functions. In particular, similar results could be obtained for an exponential utility function instead of CRRA. The method we use in order to solve the stochastic optimal control problem above is the so called martingale method. Certainly the economic literature is dominated by the stochastic dynamic programming approach, which has the advantage that it identifies the optimal strategy automatically as a function of the underlying observables, which is sometimes called feedback form. On the other side, it often turns out that the corresponding Hamilton-Jacobi-Bellman equation, which in general is a second order non-liner partial differential equation, does not admit a closed form solution. Our approach based on the martingale method and Proposition 3 leads us to a closed form solution of the optimal investment problem of the insurance company. For a general discussion of the martingale method in stochastic optimal control see [11] chapter 5. The key feature of the martingale method is that the dynamic optimization problem is decomposed into a a static optimization problem and a hedging problem. The static optimization problem in our case is the following

$$\max_B \mathbb{E}[U(B))] \textit{ subject to} \tag{32}$$

$$\mathbb{E}[H(T)B] = x + d > 0 \tag{33}$$

where B is an $\mathcal{F}_T$ measurable random variable. The Lagrangian of this problem is given by

$$L(B, \lambda) := \mathbb{E}\left[U(B) - \lambda\left(H(T)B - x - d\right)\right] \tag{34}$$

where, λ is the Lagrangian multiplier. Equating the derivatives of the Lagrangian L with respect to B and λ respectively to zero, we obtain:

$$\frac{\partial L}{\partial B} = \mathbb{E}\left[U'(B) - \lambda H(T)\right] = 0 \tag{35}$$

$$\frac{\partial L}{\partial \lambda} = [H(T)B] - x - d = 0 \tag{36}$$

Equation (35) is obviously solved by $B^* = (U')^{-1}(\lambda H(T))$. For our choice of CRRA utility function we have $(U')^{-1}(x) = x^{\frac{1}{\gamma-1}}$. This leads us to

$$B^* = \lambda^{\frac{1}{\gamma-1}}(H(T))^{\frac{1}{\gamma-1}} \tag{37}$$

while the Lagrange multiplier λ is determined by the constraint

$$\mathbb{E}\left[\lambda^{\frac{1}{\gamma-1}}(H(T))^{\frac{\gamma}{\gamma-1}}\right] = x + d$$

which is satisfied by setting

$$\lambda^{\frac{1}{\gamma-1}} = \frac{x+d}{\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right]} \tag{38}$$

Substitution of (38) in (37) gives us the optimal terminal pension fund value via the following formula

$$B^* = (x+d)\frac{(H(T))^{\frac{1}{\gamma-1}}}{\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right]} \tag{39}$$

From Proposition 3 it follows that the value process of the optimal pension plan satisfies

$$H(t)P^*(t) \quad = \quad \mathbb{E}\left[H(T)B^* \middle| \mathcal{F}_t\right] = (x+d)\frac{\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\middle|\mathcal{F}_t\right]}{\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right]} \tag{40}$$

A theoretical justification for this heuristic use of the Lagrangian can be found in [11], chapter 5.

Lemma 1. *Introducing the martingale* $Z(t) = e^{-\frac{\gamma}{\gamma-1}\theta^\top \mathbb{W}(t) - \frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^2 \|\theta\|^2 t}$ *we obtain*

$$H(t)P^*(t) = (x+d)Z(t) \tag{41}$$

Proof. A straightforward computation shows that

$$H(t)^{\frac{\gamma}{\gamma-1}} = e^{-r_N t\frac{\gamma}{\gamma-1} + \frac{1}{2}\frac{\gamma}{(\gamma-1)^2}\|\theta\|^2 t} \cdot Z(t)$$

which shows that $H(t)$ can be written as $H(t) = f(t) \cdot Z(t)$ with a deterministic function $f(\cdot)$ and a martingale $Z(\cdot)$. Now we obtain

$$\frac{\mathbb{E}[(H(T))^{\frac{\gamma}{\gamma-1}}|\mathcal{F}_t]}{\mathbb{E}[(H(T))^{\frac{\gamma}{\gamma-1}}]} = \frac{\mathbb{E}[f(T)Z(T)|\mathcal{F}_t]}{\mathbb{E}[f(T)Z(T)]} = \frac{f(T)\mathbb{E}[Z(T)|\mathcal{F}_t]}{f(T)\mathbb{E}[Z(T)]} = \frac{Z(t)}{Z(0)} = Z(t).$$

The statement now follows from equation (40)

Note that Lemma 1 does not provide a new proof that $H(\cdot)P^*(\cdot)$ is a martingale which would make Proposition 3 obsolete. The martingale property of $H(\cdot)P^*(\cdot)$ was already used in equation (40) on which the proof of the lemma depends. Nevertheless, using the fact that

$$dZ(t) = Z(t)\frac{\gamma}{1-\gamma}\theta^\top d\mathbb{W}(t) \tag{42}$$

we also obtain

$$\begin{aligned} d\left(H(t)P^*(t)\right) &= \frac{\gamma}{1-\gamma}(x+d)Z(t)\theta^T d\mathbb{W}(t) \\ &= \underbrace{\frac{\gamma}{1-\gamma}H(t)P^*(t)\theta^T}_{:=\rho^T(t)} d\mathbb{W}(t) \end{aligned} \tag{43}$$

with $\rho(\cdot)$ being a martingale as well. From this we obtain an expression for the optimal portfolio for the insurance company.

Proposition 5. *The optimal portfolio process of the dynamic optimization problem (32),(33) is given by*

$$\pi^*(t) = (\sigma^{-1})^T \left(\frac{\rho(t) - \psi(t)}{H(t)X^*(t)} + \theta\right) \tag{44}$$

for $X^*(t) > 0$ and 0 otherwise. Here $\rho(\cdot)$ and $\psi(\cdot)$ are given by

$$\begin{aligned} \rho(t) &= \frac{\gamma}{1-\gamma}H(t)P^*(t)\theta \\ \psi(t) &= H(t)D(t)(\sigma_Y - \theta) \end{aligned}$$

and $X^*(t)$ is the wealth process, corresponding to the optimal portfolio $\pi^*(t)$, which can be derived from the optimal pension fund process

$$H(t)X^*(t) = H(t)P^*(t) - H(t)D(t)$$

Proof. A comparison of the $d\mathbb{W}(t)$ term in (43) and (26) leads to

$$H(t)X^*(t)\left(\pi^{*\top}(t)\sigma - \theta^\top\right) + \psi^\top(t) = \rho^\top(t)$$

which implies that

$$\pi^{*\top}(t)\sigma = \frac{\rho^\top(t) - \psi^\top(t)}{H(t)X^*(t)} + \theta^\top$$

and from which the statement follows by transposition and multiplication with $(\sigma^{-1})^\top$.

It is certainly up to discussion in how explicit the formula for the obtained optimal investment strategy in terms of observable variables really is. Substituting $\rho(t)$, $\psi(t)$ and $H(t)X^*(t)$, the optimal portfolio process can be rewritten as

$$\pi^*(t) = (\sigma^{-1})^T \left(\frac{\frac{1}{1-\gamma} P^*(t)\theta - D(t)\sigma_Y}{P^*(t) - D(t)} \right) \tag{45}$$

This formula depends on the optimal pension fund value, which consists of the optimal pension fund level $X^*(t)$ and the expected future contributions $D(t)$. The first one is observable, while the second one in a way reflects the expectation of the insurance company on the future contribution of their plan members. The insurance company may have a clear idea about what the expected future contributions of the plan members will be and in this sense it is reasonable to express the optimal strategy in terms of $D(t)$. Nevertheless we will give a description of the optimal investment strategy in terms of the asset price $S(t)$ and the current salary of the plan member $Y(t)$ and alternatively in terms of $S(t)$ and the level of the inflation index $I(t)$ later on in our discussion.

Remark 1. The optimal investment strategy given in the Proposition 5 can be decomposed into two parts:

- part A solves the classical optimal investment problem with initial investment $x+d$
- part B hedges the future contribution stream, whose present value d has been invested while setting up the pension fund

In mathematical terms we have

$$\pi^*(t) = \underbrace{\frac{1}{1-\gamma}(\sigma^{-1})^T\theta}_{=A} + \underbrace{\frac{H(t)D(t)}{H(t)P^*(t) - H(t)D(t)}(\sigma^{-1})^T \left(\frac{\theta}{1-\gamma} - \sigma_Y\right)}_{=B} \tag{46}$$

for $X(t) > 0$ and 0 otherwise. In particular, at the initial date $t = 0$, we have

$$\pi^*(0) = \frac{1}{1-\gamma}(\sigma^{-1})^T\theta + \frac{d}{x}(\sigma^{-1})^T\left(\frac{\theta}{1-\gamma} - \sigma_Y\right) \text{ for } x > 0 \tag{47}$$

The utility obtained by following the optimal investment strategy is computed in the next proposition.

Proposition 6. *The optimal expected utility obtainable by the insurance company is given by*

$$\mathbb{E}\left[\frac{1}{\gamma}(B^*)^\gamma\right] = \frac{(x+d)^\gamma}{\gamma}e^{\gamma(r_N + \frac{1}{2(1-\gamma)}\|\theta\|^2)T} \tag{48}$$

with $d = cy\frac{1}{\beta}(e^{\beta T} - 1)$ *and* β *as defined in Proposition 2.*

Proof. We have that

$$\begin{aligned}\mathbb{E}[U(B^*)] &= \mathbb{E}\left[\frac{(x+d)^\gamma}{\gamma}\left(\frac{(H(T))^{\frac{1}{\gamma-1}}}{\mathbb{E}[(H(T))^{\frac{\gamma}{\gamma-1}}]}\right)^\gamma\right] \\ &= \frac{(x+d)^\gamma}{\gamma}\left(\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right]\right)^{1-\gamma}\end{aligned}$$

On the other side, we showed in the proof of Lemma 1 that $\mathbb{E}\left[(H(T))^{\frac{\gamma}{\gamma-1}}\right] = f(T)$ with $f(T) = e^{-r_N T\frac{\gamma}{\gamma-1} + \frac{1}{2}\frac{\gamma}{(\gamma-1)^2}\|\theta\|^2 T}$. Taking the $1-\gamma$-th power gives the desired result.

Let us now express the optimal portfolio strategy computed in Proposition 5 in terms of the primary observable variables $S(t), Y(t)$ and $I(t)$. The discussion concluding Proposition 1 showed that in order to do this, it is enough to express the terms $P^*(t)$ and $D(t)$ in terms of t, $S(t)$ and $Y(t)$. In Proposition 5 we already demonstrated how to obtain $D(t)$ from $Y(t)$. A tedious but straightforward computation shows that

$$H(t) = e^{m_0 t}\left(\frac{S(t)}{S(0)}\right)^{m_1}\left(\frac{Y(t)}{Y(0)}\right)^{m_2}$$

where m_1 and m_2 are given as follows :

$$m_1 = -\frac{\sigma_Y^1\left(b - r_N - \theta_I\sigma_S^1\right) - \theta_I\sigma_S^2\sigma_Y^2}{\sigma_S^2\,|\Sigma^{S,Y}|} \tag{49}$$

$$m_2 = \frac{\sigma_S^1\,(b - r_N) - \theta_I\|\sigma_S\|^2}{|\Sigma^{S,Y}|} \tag{50}$$

while m_0 is of the following type

$$m_0 := n_1 r_N + n_2\theta_I + n_3\|\theta\|^2 + n_4 b \tag{51}$$

with constants n_i determined below

$$\begin{aligned}
n_1 &= 4\sigma_S^1\left(\kappa - r_R - \sigma_S^2\sigma_Y^2 - b - r_N - 2\|\sigma_Y\|^2\right) + \theta_I\|\sigma_S\|^2 + \sigma_Y^1\left(\left(\sigma_S^2\right)^2 + \frac{1}{4} - b\right)\\
n_2 &= (1-4b) < \sigma_S, \sigma_Y > + \left(4\left(\kappa - r_R\right) - 2\|\sigma_Y\|^2\right) < \sigma_S, \mathbf{1} >\\
n_3 &= 2\sigma_S^2\left|\Sigma^{S,Y}\right|\\
n_4 &= \sigma_Y^1(4b-1) + \sigma_S^1\left(2\|\sigma_Y\|^2 - 4\left(\kappa - r_R\right)\right)
\end{aligned}$$

Let us now determine $P^*(t)$ in terms of t,$S(t)$ and $Y(t)$. It follows from equation (41) in Lemma 1 and the concluding equation that

$$H(t)\cdot P^*(t) = (x+d)\cdot e^{r_N t\frac{\gamma}{\gamma-1} - \frac{1}{2}\frac{\gamma}{(\gamma-1)^2}\|\theta\|^2 t}\cdot H(t)^{\frac{\gamma}{\gamma-1}}$$

Division by $H(t)$ and expressing $H(t)$ in terms of t,$S(t)$ and $Y(t)$ allows us to write

$$P^*(t) = (x+d)\cdot e^{\left(\frac{m_0}{\gamma-1}+q\right)t}\left(\frac{S(t)}{S(0)}\right)^{\frac{m_1}{\gamma-1}}\left(\frac{Y(t)}{Y(0)}\right)^{\frac{m_2}{\gamma-1}} \tag{52}$$

where q is defined by

$$q = r_N\frac{\gamma}{\gamma-1} - \frac{1}{2}\frac{\gamma}{(\gamma-1)^2}\|\theta\|^2. \tag{53}$$

We now obtain the following form of the optimal investment strategy in feedback form, i.e. $\pi^*(t) = \pi^*(t, S(t), Y(t))$ with

$$\pi^*(t,S,Y) = F_1(t,S,Y)\cdot\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top\theta + F_2(t,S,Y)\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top\sigma_Y \tag{54}$$

with

$$\begin{aligned}
F_1(t,S;Y) &= \frac{1}{1 - \frac{c}{x+d}\cdot\frac{1}{\beta}\left(e^{\beta(T-t)} - 1\right)e^{-\left(\frac{m_0}{\gamma-1}+q\right)t}\left(\frac{S}{s}\right)^{-\frac{m_1}{\gamma-1}}\left(\frac{Y}{y}\right)^{1-\frac{m_2}{\gamma-1}}}\\
F_2(t,S,Y) &= \frac{(\gamma-1)c\cdot\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)}{(x+d)e^{\left(\frac{m_0}{\gamma-1}+q\right)t}\left(\frac{S}{s}\right)^{\frac{m_1}{\gamma-1}}\left(\frac{Y}{y}\right)^{\frac{m_2}{\gamma-1}-1} - c\cdot\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)}
\end{aligned}$$

with $s = S(0)$ and $y = Y(0)$ denoting the initial stock price and the initial salary. This representation also shows that the optimal investment strategy is a linear combination of two strategies. The first one $\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top \theta$ is the optimal self financing investment strategy, i.e. without exterior financing from the stream of contributions, while the second one $\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top \sigma_Y$ is the optimal strategy in a fictitious market with two assets that have a market price of risk as specified by the volatility of the contribution stream.

Let us alternatively express the optimal strategy in terms of $S(t)$ and $I(t)$. This is not very difficult, as we already know from equation (14) that

$$\frac{Y(t)}{Y(0)} = e^{l_0 t}\left(\frac{S(t)}{S(0)}\right)^{l_1}\left(\frac{I(t)}{I(0)}\right)^{l_2}$$

By simple substitution of this expression in the functions F_1 and F_2 we obtain the optimal strategy in feedback form $\pi^*(t) = \pi^*(t, S(t), I(t))$ with

$$\pi^*(t,S,I) = G_1(t,S,I)\cdot\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top\theta + G_2(t,S,I)\frac{1}{1-\gamma}\left(\sigma^{-1}\right)^\top\sigma_Y \tag{55}$$

with

$$\begin{aligned}
G_1(t,S,I) &= \frac{1}{1-\frac{c}{x+d}\cdot\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)e^{-\left(\frac{m_0+m_2l_0}{\gamma-1}+q-l_0\right)t}\left(\frac{S}{s}\right)^{l_1-\frac{m_1+m_2l_1}{\gamma-1}}\left(\frac{I}{i}\right)^{l_2-\frac{m_2l_2}{\gamma-1}}}\\
G_2(t,S,I) &= \frac{(\gamma-1)c\cdot\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)}{(x+d)e^{\left(\frac{m_0+m_2l_0}{\gamma-1}+q-l_0\right)t}\left(\frac{S}{s}\right)^{\frac{m_1+m_2l_1}{\gamma-1}-l_1}\left(\frac{I}{i}\right)^{\frac{m_2l_2}{\gamma-1}-l_2}-c\cdot\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)}
\end{aligned}$$

A similar remark concerning the decomposition of the portfolio into two parts as pointed out before holds. This last representation allows the insurance company to dynamically change its portfolio, depending on the values of the stock and the current level of inflation. We consider this representation as the most natural one. We will give a quantitative analysis of these strategies in the next section.

5. Numerical Example

For our numerical example we use the following parameters which are displayed in table 1. One may argue with the exact quantities of the volatility parameters, but we

Symbol	Text reference/interpretation	Numerical value
γ	parameter for risk aversion	0.5
μ	expected stock return	0.06
r_N	nominal interest rate	0.03
r_R	real interest rate	0.015
κ	expected growth of economy	0.03
σ_I	volatility of inflation index	0.2
$(\sigma_S^1, \sigma_S^2)^\top$	volatility of stock	$(0.1, 1)^\top$
$(\sigma_Y^1, \sigma_Y^2)^\top$	volatility of stock	$(0.01, 0.5)^\top$
θ_I	market price of inflation risk	0.3
c	contribution rate	0.14

TABLE 1: Parameters for numerical experiment.

did not find a thorough quantitative analysis in the literature. The parameters chosen, display our assumption that the salary process is more correlated to the stock than it is to the inflation index. In fact our model already contains a compensation in salary for inflation by the choice of the drift term of the salary process. In Figures 1.-3. we display the optimal strategy as a function of S and I, which we assume to be normalized. The scales must therefore be understood as relative scales. Figures 1. and 2. show that on a large parameter range including $S \in [0.5, 1.5]$ and $I \in [0.5, 1]$ the optimal strategy is approximately equal to the static strategy $\pi^* = (-2, 3, 0)^\top$, which for the chosen parameters coincides with the Merton strategy, without inflation linked bond and income stream. This of course is also displayed in Figure 3. as in the latter parameter range, investment into the inflation linked bond appears to be rather unattractive. For inflation indices higher then 1, Figure 3. shows that short selling of inflation linked bonds becomes very attractive for the insurance company. Figure 2 also shows that in times of high inflation, the proposition of the wealth invested in the stock depends significantly on the stock price and is in fact higher for low stock prices, then for high stock prices.

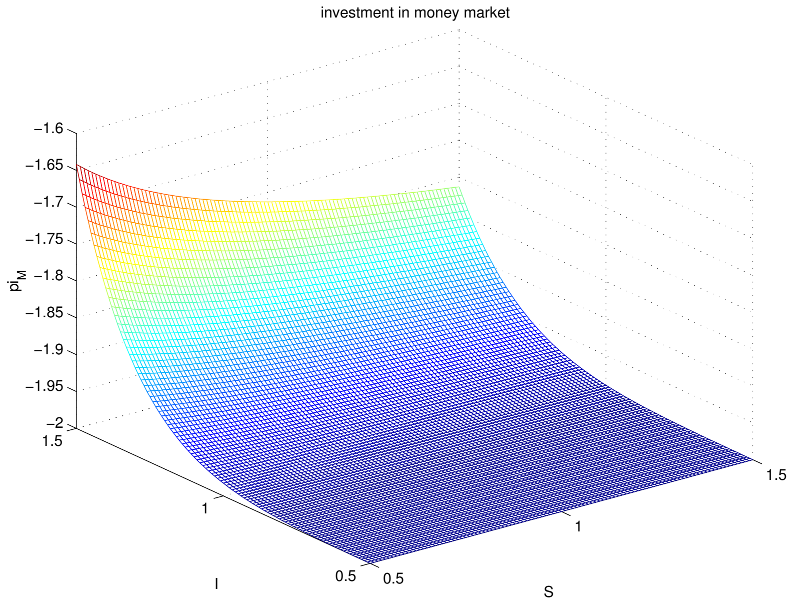

FIGURE 1:

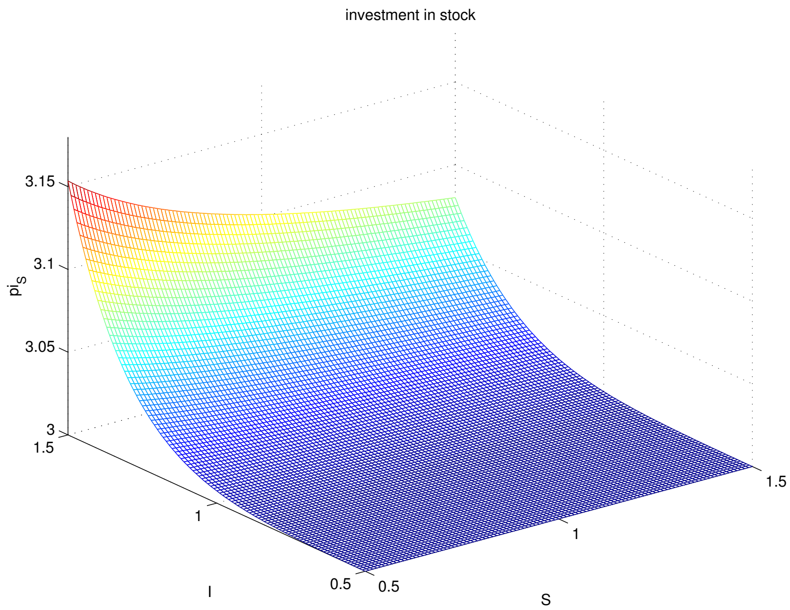

FIGURE 2:

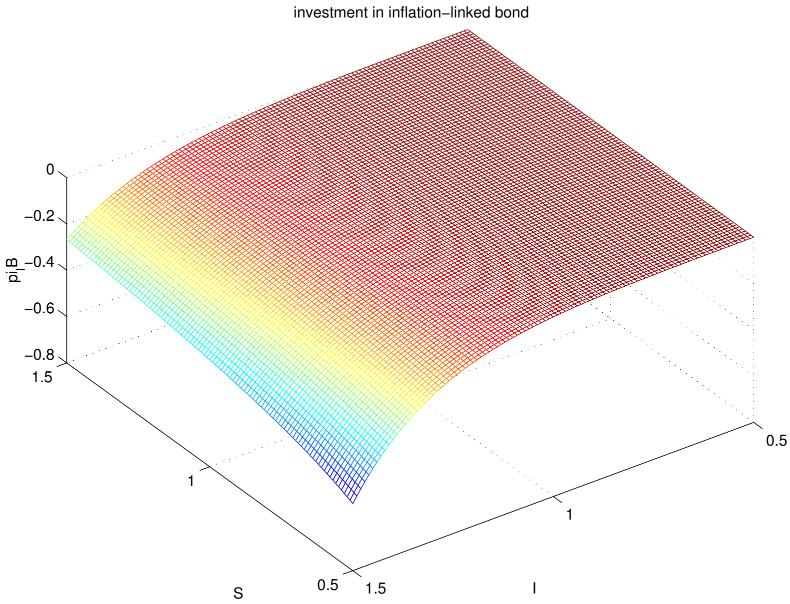

Figure 3:

6. Conclusion

We consider the case where an insurance company, which is selling inflation-linked pension products is managing a portfolio consisting of positions in a money market account, an ordinary stock and an inflation-linked bond. With the position in the latter derivative, the insurance company is able to hedge some of the risk associated to its inflation-linked pension products. We compute the optimal asset allocation rule where the criterion is optimal expected utility from terminal wealth, given that the insurance company receives a continuous but stochastic income stream, the contributions of the pension plan member. By means of the martingale method from stochastic optimal control we are able to find a closed form expression for this asset allocation rule, which we represent in various feedback forms. Of course, for our analysis to go through we have to assume that our plan member survives until the time horizon T. As however, we consider our member only as a representative member and not as a physical individual, this assumption causes no problems.We also provide a numerical example which illustrates quantitative and qualitative features of this rule.

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