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Abstract We revisit the study of the dynamics of a duopoly game à la Bertrand with horizontal product differentiation and bounded rational firms analysed by Zhang et al. (2009), (Zhang, J., Da, Q., Wang, Y., 2009. The dynamics of Bertrand model with bounded rationality. Chaos, Solitons and Fractals 39, 2048–2055), by introducing sound microeconomic foundations. We study how an increase in the relative degree of product differentiation affects the stability of the unique positive Bertrand-Nash equilibrium, in the case of both linear and non-linear costs. We show that an increase in either the degree of substitutability or complementarity between goods of different variety may destabilise the equilibrium of the two-dimensional system through a period-doubling bifurcation. Moreover, by using numerical simulations (i.e., phase portraits, sensitive dependence on initial conditions and Lyapunov exponents), we find that a "quasi-periodic" route to chaos and a large gamma of strange attractors for the cases of both substitutability and complementarity can occur.

Keywords Bifurcation; Chaos; Differentiated products; Duopoly; Price competition

JEL Classification C62; D43; L13

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1. Introduction

The present paper revisits the analyses of the dynamics of a duopoly game à la Bertrand (1883) augmented with bounded rational players (firms) and horizontal product differentiation by Zhang et al. (2009).[1]

Given the importance of product differentiation in the current industrial organization literature (see Singh and Vives, 1984), in this paper we develop, different from Zhang et al. (2009), a model with sound microeconomic foundations that determine the demand of differentiated goods and services faced by each firm in the market. This leads to discover, by using the degree of product differentiation as the key factor, some new dynamical behaviours which are of importance in both mathematics and economics. From the former point of view, we establish that the unique positive fixed point of the two-dimensional system (the sole which is relevant from an economic point of view): (1) may lose stability exclusively through a flip or period-doubling bifurcation, and (2) the route of chaos is of the "quasi-periodic" type. From the latter point of view, we show that if firms work in such a way that products tend to become more homogeneous or complements, then the Bertrand-Nash equilibrium is more likely to be destabilised.[2] Therefore, policies aiming at reducing the degree of product market differentiation tend to destabilise the economy.

Moreover, we extend the model to incorporate another sound microeconomic foundation that concerns the cost of production of each firm, that is the case of non-linear (quadratic) costs. Both the "mathematical" and "economic" results above mentioned are qualitatively confirmed in this case, but the parametric stability region is smaller than when production costs are linear, so that the loss of stability when either the degree of substitutability or degree of complementarity between goods and services increases is more likely to be observed in such a case.

The rest of the paper is organised as follows. Section 2 develops the model and discusses the main analytical results. Section 3 performs numerical simulations showing the occurrence of a "quasi-periodic" route to chaos as well as a large gamma of strange attractors for the cases of both substitutability or complementarity. Section 4 concludes.

2. The economy

Since our dynamic analysis focuses on the effects on stability of a duopolistic market à la Bertrand (1883) with product differentiation, it is of importance to set up the microeconomic foundations of the differentiated commodity setting and clarify the economic reasons why we assume specific demand and cost functions.

We assume the existence of an economy with two types of agents: firms and consumers. The economy is bi-sectorial, i.e. there exist a competitive sector that produces the numeraire good y, and a duopolistic sector with two firms, namely firm 1 and firm 2, each of which produces a differentiated good or service. Let p_i and q_i denote the firm i's price and quantity, respectively, with $i = \{1,2\}$.

[1] For the notion of differentiated goods and services see the original contributions by Hotelling (1929) and Chamberlin (1933).

[2] By passing, we note that neither the mathematical findings of the present paper nor their economic interpretation do appear in Zhang et al. (2009). Moreover they do not consider the case of non-linear production costs.

There exists a continuum of identical consumers which have preferences towards q and y represented by a separable utility function $V(q;y)$, which is linear in the numeraire good. The representative consumer maximises $V(q;y)=U(q)+y$ with respect to quantities subject to the budget constraint $p_1q_1+p_2q_2+y=M$, where $q=(q_1,q_2)$, q_1 and q_2 are non-negative and M denotes the consumer's exogenously given income. The utility function $U(q)$ is assumed to be continuously differentiable and satisfies the standard properties required in consumer theory (see, e.g., Singh and Vives, 1984, pp. 551–552). Since $V(q;y)$ is separable and linear in y, there are no income effects on the duopolistic sector. This implies that for a large enough level of income, the representative consumer's optimisation problem can be reduced to choose q_i to maximise $U(q)-p_1q_1-p_2q_2+M$. Utility maximization, therefore, yields the inverse demand functions (i.e., the price of good i as a function of quantities): $p_i=\frac{\partial U}{\partial q_i}=P_i(q)$, for $q_i>0$ and $i=\{1,2\}$. Inverting the inverse demand system gives the direct demand functions (i.e., the quantity of good i as a function of prices): $q_i=Q_i(p)$, where $p=(p_1,p_2)$ and p_1 and p_2 are non-negative.

To proceed further with the analysis of the duopolistic market, it is required to have explicit demand functions for goods and services of variety 1 and 2. Then, specific utility functions should be assumed. To this end, the usual specification in the economic literature is the quadratic utility function proposed by Dixit (1979) and subsequently used, amongst many others, by Singh and Vives (1984), Vives (1985), Qiu (1997), Häckner (2000), Correa-López and Naylor (2004) and Gosh and Mitra (2010). The important feature of such a utility function is that it generates a system of linear demand functions.

Therefore, we assume that preferences of the representative consumer over q are given by:

$$U(q_i,q_j)=a(q_i+q_j)-\frac{1}{2}\left(q_i^{\,2}+q_j^{\,2}+2d\,q_iq_j\right), \tag{1}$$

where $a>0$ is a parameter that captures the size of the market demand and $-1<d<1$ represents the degree of horizontal product differentiation. Some clarifications on the parameter d are now in order. If $d=0$, then goods of variety 1 and 2 are independent. This implies that each firm behaves as if it were a monopolist in its specific market; if $d=1$, then goods 1 and 2 are perfect substitutes or, alternatively, homogeneous (in that case, the Bertrand's model with price competition implies that the unique Nash equilibrium of the economy is determined in such a way that every firms in the market sets the price to be equal to the marginal cost); $0<d<1$ describes the case of imperfect substitutability between goods. The degree of substitutability increases, or equivalently, the extent of product differentiation decreases as the parameter d raises; a negative value of d instead implies that goods 1 and 2 are complements, while $d=-1$ reflects the case of perfect complementarity.

The inverse demand functions of products of variety 1 and 2 that come from the maximisation by the representative consumer of Eq. (1) with respect to quantities, subject to the budget constraint $p_1q_1+p_2q_2+y=M$, are given by $p_i(q_i,q_j)=a-q_i-d\,q_j$. Therefore, from the inverse demands we can easily obtain the following system of direct demand functions of products of variety 1 and 2, that is:

$$q_1(p_1,q_2)=a-p_1-d\,q_2, \tag{2.1}$$

$$q_2(p_2, q_1) = a - p_2 - d q_1 . \tag{2.2}$$

Combining now Eqs. (2.1) and (2.2) definitely gives the system of direct demands as a function of prices of both products:

$$q_1(p_1, p_2) = \frac{1}{1-d^2}\left[a(1-d) - p_1 + d\, p_2\right], \tag{3.1}$$

$$q_2(p_1, p_2) = \frac{1}{1-d^2}\left[a(1-d) - p_2 + d\, p_1\right]. \tag{3.2}$$

Following Correa-López and Naylor (2004), we assume that firm i produces output of variety i through the following production function with constant (marginal) returns to labour: $q_i = L_i$, where L_i represents the labour force employed by firm i. Firms face the same (constant) average and marginal cost $w \geq 0$ for every unit of output produced. Therefore, the firm i's cost function is linear and described by:

$$C_i(q_i) = w L_i = w q_i . \tag{4}$$

Profits of firm i in every period can be written as follows:

$$\pi_i = p_i\, q_i - w q_i = (p_i - w) q_i . \tag{5}$$

In the Bertrand's model, each player (firm) chooses the price to maximise profits given the expectation about the price sets by the rival. In a dynamic setting, each firm must form at every date t an expectation about the next period rival's price to compute its own profit-maximising price at time $t+1$.

Therefore, let $p_{i,t}$ be the firm i's price at time $t = 0,1,2,\ldots$, where $i = \{1,2\}$. Then, $p_{i,t+1}$ is obtained through the following optimisation programmes:

$$p_{1,t+1} = \arg\max_{p_{1,t}} \pi_1\left(p_{1,t}, p^{e}_{2,t+1}\right), \tag{6.1}$$

$$p_{2,t+1} = \arg\max_{p_{2,t}} \pi_2\left(p^{e}_{1,t+1}, p_{2,t}\right). \tag{6.2}$$

where $p^{e}_{i,t+1}$ represents the price that the rival (firm j) today (time t) expects will be set by firm i in the future (time $t+1$).

Substituting Eqs. (3.1) and (3.2) into Eq. (5) to eliminate q_i, profit maximisation by firm $i = \{1,2\}$ are given by:

$$\max_{p_1} \pi_1(p_1, p_2) = \frac{p_1 - w}{1-d^2}\left[a(1-d) - p_1 + dp_2\right], \tag{7.1}$$

$$\max_{p_2} \pi_2(p_1, p_2) = \frac{p_2 - w}{1-d^2}\left[a(1-d) - p_2 + dp_1\right]. \tag{7.2}$$

Therefore, marginal profits are obtained as:

$$\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = \frac{a(1-d) - 2p_1 + dp_2 + w}{1-d^2}, \tag{8.1}$$

$$\frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = \frac{a(1-d) - 2p_2 + dp_1 + w}{1-d^2}. \tag{8.2}$$

The reaction or best reply functions of firms 1 and 2 are computed as the unique solution of Eqs. (8.1) and (8.2) for p_1 and p_2, respectively, and they are given by:

$$\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = 0 \Leftrightarrow p_1(p_2) = \frac{1}{2}\left[a(1-d) + dp_2 + w\right], \tag{9.1}$$

$$\frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = 0 \Leftrightarrow p_2(p_1) = \frac{1}{2}\left[a(1-d) + dp_1 + w\right]. \tag{9.2}$$

We assume that both firms have bounded rational expectations about the level of the price that should be set in the future to maximise profits (see Zhang et al., 2009).

Therefore, each player uses information on current profit in such a way to increase or decrease the price at time $t+1$ depending on whether marginal profits are either positive or negative. Following Dixit (1986), the adjustment mechanism of prices over time of the ith bounded rational player is described by:

$$p_{i,t+1} = p_{i,t} + \alpha_i p_{i,t} \frac{\partial \pi_i}{\partial p_{i,t}}, \tag{10}$$

where $\alpha_i > 0$ is a coefficient that captures the speed of adjustment of firm i's price with respect to a marginal change in profits when p_i varies.

Using Eq. (10), the two-dimensional system that describes the dynamics of this simple Bertrand duopoly game with horizontal product differentiation is the following:

$$\begin{cases} p_{1,t+1} = p_{1,t} + \alpha p_{1,t} \dfrac{\partial \pi_1}{\partial p_{1,t}} \\ p_{2,t+1} = p_{2,t} + \alpha p_{2,t} \dfrac{\partial \pi_2}{\partial p_{2,t}} \end{cases}, \tag{11}$$

where, for simplicity, we set $\alpha_1 = \alpha_2 = \alpha$. Combining now Eqs. (8.1), (8.2) and (11) gives:

$$\begin{cases} p_{1,t+1} = p_{1,t} + \dfrac{\alpha p_{1,t}}{1-d^2}\left[a(1-d) - 2p_{1,t} + dp_{2,t} + w\right] \\ p_{2,t+1} = p_{2,t} + \dfrac{\alpha p_{2,t}}{1-d^2}\left[a(1-d) - 2p_{2,t} + dp_{1,t} + w\right] \end{cases}. \tag{12}$$

The equilibrium or fixed points of the two-dimensional system (12) are obtained when $p_{1,t+1} = p_{1,t} = p_1$ and $p_{2,t+1} = p_{2,t} = p_2$. Therefore, the fixed points $E\left(p^*_1, p^*_2\right)$ of (12) are defined by the non-negative solutions of the following system:

$$\begin{cases} \dfrac{\alpha p_1}{1-d^2}\left[a(1-d) - 2p_1 + dp_2 + w\right] = 0 \\ \dfrac{\alpha p_2}{1-d^2}\left[a(1-d) - 2p_2 + dp_1 + w\right] = 0 \end{cases}, \tag{13}$$

and they are given by:

$$E_0 = (0,0), \quad E_1 = \left(0, \frac{1}{2}\left[a(1-d) + w\right]\right), \quad E_2 = \left(\frac{1}{2}\left[a(1-d) + w\right], 0\right), \tag{14.1}$$

and

$$E_3 = \left(\frac{a(1-d) + w}{2-d}, \frac{a(1-d) + w}{2-d}\right). \tag{14.2}$$

Eq. (14.2) defines the unique interior Nash equilibrium of this simple duopoly game à la Bertrand. Of course, since every firm faces the same linear cost function, then $p^*_1 = p^*_2 = p^*$. Substituting out the equilibrium price p^* into the direct demand functions Eqs. (3.1) and (3.2), and profit functions Eq. (5), yields the equilibrium values of both quantities and profits of both firms, respectively:

$$q^* = \frac{a-w}{(2-d)(1+d)}, \tag{15}$$

$$\pi^* = \frac{(a-w)^2(1-d)}{(2-d)^2(1+d)}. \tag{16}$$

From Eq. (15) it can easily be seen that $a > w$ must hold to ensure $q^* > 0$, while from Eqs. (14.2) and (16) we observe that the case of perfect substitutability between goods of variety 1 and 2 ($d = 1$) perfectly replicates the original result by Bertrand (1883), as prices of both firms equal the average and marginal cost w and profits are zero in such a case.[3]

2.1. Local stability analysis

In this section we exclusively restrict attention to the analysis of the local stability properties of the positive Bertrand-Nash equilibrium E_3 of the two-dimensional dynamic system (12), which is the sole economically meaningful fixed point because prices should be strictly positive. For doing so, we build on the Jacobian matrix J evaluated at the equilibrium E_3, that is:

$$J(E_3) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\alpha}{1-d^2}\left[a(1-d) + w - p^*(4-d)\right] & \frac{\alpha d p^*}{1-d^2} \\ \frac{\alpha d p^*}{1-d^2} & 1 + \frac{\alpha}{1-d^2}\left[a(1-d) + w - p^*(4-d)\right] \end{pmatrix} . \tag{17}$$

where $p^* = \frac{a(1-d) + w}{2-d}$, $J_{11} = J_{22}$ and $J_{12} = J_{21}$. The trace and determinant of the Jacobian matrix (17) are respectively given by:

$$T := Tr(J) = 2J_{11} = 2 + \frac{2\alpha}{1-d^2}\left[a(1-d) + w - p^*(4-d)\right], \tag{18}$$

$$D := Det(J) = J^2_{11} - J^2_{12} = \frac{\left[d^2 + \alpha a d - \alpha(a+w) + 4\alpha p^*\right]\left[d^2 + \alpha a d - 2\alpha d p^* - \alpha(a+w) - 1 + 4\alpha p^*\right]}{\left(1-d^2\right)^2} . \tag{19}$$

The characteristic polynomial of (17) is the following:

$$G(\lambda) = \lambda^2 - T\lambda + D, \tag{20}$$

whose discriminant is $Z := T^2 - 4D$.

As is known, bifurcation theory describes the way topological features of the system (such as the number of stationary points or their stability) vary as some parameter values continuously change (the Jury's conditions, see, e.g., Medio, 1992; Gandolfo, 2010). For a system in two dimensions, the stability conditions ensuring that both eigenvalues remain within the unit circle[4] are the following:

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[3] Note that this result cannot be observed in Zhang et al. (2009), since they build on both the direct and inverse market demand functions with no microeconomic foundations, and without restricting the parameter d to vary between -1 and 1. Indeed, if one computes the equilibrium values of prices, quantities and profits in the model by Zhang et al. (2009) under the case of perfect substitutability (by assuming, for simplicity, the same average and marginal costs for both firms, and setting the parameter $b = 1$, which represents the slope of the direct linear demand functions of goods of variety 1 and 2, see Zhang et al., 2009, Eq. 1, p. 2049), one obtains the following results: $p^* = a + w$, $q^* = a$ and $\pi^* = a(a + w) > 0$. As can easily be ascertained, prices are higher than the average and marginal cost w and profits are positive. This result is at odds with the original Bertrand's findings and does not have economic sense.

[4] If no eigenvalues of the linearised system around the fixed points of a first order discrete system lie on the unit circle, then such points are defined *hyperbolic*. Roughly speaking, at non-hyperbolic points topological features are not structurally stable.

$$\begin{cases} (i): & F := 1 + T + D > 0 \\ (ii): & TC := 1 - T + D > 0 \\ (iii): & H := 1 - D > 0 \end{cases}. \tag{21}$$

The violation of any single inequality in (21), with the other two being simultaneously fulfilled leads to: (*i*) a flip bifurcation (a real eigenvalue that passes through -1) when $F = 0$; (*ii*) a fold or transcritical bifurcation (a real eigenvalue that passes through $+1$) when $TC = 0$; (*iii*) a Neimark-Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when $H = 0$, namely $D = 1$, and $|T| < 2$.

For the particular case of the Jacobian matrix defined by (17), the stability conditions in (21) become the following:

$$\begin{cases} (i): \quad F = \dfrac{\left[-2d^2 + \alpha ad - \alpha(a+w) + 2\right]\left[2d^3 - (4-\alpha a)d^2 + [\alpha(a-w) - 2]d + 2\alpha(a+w) + 4\right]}{\left(1-d^2\right)^2(2-d)} > 0 \\ (ii): \quad TC = \dfrac{\alpha^2[a(1-d)+w]^2(2+d)}{\left(1-d^2\right)^2(2-d)} > 0 \\ (iii): \quad H := \dfrac{\alpha[a+w-ad]\left[-(4-\alpha a)d^2 + \alpha(a-w)d - 2\alpha(a+w) + 4\right]}{\left(1-d^2\right)^2(2-d)} > 0 \end{cases}. \tag{22}$$

From (22) it is clear that the Nash equilibrium E_3 of the two-dimensional dynamic system (12) cannot loose stability through a transcritical or fold bifurcation, as condition (*ii*) in (22) is always satisfied. However, conditions (*i*) and (*iii*) can be violated.

Since in the present paper we are wondering about the stability effects of horizontal product differentiation, in what follows we take d as the parameter of interest. It is known from the existing literature on the dynamics of oligopoly models (see, amongst many others, Bischi and Naimzada, 1999; Agiza and Elsadany, 2003, 2004; Zhang et al., 2007; 2009; Tramontana, 2009) that when at least one of the two players have bounded rational expectations, the higher the speed of adjustment α, the more likely the destabilisation of an equilibrium of a two-dimensional map. Therefore, by fixing the coefficient α, we now analyse how the fixed point E_3 of system (12) can loose stability when d continuously changes.

From the flip and Neimark-Sacker bifurcation surfaces F and H in (22), it is clear that several values of d exist that makes $F = 0$ and $H = 0$ in the (F, d) and (H, d) planes, respectively. In particular, the numerator of F is a fifth order polynomial in d, while the numerator of H is a third order polynomial in d.[5] Therefore, the

[5] This result strongly differs from Zhang et al. (2009). Indeed, by assuming only for comparison purposes that firms face the same constant average and marginal cost, namely $w_1 = w_2 = w$, $b = 1$ and $\alpha_1 = \alpha_2 = \alpha$, the stability condition by Zhang et al. (2009) that correspond to those stated in (22) in the present paper, are the following: $F = \dfrac{[\alpha(a+w)-2]\{d[\alpha(a+w)+2] + 2\alpha(a+w) - 4\}}{2-d}$, $TC = \dfrac{\alpha^2(a+w)^2(2+d)}{2-d} > 0$ and $H = \dfrac{\alpha(a+w)[\alpha d(a+w) + 2\alpha(a+w) - 4]}{2-d}$. Since the numerators of the bifurcation surfaces F and H are first order polynomials in d, there exists one and only one

existence of horizontal product differentiation introduces the possibility to observe very interesting dynamical events.

Below we develop the usual one-parameter bifurcation analysis by studying how the stability properties of the unique positive equilibrium E_3 changes when the degree of product market differentiation varies within its domain of definition $-1 < d < 1$. Unfortunately, since closed form solutions for d from F and H in (22) cannot be dealt with in a neat analytical form, we now perform numerical simulations to study the dynamics of the model. Indeed, since E_3 can a priori be destabilised through either a flip or Neimark-Sacker bifurcation, we have to check, by starting from a stability situation, which of them occurs before the other one when d varies.

Since from an economic point of view the case $d = 0$ implies independence of goods of variety 1 and 2 (that is, there exist two separate monopolistic firms that produce distinct goods), it is interesting to study the stability properties of E_3 when strategic interaction between firms exists either by increasing the degree of substitutability or complementarity between goods, that is when d varies either from 0 to 1 or from 0 to -1, respectively.

The fixed point E_3 of the dynamical system (12) may undergo a flip or Neimark-Sacker bifurcation depending on the ranking of the bifurcation values of the parameter d, i.e., the roots of $F = 0$ and $H = 0$. Indeed, although the bifurcation values of d crucially depend on the other parameters of the problem, namely α, a and w, we are able to set up a ranking of bifurcation values of it, which is always preserved at least when the fixed point E_3 of the two-dimensional system (12) is stable when the two firms act as two separate monopolists that independently produce goods of variety 1 and 2 in their own markets, starting from the case $d = 0$ and moving either towards the case of perfect substitutability ($d = 1$) or perfect complementarity ($d = -1$). Therefore, we are able to give general results as regards the role of the degree of horizontal product differentiation on stability of the Bertrand-Nash equilibrium of a Bertrand duopoly market.

As an example, we choose the following parameter set: $\alpha = 0.5$, $a = 3$ and $w = 0.5$.[6]

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solution to each of them for d in such a case, namely $d = d^z{}_F := \dfrac{2[2-\alpha(a+w)]}{2+\alpha(a+w)}$ and $d = d^z{}_H := \dfrac{2[2-\alpha(a+w)]}{\alpha(a+w)}$, where $d^z{}_F < d^z{}_H$. Therefore, the unique interior Nash equilibrium in their model can loose stability exclusively through a flip bifurcation, while Zhang et al. (2009, p. 2055) refer to the case of Neimark-Sacker bifurcations, which cannot occur because the discriminant Eq. (15) of the Jacobian matrix (13) (see, Zhang et al., 2009, p. 2053) is always positive.

[6] However, through extensive numerical experiments not reported here for economy of space, we feel confident that the shape of the bifurcation loci F and H in (22), depicted in Figures 1 and 2, respectively, holds true for any meaningful parameter sets such that the positive fixed point E_3 is stable when $d = 0$.

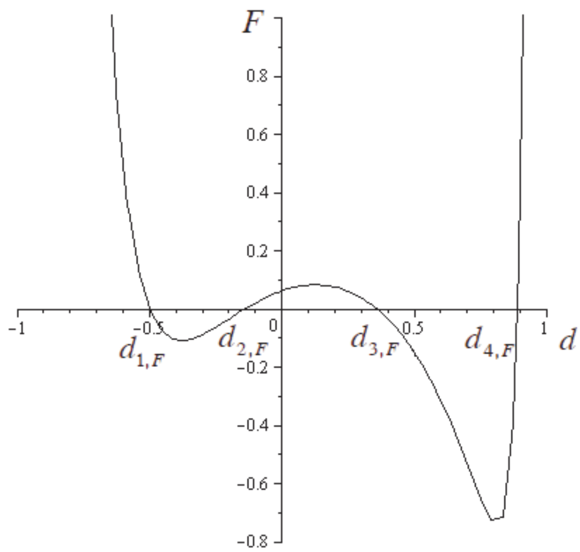


Figure 1. Flip bifurcation locus F when d varies (Parameter values: $\alpha = 0.5$, $a = 3$ and $w = 0.5$).

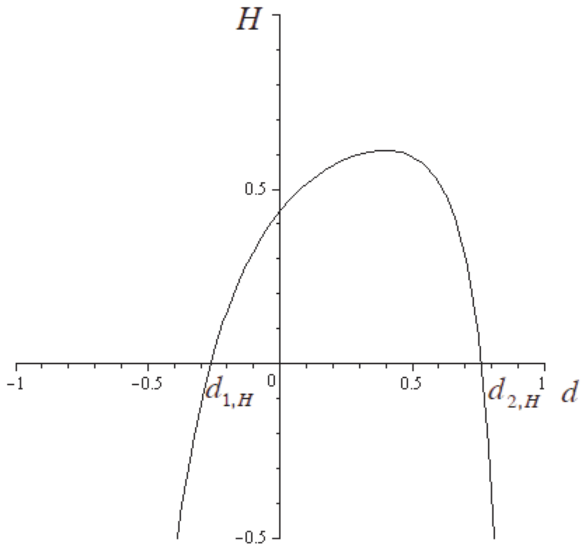


Figure 2. Neimark-Sacker bifurcation locus H when d varies (Parameter values: $\alpha = 0.5$, $a = 3$ and $w = 0.5$).

Figure 1 (2) depicts the flip (Neimark-Sacker) bifurcation locus when d varies within the range $(-1,1)$ and shows the existence of four (two) roots of d in the (F,d) ((H,d)) planes. Starting from the left (negative values of d) and moving towards the right (higher values of d), we define the flip bifurcation values as $d_{1,F}$, $d_{2,F}$, $d_{3,F}$ and $d_{4,F}$, and the Neimark-Sacker bifurcation values as $d_{1,H}$ and $d_{2,H}$. For the configuration of parameters chosen above, Figures 1 and 2 show, respectively, that $d_{1,F} = -0.5$, $d_{2,F} = -0.1403$, $d_{3,F} = 0.3596$ and $d_{4,F} = 0.8903$, and $d_{1,H} = -0.2623$ and $d_{2,H} = 0.7623$. Therefore, the following results generically hold.

Result 1. *For any meaningful configuration of parameters such that the fixed point E_3 of the two-dimensional system (12) is stable when $d = 0$ (i.e., firms 1 and 2 behave as two separate monopolists), the existence of strategic interaction with horizontal product* *differentiation implies the existence of two negative flip bifurcation values of* d*, i.e.* $d_{1,F} < d_{2,F}$ *and two positive flip bifurcation values of* d*, i.e.* $d_{3,F} < d_{4,F}$.

Result 2. *For any meaningful configuration of parameters such that the fixed point* E_3 *of the two-dimensional system (12) is stable when* $d = 0$*, an increase in the degree of substitutability between goods of variety* 1 *and* 2*, i.e. the parameter* d *moves from* 0 *to* 1*, implies that the Bertrand-Nash equilibrium* E_3 *looses stability exclusively through a flip or period-doubling bifurcation, as* $d_{3,F} < d_{2,H}$ *always holds.*

Result 3. *For any meaningful configuration of parameters such that the fixed point* E_3 *of the two-dimensional system (12) is stable when* $d = 0$*, an increase in the degree of complementarity between goods of variety* 1 *and* 2*, i.e. the parameter* d *moves from* 0 *to* -1*, implies that the Bertrand-Nash equilibrium* E_3 *looses stability exclusively through a flip or period-doubling bifurcation, as* $|d_{2,F}| < |d_{1,H}|$ *always holds.*

3. Numerical simulations

In this section the dynamical behaviours of a duopoly model with product differentiation are investigated through numerical simulations. To provide some numerical evidence for the existence of chaotic motions, we use several standard tools, such as bifurcations diagrams, phase portraits with basin of attractions and shape of attractors, Lyapunov exponents, sensitive dependence on initial conditions and so on. In order to study the local and global dynamical properties of the map defined by Eq. (12) conveniently, in dependence of the different characteristics of the variety of goods 1 and 2, we take the degree of differentiation d as the bifurcation parameter and the illustrative parameter values $\alpha = 0.5$, $a = 3$ and $w = 0.5$.

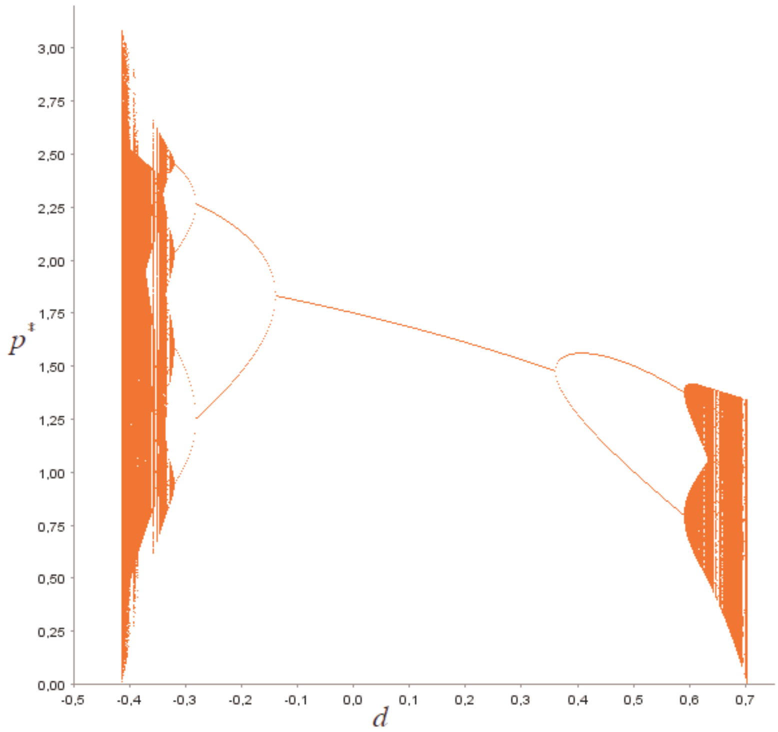


Figure 3.a. Bifurcation diagram for d. Initial conditions: $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$ (Parameter values: $\alpha = 0.5$, $a = 3$ and $w = 0.5$).

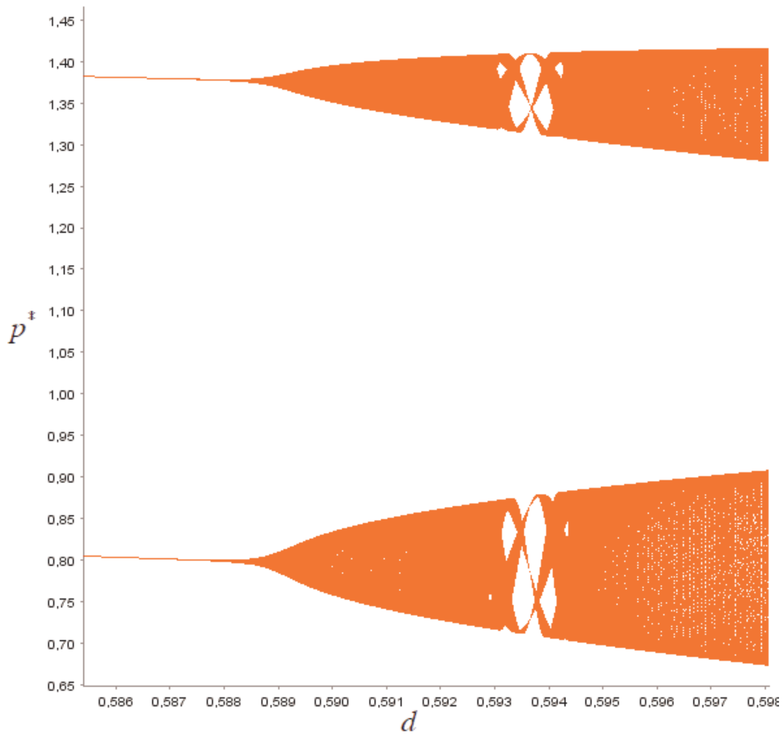


Figure 3.b. Bifurcation diagram for d. An enlarged view for $0.585 < d < 0.598$ and $0.65 < p^* < 1.46$.

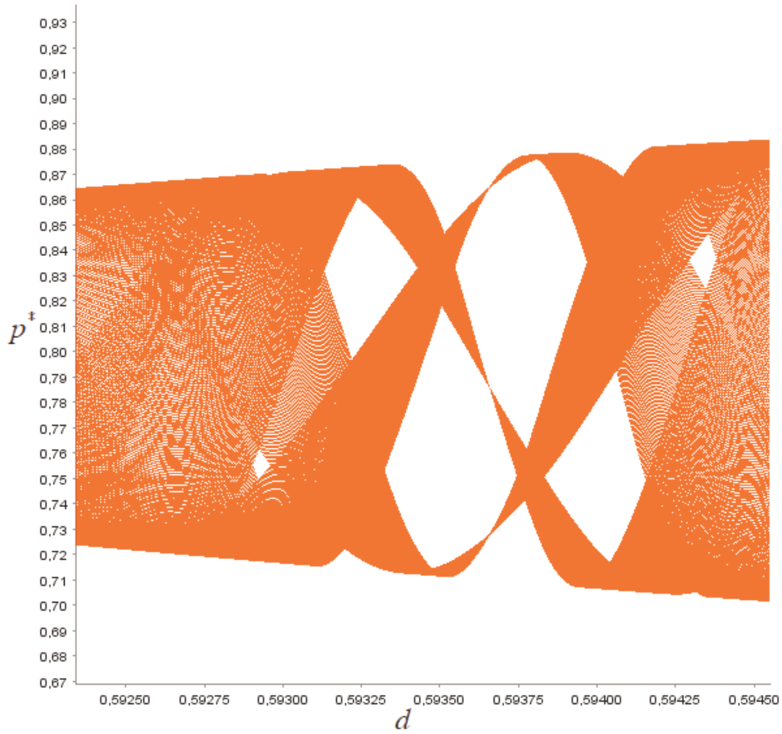


Figure 3.c. Bifurcation diagram for d. An enlarged view $0.592 < d < 0.5946$ and $0.67 < p^* < 0.937$.

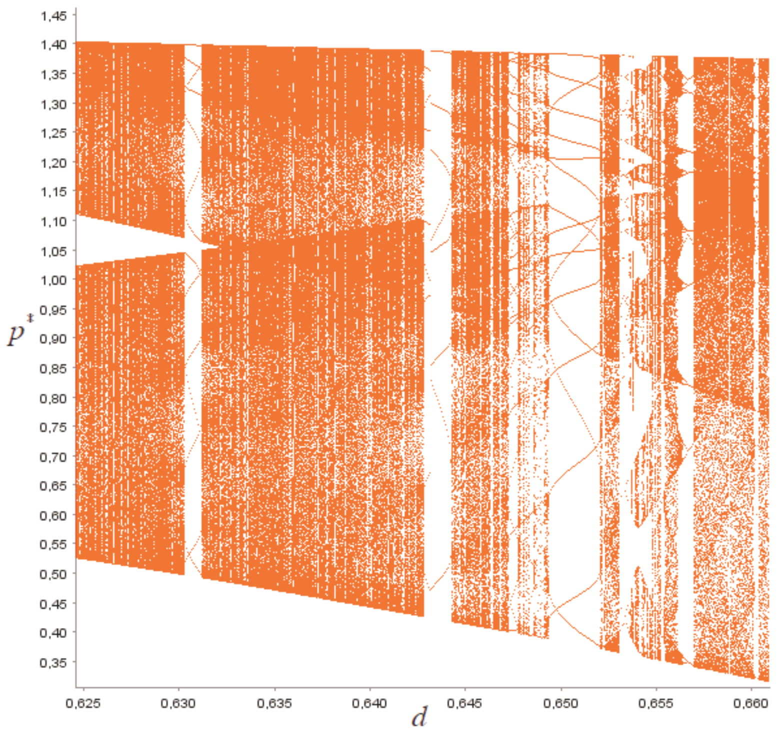


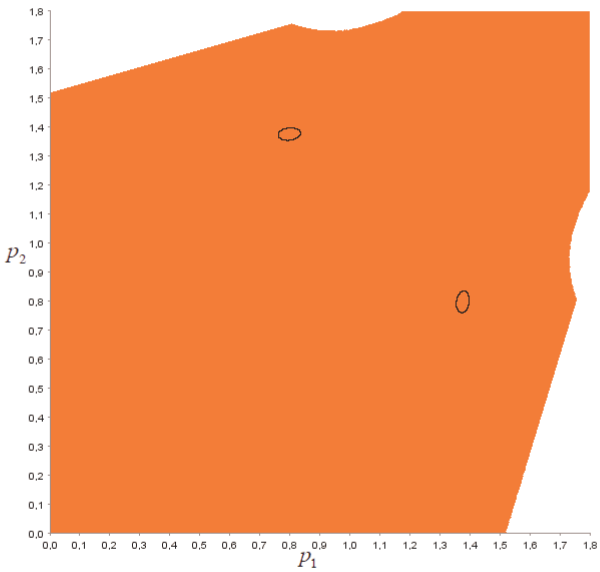
Figure 3.d. Bifurcation diagram for d. An enlarged view for $0.624 < d < 0.662$ and $0.3 < p^* < 1.46$.

Starting from a stability situation when $d = 0$, Figure 3.a shows that an increase in the degree of substitutability, or equivalently, a reduction in the extent of product differentiation (i.e., the parameter d moves from 0 to 1), implies that the map (12) converges to a fixed point for $-0.1403 < d < 0.3596$. Starting from this interval, in which the positive equilibrium of system (12) is stable, Figure 3a shows that E_3 undergoes a flip bifurcation both at $d_{1,F} = -0.1403$ (when the parameter d decreases or, alternatively, the degree of complementarity increases) and $d_{2,F} = 0.3596$ (when the parameter d increases or, alternatively, the degree of substitutability increases).

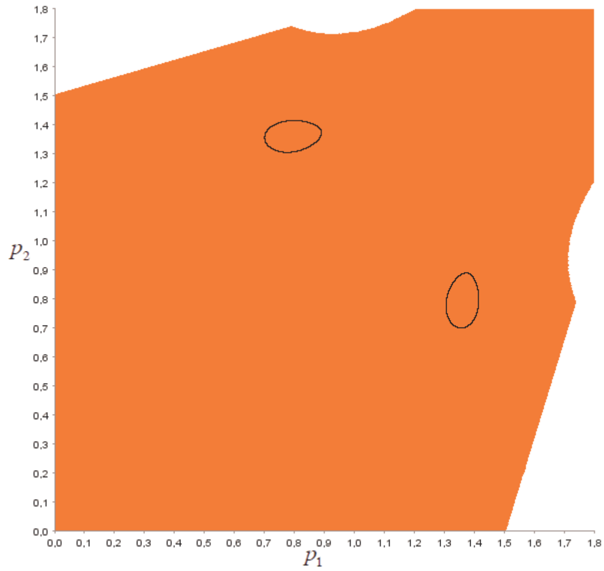
On the one hand, starting from $d = -0.1403$, a further decrease in d, i.e. an increase in the degree of complementarity between products of variety 1 and 2, implies that: for $-0.2823 < d < -0.1403$ a two-period cycle emerges; for $-0.3207 < d < -0.2823$ a four-period cycle emerges; for $-0.36 < d < -0.3207$ the map (12) converges to a quasi periodic attractor, while within the interval $-0.415 < d < -0.36$ it converges to a chaotic attractor, except for a number of periodic windows; finally, when $d < -0.415$ the map (12) does not converge. On the other hand, starting from $d = 0.3596$, a further increase in d, i.e. an increase in the degree of substitutability between products of variety 1 and 2, implies that: for $0.3596 < d < 0.588$ a two-period cycle emerges; for $0.588 < d < 0.65$ the map (12) converges to a quasi periodic attractor, while within the interval $0.65 < d < 0.7021$ it converges to a chaotic attractor, except for a number of periodic windows; for $d > 0.7021$ the map (12) does not converge.

Figures 3.b-3.d depict several enlarged views of the bifurcation diagram 3.a, where it is clearly shown both a quasi-period route to chaos (and not a period-doubling cascade, as argued by Zhang et al. (2009, p. 2055) (Figures 3.b and 3.c) and a complex dynamic behaviour interspersed by parametric intervals of periodic behaviour (Figure 3.d).

In Figures 4 and 5 we depict the phase portraits for several different values of d, corresponding to the bifurcation diagram plotted in Figure 3.a, when: (*i*) the extent of the degree of substitutability between goods of variety 1 and 2 increases for positive values of d and (*ii*) the extent of the degree of complementarity between goods of variety 1 and 2 increases for negative values of d.



(a)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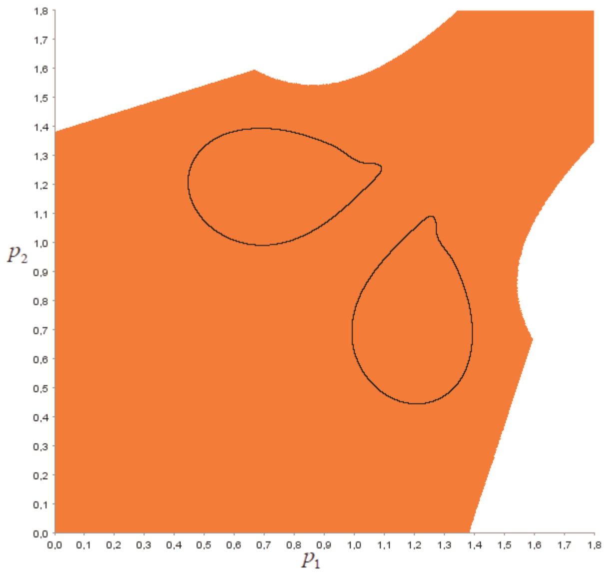
(b)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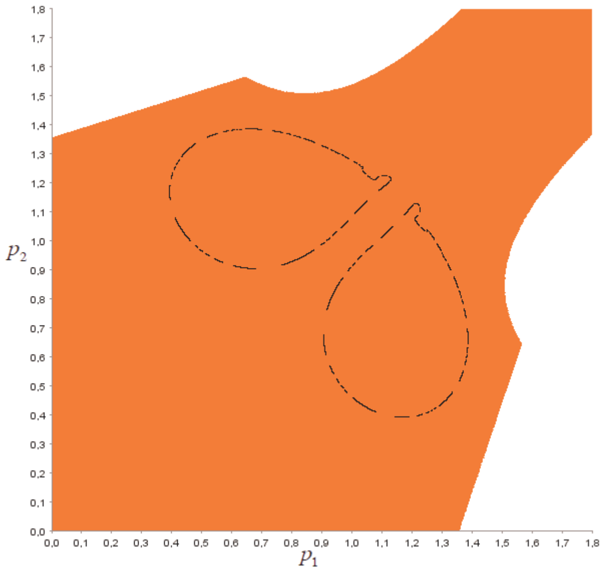
(c)



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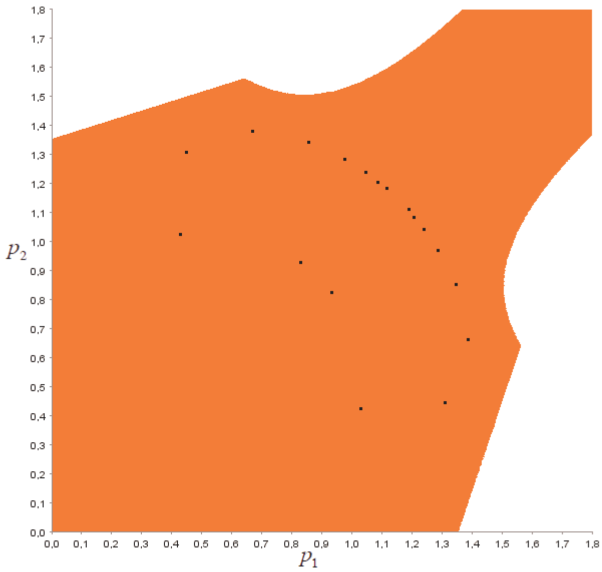

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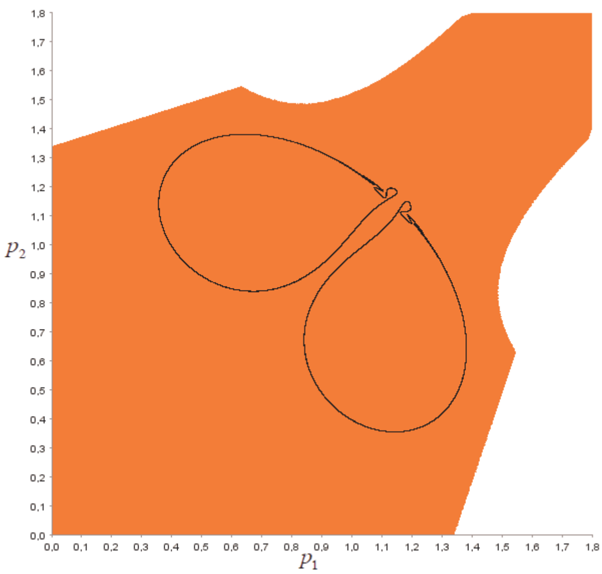

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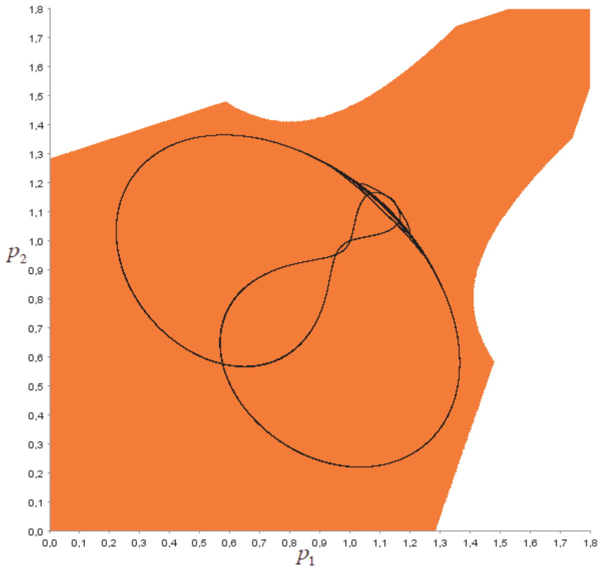
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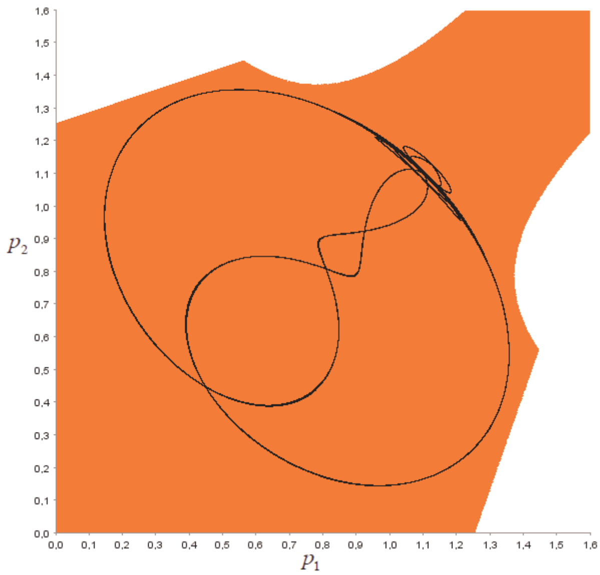
(g)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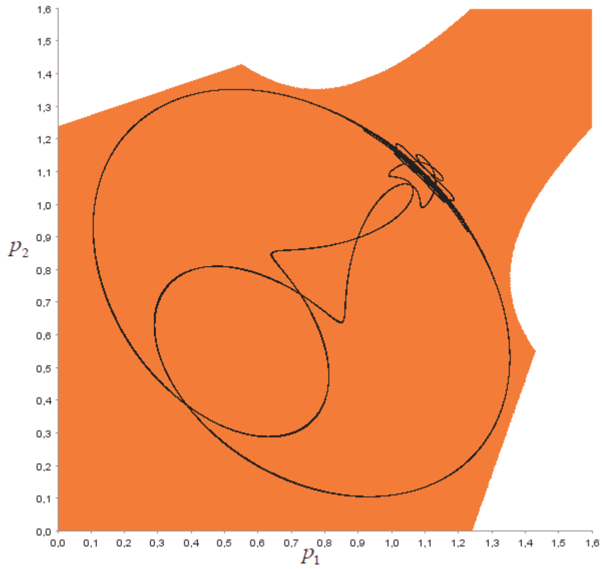
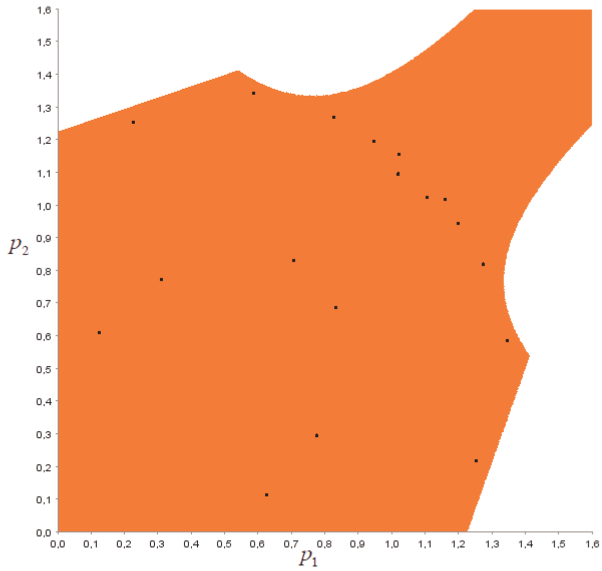
p_1

(h)



(i)



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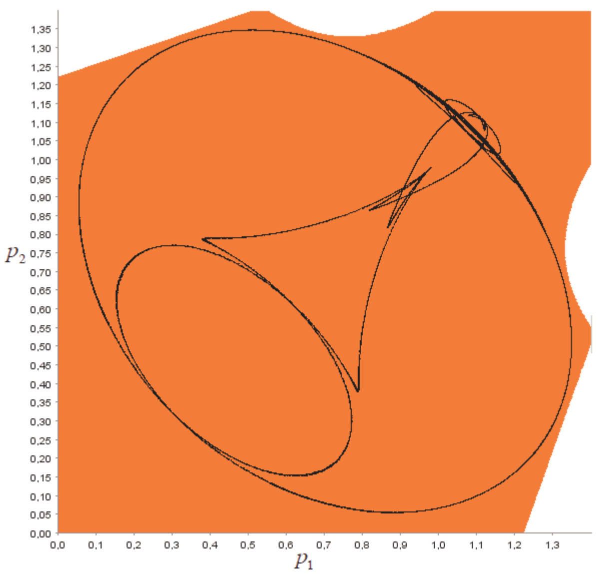
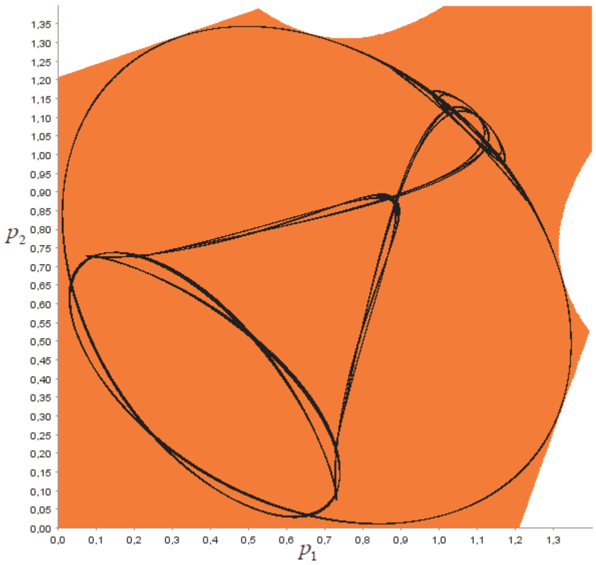


(k)



(l)

Figure 4. Case $0 < d < 1$. Phase portrait for different values of d (the degree of substitutability between goods of variety 1 and 2 increases): (a) $d = 0.59$, (b) $d = 0.595$, (c) $d = 0.64$, (d) $d = 0.6491$ (e) $d = 0.65$, (f) $d = 0.655$, (g) $d = 0.675$, (h) $d = 0.685$, (i) $d = 0.69$, (j) $d = 0.695$, (k) $d = 0.696$ and (l) $d = 0.701$ (Parameter values: $\alpha = 0.5$, $a = 3$ and $w = 0.5$).

A main result of the present paper, which is detailed below, is that the bifurcation diagrams (Figures 3.a-3.d), phase portraits (Figures 4-5), Lyapunov exponents (Figure 6) and time trajectories for arbitrarily close initial conditions (Figures 7a and 7.b), are all convergent in identifying a "quasi-periodic" route to chaos as the typical route to chaos of a Bertrand duopoly economy with product differentiation (see Eq. 12).

We now briefly recall that nonlinear dynamical systems with chaotic behaviour exhibit a number of well characterised routes to chaotic behaviour (see Bergé et al., 1986; Medio, 1992). If we limit us to consider the generic "co-dimension one route to chaos" (i.e., transitions to chaos characterised by local or global bifurcations occurring when only a single parameter varies) we may, loosely speaking, distinguish four types of routes: (1) period-doubling; (2) intermittency (or explosion); (3) saddle connection (or "blue-sky catastrophe"); (4) quasi-periodic. These routes present universal properties, that allow to classify nonlinear systems in universality classes formed by systems with completely different microscopic interactions. Transitions to chaos through quasi-periodic motions have deeply been investigated, at least starting from Ruelle and Takens (1971), in particular the transition from two-frequency quasi-periodic behaviour to low-dimensional chaotic behaviour, which is the typical route identified in the present economic dynamic model. Typically, the literature (see, e.g., Curry and Yorke, 1977 and, in particular, the studies on the simplified circle map, Feigenbaum et al., 1982) has shown that this transition to chaos usually proceeds through the interaction of resonances (mode-locking),[7] that lead to a wrinkling or corrugation of the torus, and ultimately to a strange attractor.

[7] However, it has also been shown that 2D-tori can lead to a strange attractor without mode-locked states (see, e.g. Moon, 1997).

Figure 4 shows the attractor and basin of attraction in the case $0 < d < 1$ and for increasing values of the degree of substitutability between goods of variety 1 and 2. The figure shows the different shape of the attractor when d increases. Figures 4.a, 4.b and 4.c clearly show two increasing invariant closed curves when $d = 0.59$, $d = 0.595$ and $d = 0.64$. Such figures illustrate, in addition to the local flip bifurcation above mentioned in Results 2 and 3, that we can observe that the initially stable two-period cycle (which is born when the fixed point E_3 undergoes a flip bifurcation at $d = 0.3596$) loses stability almost at $d = 0.589$ through the so-called Neimark-Sacker bifurcation (the pair of complex conjugate eigenvalues cross the unit circle) and two limit cycles are born.

If we interpret, in order to better understand our numerical results, our two-dimensional plots as Poincaré surfaces of sections of continuous solutions of differential equations, we know that the fixed point corresponds to a stable limit cycle. This limit cycle evolves into a torus through a Neimark-Sacker bifurcation.[8] The torus represents a quasi-periodic (one-frequency periodic) behaviour of continuous solutions and is responsible for the invariant orbits on the Poincaré surface of sections. This is illustrated in Figure 4.a, where the two invariant symmetric circles represent the 2D-torus on the Poincaré map.

The invariant circles grow as the parameter d continues to change, and then lose their smoothness (see Figure 4.d, where several chaotic pieces appears at $d = 0.6491$ and where the attractors now cease to be even topologically a circle).[9] When $d = 0.65$, Figure 4.d shows that a cycle of period 18 emerges. As long as the parameter d increases the figures show a double ring chaotic area, unless a cycle of period 18 that appears when $d = 0.695$ (see Figure 4.j).

Another interesting observation concerns the bifurcation of attracting invariant closed curves from many periodic points. For instance, for the case of complementarity between products of variety 1 and 2, Figure 5.c shows a 20-periodic orbit, which is created by so-called mode-locking (and for which the rotation number is rational), and for slight increases in the (modulus) of d a 60-period cycle occurs, which is not shown here for economy of space). Indeed, when the modulus of d increases further on, we initially observe the bifurcation of a 60-periodic orbit with the emergence of 60 invariant “circles” (see Figure 5.e), clustered in four “archipelagos” and subsequently

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[8] More specifically, the co-dimension-one bifurcations concerning a periodic orbit (such as the stable two period-cycle surrounding the positive equilibrium point emerged through a flip bifurcation), which are responsible for the loss of stability or disappearance of such an orbit, can briefly be classified according to whether the periodic orbit: (1) collapses into an equilibrium state through a supercritical Neimark-Sacker bifurcation; (2) collides with an unstable periodic orbit (acquiring a multiplier equal to $+1$) and then vanishes; (3) becomes a homoclinic loop to a saddle equilibrium state; (4) transforms itself into a homoclinic loop of a saddle-node equilibrium state; (5) a period doubling or flip bifurcation, where a multiplier of the orbit decreases through -1, occurs (note that the stability of the original orbit is inherited by an orbit of doubled period); (6) a secondary Neimark-Sacker bifurcation, where a pair of complex-conjugate multipliers of the periodic orbit crosses the unit circle outwards, occurs and the periodic orbit becomes a two-dimensional invariant torus. Note that the fifth and sixth possibilities (*i*) require at least a continuous-time 3D system (a 2D Poincaré map), and (*ii*) are characterised by the fact that the periodic orbit no longer disappears at the bifurcation but only loses its stability through either a flip or Hopf-Neimark-Sacker bifurcation (e.g., Andronov et al. 1971; Shilnikov et al., 2001).

[9] As noted by Kopel (1996, p. 2043), the loss of smoothness of the invariant circles can often be related to behaviour associated with the existence of homoclinic and heteroclinic points, and differently from the local bifurcations discussed in our model in Results 1-3, to the occurrence of global bifurcation phenomena (in particular homoclinic bifurcations).

the bifurcation of the set of 60 closed curves which lose their smoothness and result in a strange attractor (see Figures 5.f and 5.g).[10]

Similar comments concern Figures 4.i, 4.k, 4.l and 5.h-5-l (reported here for the remarkable aesthetic beauty of these attractors).

In order to investigate if an aperiodic behaviour occurs, we study the largest Lyapunov exponent ($Le1$) as a function of d. If $Le1$ lies on zero (is positive), then there is evidence for "quasi-periodicity" (chaos). Figure 6 displays, for the case of substitutability between products of variety 1 and 2, the intervals of the (positive values of the) parameter d for which the two-dimensional system (12) converges to cycles, quasi-periodicity and chaotic behaviour. Note the agreement of the predictions of Figure 6 with those of Figures 3, 4 and 5 together with corresponding comments.

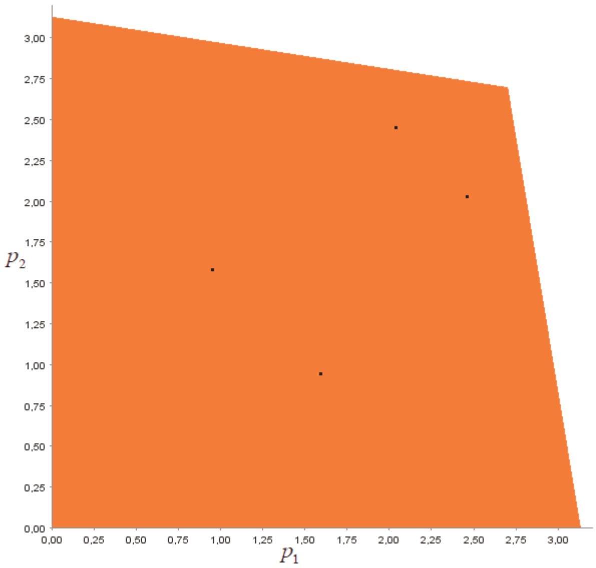
As known, the sensitivity to initial conditions is a characteristic of deterministic chaos. In order to show the sensitivity to initial conditions of system (12), we have computed two orbits of the variable p_1 whose coordinates of initial conditions differ by 0.000001. Figures 7.a and 7.b depict the orbits of p_1 with initial conditions $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$, and $p_{1,0} = 0.200001$ and $p_{1,0} = 0.300001$ at $d = 0.62$ (corresponding to the quasi-periodic parametric region), and $d = 0.701$ (corresponding to the chaotic parametric region), respectively. As expected, while when $d = 0.62$ the orbits remain similar irrespective of initial conditions, when $d = 0.701$ the orbits rapidly separate each other, thus indicating the existence of chaotic motions.

Furthermore, since strange attractors must typically be characterised by fractal dimensions, we have calculated the attractor dimension according to the Kaplan-Yorke conjecture obtaining, as a confirmation of the presence of chaos, a dimension smaller than 2. The values are not reported here for brevity, but are of course available on request.

Finally, we note that the dynamic behaviour discussed above is at all typical of quadratic 2D maps (of which the present model (eq. 21) is a case – typical for duopoly models with bounded rational firms – with four non-linearity), such as the coupled logistic equations dynamic behaviour analysed by Kopel (1996). For recent papers offering an overview of some important issues related to the general case of the dynamics of general 2D quadratic maps), especially predicting the strange attractors and their properties, see, for instance, Elhadj and Sprott, 2008, 2010).[11]

[10] As is known, the intermittent periodic behaviour (mode-locking) comes about when the two characteristic frequencies on the torus are in the ratio of two integers. Higher bifurcations of the torus occur as the system moves out of the quasi-periodic region by further increase in d. These bifurcations are characterised by the folding of the boundaries onto itself and eventually the break-up of the torus (see Kopel, 1996, p. 2045).

[11] See also the papers by Bischi et al. (1998), Bischi et al. (1999) and Bischi and Gardini (2000).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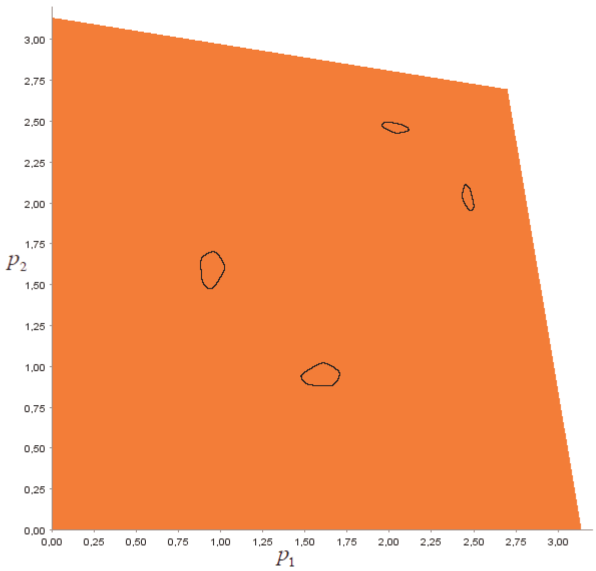
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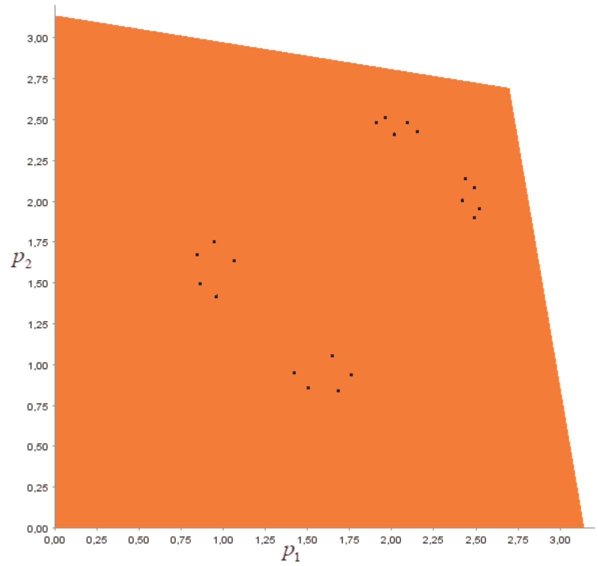
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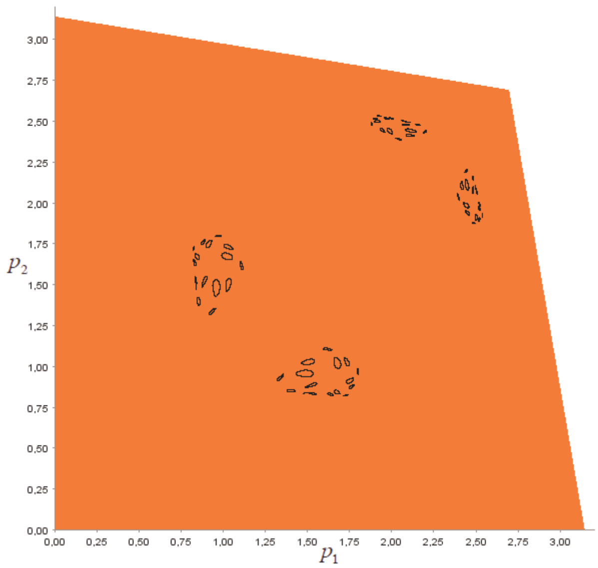
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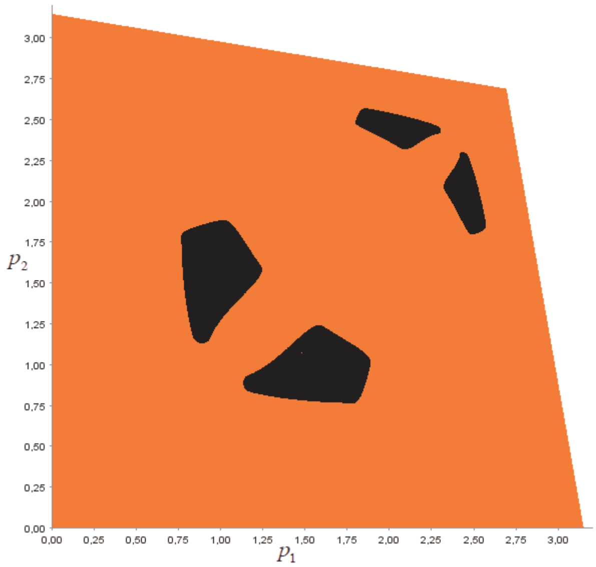
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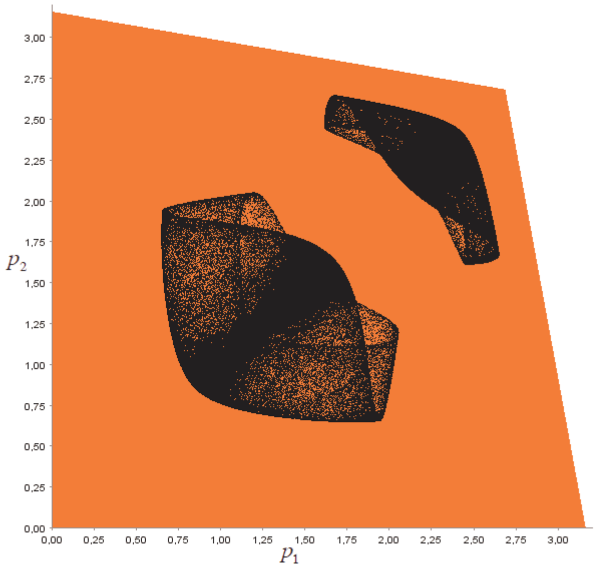
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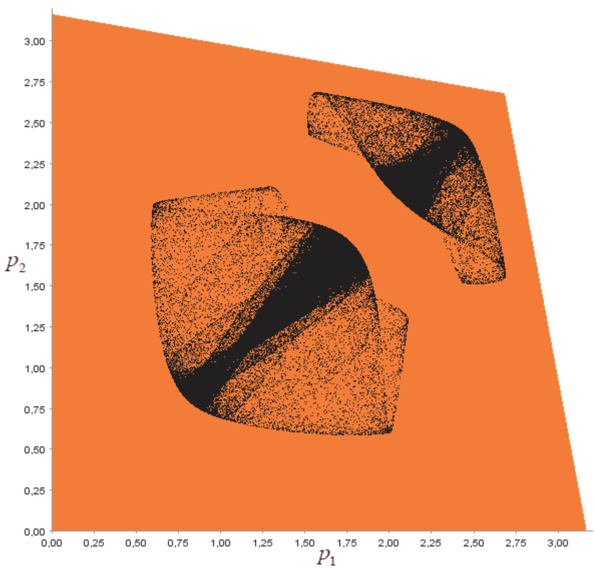
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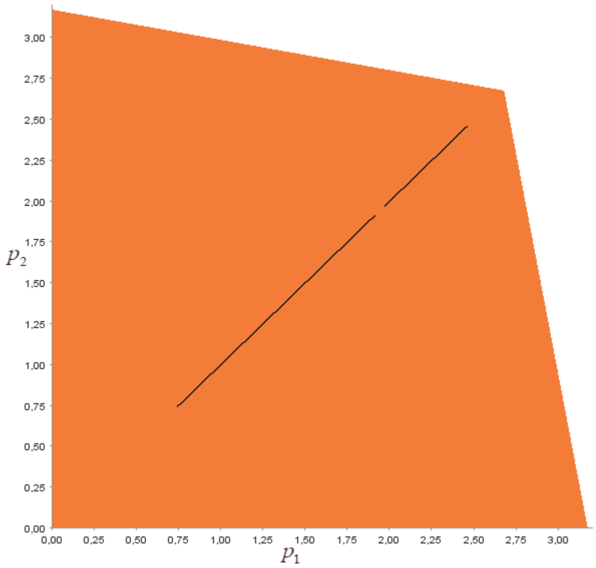
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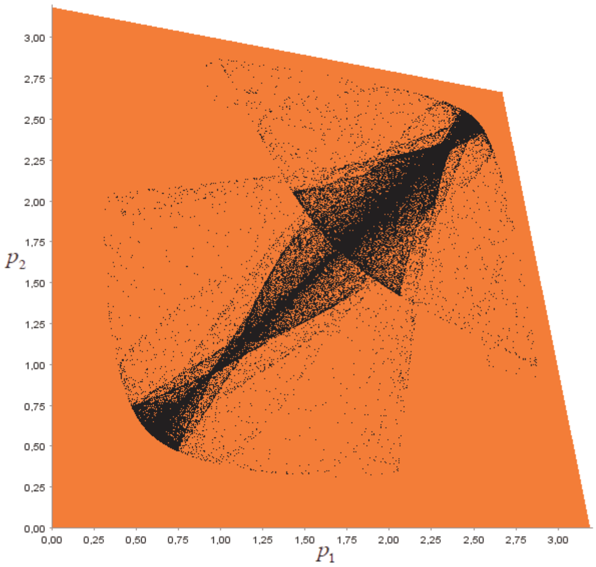
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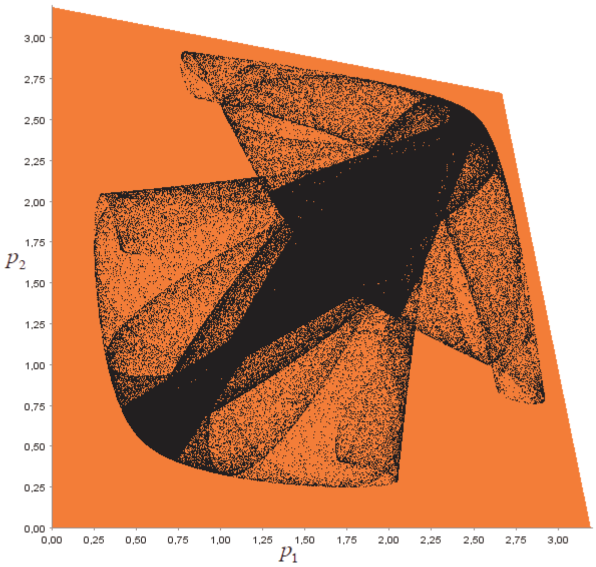
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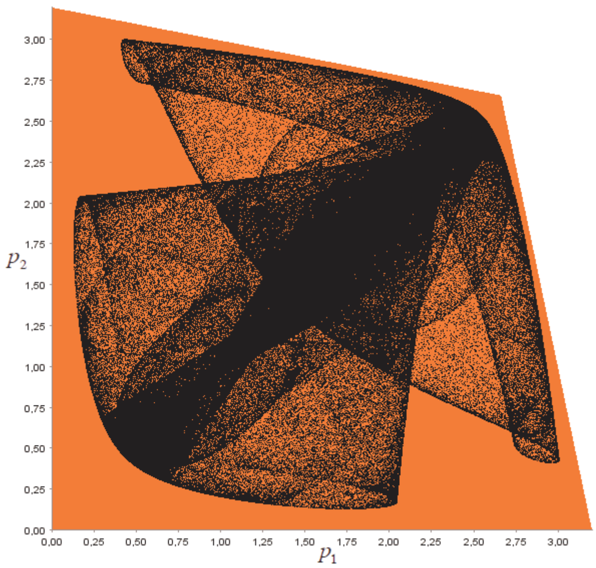
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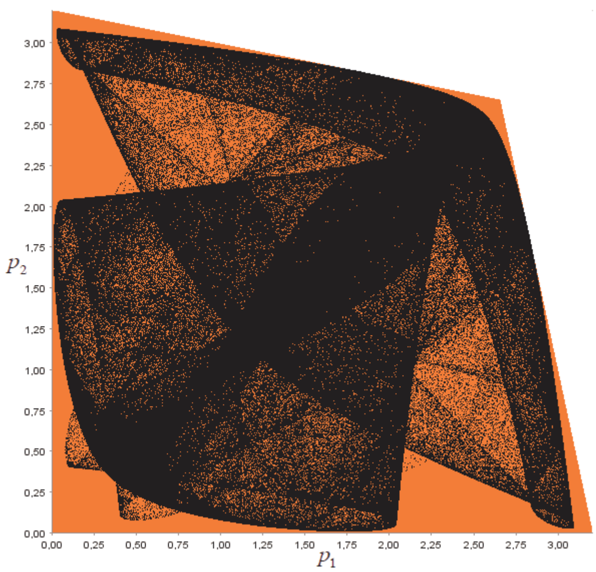
p_1

(k)


3,00
2,75
2,50
2,25
2,00
1,75
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1,25
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0,25
0,00
p_2
0,00 0,25 0,50 0,75 1,00 1,25 1,50 1,75 2,00 2,25 2,50 2,75 3,00
p_1

(l)

Figure 5. Case $-1 < d < 0$. Phase portrait for different values of d (the degree of complementarity between goods of variety 1 and 2 increases): (a) $d = -0.32$, (b) $d = -0.325$, (c) $d = -0.33$, (d) $d = -0.33326348$, (e) $d = -0.34$, (f) $d = -0.355$, (g) $d = -0.362$, (h) $d = -0.37$, (i) $d = -0.39$, (j) $d = -0.395$, (k) $d = -0.405$ and (l) $d = -0.414$ (Parameter values: $\alpha = 0.5$, $a = 3$ and $w = 0.5$).

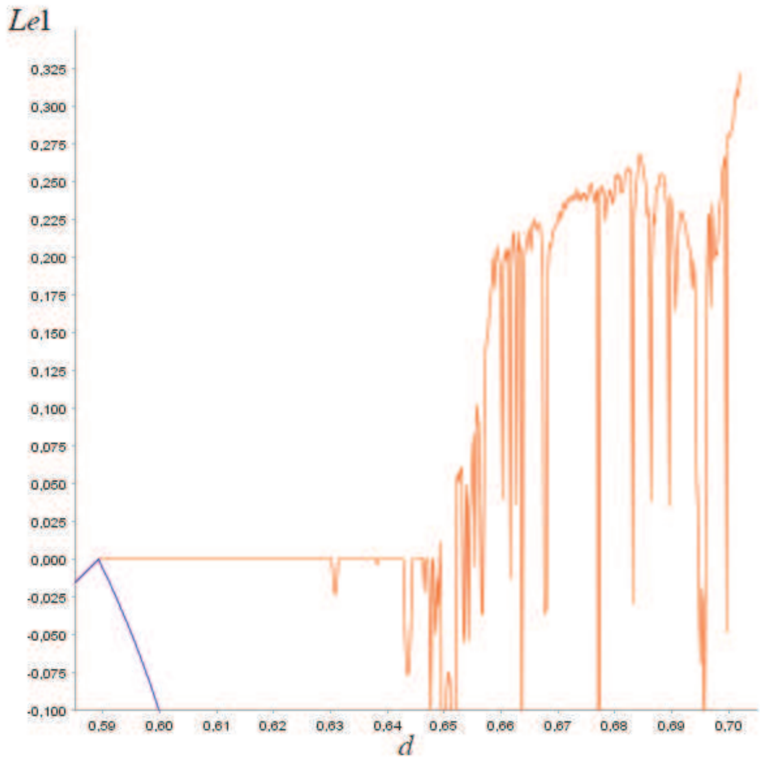

Figure 6. The largest Lyapunov exponent for $0.585 < d < 0.705$ (one million iterations).

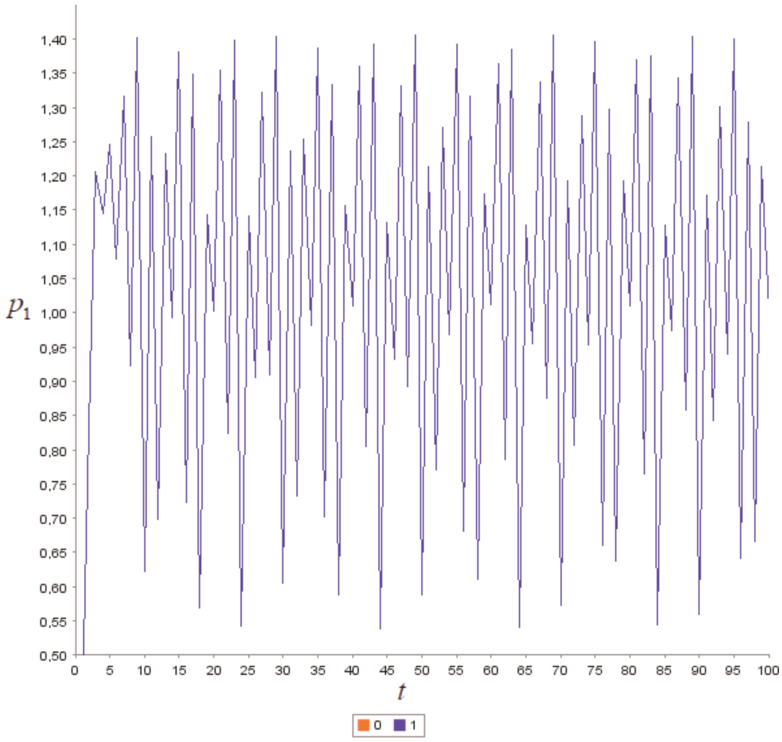

Figure 7.a. Sensitivity dependence to initial conditions (p_1 versus time). Initial conditions: $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$ red and $p_{1,0} = 0.200001$ and $p_{2,0} = 0.300001$ blue (Parameter values: $\alpha = 0.5$, $a = 3$, $w = 0.5$ and $d = 0.622$).

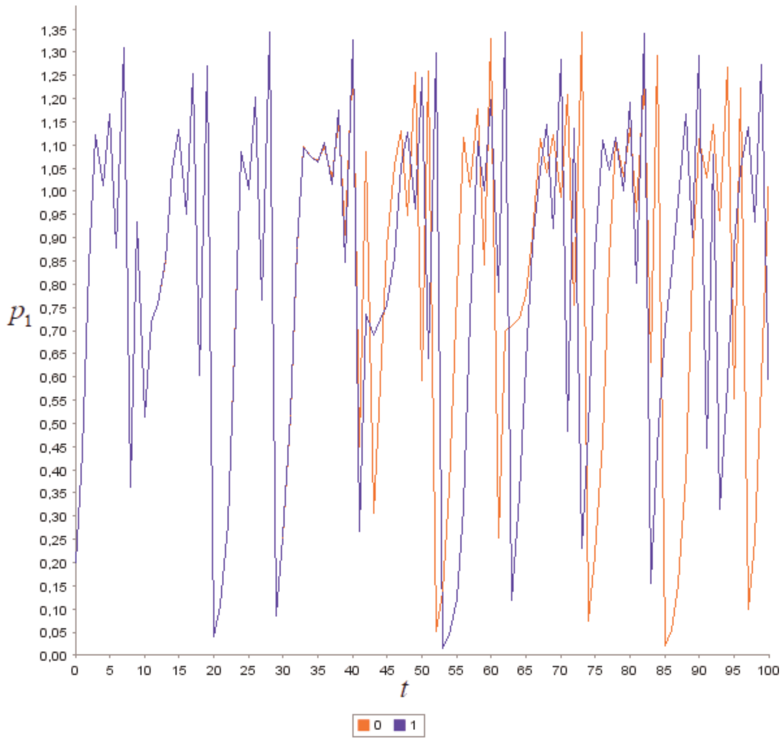

Figure 7.b. Sensitivity dependence to initial conditions (p_1 versus time). Initial conditions: $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$ red and $p_{1,0} = 0.200001$ and $p_{2,0} = 0.300001$ blue (Parameter values: $\alpha = 0.5$, $a = 3$, $w = 0.5$ and $d = 0.701$).

2.2. The case with quadratic costs

In this section we relax the hypothesis of constant marginal returns to labour, which leads each firm facing with a linear cost function, and assume, following Fanti and Meccheri (2011), that decreasing returns to labour exist. The production function of firm i that produces output of variety i therefore becomes $q_i = \sqrt{L_i}$, where $L_i = q_i^2$ is the labour force employed by firm i. The hypothesis of decreasing returns to labour implies that firms have quadratic costs, that is:

$$C_i(q_i) = wL_i = wq_i^2, \tag{23}$$

so that average and marginal costs do not coincide and, in particular, marginal costs are higher than average costs for every $q_i > 0$.

Profits of firm i in every period can be written as follows:

$$\pi_i = p_i q_i - wq_i^2. \tag{24}$$

Following the same line of reasoning as in Section 2, profit maximisation with respect to prices gives the following marginal profits:

$$\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = \frac{[a(1-d)+dp_2](1-d^2+2w)-2(1-d^2+w)p_1}{(1-d^2)^2}, \tag{25.1}$$

$$\frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = \frac{[a(1-d)+dp_1](1-d^2+2w)-2(1-d^2+w)p_2}{(1-d^2)^2}. \tag{25.2}$$

Therefore, the reaction- or best-reply functions of firms 1 and 2 are obtained as the unique solution of Eqs. (25.1) and (25.2) for p_1 and p_2, respectively, and they are given by:

$$\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = 0 \Leftrightarrow p_1(p_2) = \frac{[a(1-d)+dp_2+w](1-d^2+2w)}{2(1-d^2+w)}, \tag{26.1}$$

$$\frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = 0 \Leftrightarrow p_2(p_1) = \frac{[a(1-d)+dp_1+w](1-d^2+2w)}{2(1-d^2+w)}. \tag{26.2}$$

Under the hypothesis of bounded rational expectations by firms, the two-dimensional system that describes the dynamics of the Bertrand duopoly economy with horizontal product differentiation and quadratic costs is the following:

$$\begin{cases} p_{1,t+1} = p_{1,t} + \dfrac{\alpha p_{1,t}}{(1-d^2)^2}\{[a(1-d)+dp_{2,t}](1-d^2+2w)-2(1-d^2+w)p_{1,t}\} \\ p_{2,t+1} = p_{2,t} + \dfrac{\alpha p_{2,t}}{(1-d^2)^2}\{[a(1-d)+dp_{1,t}](1-d^2+2w)-2(1-d^2+w)p_{2,t}\} \end{cases}. \tag{27}$$

The fixed points of the two-dimensional time map (27) are obtained when $p_{1,t+1} = p_{1,t} = p_1$ and $p_{2,t+1} = p_{2,t} = p_2$ hold. Therefore, the fixed points $E(p^*_1, p^*_2)$ of (27) are defined by the non-negative solutions of the following system:

$$\begin{cases} \dfrac{\alpha p_1}{(1-d^2)^2}\{[a(1-d)+dp_2](1-d^2+2w)-2(1-d^2+w)p_1\} = 0 \\ \dfrac{\alpha p_2}{(1-d^2)^2}\{[a(1-d)+dp_1](1-d^2+2w)-2(1-d^2+w)p_2\} = 0 \end{cases}, \tag{28}$$

and they are given by:

$$E_0=(0,0),\quad E_1=\left(0,\frac{a(1-d)(1-d^2+2w)}{2(1-d^2+w)}\right),\quad E_2=\left(\frac{a(1-d)(1-d^2+2w)}{2(1-d^2+w)},0\right),\tag{29.1}$$

and

$$E_3=\left(\frac{a(1-d^2+2w)}{2(1+w)+d(1-d)},\frac{a(1-d^2+2w)}{2(1+w)+d(1-d)}\right).\tag{29.2}$$

where E_3 is the unique interior Nash equilibrium and $p^*_1=p^*_2=p^*$. Substituting out the equilibrium price p^* into the direct demand functions Eqs. (3.1) and (3.2), and profit functions Eq. (24), yields the equilibrium values of both quantities and profits of both firms, respectively:

$$q^*=\frac{a}{2(1+w)+d(1-d)},\tag{30}$$

$$\pi^*=\frac{a^2(1-d^2+w)}{[2(1+w)+d(1-d)]^2}.\tag{31}$$

Building on the Jacobian matrix from the two-dimensional system (27) and computing both the trace and determinant, it is possible to show that Results 1-4 above qualitatively hold even in the case of quadratic costs, that is the Nash equilibrium E_3 of the dynamical system (27) always undergoes flip or period-doubling bifurcation when, starting from a stability situation when $d=0$, either the degree of substitutability or the degree of complementarity between goods of variety 1 and 2 increases.

What is interesting now is to compare the role of the relative degree of product differentiation when firms face linear or quadratic costs. In what follow, therefore, we contrast the flip and Neimark-Sacker bifurcation loci (Figures 8 and 9), while also showing the bifurcation diagrams (Figures 10 and 11), in the case of both linear and quadratic costs.

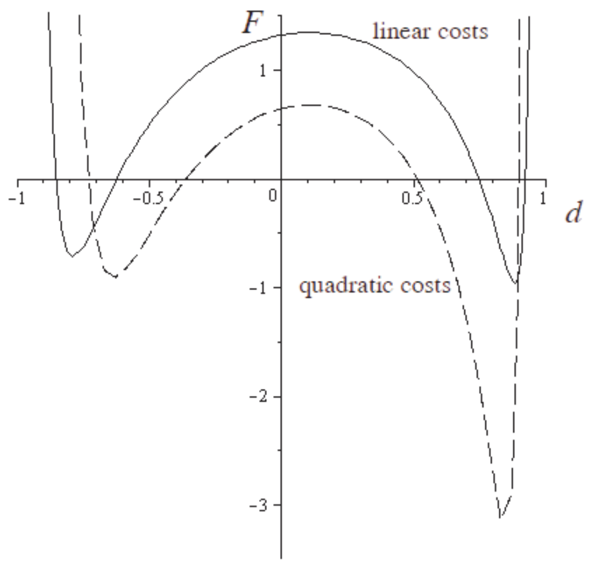

Figure 8. Flip bifurcation loci F when d varies in the case of linear (solid line) and quadratic (dashed line) costs ($\alpha=0.5$, $a=1.2$ and $w=0.5$).

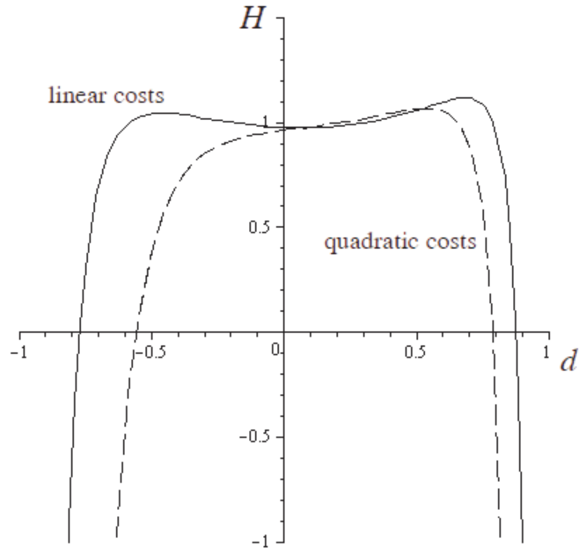

Figure 9. Neimark-Sacker bifurcation loci H when d varies in the case of linear (solid line) and quadratic (dashed line) costs ($\alpha = 0.5$, $a = 1.2$ and $w = 0.5$).

Therefore, the following results hold.

Result 4. *For any meaningful configuration of parameters such that the fixed points E_3 described by Eqs. (14.2) and (29.2) of the two-dimensional systems (12) and (27) are stable when $d = 0$, an increase in the degree of substitutability between goods of variety 1 and 2, i.e. the parameter d moves from 0 to 1, implies that the Bertrand-Nash equilibrium E_3 is more likely to undergo a flip or period-doubling bifurcation under quadratic costs than under linear costs.*

Result 5. *For any meaningful configuration of parameters such that the fixed points E_3 described by Eqs. (14.2) and (29.2) of the two-dimensional systems (12) and (27) are stable when $d = 0$, an increase in the degree of complementarity between goods of variety 1 and 2, i.e. the parameter d moves from 0 to -1, implies that the Bertrand-Nash equilibrium E_3 is more likely to undergo a flip or period-doubling bifurcation under quadratic costs than under linear costs.*

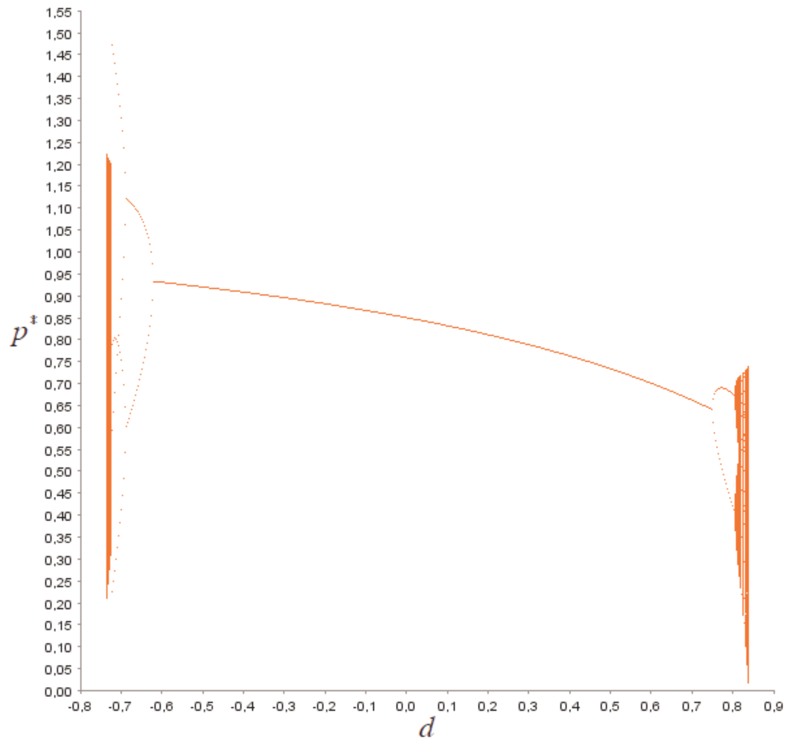

Figure 10. Bifurcation diagram for d in the case of linear costs. Initial conditions: $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$ (Parameter values: $\alpha = 0.5$, $a = 1.2$ and $w = 0.5$).

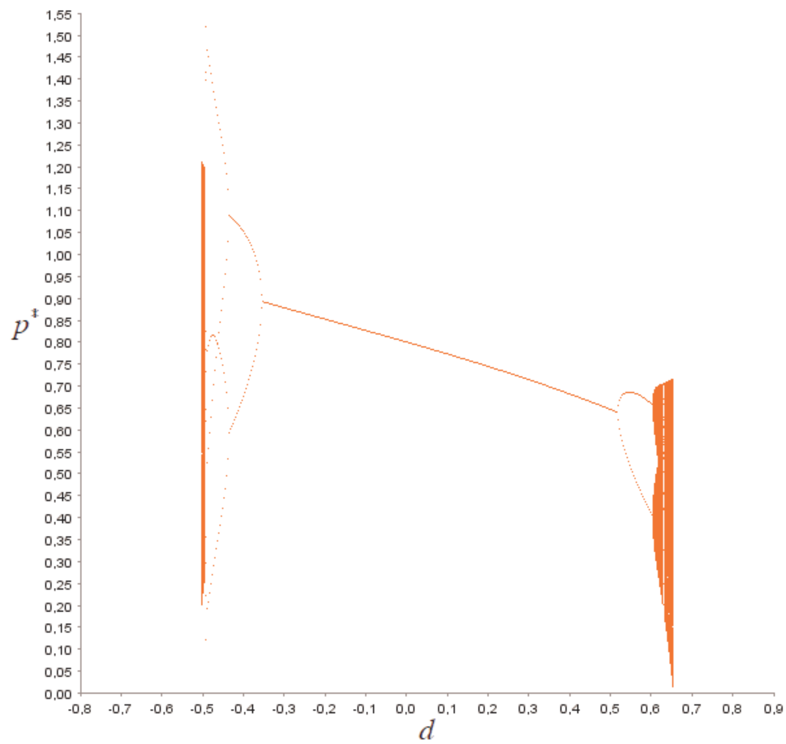

Figure 11. Bifurcation diagram for d in the case of quadratic costs. Initial conditions: $p_{1,0} = 0.2$ and $p_{2,0} = 0.3$ (Parameter values: $\alpha = 0.5$, $a = 1.2$ and $w = 0.5$).

3. Conclusions

We revisited the dynamics of a Bertrand duopoly and bounded rational firms analysed by Zhang et al. (2009), by introducing the mainstream microeconomic foundations of the demand of differentiated products faced by each firm in the market.

Some new results as regards dynamical behaviours – which are of importance in both mathematics and economics – are established. By using the degree of product differentiation as the key factor, we show that: (*i*) from a mathematical point of view, the unique interior fixed point: (1) may loss stability only through a flip bifurcation, (2) the route of chaos is of the "quasi-periodic" type, (3) a wide variety of possible attractors exists (most of which are "aesthetically" interesting); (*ii*) from an economic point of view, we have established the relationship between market stability and the degree of product market differentiation, i.e. the variety of goods and services, showing that if firms work to make products either homogeneous or complements between them, the market equilibrium tends to be destabilised.[12] Moreover, extending the model to consider non-linear costs, we also show that, while both the mathematical and economic results obtained for the linear cost case qualitatively hold, the market instability is more likely to occur in the latter case.

References

Agiza, H.N., Elsadany, A.A., 2003. Nonlinear dynamics in the Cournot duopoly game with heterogeneous players. Physica A 320, 512–524.

Agiza, H.N., Elsadany, A.A., 2004. Chaotic dynamics in nonlinear duopoly game with heterogeneous players. Applied Mathematics and Computation 149, 843–860.

Andronov, A.A., Leontovich, E.A., Gordon, I.E., Maier, A.G., 1971. The Theory of Bifurcations of Dynamical Systems on a Plane. New York (NY): Wiley.

Bertrand, J., 1883. Théorie mathématique de la richesse sociale. Journal des Savants 48, 499–508.

Bergé, P., Pomeau, Y., Vidal, C., 1986. Order Within Chaos. Towards a Deterministic Approach to Turbulence. New York (NY): Wiley-Interscience.

Bischi, G.I., Gardini, L., 2000. Global properties of symmetric competition models with riddling and blowout phenomena. Discrete Dynamics in Nature and Society 5, 149–160.

Bischi, G.I., Naimzada, A., 1999. Global analysis of a dynamic duopoly game with bounded rationality. Advanced in Dynamics Games and Application, vol. 5. Birkhauser, Basel.

Bischi, G.I., Gallegati, M., Naimzada, A., 1999. Symmetry-breaking bifurcations and representative firm in dynamic duopoly games. Annals of Operations Research 89, 253–272.

Bischi, G.I., Stefanini, L., Gardini, L., 1998. Synchronization, intermittency and critical curves in a duopoly game. Mathematics and Computers in Simulation 44, 559–585.

Chamberlin, E., 1933. The Theory of Monopolistic Competition. Cambridge (MA): Harvard University Press.

Correa-López, M., Naylor, R.A., 2004. The Cournot-Bertrand profit differential: a reversal result in a differentiated duopoly with wage bargaining. European Economic Review 48, 681–696.

[12] By passing we note that neither both "mathematical" results nor the "economic" interpretation are not appeared in Zhang et al. (2009).

Curry, J., Yorke, J., 1977. A transition from Hopf bifurcation to chaos: computer experiments with maps in R2. In: The Structure of Attractors in Dynamical Systems, Springer Notes in Mathematics, vol. 668, 48–56.

Dixit, A.K., 1979. A model of duopoly suggesting a theory of entry barriers. Bell Journal of Economics 10, 20–32.

Dixit, A.K., 1986. Comparative statics for oligopoly. International Economic Review 27, 107–122.

Elhadj Z., Sprott, J.C., 2008. A minimal 2-D quadratic map with quasi-periodic route to chaos. International Journal of Bifurcations & Chaos, 1567–1577.

Elhadj Z., Sprott, J.C., 2010. Quadratic maps of the plane: tutorial and review. In: 2-D Quadratic Maps and 3-D ODE Systems: A Rigorous Approach. World Scientific Series on Nonlinear Science Series A.

Fanti, L., Meccheri, N., 2011. The Cournot-Bertrand profit differential in a differentiated duopoly with unions and labour decreasing returns. Economics Bulletin 31, 233–244.

Feigenbaum, M.J., Kadanoff, L.P., Shenker, S.J., 1982. Quasiperiodicity in dissipative systems: a renormalization group analysis. Physica D, 370–386.

Gandolfo, G., 2010. Economic Dynamics. Forth Edition. Heidelberg: Springer.

Gosh, A., Mitra, M., 2010. Comparing Bertrand and Cournot in mixed markets. Economics Letters 109, 72–74.

Häckner, J., 2000. A note on price and quantity competition in differentiated oligopolies. Journal of Economic Theory 93, 233–239.

Hotelling, H., 1929. Stability in competition. Economic Journal 39, 41–57.

Qiu, L.D., 1997. On the dynamic efficiency of Bertrand and Cournot equilibria. Journal of Economic Theory 75, 213–229.

Medio, A., 1992. Chaotic Dynamics. Theory and Applications to Economics. Cambridge (UK): Cambridge University Press.

Moon, H.T., 1997. Two-frequency motion to chaos with fractal dimension $d > 3$. Physical Review Letters 79, 403–406.

Ruelle, D., Takens, F., 1971. On the nature of turbulence. Communications in Mathematical Physics 20, 167–192.

Shilnikov, L., Shilnikov, A., Turaev, D., Chua, L., 2001. Methods of Qualitative Theory in Nonlinear Dynamics. Parts II. Singapore: World Scientific.

Singh, N., Vives, X., 1984. Price and quantity competition in a differentiated duopoly. RAND Journal of Economics 15, 546–554.

Vives, X., 1985. On the efficiency of Bertrand and Cournot equilibria with product differentiation. Journal of Economic Theory 36, 166–175.

Tramontana, F., 2009. Heterogeneous duopoly with isoelastic demand function. Economic Modelling 27, 350–357.

Zhang, J., Da, Q., Wang, Y., 2007. Analysis of nonlinear duopoly game with heterogeneous players. Economic Modelling 24, 138–148.

Zhang, J., Da, Q., Wang, Y., 2009. The dynamics of Bertrand model with bounded rationality. Chaos, Solitons and Fractals 39, 2048–2055.