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Abstract

Interaction terms are often misinterpreted in the empirical economics literature by assuming that the coefficient of interest represents unconditional marginal changes. I present the correct way to estimate conditional marginal changes in a series of non-linear models including (ordered) logit/probit regressions, censored and truncated regressions. The linear regression model is used as the benchmark case.

Keywords: interaction terms, ordered probit, ordered logit, truncated regression, censored regression, nonlinear models

JEL codes: C12, C24, C25, C51

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1. Introduction

Applied economists often employ interaction terms to test conditional hypothesis, that is, to test how a dependent variable changes when an independent variable changes *conditional* on the magnitude of another independent variable. Although this notion seems pretty basic, interpretation and estimation of models with interaction terms is still flawed in the empirical economics literature. As a reviewer and a reader I often come across flawed empirical analysis which has committed one of the errors described in Brambor et al. (2006), that is, constitutive terms are often not included in the empirical specification and even if they are included they are interpreted as unconditional marginal changes. Both points are explained in detail in the pedagogical paper of Brambor et al. (2006).

The points raised by Brambor et al. (2006) seem to have had a significant effect in the literature as evidenced by the more than 1000 citations in Google scholar as of July 2011, although these citations come predominantly from the Political science literature. In the economics profession, however, this paper has largely remained unnoticed. However, Ai and Norton (2003) raised this issue early on. They showed that coefficients of interaction terms are often misused in nonlinear models as well (i.e., probit and logit models). Their points seem to be well taken in the literature collecting more than 1250 citations in Google scholar (as of July 2011). However, econometric softwares are still likely to report mis-measured effects which are typically computed as separate "partial effects" for each variable that appears in the model (Greene 2010). The software is in essence incapable of knowing which variables are interacted. Therefore, many applied economists ignore the warnings raised by the relevant literature and continue reporting effects as if these are unconditional marginal changes. An exception to this standard software output is the latest release of Stata (version 11 and forth). The `mfx` command used by Stata ver. 10 and earlier has been superseded by `margins` which is now capable of estimating marginal changes for interacted variables along the lines of this paper. However, researchers use various other software packages in their empirical research which are likely not to have embedded automated procedures for estimating conditional effects of interactions terms.

Although the points laid above seem trivial, these are often ignored in practice. I reviewed five of the most prestigious journals in the profession, namely the *American Economic Review* (AER), *Journal of Economic Literature* (JEL), *Journal of Economic Perspectives* (JEP), *Journal of Political Economy* (JPE) and *Quarterly Journal of Economics* (QJE) for papers with empirical applications employing interaction terms. I only reviewed papers published between 2005-2010, which allowed for a two-year time lag since Ai and Norton's (2003) paper. In the search criteria I specified "interaction term(s)" as the keyword and then excluded papers for which the empirical specification is posed on a theoretical basis[1]. I also excluded papers published in the *Papers & Proceedings* issue of AER which are not subject to standard peer review process. The list of qualified papers was therefore trimmed down to 24 papers (10 AER, 9 QJE, 4 JEP, 1JPE) and is listed in an online Appendix (https://sites.google.com/site/interactionterms/). As expected, interaction terms were more prevalent in articles published in AER and QJE and were less often used in papers in JEP and JPE. Not surprisingly, papers utilizing interaction terms were completely absent from journals such as JEL which mostly publish comprehensive literature review papers[2].

Only 9 out of 24 papers (37.5%) clearly used all constitutive terms in every single specification that was employed in the paper. Some of the rest of the papers used all constitutive terms in some specifications but then removed one or two of the constitutive terms in the rest of the specifications. The majority of the reviewed papers either included one of the constitutive terms or only included the interaction term for at least one specification of the estimated models. While there are a couple of circumstances where omitting a constitutive term would have not lead to a significant inferential error (Brambor et al., 2006), none of these circumstances was obvious to this author. Most of these papers may have miss-specified their empirical models by omission of constitutive terms.

The fraction of the papers that interpreted constitutive terms as conditional marginal changes in an obvious way is even smaller. Five out of nine papers that included all constitutive terms explicitly mentioned how effects are to be evaluated. It is worth noting that none of the papers calculated standard errors for the conditional marginal changes besides those reported by the software; whether this was the case in some of the papers, it was not reported explicitly. While these results are not completely discouraging it is worth noting that not even one of the papers employing a non-linear model (e.g., probit) cited the Ai and Norton (2003) article, which indicates that appropriate calculations for marginal changes of the interaction terms (and their respective standard errors) were mostly ignored.

[1] There were several papers omitting constitutive terms from the empirical specification, however, this practice was dictated by theory. Thus, these papers were not included in the review list.

[2] One JEL paper was finally excluded from the review list since the empirical model was largely based on a priori assumptions stemming out from a theoretical model.

If this largely depicts how empirical research on evaluation of conditional hypothesis is applied in five of the most prestigious journals in the profession, policing standards in lower tier journals certainly seems warranted. To help novice researchers avoid repeating errors, I derive and provide the necessary formulas that would allow the estimation of marginal changes for linear and non-linear models. I start by demonstrating the calculations in OLS regression as the benchmark case and then gradually proceed to binary choice models, ordered choice models, censored and truncated regressions. To my knowledge, the only formulas that appear in the literature are those associated with linear models (Brambor et al., 2006), probit/logit models (Ai and Norton 2003) and random parameters ordered response models (Drichoutis and Nayga 2011) but never all in one place. The censored and truncated regression model have not garnered any attention in the literature.

2. OLS regression models

In general, researchers should include interaction terms in their models whenever they think that the relationship between two or more variables depends on the value of one or more other variables. We will only consider the simplest conditional hypothesis as more complicated cases can be straightforwardly extended. Assume the below specification:

$$Y_i = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \tag{1}$$

The constant term b_0 is included in $\mathbf{b}'$. Standard software output will report the lower order coefficients as exhibited in equation (1). It should be clear that all b_j captures (for *j*=1,2), is the effect of a one unit change (given X_j is a continuous variable; similar interpretations are in place if X_j is a dummy) on *Y when* X_k *is zero* (*k*=2 if *j*=1 and vice versa), that is, in the absence of the condition defined by X_k. When the condition defined by X_k is present then one need to evaluate the joint effect of b_j and b_{12}. Formally, the marginal effect of any of the interacted variables X_j can be written as:

$$\frac{\partial Y}{\partial X_{ji}} = b_j + b_{12} X_{ki} \quad \text{for } k\text{=1, 2 when } j\text{=2, 1} \tag{2}$$

Equation (2) can be evaluated at various values of the X_k variable. If X_k is a dummy then the marginal effect can be evaluated for values of 0 and 1 respectively (and at the means of all other variables as standard practice dictates), which gives rise to:

$$\left.\frac{\partial Y}{\partial X_j}\right|_{X_k=0} = b_j \quad \text{and} \quad \left.\frac{\partial Y}{\partial X_j}\right|_{X_k=1} = b_j + b_{12} \tag{3}$$

Equation (3) shows that the effect of a change in X_j on *Y* when the condition defined by X_k is present is $b_j + b_{12}$ while when the condition is absent the effect is b_j.

If, instead, X_k is a continuous variable then one can evaluate several conditional hypothesis e.g., for various percentiles[3] or the mean of the variable:

$$\left.\frac{\partial Y}{\partial X_j}\right|_{X_{k,Qq}} = b_j + b_{12} X_{k,Qq} \quad \text{or} \quad \left.\frac{\partial Y}{\partial X_j}\right|_{\bar{X}_k} = b_j + b_{12} \bar{X}_k \quad \text{for } k\text{=1,2 when } j\text{=2,1 \& } q\text{=1,2,3} \tag{4}$$

Equation (4) would then produce estimates for the effect of a change in X_j on *Y when* X_k increases. Standard errors for the above expressions might not be that hard to write down and compute but will be tedious for the non-linear models in the later sections. Fortunately all popular software packages (including Stata and Limdep) provide post-estimation canned routines that allow computation of standard errors for linear or non-linear expressions involving beta coefficients using the delta method (`lincom` and `nlcom` in Stata and `wald` in Limdep).

Before proceeding to the non-linear models a warning is warranted. Greene (2007, p. E18-23) argues that significance tests of the influence of a variable for non-linear models should be based on the coefficients alone. Marginal changes in the case of non-linear models are a highly non-linear function of all the coefficients in the model and the hypothesis that this

[3] I will only consider the 25th, 50th and 75th percentile indexed as Q1, Q2 and Q3. Obviously, various other percentiles can be used. Stata ver. 11 & 12 provide the option to use any *nth* percentile in the evaluation of the effect.

function equals zero is not equivalent to the hypothesis that the coefficient is zero or that the variable in question is not a significant determinant of the outcome.

3. Probit and Logit regression models:

Assume the latent regression:

$$Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \tag{5}$$

with the observed counterpart being $Y = 1$ if and only if $Y^* > 0$. The conditional mean function for the observed binary Y is:

$$F\left(\boldsymbol{\beta}'\mathbf{x}\right) = 0 \times \text{Prob}\left[Y = 0 \mid \mathbf{x}\right] + 1 \times \text{Prob}\left[Y = 1 \mid \mathbf{x}\right] \tag{6}$$

where $\mathbf{x}$ is a column vector of X_1, X_2, $X_1 X_2$ and $\mathbf{X}$, and $\boldsymbol{\beta}$ is a column vector of b_1, b_2, b_{12} and $\mathbf{b}$. The probit model arises by assuming that F is the Cumulative Distribution Function (CDF) of the normal distribution (typically denoted as Φ) while the logit model arises by assuming that F is the CDF of the logistic distribution (typically denoted as Λ).

Marginal changes in binary choice models can be obtained as:

$$ME = \frac{\partial E\left(Y \mid \mathbf{x}\right)}{\partial \mathbf{x}} = \frac{\partial F\left(\boldsymbol{\beta}'\mathbf{x}\right)}{\partial \mathbf{x}} = \beta f\left(\boldsymbol{\beta}'\mathbf{x}\right) \tag{7}$$

and

$$DC = \text{Prob}\left[Y = 1 \mid x_j = 1\right] - \text{Prob}\left[Y = 1 \mid x_j = 0\right] = F\left(\boldsymbol{\beta}'\mathbf{x} \mid x_j = 1\right) - F\left(\boldsymbol{\beta}'\mathbf{x} \mid x_j = 0\right) \tag{8}$$

for marginal effects (estimated for continuous variables)[4] and discrete changes (estimated for dummy variables) respectively. In contrast to the linear regression model presented in the previous section, in non-linear models it makes a difference whether the evaluated change is for a continuous or a dummy variable. If X_j is a continuous variable then the marginal effect based on the functional form of (5) would be:

[4] *f* denotes the probability density function (pdf).

$$ME_{X_j} = \left(b_j + b_{12} X_k\right) f\left(\boldsymbol{\beta}'\mathbf{x}\right) \text{ for } k\text{=1,2 when } j\text{=2,1} \qquad (9)$$

One can then compute estimates for the effect of a change in X_j on the probability of $Y=1$ *when* X_k increases, for various percentiles (or the mean) of the *k* variable (*if* X_k is a continuous variable), e.g.,:

$$ME_{X_j}\Big|_{X_{k,Qq}} = \left(b_j + b_{12} X_{k,Qq}\right) f\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_{k,Qq}\right) \quad \text{or} \quad ME_{X_j}\Big|_{\overline{X}} = \left(b_j + b_{12} \overline{X}_k\right) f\left(\boldsymbol{\beta}'\overline{\mathbf{x}}\right) \qquad (10)$$

or for values of 0 or 1 of the *k* variable (*if* X_k is a dummy):

$$ME_{X_j}\Big|_{X_{k=0}} = b_j f\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 0\right) \quad \text{or} \quad ME_{X_j}\Big|_{X_{k=1}} = \left(b_j + b_{12}\right) f\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 1\right) \qquad (11)$$

However, if X_j is a dummy variable then the appropriate marginal change is a discrete change which is given by:

$$DC_{X_j} = \text{Prob}\left[Y=1 \mid X_j = 1\right] - \text{Prob}\left[Y=1 \mid X_j = 0\right] = F\left(\boldsymbol{\beta}'\mathbf{x} \mid X_j = 1\right) - F\left(\boldsymbol{\beta}'\mathbf{x} \mid X_j = 0\right) \text{ for } j\text{=1,2} \qquad (12)$$

The effect given by (12) can then be evaluated conditional on various percentiles (or the mean) of the *k* variable (*if* X_k is a continuous variable), e.g.,:

$$\begin{aligned} DC_{X_j}\Big|_{X_{k,Qq}} &= F\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 1, X_{k,Qq}\right) - F\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 0, X_{k,Qq}\right) = \\ &= F\left(b_j + \left(b_{12} + b_k\right) X_{k,Qq} + \mathbf{b}'\overline{\mathbf{X}}\right) - F\left(b_k X_{k,Qq} + \mathbf{b}'\overline{\mathbf{X}}\right) \end{aligned} \text{ for } k\text{=1,2 when } j\text{=2,1 \& } q\text{=1,2,3} \qquad (13)$$

or

$$\begin{aligned} DC_{X_j}\Big|_{\overline{X}_k} &= F\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 1, \overline{X}_k\right) - F\left(\boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 0, \overline{X}_k\right) = \\ &= F\left(b_j + \left(b_{12} + b_k\right) \overline{X}_k + \mathbf{b}'\overline{\mathbf{X}}\right) - F\left(b_k \overline{X}_k + \mathbf{b}'\overline{\mathbf{X}}\right) \end{aligned} \text{ for } k\text{=1,2 when } j\text{=2,1} \qquad (14)$$

or for values of 0 or 1 of the *k* variable (*if* X_k is a dummy), e.g.,:

$$\begin{aligned} DC_{X_j}\Big|_{X_k=1} &= F\left(\boldsymbol{\beta}'\mathbf{x} \mid X_j = 1, X_k = 1\right) - F\left(\boldsymbol{\beta}'\mathbf{x} \mid X_j = 0, X_k = 1\right) = \\ &= F\left(b_1 + b_2 + b_{12} + \mathbf{b}'\overline{\mathbf{X}}\right) - F\left(b_k + \mathbf{b}'\overline{\mathbf{X}} \mid X_j = 0, X_k = 1\right) \end{aligned} \text{ for } k\text{=1,2 when } j\text{=2,1} \qquad (15)$$

4. Ordered Probit and ordered Logit models

The rational in evaluating conditional marginal changes can be extended to the ordered response model in a straightforward manner. In the ordered response model we assume there is a latent variable Y^* ranging from $-\infty$ to $+\infty$ which is mapped to an observed variable Y. The Y variable is providing incomplete information about the underlying Y^* according to:

$$Y_i = m \ \text{ if } \tau_{m-1} \le Y_i^* < \tau_m \qquad \text{for} \qquad m=1 \text{ to } J \tag{16}$$

The τ's are called thresholds and the extreme categories 1 and J are defined by open-ended intervals with $\tau_0 = -\infty$ and $\tau_J = +\infty$. The standard structural model of the ordered choice model is:

$$Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \tag{17}$$

The observation mechanism results from a complete censoring of the latent dependent variable as follows:

$$\begin{aligned} Y_i \ &= 0 \text{ if } \mathrm{Y}_i^* \le \tau_0, \\ &= 1 \text{ if } \tau_0 < Y_i^* \le \tau_1, \\ &= 2 \text{ if } \tau_1 < Y_i^* \le \tau_2, \\ &\cdots \\ &= J \text{ if } \mathrm{Y}_i^* \le \tau_{J-1}. \end{aligned} \tag{18}$$

The main focus in ordered data is on the conditional cell probabilities given by:

$$\Pr\left(Y_i = m \,|\, \mathbf{x}_i\right) = F\left(\tau_m - \boldsymbol{\beta}_i'\mathbf{x}_i\right) - F\left(\tau_{m-1} - \boldsymbol{\beta}_i'\mathbf{x}_i\right) \tag{19}$$

If one of the interacted variables is continuous then taking the partial derivative of (19) with respect to X_j yields the marginal effect,

$$\begin{aligned} ME_{m,X_j} &= \frac{\partial \Pr\left(Y_i = m \,|\, \mathbf{x}_i\right)}{\partial X_j} = \frac{\partial F\left(\tau_m - \boldsymbol{\beta}_i'\mathbf{x}_i\right)}{\partial X_j} - \frac{\partial F\left(\tau_{m-1} - \boldsymbol{\beta}_i'\mathbf{x}_i\right)}{\partial X_j} = \\ &= \left(b_j + b_{12} X_k\right)\left[f\left(\tau_{m-1} - \boldsymbol{\beta}_i'\mathbf{x}_i\right) - f\left(\tau_m - \boldsymbol{\beta}_i'\mathbf{x}_i\right)\right] \end{aligned} \qquad \text{for } k=1,2 \text{ when } j=2,1 \tag{20}$$

One can then compute estimates for the effect of a change in X_j on the probability of $Y=m$ when X_k increases, for various percentiles (or the mean) of the *k* variable (*if* X_k is a continuous variable), e.g.,:

$$ME_{m,X_j}\Big|_{X_{k,Qq}} = \left(b_j + b_{12}X_{k,Qq}\right)\left[f\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_{k,Qq}\right) - f\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_{k,Qq}\right)\right]$$ for *k*=1,2 when *j*=2,1 & *q*=1,2,3 (21)

or $$ME_{m,X_j}\Big|_{\overline{X}_k} = \left(b_j + b_{12}\overline{X}_k\right)\left[f\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}}\right) - f\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}}\right)\right] \quad (22)$$

On the other hand, if X_k is a dummy then expression (20) can be evaluated at values of 0 or 1 of the X_k variable, e.g.,:

$$ME_{m,X_j}\Big|_{X_k=1} = \left(b_j + b_{12}\right)\left[f\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 1\right) - f\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 1\right)\right]$$ for *k*=1,2 when *j*=2,1 (23)

or $$ME_{m,X_j}\Big|_{X_k=0} = b_j\left[f\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 0\right) - f\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_k = 0\right)\right]$$ for *k*=1,2 when *j*=2,1 (24)

Interpretation using the marginal effects can be misleading when an independent variable is a dummy variable. Hence, it is more appropriate to calculate the discrete change which is the change in the predicted probability for a change in X_j from the start value 0 to the end value 1:

$$DC_{m,X_j} = \frac{\Delta \Pr\left(Y=m \mid \mathbf{x}\right)}{\Delta X_j} = \Pr\left(Y=m \mid \mathbf{x}, X_j = 1\right) - \Pr\left(Y=m \mid \mathbf{x}, X_j = 0\right) \quad (25)$$

Hence, expression (25) can be written as:

$$\begin{aligned} DC_{m,X_j} &= F\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 1\right) - F\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 1\right) - \\ &- F\left(\tau_m - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 0\right) + F\left(\tau_{m-1} - \boldsymbol{\beta}'\overline{\mathbf{x}} \mid X_j = 0\right) \end{aligned} \quad (26)$$

Expression (26) can then be evaluated at various percentiles and/or the mean of the *k* variable (*if* X_k is a continuous variable) or for values of 0 or 1 of the *k* variable (*if* X_k is a dummy) giving rise to the formulas:

$$DC_{m,X_j}\Big|_{X_{k,Qq}} = F\left(\tau_m - b_j - \left(b_k + b_{12}\right) X_{k,Qq} - \mathbf{b'\bar{X}}\right) - F\left(\tau_{m-1} - b_j - \left(b_k + b_{12}\right) X_{k,Qq} - \mathbf{b'\bar{X}}\right) - F\left(\tau_m - b_k X_{k,Qq} - \mathbf{b'\bar{X}}\right) + F\left(\tau_{m-1} - b_k X_{k,Qq} - \mathbf{b'\bar{X}}\right) \quad \text{for } k=1,2$$

when j=2,1 & q=1,2,3

(27)

or
$$DC_{m,X_j}\Big|_{X_{k,Qq}} = F\left(\tau_m - b_j - \left(b_k + b_{12}\right) \bar{X}_k - \mathbf{b'\bar{X}}\right) - F\left(\tau_{m-1} - b_j - \left(b_k + b_{12}\right) \bar{X}_k - \mathbf{b'\bar{X}}\right) - F\left(\tau_m - b_k \bar{X}_k - \mathbf{b'\bar{X}}\right) + F\left(\tau_{m-1} - b_k \bar{X}_k - \mathbf{b'\bar{X}}\right) \quad \text{for } k=1,2$$

when j=2,1 (28)

$$DC_{m,X_j}\Big|_{X_k=1} = F\left(\tau_m - b_1 - b_2 - b_{12} - \mathbf{b'\bar{X}}\right) - F\left(\tau_{m-1} - b_1 - b_2 - b_{12} - \mathbf{b'\bar{X}}\right) - F\left(\tau_m - b_k - \mathbf{b'\bar{X}}\right) + F\left(\tau_{m-1} - b_k - \mathbf{b'\bar{X}}\right) \quad \text{for } k=1,2 \qquad (29)$$

or
$$DC_{m,X_j}\Big|_{X_k=0} = F\left(\tau_m - b_j - \mathbf{b'\bar{X}}\right) - F\left(\tau_{m-1} - b_j - \mathbf{b'\bar{X}}\right) - F\left(\tau_m - \mathbf{b'\bar{X}}\right) + F\left(\tau_{m-1} - \mathbf{b'\bar{X}}\right) \quad \text{for } k=1,2 \text{ when } j=2,1 \qquad (30)$$

5. Censored regression models

In the censored regression model the latent underlying regression is:

$$Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \qquad (31)$$

while the observed dependent variable is:

$Y_i = L_i$ if $Y_i^* \le L_i$ (lower tail censoring)

$Y_i = U_i$ if $Y_i^* \ge U_i$ (upper tail censoring) and

$Y_i = Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i$ if $L_i < Y_i^* < U_i$

There are three conditional mean functions to consider. For the latent variable it is $E\left(Y_i^* \mid \mathbf{x}_i\right) = \boldsymbol{\beta}'\mathbf{x}$. If one is interested in obtaining marginal changes from the interacted variables then similar estimations to the OLS section of this paper are in place i.e., one just needs to consider the various linear combinations of coefficients. For an observation randomly drawn from the population, which may or may not be censored, the conditional mean function is (Greene 2007, 2003):

$$E\left(Y_i \mid \mathbf{x}_i\right) = L_i \Phi\left(\alpha_i\right) + U_i\left(1-\Phi\left(\gamma_i\right)\right) + \left(\Phi\left(\gamma_i\right)-\Phi\left(\alpha_i\right)\right)\left(\boldsymbol{\beta}'\mathbf{x} + \sigma \frac{\phi\left(\alpha_i\right)-\phi\left(\gamma_i\right)}{\Phi\left(\gamma_i\right)-\Phi\left(\alpha_i\right)}\right) \qquad (32)$$

where $\alpha_i = \dfrac{L_i - \boldsymbol{\beta}'\mathbf{x}}{\sigma}$ and $\gamma_i = \dfrac{U_i - \boldsymbol{\beta}'\mathbf{x}}{\sigma}$

The most familiar censored regression model is the Tobit model which arises by setting $U_i = +\infty$ and $L_i = 0$ in (32). Marginal effects of a continuous interacted variable are given by differentiating (32) with respect to some X_j (see online Appendix at https://sites.google.com/site/interactionterms/):

$$ME_{X_j} = \partial E\left(Y_i \mid \mathbf{x}_i\right)/\partial X_j = \left(b_j + b_{12}X_k\right)\left(\Phi\left(\gamma_i\right)-\Phi\left(\alpha_i\right)\right) \qquad \text{for } k\text{=1,2 when } j\text{=2,1} \qquad (33)$$

If X_k is continuous then the above expression can be evaluated for various percentiles of the X_k variable as well as its mean:

$$ME_{X_j}\Big|_{X_{k,Qq}} = \left(b_j + b_{12}X_{k,Qq}\right)\left(\Phi\left(\gamma_i \mid X_{k,Qq}, \bar{\mathbf{x}}\right)-\Phi\left(\alpha_i \mid X_{k,Qq}, \bar{\mathbf{x}}\right)\right) \text{ for } k\text{=1,2 when } j\text{=2,1 \& } q\text{=1,2,3} \quad (34)$$

or $$ME_{X_j}\Big|_{\bar{X}_k} = \left(b_j + b_{12}\bar{X}_k\right)\left(\Phi\left(\gamma_i \mid \bar{\mathbf{x}}\right)-\Phi\left(\alpha_i \mid \bar{\mathbf{x}}\right)\right) \qquad \text{for } k\text{=1, 2 when } j\text{=2, 1} \qquad (35)$$

Otherwise, if X_k is a dummy, (33) can be evaluated conditional on the X_k values being 1 or 0:

$$ME_{X_j}\Big|_{X_k=1} = \left(b_j + b_{12}\right)\left(\Phi\left(\gamma_i \mid X_k = 1, \bar{\mathbf{x}}\right)-\Phi\left(\alpha_i \mid X_k = 1, \bar{\mathbf{x}}\right)\right) \text{ for } k\text{=1,2 when } j\text{=2,1} \qquad (36)$$

or $$ME_{X_j}\Big|_{X_k=0} = b_j\left(\Phi\left(\gamma_i \mid X_k = 0, \bar{\mathbf{x}}\right)-\Phi\left(\alpha_i \mid X_k = 0, \bar{\mathbf{x}}\right)\right) \text{ for } k\text{=1,2 when } j\text{=2,1} \qquad (37)$$

If X_j is a dummy then marginal effects are only approximately correct and discrete changes should be estimated. From (32) we have:

$$DC_{X_j} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0\right) \tag{38}$$

Expression (38) can then be evaluated for various percentiles (or the mean) of the X_k variable (if X_k is continuous) or for values of 0 or 1 (if X_k is a dummy). Appropriate formulas are given by:

$$DC_{X_j}\Big|_{X_{k,Qq}} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, X_{k,Qq}\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, X_{k,Qq}\right) \text{ for } k\text{=1,2 when } j\text{=2,1 \& } q\text{=1,2,3} \tag{39}$$

or $DC_{X_j}\Big|_{X_{k,Qq}} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, \bar{X}_k\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, \bar{X}_k\right)$ for k=1,2 when j=2,1 (40)

and

$$DC_{X_j}\Big|_{X_k=1} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, X_k = 1\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, X_k = 1\right) \text{ for } k\text{=1,2 when } j\text{=2,1} \tag{41}$$

or $DC_{X_j}\Big|_{X_k=0} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, X_k = 0\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, X_k = 0\right)$ for k=1,2 when j=2,1 (42)

6. Truncated regression models:

The truncated regression model can be seen as a special case of the censored regression model where only data from the third group are observed. The latent regression is:

$$Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \tag{43}$$

while the observed dependent variable is:

$$Y_i = Y_i^* = b_1 X_{1i} + b_2 X_{2i} + b_{12} X_{1i} X_{2i} + \mathbf{b}'\mathbf{X} + u_i \quad \text{if} \qquad L_i < Y_i^* < U_i$$

The conditional mean function in this case is:

$$E\left(Y_i \mid \mathbf{x}_i, L_i < Y_i < U_i\right) = \boldsymbol{\beta}'\mathbf{x} + \sigma \frac{\phi(\alpha_i) - \phi(\gamma_i)}{\Phi(\gamma_i) - \Phi(\alpha_i)} \tag{44}$$

where $\alpha_i = \dfrac{L_i - \boldsymbol{\beta}'\mathbf{x}}{\sigma}$ and $\gamma_i = \dfrac{U_i - \boldsymbol{\beta}'\mathbf{x}}{\sigma}$.

If X_j is continuous then marginal effects can be derived by differentiating (44) (see online Appendix at https://sites.google.com/site/interactionterms/):

$$ME_{X_j} = \left(b_j + b_{12}X_k\right)\left[1 + \left(\frac{\alpha_i\phi(\alpha_i) - \gamma_i\phi(\gamma_i)}{\Phi(\gamma_i) - \Phi(\alpha_i)}\right) - \left(\frac{\phi(\alpha_i) - \phi(\gamma_i)}{\Phi(\gamma_i) - \Phi(\alpha_i)}\right)^2\right] = \left(b_j + b_{12}X_k\right)h(\beta,\sigma) \quad (45)$$

where $h(\beta,\sigma)$ is the scale factor.

Conditional effect can be evaluated at various percentiles and/or the mean of X_k (*if* X_k is a continuous variable):

$ME_{X_j}\Big|_{X_{k,Qq}} = \left(b_j + b_{12}X_{k,Qq}\right)h\left(\beta,\sigma \mid \bar{\mathbf{x}}, X_{k,Qq}\right)$ or $ME_{X_j}\Big|_{\bar{X}_k} = \left(b_j + b_{12}\bar{X}_k\right)h\left(\beta,\sigma \mid \bar{\mathbf{x}}, \bar{X}_k\right)$ for *k*=1,2 when *j*=2,1 & *q*=1,2,3 (46)

If, on the other hand, X_k is a dummy, expression (45) can be evaluated at values of $X_k = 0$ or 1:

$ME_{X_j}\Big|_{X_{k=1}} = \left(b_j + b_{12}\right)h\left(\beta,\sigma \mid \bar{\mathbf{x}}, X_k = 1\right)$ or $ME_{X_j}\Big|_{X_k=0} = b_j h\left(\beta,\sigma \mid \bar{\mathbf{x}}, X_k = 0\right)$ for *k*=1,2 when *j*=2,1 (47)

If X_j is a dummy then discrete changes should be estimated as:

$$DC_{X_j} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, L_i < Y_i < U_i\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, L_i < Y_i < U_i\right) \quad (48)$$

Expression (48) can then be evaluated for various percentiles and/or the mean of X_k (provided X_k is a continuous variable):

$DC_{X_j}\Big|_{X_{k,Qq}} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, X_{k,Qq}, L_i < Y_i < U_i\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, X_{k,Qq}, L_i < Y_i < U_i\right)$ for *k*=1,2 when *j*=2,1 & q=1,2,3 (49)

or $DC_{X_j}\Big|_{X_{k,Qq}} = E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 1, L_i < Y_i < U_i\right) - E\left(Y_i \mid \bar{\mathbf{x}}, X_j = 0, L_i < Y_i < U_i\right)$ for *k*=1,2 when *j*=2,1(50)

or at values of $X_k = 0$ or 1 (provided X_k is a dummy), e.g.,:

$$DC_{X_j}\Big|_{X_k=1} = E\left(Y_i \mid \overline{\mathbf{x}}, X_j = 1, X_k = 1, L_i < Y_i < U_i\right) - E\left(Y_i \mid \overline{\mathbf{x}}, X_j = 0, X_k = 1, L_i < Y_i < U_i\right) \quad \text{for} \quad k=1,2$$

when j=2,1 (51)

7. Conclusions

In this paper I argue that empirical practice with respect to the use of interaction terms is still flawed even though a couple of papers have raised the flag early on (Ai and Norton 2003; Brambor, Clark, and Golder 2006). Unavailability of automated routines in popular software packages that would appropriately estimate marginal changes from interacted variables may have exacerbated the problem. Some software packages have recently moved forward on this point by making available modules that estimate conditional marginal changes from interacted variables (i.e., the *margins* module in Stata ver. 11 and forth). For the rest of the practitioners that are not accustomed to Stata, it is necessary to understand that the standard output in software packages in linear regression is not to be interpreted as unconditional marginal changes while in nonlinear models marginal changes of interacted variables are incorrect. Extra effort is required to estimate marginal effects or discrete changes and the corresponding standard errors that take into account the interaction terms. This paper has opted to reduce the estimation burden for researchers by providing the necessary formulas that allow for estimation of conditional marginal changes in a series of linear and non-linear models.

8. References

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