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Abstract: The class of defensive online portfolio selection algorithms, designed for finite investment horizon, is introduced. The Game Constantly Rebalanced Portfolio and the Worst Case Game Constantly Rebalanced Portfolio, are presented and theoretically analyzed. The analysis exploits the rich set of mathematical tools available by means of the connection between Universal Portfolios and the Game-Theoretic framework. The empirical performance of the Worst Case Game Constantly Rebalanced Portfolio algorithm is analyzed through numerical experiments concerning the FTSE 100, Nikkei 225, Nasdaq 100 and S&P500 stock markets for the time interval, from January 2007 to December 2009, which includes the credit crunch crisis from September 2008 to March 2009. The results emphasize the relevance of the proposed online investment algorithm which significantly outperformed the market index and the minimum variance Sharpe-Markowitz's portfolio.

Keywords: Constant rebalanced portfolio; Online investment; Portfolio selection; Defensive forecasting.

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Alfonso Ventura is short biography for the second author, short biography for the second author, short biography for the second author, short biography for the second author.

1 Introduction

Online portfolio selection has been introduced in the seminal paper of Cover (1991a). The main characteristic of Cover's approach is that no distributional assumptions on the sequence of assets prices are required. Indeed, within Cover's investment framework, the portfolio selection is based completely on the sequence of past prices which is taken *as is* with few, if any, statistical processing. No assumptions are made not only on the family of probability distributions which describes assets prices, but even on the existence of such distributions. To stress this independence of statistical assumptions, the portfolio has been called *Universal Portfolio* (UP). Cover has shown that *Universal Portfolios* (UPs) possess important theoretical properties concerning their asymptotic behavior and exhibit reasonable finite time behavior (Algoet and Cover, 1988; Bell and Cover, 1980; Cover, 1991b). In the last years, UPs have received increasing attention, and several contributions have been made available in the specialized literature (Singer, 1997; Browne, 1998; Borodin et al., 2000; Helmbold et al., 1996; Auer and Warmuth, 1998; Herbster and Warmuth, 1995; Gaivoronski and Stella, 2000). UPs have also been investigated in the case when *side-information*, i.e. additional information about the stock market, is available before each trading period takes place (Cover and Ordentlich, 1996; Herbster and Warmuth, 1995; Fagiuoli et al., 2007). Contributions concerned with UPs that allow investors to take both long and short positions were first made available by Vovk and Watkins (1998) and subsequently by Cross and Barron (2003). Furthermore, UPs have been also studied when transaction costs are considered (Blum and Kalai, 1998; Evstigneev and Schenk-Hoppè, 2002; Gaivoronski and Stella, 2003). Shafer and Vovk (2001) introduced a new and original mathematical framework for computational finance, called *Game-Theoretic Framework*, that rests more on game theory than on measure theory. The main advantage of the Game-Theoretic Framework is that it captures the basic intuitions of probability in a simple and effective manner and clarifies the close relationship between probability theory and finance theory while offering a natural adaptability to many practical problems.

In this paper the *Online Portfolio Selection Game* (OPSG) protocol, is introduced. It allows to interpret the online portfolio selection problem as a *Bounded Forecasting Game II* (Ventura, 2006) and to connect the class of UPs with the Game-Theoretic Framework. This connection is exploited to define the class of *Defensive Online Portfolio Selection* (DOPS) algorithms, which deals with finite investment horizons. The *Game Constantly Rebalanced Portfolio* and *Worst Case Game Constantly Rebalanced Portfolio* algorithms, belonging to the DOPS class, are presented and theoretically analyzed. Furthermore, the empirical performance of the *Worst Case Game Constantly Rebalanced Portfolio* algorithm is investigated through a set of numerical experiments concerning the following stock market data sets; FTSE 100, Nikkei 225, Nasdaq 100 and S&P500.

The rest of the paper is organized as follows. Section 2 introduces the notations and main definitions. The features of the GTF are described through Section 3. Section 4 introduces and analyzes the OPSG protocol and the DOPS class of investment strategies. Finally, Section 5 presents the results of a set of numerical experiments related to the FTSE 100, Nikkei 225, Nasdaq 100 and S&P500 stock markets.

2 Portfolio Selection and Universal Portfolios

Following the paper from Cover (1991a), a *stock market vector* is represented as a vector $\mathbf{z} = (z_1, ..., z_m)$ such that $z_i \geq 0, \ \forall \ i = 1, ..., m$, where m is the number of stocks and z_i is the *price relative*, i.e. represents the ratio of the price at the end of the trading period to the price at the beginning of the trading period. A *portfolio* is described by a vector $\mathbf{x} = (x_1, ..., x_m)$ such that $x_i \geq 0, \ \forall \ i = 1, ..., m, \ \sum_{i=1}^{m} x_i = 1$ and is an allocation of *wealth* across the stocks in the sense that x_i represents the fraction of the wealth invested in the i^{th} stock. By assuming that $\mathbf{x}$ and $\mathbf{z}$ represent respectively the portfolio and the stock market vector for one investment period, the *wealth relative* (i.e. the ratio of the wealth at the end of the trading period to the wealth at the beginning of the trading period), given by $S = \mathbf{x}^T \mathbf{z}$, represents the factor by which the wealth increases/decreases in one investment period by using portfolio $\mathbf{x}$.

The problem of portfolio selection consists of selecting a portfolio $\mathbf{x}$ which would maximize S in some sense. The financial theory has developed various notions of optimality for the portfolio selection problem. One possibility is to maximize the expected value of S subject to constraint on the variance as proposed by the Sharpe-Markowitz theory of investment (Markowitz, 1952; Merton, 1990) that deals with long term behavior of fixed portfolios. However, the *mean-variance investment framework* does not take into proper consideration the possibility of frequent portfolio rebalances, which is one of the most important features that characterize a stock market.

To overcome this limitation, another possibility has been proposed by Cover (1991b), which exploits the concept of *Constant Rebalanced Portfolio* (CRP), i.e. a portfolio which at each trading period keeps the fraction of wealth invested in every stock constant. By considering an arbitrary non random sequence of n stock market vectors $\mathbf{z}^n = \mathbf{z}_1, ..., \mathbf{z}_n$, a CRP $\mathbf{x}$ achieves the wealth $S_n(\mathbf{x}) = \prod_{t=1}^{n} \mathbf{x}^T \mathbf{z}_t$ where it is assumed that the initial wealth is normalized to one ($S_0(\mathbf{x}) = 1$). Within the class of *Constant Rebalanced Portfolios* (CRPs) the best one is called *Best Constant Rebalanced Portfolio* (BCRP). The BCRP is determined in hindsight, i.e. it is the CRP computed by assuming perfect knowledge of future stock prices, and possesses interesting properties. Indeed, Cover (1991a) showed that the achieved wealth by means of the BCRP is non inferior to the one achieved by the best stock, the one associated with the value line and the one associated with the arithmetic mean. These properties have motivated increasing interest to study and to analyze the main features of this investment strategy and to use this portfolio as the reference benchmark to evaluate and to compare online portfolio selection algorithms. Let us now formally introduce the BCRP by considering the vector $\overset{*}{\mathbf{x}}$ that solves the following optimization problem

$$\max_{x \in X} S_n(\mathbf{x}) \tag{1}$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0, \ \forall \ i = 1, ..., m.$$

The vector $\overset{*}{\mathbf{x}}$ maximizes the wealth $S_n(\mathbf{x})$ across the stock market vector sequence $\mathbf{z}^n = \mathbf{z}_1, ..., \mathbf{z}_n$, and therefore it is defined as the BCRP for the stock market vector sequence $\mathbf{z}^n$. However, the portfolio $\overset{*}{\mathbf{x}}$ cannot be used for actual stock selection at trading periods $t = 1, ..., n$ because it explicitly depends on the sequence $\mathbf{z}^n$ which becomes known only after the expiration of this time interval. A reasonable objective might be, therefore, to construct a sequence of portfolios $\mathbf{x}^t = \mathbf{x}_1, ..., \mathbf{x}_t$ which depends on the sequence $\mathbf{z}^{t-1} = \mathbf{z}_1, ..., \mathbf{z}_{t-1}$ and uses portfolio $\mathbf{x}_t$ for stock selection at trading period t. Let us denote by $S_n(\mathbf{x}^n)$ the wealth generated after n trading periods by successive applications of the sequence of portfolios $\mathbf{x}^n$, then $S_n(\mathbf{x}^n) = \prod_{t=1}^{n} \mathbf{x}_t^T \mathbf{z}_t$ while $\log S_n(\mathbf{x}^n)$ is called *logarithmic wealth*. It would be desirable if such sequential investment strategy $\mathbf{x}^n$ would yield a wealth in some sense *close* to the achieved wealth by means of the BCRP $\overset{*}{\mathbf{x}}$. One such strategy was proposed in (Cover, 1991a), under the name of *Universal Portfolio* (UP) and consists of selecting the investment portfolio as follows

$$\widehat{\mathbf{x}}_1 = \left(\frac{1}{m}, ..., \frac{1}{m}\right), \qquad \widehat{\mathbf{x}}_{t+1} = \frac{\int_X \mathbf{x} S_t(\mathbf{x}) d\mathbf{x}}{\int_X S_t(\mathbf{x}) d\mathbf{x}}. \tag{2}$$

The UP (2) has been shown to possess a very interesting property (Cover, 1991b): it has the same exponent to the first order as the BCRP. Formally, by letting $S_n(\widehat{\mathbf{x}}^n) = \prod_{t=1}^{n} \widehat{\mathbf{x}}_t^T \mathbf{z}_t$ be the achieved wealth by means of the UP, then it has been shown that $\lim_{n\to\infty} \left(\frac{1}{n}\log S_n(\widehat{\mathbf{x}}^n) - \frac{1}{n}\log S_n(\overset{*}{\mathbf{x}})\right) = 0$ with the following inequality holding $S_n(\widehat{\mathbf{x}}^n) \geq S_n(\overset{*}{\mathbf{x}})\ C_n\, n^{-(m-1)/2}$ where C_n tends to some limit along subsequences for which $W_n(\overset{*}{\mathbf{x}}) = \frac{1}{n}\log S_n(\overset{*}{\mathbf{x}}) \to W(\overset{*}{\mathbf{x}})$ for some strictly concave function $W(\mathbf{x})$. (See Theorem 6.1 from Cover (1991b)).

Another example of investment strategy, which exploits the definition of BCRP, has been proposed in (Gaivoronski and Stella, 2000) under the name of *Successive Constant Rebalanced Portfolio* (SCRP). The SCRP selects the investment portfolio as follows:

$$\widetilde{\mathbf{x}}_1 = \left(\frac{1}{m}, ..., \frac{1}{m}\right), \qquad \widetilde{\mathbf{x}}_{t+1} = \arg\max_{\mathbf{x}\in\mathbf{X}} S_t(\mathbf{x}) \tag{3}$$

where $\mathbf{X} = \{\mathbf{x} :\ x_i \geq 0,\ \forall\, i = 1, ..., m,\ \sum_{i=1}^{m} x_i = 1\}$. The SCRP $\widetilde{\mathbf{x}}^n$ possesses interesting properties. Its asymptotic wealth $S_n(\widetilde{\mathbf{x}}^n)$ coincides with the wealth obtained by the BCRP to the first order in the exponent, i.e. $\lim_{n\to\infty} \left(\frac{1}{n}\log S_n(\widetilde{\mathbf{x}}^n) - \frac{1}{n}\log S_n(\overset{*}{\mathbf{x}})\right) = 0$ with the following inequality holding $S_n(\widetilde{\mathbf{x}}^n) \geq S_n(\overset{*}{\mathbf{x}})\, C\,(n-1)^{-\frac{2K^2}{\delta}}$ where $K = \sup_{t,\mathbf{x}\in\mathbf{X}} \left\| \nabla_x \left[\log\left(\mathbf{x}^T \mathbf{z}^{(t)}\right)\right] \right\|$, while C and δ are constants.

Fagiuoli et al. (2007) extended both the BCRP and SCRP to the case when *side-information* about the stock market is available. In such a framework the *Mixture Best Constant Rebalanced Portfolio* (MBCRP) and the *Mixture Successive Constant Rebalanced Portfolio* (MSCRP) have been defined and their theoretical properties studied. MBCRP and MSCRP are shown to possess interesting theoretical properties in the case when the stock market is *non-stationary*.

3 Game-Theoretic Framework

The *Game-Theoretic Framework* (GTF) (Shafer and Vovk, 2001) involves a two player (*Skeptic* and *World*) sequential game. It may have many, perhaps infinitely many, rounds of play. At each round *Skeptic* bets on what will happen and then *World* decides what will happen. *Skeptic* and *World* both have perfect information about the other's moves as soon as they are made. Following Shafer and Vovk (2001), *World* can be divided into the following virtual players:

Experimenter, who decides what each round of play will be about.
Forecaster, who sets the prices.
Reality, who decides the outcomes.

Let us now introduce the *Bounded Forecasting Game II* (BFG II) protocol, a slight modification of the *Bounded Forecasting Game* protocol (Shafer and Vovk, 2001), which will be useful in the next section.

Bounded Forecasting Game II
Parameter: $C > 0$
Players: *Forecaster, Skeptic, Reality*
Protocol
$\mathcal{K}_0 = 1$
For $t = 1, 2, ..., n$:
  Reality announces $\mathbf{y}_t \in \mathbf{Y}$.
  Skeptic announces $s_t(f_t) : [-C, C] \to \mathbb{R}$.
  Forecaster announces $f_t \in [-C, C]$.
  Reality announces $r_t \in [-C, C]$.

  $\mathcal{K}_t := \mathcal{K}_{t-1} + s_t(f_t)(r_t - f_t)$.

Winner: *Skeptic* wins if (1) $\mathcal{K}_t$ is never negative and (2) either

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} (r_i - f_i) = 0$$

or

$$\lim_{t \to \infty} \mathcal{K}_t = \infty$$

holds. Otherwise *Reality* wins.

Skeptic has to announce his strategy at each round before *Forecaster*'s move on that round and furthermore, *Skeptic*'s strategy is a function of *Forecaster*'s strategy. The protocol is a game between two players, namely *Forecaster* and *Reality*. At each round *Forecaster* predicts *Reality*'s move r_t chosen from the label space $[-C, C]$. *Forecaster*'s goal is to produce f_t that agrees with the observed r_t; it is formalized by adding a player, *Skeptic*, who is allowed to gamble at odds given by *Forecaster*'s probabilities. *Skeptic*'s gambling strategies can be used as tests of agreement between f_t and r_t, while all tests of agreement between f_t and r_t can be expressed as *Skeptic*'s gambling strategies. To help *Forecaster*, *Reality* presents him with an object $\mathbf{y}_t$ at the beginning of the round; $\mathbf{y}_t$ is chosen from an object space $\mathbf{Y}$. The protocol is *coherent* and *symmetric* (Shafer and Vovk, 2001). A protocol is said *coherent* if the gambles *Skeptic* is offered do not guarantee him an opportunity to make money. The protocol complies with this property because *Reality* can supply prices r_t so that each term $(r_t - f_t)$ of (4) equals zero. Furthermore, the protocol is *symmetric* because the function $s_t(f_t)$ takes values on $[-C, C]$. According to Murphy and Epstein (1967) and Vovk et al. (2005b), forecasts f_t, such that the following condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} (r_t - f_t) = 0 \tag{4}$$

holds, are said to share the *unbiasedness in the large property*. However, condition (4) is quite easy to achieve while not ensuring that forecasts f_t are useful. Therefore, in (Vovk et al., 2005b) a subtler requirement that forecasts f_t should satisfy has been given under the name of *unbiasedness in the small property*. Forecasts f_t are *unbiased in the small* if, for any $f' \in [-C, C]$, the condition

$$\frac{\sum_{t=1,\dots,n: f_t \cong f'} (r_t - f_t)}{\sum_{t=1,\dots,n: f_t \cong f'} 1} \cong 0 \tag{5}$$

holds, provided $\sum_{t=1,\dots,n: f_t \cong f'} 1$ is not too small, where $\cong$ stands for approximate equality.

4 Defensive Online Portfolio Selection

Defensive Online Portfolio Selection (DOPS) is a specialization of the BFG II protocol in the *Online Portfolio Selection Game*(OPSG) protocol (Ventura, 2006).

Online Portfolio Selection Game
Parameters: $0 < z^- \leq z^+$
Players: *Reality*, *Skeptic*, *Forecaster*
Protocol:
$\mathcal{K}_0 = 1$
For $t = 1, 2, .., n$:
  Reality announces $\mathbf{y}_t \in \mathbf{Y}$.
  Skeptic announces $s_t(\mathbf{x}) : \mathbf{X} \to \mathbb{R}$.
  Forecaster announces $\mathbf{x}_t = (x_{t1}, ..., x_{tm}) \in \mathbf{X}$.
  Reality announces $\mathbf{z}_t : z^- \leq z_{tj} \leq z^+$.

  $\mathcal{K}_t := \mathcal{K}_{t-1} + s_t(\mathbf{x}_t)(\log\left(\bar{\mathbf{x}}_t^T \mathbf{z}_t\right) - \log\left(\mathbf{x}_t^T \mathbf{z}_t\right))$.

Winner: *Skeptic* wins if (1) $\mathcal{K}_t$ is never negative and (2) either

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left(\log\left(\bar{\mathbf{x}}_t^T \mathbf{z}_t\right) - \log\left(\mathbf{x}_t^T \mathbf{z}_t\right)\right) = 0 \tag{6}$$

or

$$\mathcal{K}_n = \Omega(e^{nk}) \qquad \text{for a } k > 0 \tag{7}$$

holds. Otherwise *Reality* wins.

The OPSG is a perfect-information protocol in the sense that each player can see the other players' moves and therefore can freely choose his own move based on what he sees. It has the following interpretation: at each trading period t, *Forecaster* invests in the stock market by using portfolio $\mathbf{x}_t$ with the goal to track *Benchmark* that invests in the stock market by using portfolio $\bar{\mathbf{x}}_t$. It is worthwhile to notice that the OPSG protocol, differently from the BFG II protocol, includes a virtual player called *Benchmark* which is used here only to decide the winner of the game. *Benchmark* selects the portfolio $\bar{\mathbf{x}}_t$ by knowing in advance, i.e. at each trading period $(t-1)$, either the next stock market vector $\mathbf{z}_t$ or the sequence $\mathbf{z}^n$ of stock market vectors for the entire investment horizon. *Skeptic* tries to become rich by repeatedly betting an amount $s_t(\mathbf{x}_t)$ on the disagreement between *Forecaster* and *Benchmark*. He wins the game if his capital $\mathcal{K}_t$ is never negative and if condition (6) holds or he achieves a capital which is exponential in n (7), otherwise *Reality* wins. The vector $\mathbf{y}_t$ introduced in (Vovk et al., 2005b) is associated with additional information available to *Forecaster* at the beginning of each trading period t. In the OPSG protocol this kind of information is related to the stock market and can be conveniently summarized through the concept of *side-information.* The OPSG protocol is *coherent* because *Reality* can supply a sequence of stock market vectors $\mathbf{z}^n$ such that $\mathbf{x}_t$ is the best portfolio at each trading period, i.e.

$$\mathbf{x}_t = \arg\max \log\left(\mathbf{x}^T \mathbf{z}_t\right), \qquad \forall t = 1, ..., n.$$

Moreover the OPSG complies with the *fundamental interpretative hypothesis* (Shafer and Vovk, 2001), which states that no strategy for *Skeptic* can both (1) be sure of avoiding bankruptcy and (2) have a reasonable chance of multiplying the initial capital by a large factor. It is worthwhile to notice that requirement (2) has been translated into the formal condition (7). Furthermore, the OPSG protocol, as well as the BFG II protocol, is *symmetric.* In fact, at each trading period t the achieved wealth, by means of portfolio $\mathbf{x}_t$, is bounded as follows

$$-C \leq \log z^- \leq \log\left(\mathbf{x}_t^T \mathbf{z}_t\right) \leq \log z^+ \leq C.$$

It is worthwhile to notice that condition (6), for the OPSG protocol, is equivalent to condition (4), for the BFG II protocol, and is concerned with the unbiasedness in the large property of *Forecaster*'s investment portfolio $\mathbf{x}_t$.

Let us now take into account, for the OPSG protocol, two possible choices for *Benchmark*'s portfolio $\bar{\mathbf{x}}_t$. The first choice consists of selecting, at each trading period t, the BCRP $\overset{*}{\mathbf{x}}$ as *Benchmark*'s portfolio $\bar{\mathbf{x}}_t$, i.e. it consists of setting $\bar{\mathbf{x}}_t = \overset{*}{\mathbf{x}}_n$, $\forall t = 1, ..., n$. The second choice consists of selecting, at each trading period t, the *Sequential Best Constant Rebalanced Portfolio* (SBCRP) $\overset{*}{\mathbf{x}}^n$ as *Benchmark*'s portfolio $\bar{\mathbf{x}}_t$, i.e. it consists of setting

$$\bar{\mathbf{x}}_t = \overset{*}{\mathbf{x}}_t, \qquad \forall t = 1, ..., n. \tag{8}$$

The choice (8) is particularly interesting: the following corollary shows that the SBCRP achieves a wealth that is never less than the achieved wealth by means of the BCRP.

Corollary 4.1: *(Ventura, 2006) Given any stock market vector sequence* $\mathbf{z}^n$*, the achieved logarithmic wealth by means of the SBCRP* $\overset{*}{\mathbf{x}}^n$ *(8) is never less than the achieved logarithmic wealth by means of the BCRP* $\overset{*}{\mathbf{x}}_n$*, formally:*

$$\sum_{t=1}^{n} \log(\overset{*}{\mathbf{x}}_n^T \mathbf{z}_t) \leq \sum_{t=1}^{n} \log(\overset{*}{\mathbf{x}}_t^T \mathbf{z}_t).$$

4.1 Defeating the Skeptic

The next theorem is concerned with the choice of the SBCRP, i.e. $\bar{\mathbf{x}}_t = \overset{*}{\mathbf{x}}_t,\ \forall t = 1, ..., n$, as *Benchmark* and with the case when the side-information $\mathbf{y}_t$ is not available. The theorem shows that *Forecaster* can prevent *Skeptic*'s capital from growing exponentially, which is equivalent to ensure that condition (7) does not hold.

Theorem 4.1: *(Ventura, 2006) If, for the OPSG protocol,* $\bar{\mathbf{x}}_t = \overset{*}{\mathbf{x}}_t$, $\forall t = 1, ..., n$, *then Forecaster has a strategy ensuring* $\mathcal{K}_n = O(n^k)$*.*

Theorem 4.1 for the OPSG protocol states that if the SBCRP is selected as *Benchmark* (8), then any strategy for *Skeptic* admits a strategy for *Forecaster* that does not allow *Skeptic*'s capital to grow indefinitely in the sense of (7), regardless of what *Reality* is doing. The strategy used by *Forecaster* is the SCRP investment scheme (3). The class of *Forecaster*'s investment strategies that prevent *Skeptic*'s capital from growing exponentially is the class of DOPS investment strategies.

4.2 Unbiasedness in the Large and Universality

Let us now introduce a theorem which in the case when the *Forecaster*'s strategy is DOPS guarantees that the *Skeptic* has a strategy which ensures (6). Following the procedure described in (Shafer and Vovk, 2001) (p. 69), in the case when the OPSG protocol is considered, if we take into account the following strategy for *Skeptic*

$$s_t(\mathbf{x}) := s_t^{\epsilon}(\mathbf{x}) = \epsilon \mathcal{K}_{t-1} \tag{9}$$

and let ϵ to be a small number, it is possible to formulate the following theorem.

Theorem 4.2: *(Ventura, 2006) If, for the OPSG protocol, Forecaster has a strategy ensuring* $\mathcal{K}_n \leq f(n)$*, where* $f(n)$ *is sub-exponential with respect to* n*, i.e.*

$$\lim_{n \to \infty} \frac{1}{n} \log f(n) = 0,$$

then Skeptic can force (6) and Forecaster's strategy is said to be unbiased in the large.

Theorem 4.1 states that the SCRP online portfolio selection algorithm (3) does not allow *Skeptic*'s wealth to grow exponentially if the SBCRP is chosen as *Benchmark* (8).

However, from the proof of Theorem 4.1 (Ventura, 2006), emerges that the same property holds for any investment strategy that achieves a wealth close to the *Benchmark's* wealth. Portfolio selection algorithms sharing property (6) are called UPs. Therefore, UPs and the GTF are connected by the OPSG protocol. The following theorem states that an investment strategy which shares the unbiasedness in the large property also shares the universality property.

Theorem 4.3: *(Ventura, 2006) If Forecaster has a strategy that guarantees Skeptic's wealth is sub-exponential in the number of trading periods, then Skeptic has a strategy such that the unbiasedness in the large property holds. If Forecaster has a strategy for the OPSG protocol, that guarantees that Skeptic's wealth is sub-exponential in the number of trading periods, then such a strategy is a UP with respect to the selected Benchmark.*

Theorem 4.3 allows to connect the class of UP investment strategies with the GTF. This connection is particularly important: it offers the possibility to exploit the rich set of mathematical tools belonging to the GTF for developing and analyzing the class of DOPS investment algorithms.

4.3 Unbiasedness in the Small

Let us now consider a subtler requirement for the OPSG protocol, equivalent to condition (5) for the BFG II protocol that forecasts should satisfy, which could be useful to improve the investment performance when considering a finite investment horizon. Forecasts $\mathbf{x}_t$ are said to be unbiased in the small (or well calibrated) (Murphy and Epstein, 1967; Vovk et al., 2005b) if, for any $\acute{\mathbf{x}} \in \mathbf{X}$,

$$\frac{\sum_{t=1,\ldots,n:\mathbf{x}_t \cong \acute{\mathbf{x}}} \left(\log\left(\bar{\mathbf{x}}_t^T \mathbf{z}_t\right) - \log\left(\mathbf{x}_t^T \mathbf{z}_t\right)\right)}{\sum_{t=1,\ldots,n:\mathbf{x}_t \cong \acute{\mathbf{x}}} 1} \cong 0 \tag{10}$$

provided $\sum_{t=1,\ldots,n:\mathbf{x}_t \cong \acute{\mathbf{x}}} 1$ is not too small, where $\cong$ stands for approximate equality.

Let us now consider a given value for $\acute{\mathbf{x}}$ and following what suggested by Vovk et al. (2005b); instead of the *crisp* point $\acute{\mathbf{x}}$, consider a fuzzy point $I : \mathbf{X} \to [0, 1]$ such that $I(\acute{\mathbf{x}}) = 1$ and $I(\mathbf{x}) = 0$ for all $\mathbf{x}$ lying outside a small neighborhood of $\acute{\mathbf{x}}$. A standard choice, proposed by Vovk et al. (2005a), would be something like $I := \mathbb{I}_E$, where $E \in \mathbf{X}$ is a small neighborhood of $\acute{\mathbf{x}}$ while $\mathbb{I}_E$ is its indicator function where it is required for I to be continuous. Then, the investment strategy for *Skeptic* (9), which ensures that condition (6) holds, can be properly modified as

$$s_t(\mathbf{x}) := s_t^{\epsilon}(\mathbf{x}) = \epsilon I(\mathbf{x}) \mathcal{K}_{t-1}$$

to ensure that the unbiasedness in the small property holds, as stated by the following theorem.

Theorem 4.4: *(Ventura, 2006) If, for the OPSG protocol, $\mathcal{K}_n \leq f(n)$ with*

$$\lim_{n\to\infty} \frac{1}{n} \log f(n) = 0,$$

then Skeptic can force condition (10) and Forecaster's strategy is said to be unbiased in the small.

4.4 Game Constantly Rebalanced Portfolio

In binary forecasting it has been empirically observed that the performance of defensive forecasting with a fixed small ϵ, does not depend on ϵ very much (Vovk et al., 2005b), thus suggesting to let $\epsilon \to 0$. We use the Gaussian bells I_j, located densely and uniformly in the interval $[0,1]$, with standard deviation $\sigma > 0$ as test functions, j indexes the mixture's components. Following the procedure described by Vovk et al. (2005b) it is possible to write $s_t(\mathbf{x})$ as follows:

$$s_t(\mathbf{x}) = \sum_{i=1}^{t-1} \boldsymbol{K}\left((\mathbf{x}, \mathbf{y}_t), (\mathbf{x}_i, \mathbf{y}_i)\right) (\log(\overset{*}{\mathbf{x}}{}_t^T \mathbf{z}_t) - \log\left(\mathbf{x}_t^T \mathbf{z}_t\right)) \tag{11}$$

where $\boldsymbol{K}\left((\mathbf{x}, \mathbf{y}_t), (\mathbf{x}_i, \mathbf{y}_i)\right)$ is the Mercer kernel which ensures that (11) possesses the unbiasedness in the small property. It is approximated as follows

$$\boldsymbol{K}\left((\mathbf{x}, \mathbf{y}_t), (\mathbf{x}_i, \mathbf{y}_i)\right) \cong \exp\left(-\frac{\|\mathbf{y}_t - \mathbf{y}_i\|^2 + \|\mathbf{x} - \mathbf{x}_i\|^2}{2\sigma^2}\right). \tag{12}$$

Let us now exploit the rich set of mathematical tools belonging to the GTF to introduce a new algorithm, originating from the OPSG protocol that will be called *Game Constantly Rebalanced Portfolio* (G-CRP) algorithm (Ventura, 2006).

Game Constantly Rebalanced Portfolio

Parameter: forecast-continuous Mercer kernel $\boldsymbol{K}$ on $(\mathbf{X} \times \mathbf{Y})^2$.

For $t = 1, 2, .. :$

Read $\mathbf{y}_t \in \mathbf{Y}$.

Set $s_t(\mathbf{x}) = \sum_{i=1}^{t-1} \boldsymbol{K}\left((\mathbf{x}, \mathbf{y}_t), (\mathbf{x}_i, \mathbf{y}_i)\right) (\log(\overset{*}{\mathbf{x}}{}_i^T \mathbf{z}_i) - \log\left(\mathbf{x}_i^T \mathbf{z}_i\right))$.

Search any root $\overset{\bullet}{\mathbf{x}}$ for $s_t(\mathbf{x}) = 0$.

If a root $\overset{\bullet}{\mathbf{x}}$ exists Then

$\mathbf{x}_t := \overset{\bullet}{\mathbf{x}}$. */uses the root $\overset{\bullet}{\mathbf{x}}$ as the investment portfolio/*

Else

$\mathbf{x}_t := \widetilde{\mathbf{x}}_t$ */uses the SCRP $\widetilde{\mathbf{x}}_t$ as the investment portfolio/*

Read $\mathbf{z}_t$.

End

The G-CRP algorithm approximates instances belonging to the class of online portfolio selection strategies introduced by means of Theorem 4.4. Let us analyze the G-CRP algorithm to show that it possesses the unbiasedness in the large property (6), as well as the unbiasedness in the small property (10), depending on the choice of the Mercer kernel $\boldsymbol{K}$. It is worthwhile to notice that within the G-CRP algorithm the portfolio $\widetilde{\mathbf{x}}_t$ is the SCRP (3) at trading period t while the forecast-continuous Mercer kernel $\boldsymbol{K}$ is defined according to formula (12). The properties shared by the G-CRP algorithm are presented through the following three theorems.

Theorem 4.5: *(Ventura, 2006) The G-CRP algorithm, with parameter $\boldsymbol{K}$, ensures the following bound holds*

$$\frac{1}{n}\left\|\sum_{t=1}^{n}\Phi(\mathbf{x}_t,\mathbf{y}_t)(\log(\overset{*}{\mathbf{x}}{}_i^T\mathbf{z}_i)-\log\left(\mathbf{x}_i^T\mathbf{z}_i\right))\right\|\leq\frac{T}{n}\sqrt{\overset{\bullet}{n}\frac{8K^2}{\delta}C+n\left(2C\right)^2} \tag{13}$$

where Φ is a function such as $\boldsymbol{K}(a,b)=\Phi(a)\Phi(b)$ while $\overset{\bullet}{n}$ is the number of trading periods such that a root $\overset{\bullet}{\mathbf{x}}$ for $s_t(\mathbf{x})=0$ does not exist.

The unbiasedness in the large property is considered by the following theorem.

Theorem 4.6: *(Ventura, 2006) The investment strategy associated with the G-CRP algorithm when setting $\Phi(\mathbf{x}_t,\mathbf{y}_t)=1$, is unbiased in the large, therefore condition (6) holds.*

Let us now focus on the unbiasedness in the small property and let $(\acute{\mathbf{x}},\acute{\mathbf{y}})$ be a point in $\mathbf{X}\times\mathbf{Y}$. We would like portfolio $\mathbf{x}$ and portfolio $\overset{*}{\mathbf{x}}$ to achieve close wealth values, also whenever $(\mathbf{x}_t,\mathbf{y}_t)$ is close to $(\acute{\mathbf{x}},\acute{\mathbf{y}})$. The forecast-continuous Mercer kernel $\boldsymbol{K}:(\mathbf{X}\times\mathbf{Y})^2\rightarrow\mathbb{R}$ is fixed and the *soft neighborhood* is defined as follows:

$$I_{(\acute{\mathbf{x}},\acute{\mathbf{y}})}\left(\mathbf{x},\mathbf{y}\right):=\boldsymbol{K}\left((\acute{\mathbf{x}},\acute{\mathbf{y}}),(\mathbf{x},\mathbf{y})\right)$$

for some point $(\acute{\mathbf{x}},\acute{\mathbf{y}})$. Then, the following theorem can be formulated.

Theorem 4.7: *(Ventura, 2006) The investment strategy associated with the G-CRP algorithm when setting $\sum_{t=1}^{n}I_{(\acute{\mathbf{x}},\acute{\mathbf{y}})}\left(\mathbf{x}_t,\mathbf{y}_t\right)\gg\sqrt{n}$, is unbiased in the small, therefore condition (10) holds.*

The G-CRP algorithm is numerically difficult to compute because it requires to solve a multidimensional root finding problem ($s_t(\mathbf{x})=0$.) which is known to be quite hard (Acton, 1970). It is worthwhile to mention that this problem becomes even more difficult due to the presence of the non-negativity constraints associated with the components of the portfolio vector $\mathbf{x}$ and with the linear constraint required to ensure the solution vector $\mathbf{x}$ to be a portfolio. Therefore, a different investment strategy, less computationally demanding, is described in the following subsection.

4.5 Worst Case Game Constantly Rebalanced Portfolio

The GTF requires the *Skeptic*'s capital, generated accordingly to:

$$\mathcal{K}_t:=\mathcal{K}_{t-1}+s_t(\mathbf{x}_t)(\log(\overset{*}{\mathbf{x}}{}_t^T\mathbf{z}_t)-\log\left(\mathbf{x}_t^T\mathbf{z}_t\right)), \tag{14}$$

to be sub-exponential in n. The SCRP algorithm (3) ignores the $s_t(\mathbf{x}_t)$ term in (14) while focuses its attention to minimize the term $(\log(\overset{*}{\mathbf{x}}{}_t^T\mathbf{z}_t)-\log\left(\mathbf{x}_t^T\mathbf{z}_t\right))$ and therefore to go as close as possible to the logarithmic wealth achieved by means of the SBCRP. The SCRP algorithm exploits the fact that when the number of trading periods n becomes sufficiently large, the difference between the achieved wealth by means of the SCRP and SBCRP algorithms becomes negligible, as emerges by combining conditions (22) and (23) from Gaivoronski and Stella (2000) to obtain

$$\log(\overset{*}{\mathbf{x}}{}_n^T \mathbf{z}_n) - \log(\overset{*}{\mathbf{x}}{}_{n-1}^T \mathbf{z}_n) \le \frac{2K^2}{\delta}\frac{1}{n-1}.$$

The G-CRP algorithm tries to find a root $\overset{\bullet}{\mathbf{x}}$ for $s_t(\mathbf{x}) = 0$ to be used as investment portfolio for trading period t; when such a root does not exist it uses the SCRP (3) as investment portfolio. A natural question to be asked, in the case when a root $\overset{\bullet}{\mathbf{x}}$ does not exist, whether to select the investment portfolio $\mathbf{x}_t$ which maximizes $s_t(\mathbf{x})$ is a good strategy or not when compared with the SCRP. The main problem of this strategy, which when a root $\overset{\bullet}{\mathbf{x}}$ for $s_t(\mathbf{x}) = 0$ exists sets $\mathbf{x}_t = \overset{\bullet}{\mathbf{x}}$ while otherwise uses the portfolio which minimize $s_t(\mathbf{x})$ is the limitation we have to cope with. In fact, no information, about the future behavior of the term $(\log(\overset{*}{\mathbf{x}}{}_t^T \mathbf{z}_t) - \log\left(\mathbf{x}_t^T \mathbf{z}_t\right))$, is available. To overcome this limitation a worst case approach, namely the *Worst Case Game Constantly Rebalanced Portfolio* (WCG-CRP) algorithm, is proposed. This algorithm, at each trading period t, performs the following test

$$s_t(\overset{\circ}{\mathbf{x}})2C \le s_t(\widetilde{\mathbf{x}}_t)\frac{2\bar{K}^2}{\delta}\frac{1}{t-1} \tag{15}$$

where $\overset{\circ}{\mathbf{x}} := \arg\min |s_t(\mathbf{x})|$, $\widetilde{\mathbf{x}}_t$ is the corresponding SCRP (3),

$$\bar{K} = \frac{\max\limits_{1\le j\le m, i<t}(z_{ij})\sqrt{m}}{\min\limits_{1\le j\le m, i<t}(z_{ij})}, \qquad \delta = \lambda_{\min}\left(\frac{1}{n}\sum_{i=1}^{t}\frac{\mathbf{z}_i\mathbf{z}_i^T}{\|\mathbf{z}_i\|^2}\right),$$

and selects the investment portfolio $\mathbf{x}_t$ as follows.

Worst Case Game Constantly Rebalanced Portfolio

Parameter: forecast-continuous Mercer kernel $\boldsymbol{K}$ on $(\mathbf{X}\times\mathbf{Y})^2$

For $t = 1, 2, ...$:
  If $(t == 1)$ Then
    $\mathbf{x}_t := \left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right)$.
  Else
    If $\left(s_t(\overset{\circ}{\mathbf{x}})2C \le s_t(\widetilde{\mathbf{x}}_t)\frac{2\bar{K}^2}{\delta}\frac{1}{t-1}\right)$ Then
      (a) $\mathbf{x}_t := \overset{\circ}{\mathbf{x}}$.   */uses the minimum of* $|s_t(\mathbf{x})|$ *as the investment portfolio/*
    Else
      (b) $\mathbf{x}_t := \widetilde{\mathbf{x}}_t$.   */uses the SCRP* $\widetilde{\mathbf{x}}_t$ *as the investment portfolio/*
    End
  End
End

The rationale behind the WCG-CRP algorithm is twofold; the exploitation of the asymptotic property of the SCRP (3) and the improvement of its short term performance. The left hand side of (15) represents the increase of *Skeptic*'s capital in the worst case, i.e. when $|s_t(\mathbf{x})|$ is minimized, while the right hand side of (15), estimates the worst case performance when using the SCRP. It is evident how the right hand side of (15) tends to zero when the number of trading periods t becomes large. Therefore, there is an integer N, such that the WCG-CRP invests by using the SCRP for all trading periods $t > N$. This asymptotic behavior allows to conclude that assignment (a) leads to a constant wealth, and the WCG-CRP exhibits in the long run the unbiasedness in the large property or the unbiasedness in the small property, depending on the choice of the kernel $\boldsymbol{K}$. This is easily verified when considering an investment horizon $n \gg N$ by adding a constant to the numerator of (13) which represents the starting point for the proofs of Theorem 4.6 and Theorem 4.7.

5 Numerical Experiments

The performance of the WCG-CRP algorithm has been evaluated for the following stock markets; FTSE 100, Nikkei 225, Nasdaq 100 and S&P500. The first 30 stocks, sorted by weight, have been selected for each stock market. The considered investment period goes from January 2^{nd} 2007 to December 31^{st} 2009 for the FTSE 100, from January 4^{th} 2007 to December 30^{th} 2009 for the Nikkei 225, while it goes from January 3^{rd} 2007 to December 31^{st} 2009 for both the Nasdaq 100 and the S&P500. The considered three year time interval is known to be a high volatility period which includes the *credit crunch* or *subprime mortgage* crisis which started in September 2008 and hopefully finished in March 2009. This crisis was responsible for several bank defaults and probably started the worst world economic recession since the longest and most severe economic depression, known ad the *Great Depression* which began in October 1929.

The following benchmarks have been used; Sequential Best Constant Rebalanced Portfolio ($SBCRP$), Best Constant Rebalanced Portfolio ($BCRP$), stock market index ($Index$), two instances of the Sharpe-Markowitz minimum variance portfolio (SM_0 and SM_3), Successive Constant Rebalanced Portfolio ($SCRP$), Exponentiated Gradient (Helmbold et al., 1996) with parameter $\eta = 0.01$ ($EG(0.01)$) and Switchning Portfolio (Singer, 1997) with parameter $\gamma = 0.25$ ($SP(0.25)$). It is worthwhile to mention that the first instance of the Sharpe-Markowitz minimum variance portfolio (SM_0) uses the stock market data for the years 2005 and 2006 to compute the *minimum variance portfolio* to be applied to the years 2007, 2008 and 2009. The second instance (SM_3) is a standard practice which updates the investment portfolio, i.e. the *minimum variance portfolio* computed on the stock market data of the last two years, every three months. The performances of $SBCRP$ and $BCRP$, i.e. their achieved wealths, have to be correctly taken into account. Indeed, these portfolios cannot be used for actual stock selection because they explicitly depend on the entire sequence of price relatives which becomes known only after the expiration of the investment horizon. The numerical experiments concerning the WCG-CRP have been performed by using the following σ^2 values; 0.01, 0.001, 0.0001 and 0.00001.

Table 1 Wealth comparison of the portfolios.

Portfolio	FTSE 100	Nikkei 225	Nasdaq 100	S&P500
$SBCRP$	8.68	7.76	13.24	5.64
$BCRP$	2.14	1.55	2.42	1.49
$Index$	0.87	0.61	1.06	0.79
SM_0	1.04	0.64	1.02	0.94
SM_3	1.07	0.67	1.12	1.04
$SCRP$	0.58	0.31	0.82	1.12
$EG(0.01)$	**1.20**	**0.78**	1.28	1.06
$SP(0.25)$	**1.20**	**0.78**	1.28	1.06
WCG-CRP (σ^2=0.01000)	0.98	0.63	**2.34**	**1.22**
WCG-CRP (σ^2=0.00100)	1.14	0.68	1.21	1.16
WCG-CRP (σ^2=0.00010)	1.12	0.66	1.38	1.11
WCG-CRP (σ^2=0.00001)	**1.20**	0.65	1.23	0.76

Table 1 gives the following observations:

- the $BCRP$ achieved wealth values greater than those achieved by the considered investment strategies. However, the $SBCRP$, i.e. the benchmark portfolio used by the OPSG protocol, achieved the best wealth values.

- the WCG-CRP algorithm achieved a wealth value always greater than the $Index$, except when $\sigma^2 = 0.00001$ on the S&P500 stock market. However, the difference is extremely small, i.e. 0.76 vs 0.79.

- the Sharpe-Markowitz portfolio SM_3 always outperformed the $Index$. Furthermore, SM_3 outperformed SM_0 while their wealth values did not differ much.

- there is always at least one instance of the WCG-CRP investment portfolio which outperforms the minimum variance portfolio SM_3; on the FTSE 100 data set three out of four WCG-CRP instances outperformed SM_3, the same occurred for the S&P500 data set while on the Nasdaq 100 data set all the WCG-CRP instances outperformed SM_3.

- the performance of the WCG-CRP investment portfolio on the Nikkei 225 data set is somewhat curious. Indeed, the wealth is almost flat while ranging from 0.61 to 0.68 for the $Index$ and the whole set of investment portfolios.

- the WCG-CRP is the best investment strategy on three stock markets (FTSE 100, Nasdaq 100 and S&P500) while it does not achieve the maximum wealth value on the Nikkei 225 stock market data set. Indeed, the best investment strategies are the Exponentiated Gradient ($EG(0.01)$) and the Switching Portfolio ($SP(0.25)$).

- the wealth values achieved by the Exponentiated Gradient ($EG(0.01)$) and by the Switching Portfolio ($SP(0.25)$) are the same across all the considered stock market data sets.

The timeseries of the portfolio's returns allow to evaluate the nature of an investment strategy and to compute risk measures such as Value at Risk, Expected Shortfall and conditional Value at Risk. For matters of brevity only 95% and 99% one month Value at Risk, computed with the historical simulation method, are reported in Table 2 and Table 3. The *less risky* investment strategies, at 95% and 99% confidence level, are those associated with the minimum variance investment portoflios SM_0 and SM_3. The Exponentiated Gradient ($EG(0.01)$), Switching Portfolio ($SP(0.25)$) and the four instances of the WCG-CRP algorithm achieved VaR values wich are comparable to those achieved by the minimum variance investment portoflios SM_0 and SM_3. 95% and 99% VaR values associated with the WCG-CRP instances tend to be smaller than those associated with the Exponentiated Gradient ($EG(0.01)$) and Switching Portfolio ($SP(0.25)$). Furthermore, the best WCG-CRP, for the FTSE 100 and Nikkei 225 stock markets, achieved a value of 95% VaR which does not significantly differ from the one achieved by the Exponentiated Gradient ($EG(0.01)$) and Switching

Table 2 Portfolios 95% one month Value at Risk (in percentage).

Portfolio	FTSE 100	Nikkei 225	Nasdaq 100	S&P500
$SBCRP$	14.31	13.91	17.64	17.22
$BCRP$	15.67	14.52	18.45	25.69
$Index$	10.78	13.25	14.17	12.55
SM_0	9.68	12.37	10.20	11.29
SM_3	9.14	12.42	6.74	8.47
$SCRP$	17.16	23.47	22.16	26.41
$EG(0.01)$	10.56	12.56	11.71	11.84
$SP(0.25)$	10.56	12.56	11.69	11.88
WCG-CRP (σ^2=0.01000)	12.76	13.07	8.43	8.96
WCG-CRP (σ^2=0.00100)	13.43	14.07	11.43	16.05
WCG-CRP (σ^2=0.00010)	11.26	14.01	11.34	10.90
WCG-CRP (σ^2=0.00001)	11.05	12.55	11.43	12.49

Table 3 Portfolios 99% one month Value at Risk (in percentage).

Portfolio	FTSE 100	Nikkei 225	Nasdaq 100	S&P500
$SBCRP$	18.40	18.14	23.78	31.13
$BCRP$	31.28	19.68	29.06	36.97
$Index$	16.93	28.43	24.62	22.25
SM_0	15.62	18.13	19.49	16.93
SM_3	14.07	20.37	16.87	14.96
$SCRP$	34.22	36.18	41.02	35.23
$EG(0.01)$	20.03	26.21	22.31	18.71
$SP(0.25)$	20.03	26.22	22.31	18.77
WCG-CRP (σ^2=0.01000)	19.31	26.65	13.58	15.21
WCG-CRP (σ^2=0.00100)	21.82	23.48	16.97	26.10
WCG-CRP (σ^2=0.00010)	18.73	25.76	23.54	14.28
WCG-CRP (σ^2=0.00001)	14.29	17.83	22.56	28.62

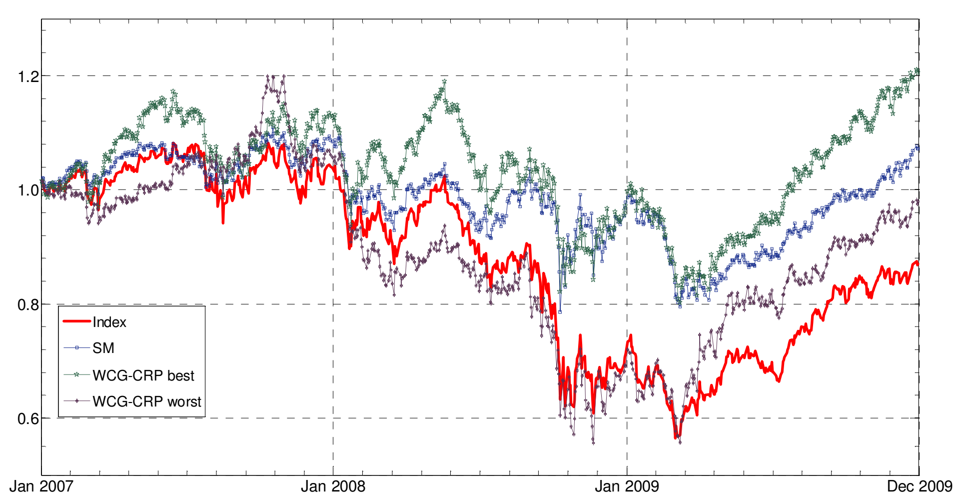
Portfolio ($SP(0.25)$), while it is significantly smaller than the one achieved by the Exponentiated Gradient ($EG(0.01)$) and Switching Portfolio ($SP(0.25)$) for the Nasdaq 100 and S&P500. The WCG-CRP moves its advantage further over the Exponentiated Gradient ($EG(0.01)$) and Switching Portfolio ($SP(0.25)$) when 99% VaR is concerned. Indeed, the WCG-CRP achieved a value of the 99% VaR on all stock market data sets which is significantly smaller than those achieved by the Exponentiated Gradient ($EG(0.01)$) and Switching Portfolio ($SP(0.25)$).

We further analyze how the WCG-CRP algorithm compares with respect to the stock market index and the Sharpe-Markowitz gold standard. For matters of brevity only two stock market data sets, namely the FTSE 100 and the Nasdaq 100, are presented. The *Index* (Index) together with the behavior of the wealth achieved by SM_3 (SM), best and worst WCG-CRP instances (WCG-CRP best and WCG-CRP worst) are depicted in Figure 1 for the FTSE 100 data set and in Figure 2 for the Nasdaq 100 data set.

The analysis of Figure 1 allows to make the following observations; the wealth achieved by the best WCG-CRP (WCG-CRP best) is almost always greater than the *Index* (Index) and greater than the wealth achieved by the minimum variance Sharpe-Markowitz portfolio SM_3 (SM). The wealth achieved by the worst WCG-CRP (WCG-CRP worst) is almost always smaller than the wealth achieved by the minimum variance Sharpe-Markowitz portfolio SM_3 (SM), while starting from April 2009 the worst WCG-CRP (WCG-CRP worst) outperformed the *Index*.

The analysis of Figure 2 allows to make the following observations; the best WCG-CRP (WCG-CRP best) significantly outperformed both the *Index* (Index) and the minimum variance Sharpe-Markowitz portfolio SM_3 (SM). Indeed, starting from January 2007 the wealth achieved by the best WCG-CRP (WCG-CRP best) is consistently greater than the wealth achieved by the minimum variance Sharpe-Markowitz portfolio SM_3 (SM). The wealth value achieved by the

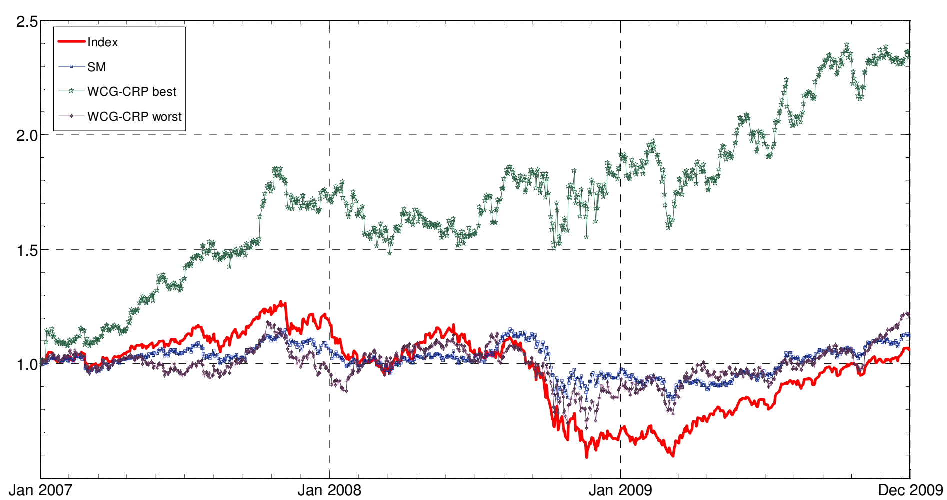
Figure 1 Index, Sharpe-Markowitz, best and worst WCG-CRP (FTSE 100).



worst WCG-CRP (WCG-CRP worst) is smaller than both the *Index* (Index) and the wealth value achieved by the minimum variance Sharpe-Markowitz portfolio SM_3 (SM) till March 2008. However, from April 2008 to December 2009 the worst WCG-CRP (WCG-CRP worst) behaved nearly the same as the minimum variance Sharpe-Markowitz portfolio SM_3 (SM), and thus it consistently outperformed the *Index*.

The result of the numerical experiments allows us to conclude that the WCG-GRP portfolio investment strategy is competitive with respect to the Sharpe-Markowitz golden standard in terms of both risk and return. For all the considered stock market data sets the WCG-GRP portfolio achieved a wealth which is greater than the wealth achieved by the Sharpe-Markowitz portfolio, while resulting in a VaR value which does not significantly differ from the one achieved by the Sharpe-Markowitz portfolio. The wealth achieved by the WCG-GRP portfolio investment strategy significantly outperformed the wealth achieved by the *Index* for all the considered stock market data sets, while the WCG-GRP portfolio and *Index* did not significantly differ in terms of risk. The wealth achieved by the WCG-GRP portfolio outperformed the wealth achieved by other portfolio investment strategies; $SCRP$, $EG(0.01)$ and $SP(0.25)$. Indeed, the wealth achieved by the WCG-GRP portfolio was not less than the wealth achieved by $SCRP$, $EG(0.01)$ and $SP(0.25)$ for three out of the four considered stock market data sets. However, the WCG-GRP portfolio achieved a 99% VaR value, which is significantly lower than the one achieved by $SCRP$, $EG(0.01)$ and $SP(0.25)$. Therefore, the WCG-GRP portfolio is confirmed to be an extremely competitive investment strategy, also when extremely stressful market conditions have to be dealt with. Its defensive nature allows to achieve VaR values which are lower or comparable to those associated with minimum variance (risk) portfolios.

Figure 2 Index, Sharpe-Markowitz, best and worst WCG-CRP (Nasdaq 100).



6 Conclusions and Research Directions

This paper links the class of *Universal Portfolios* with the *Game-Theoretic Framework* and develops a new mathematical framework for online portfolio selection. A new game protocol, called *Online Portfolio Selection Game* protocol, has been introduced. This protocol allows us to link the *universality property* with both the *unbiasedness in the large property* and *unbiasedness in the small property*. The unbiasedness in the small property has been exploited to define the class of *Defensive Online Portfolio Selection* investment strategies which is capable of taking into account the finiteness of the investment horizon. The Game Constantly Rebalanced Portfolio (G-CRP) and the Worst Case Game Constantly Rebalanced Portfolio (WCG-CRP) algorithms have been introduced. Their performance has been theoretically analyzed. The empirical performance of the WCG-CRP has been investigated through a rich set of numerical experiments concerning four major stock markets. The results emphasized the relevance of the class of *Defensive Online Portfolio Selection* investment strategies. However, the value of the σ^2 parameter influences the wealth achieved by the WCG-CRP algorithm. Therefore, it is important to understand how the value of this parameter affects the performance of the WCG-CRP. It is worthwhile to mention that the wealth achieved by the WCG-CRP is quite stable for σ^2 equal to 0.001 and 0.0001. However, a deeper analysis to find its *optimal value*, provided it exists, is required. This important aspect is out of the scope of this work and is currently under investigation through the design and analysis of adaptive optimization strategies. In light of such considerations some interesting research directions are:

- to study of the optimality of the σ^2 parameter together with a sensitivity analysis of the wealth with respect to the σ^2 parameter,
- to study and development of further kernel functions specialized for stock markets data,
- to develop of new game protocols to allow *Skeptic* to test different statistical laws, e.g. the game-theoretic *law of the iterated logarithm*.

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