Theory of the risk averse producer cooperative firm under uncertain demand

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THEORY OF THE RISK AVERSE PRODUCER COOPERATIVE FIRM FACING UNCERTAIN DEMAND

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INTRODUCTION

The literature on the theory of the firm has long recognized the special case of the producer cooperative firm as a distinct organizational structure from the capitalistic entrepreneurial firm. A producer cooperative, or labor-managed firm, is an enterprise organized and managed by workers who are the firm's proprietor-members, in contrast to the owner-controlled entrepreneurial firm which hires labor. The objective of the entrepreneurial firm is the maximization of its total profits whereas the producer cooperative firm is assumed to seek the maximization of income-per-member. WARD (1958), DOMAR (1966) and VANEK (1969) have shown that the difference between the objective functions of the two types of firm leads to different output and employment equilibrium positions as well as different behavior in response to changing market conditions. The producer cooperative firm produces less than the entrepreneurial firm, assuming identical output prices and technologies, and reduces its optimal level of output in response to an increase in fixed costs or output price.

The standard theory of the producer cooperative firm has been extended by several authors. OI and CLAYTON (1968), VANEK (1970), MAURICE and FERGUSON (1972) and MEADE (1972, 1974) have investigated the case of multiple inputs, or multiple production functions, under both perfect and imperfect competitions, and FURUBOTN (1976) has incorporated the effect of the firm's decision-making horizon. Other contributions were recently made by DREZE (1976), JONES and BACKUS (1977), and STEINHERR (1978).

The theory of the producer cooperative firm has also been criticized. A discussion of the deficiencies of the standard economics of the labor-managed firm can be found in McCAIN (1973). A recent response to these criticisms has been the development by CARSON (1977) of a new general theory of the cooperatives based on the maximization of a welfare function which incorporates each member's utility function.

In this article we attempt to extend the standard theory of the producer cooperative firm to the case of production under condition of risk. Risk is introduced into the model by assuming that the price of output is not known to the firm at the time it makes its employment and production decisions, that is, the firm faces an uncertain demand price for its product. Furthermore, it is assumed that the firm's members are risk averse, meaning that they « dislike » the risk associated with production under price uncertainty, and that the firm operating under price uncertainty seeks to maximize the expected utility of its members' income instead of just the income-per-member as in the case of certainty. To incorporate risk into the theory of the producer cooperative firm, we use the mean-variance framework developed by MARKOWITZ (1952, 1959) and TOBIN (1958) with the same methodology as in HAWAWINI (1978) and HAWAWINI and MICHEL (1979). Other attempts to examine the effect of price uncertainty on the behavior of the producer cooperative firm can be found in the works of TAUB (1974) and PAROUSH and KAHANA (1978). TAUB approaches the problem differently and reaches conclusions different from those in this paper. PAROUSH and KAHANA consider risk aversion but treat the case of perfectly competitive firm only using an analytic approach. In this article the analysis is entirely geometric and thus very simple. Besides it is general since it covers both the perfectly and imperfectly competitive structures.

Contrary to the case of the producer cooperative firm, the impact of price uncertainty on the risk averse entrepreneurial firm has been widely studied. Of particular interest are the contributions of BARON (1970, 1971), SANDMO (1971) and LELAND (1972). An important result of these authors' work is that the risk averse entrepreneurial firm produces less under price uncertainty than under complete foresight. It is shown in this paper that this result is reversed in the case of the producer cooperative firm which produces more under price uncertainty. Thus, the difference between the output level of the entrepreneurial and cooperative firms is not as significant under price uncertainty as it is in a world of certainty.

The remaining part of this article is divided into four sections. The first describes a model of the risk averse producer cooperative firm, based on the mean-standard deviation framework. The second examines the optimal employment and production decisions of the risk averse producer cooperative firm. In the third section, the model is subjected to a comparative-statics analysis in order to explore the firm's behavior in response to changes in market conditions. The last section contains a concluding summary.

THE MODEL

The methodology employed in this article to examine the effect of price uncertainty on the behavior of the producer cooperative firm relies on a geometric exposition based on the mean-standard deviation framework ; it is similar to that used by HAWAWINI and MICHEL (1979). The firm is assumed to adopt a quantity-setting strategy and to face an uncertain demand for its product written as

$$\tilde{p} = D\ (q, \tilde{u}) = D\ (q) + \tilde{u} \tag{1}$$

where the price p is a random variable, the output q is under the firm's control, and the random element $\tilde{u}$ is additive [1] and normally distributed [2] with a zero mean and a standard deviation $\sigma(\tilde{u})$. From the above specification it follows that the price is normally distributed with mean and standard deviation, respectively,

$$E\ (\tilde{p}) = D\ (q) \tag{2}$$
$$\sigma\ (\tilde{p}) = \sigma\ (\tilde{u}) \tag{3}$$

where E is the expectation operator.

Under price uncertainty, the objective function of the producer cooperative firm is to maximize the expected utility of income-per-member, denoted $\tilde{S}$, and expressed as

$$\tilde{S} = w + \frac{\tilde{\pi}}{L} = w + \frac{D\ (q, \tilde{u})q - wL - F}{L}$$
$$\tilde{S} = (D(q, \tilde{u})q - F)L^{-1} \tag{4}$$

where w is the competitive wage rate, L the number of workers or level of employment, $\tilde{\pi}$ the profit function, and F the fixed costs. The output q is assumed to be a function of the single input L and the production function is assumed to display diminishing marginal returns over the relevant production range. We have

$$q = q\ (L) \text{ with } q'\ (L) > O \text{ and } q''\ (L) < O \tag{5}$$

where the prime designates a derivative with respect to the variable in parentheses.

Because income-per-member S is not known at the time employment and production decisions are made, the cooperative firm cannot maximize S itself as in the case of certainty. Under demand uncertainty S

(1) We could have specified the demand function with a multiplicative random disturbance $\tilde{e}$, that is, $\tilde{p} = D(q)\ \tilde{e}$, normally distributed with a unit mean and a standard deviation $\sigma(\tilde{e})$, without affecting the model's conclusions.

(2) Although the analysis is restricted to normal distributions it should be pointed out that is could be easily generalized to hold for a large family of distributions called location-scale distributions. The mean is simply replaced by a location parameter and the standard deviation by a scale parameter without affecting the analysis. For details see BAWA (1975).

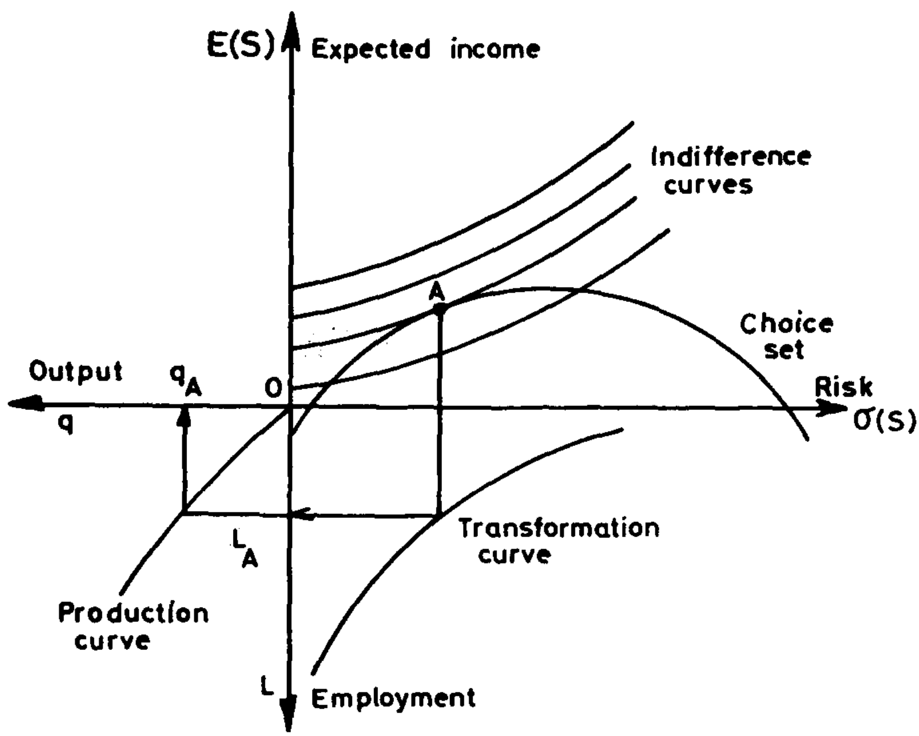

E(S)
Expected income
Indifference
curves
A
Choice
set
Output
q
q_A
O
Risk
σ(S)
L_A
Transformation
curve
Production
curve
L
Employment

FIGURE 1

becomes a normally distributed random variable and the cooperative is assumed to maximize the expected utility of $\tilde{S}$, that is, $E[U(\tilde{S})]$, where $U(.)$ represents the VON NEUMANN-MORGENSTERN utility-of-money function such as $U'(.) > O$ and $U''(.) < O$. This means that the firm is risk averse [3] in the sense that it exhibits a declining marginal utility for money; it prefers more money to less with a decreasing preference for additional units of money.

Using equation (4) we can derive the firm's expected income-per-member, $E(\tilde{S})$, and the variability of $\tilde{S}$ measured by the standard deviation of $\tilde{S}$. We have

$$E(\tilde{S}) = D(q)\, qL^{-1} - FL^{-1} \tag{6}$$

$$\sigma(\tilde{S}) = \sigma(\tilde{p})\, qL^{-1} \tag{7}$$

Because of the normality assumption, moments of the distribution of $\tilde{S}$ which are higher than the second are zero and $\sigma(\tilde{S})$ can be used as a measure of the riskiness of income-per-member. The standard deviation $\sigma(\tilde{S})$ is a risk measure in the sense that it summarizes all deviations from the expected income-per-member. The wider the deviation, the larger is $\sigma(\tilde{S})$ and the riskier is the income-per-member.

Both the firm's objective function (maximization of expected utility of $\tilde{S}$) and the revenues and costs constraints it faces (expressed in equation (4)) can be represented diagrammatically in the mean-risk plane as illustrated in the upper quadrant of figure 1, where the vertical axis indicates the expected income-per-member $E(\tilde{S})$ and the horizontal axis indicates the risk $\sigma(\tilde{S})$.

The expected utility is represented in the plane by a family of indifference curves. It is shown in the mathematical appendix that risk aversion implies that these curves are upward sloping and convex to the origin: the risk averse firm will accept additional increments of risk only if compensated with increasingly larger increments of expected income. Each indifference curve indicates a constant level of expected utility, with higher curves, away from the origin, indicating increasing levels of expected utility. This is illustrated in the upper quadrant of figure 1. Translated geometrically, the firm's objective function is to attain the highest indifference curve, and hence to maximize its expected utility.

However, the highest indifference curve may not be attainable simply because the firm faces revenues and costs constraints, that is, there exists a set of feasible combinations of $E(\tilde{S})$ and $\sigma(\tilde{S})$ from which the firm must choose. This choice or opportunity set can be derived from equations (6) and (7). We get

$$E(\tilde{S}) = (D(q)/\sigma(\tilde{p}))\, \sigma(\tilde{S}) - FL^{-1} \tag{8}$$

[3] This statement should be interpreted to mean that all the members of the cooperative are risk averse and that group preferences can be adequately described by a unique utility function.

Equation (8) can be represented in the $E(\tilde{S})$ - $\sigma(\tilde{S})$ plane by a concave curve to the origin with a maximum as illustrated in the upper quadrant of figure 1. The proof is given in the mathematical appendix.

For the sake of exposition and without loss of generality, assume that the demand function is linear in q and that the production function is of Cobb-Douglas type with an elasticity coefficient of one half. We have

$$D(q) = a - bq, \; a > o, \; b > o \tag{9}$$

$$q = L^{1/2} \tag{10}$$

Using equations (9) and (10), the choice set given by equation (8) can be expressed as

$$E(\tilde{S}) = (a/\sigma(\tilde{p}))\sigma(\tilde{S}) - b - (F/\sigma^2(\tilde{p}))\sigma^2(\tilde{S}) \tag{11}$$

This result is derived in the mathematical appendix. In the case of perfect competition, the firm faces a perfectly elastic demand curve with an expected price $\bar{p}$ and the choice set becomes

$$E(\tilde{S})^* = (\bar{p}/\sigma(\tilde{p}))\sigma(\tilde{S}) - (F/\sigma^2(\tilde{p}))\sigma^2(\tilde{S}) \tag{12}$$

In both cases the choice set is a quadratic function of risk $\sigma(\tilde{S})$.

Finally the equilibrium position for the firm is found at the tangency point between the highest attainable indifference curve and the choice set. At this point, the slope of the indifference curve, which is the firm's Marginal Rate of Substitution (MRS) of risk for expected income, is equal to the slope of the choice set, which is the firm's Marginal Expected Income (MEI), that is, the additional expected income the firm will secure if it takes on additional risk.

From the equilibrium point, the firm's optimal level of output can be determined using the lower quadrants in figure 1. In the right-lower quadrant we have drawn the relationship between risk $\sigma(\tilde{S})$ and the level of employment L as expressed in equation (7), and in the left-lower quadrant we have drawn the production function. When production exhibits diminishing marginal returns, the risk-employment Transformation Curve is convex downward (right-lower quadrant). Taking the derivative of equation (7) with respect to L, we get

$$d\sigma(\tilde{S})/dL = (dq/dL - q/L)\sigma(\tilde{p})L^{-1} < O \tag{13}$$

Under the conditions stated in equation (5), average product q/L is larger than marginal product dq/dL and thus the derivative is negative which explains the inverse relationship between risk and the level of employment. As the cooperative firm increases the number of members the variability of income-per-member drops. Referring to equation (7) we can see that under diminishing marginal returns q increases at a smaller rate than L and the ratio q/L declines which in turn reduces risk $\sigma(\tilde{S})$. Returning to figure 1, observe that to the equilibrium point A corresponds an optimal level of employment L_A to which in turn corresponds an optimal level of production q_A with the production curve drawn in the left-lower quadrant.

COMPARATIVE EMPLOYMENT AND PRODUCTION LEVELS

The analysis carried out in the previous section uses the tangency point between the indifference curve and the choice set in order to determine the firm's equilibrium position. The analysis becomes much simpler and definitely more flexible if it uses, instead, the equality between the MRS and the MEI in order to determine the equilibrium position.

The MEI for the imperfectly competitive firm, which is the additional income-per-member obtained by the firm when it takes on an additional unit of risk, is given by the derivative of equation (11) with respect to risk $\sigma(\tilde{S})$. We have

$$MEI = dE(\tilde{S})/d\sigma(\tilde{S}) = (a/\sigma(\tilde{p})) - 2(F/\sigma^2(\tilde{p}))\sigma(\tilde{S}) \quad (14)$$

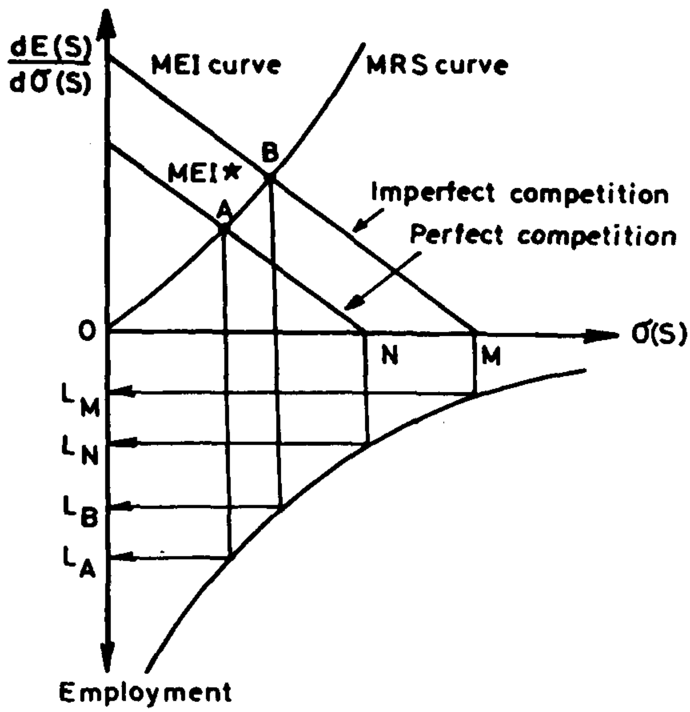
In the case of perfect competition it is

$$MEI^* = dE(\tilde{S})^*/d\sigma(\tilde{S}) = (\bar{p}/\sigma(\tilde{p})) - 2(F/\sigma^2(\tilde{p}))\sigma(\tilde{S}) \quad (15)$$

Both Marginal Expected Income curves are drawn in figure 2. Note that they are both linear in risk because of the assumptions we made in equations (9) and (10). They have the same slope but the MEI curve under imperfect competition has a larger intercept since $a > \bar{p}$. This is so because a is the price when output is zero which must exceed the expected price $\bar{p}$ prevailing at some nonzero level of production. We should point out that the MEI curves are downward sloping in general, although not always linear as in this example. Again, the linearity is the result of the assumptions made in equations (9) and (10). A proof that the MEI curves are in general downward sloping is given in the mathematical appendix.

Turning to the MRS curves, which are the derivatives of the indifference curves with respect to risk, it is shown in the appendix that under risk aversion they are upward sloping and pass through the origin as shown in figure 2.

Finally, the firm's equilibrium is determined at the intersection point between the MEI and MRS curves. Referring to figure 2, equilibrium is at point A under risk aversion and perfect competition and at point B under risk aversion and imperfect competition. To these two points correspond, respectively, the optimal level of employment L_A and L_B such as $L_A > L_B$ and, therefore, optimal production under perfect competition exceeds optimal production under imperfect competition assuming identical indifference curves and MRS curves under both perfect and imperfect competitions.



Levels of employment and production (L)

	Perfect competition		Imperfect competition
Certainty	L_N	$>$	L_M
	$\wedge$		$\wedge$
Price uncertainty	L_A	$>$	L_B

FIGURE 2

The level of employment under certainty can be determined as follows. When certainty prevails, the producer cooperative firm maximizes income-per-member rather than the expected utility of income-per-member and the certainty equilibrium is thus found at the point where $dS/dL = O$. But income-per-member is also maximized at the point where $MEI = dE(\tilde{S})/d\sigma(\tilde{S})$ is equal to zero since $dE(\tilde{S})/d\sigma(\tilde{S}) = O$ implies $dS/dL = O$ (4). This point is found at the intersection between the MEI curves and the risk axis. Referring to figure 2, equilibrium under certainty is at point N under perfect competition and point M under imperfect competition to which correspond, respectively, the level of employment L_N and L_M such as $L_N > L_M$ and therefore $q_N > q_M$. We can state the following proposition:

> **PROPOSITION I**: « Under price uncertainty as well as under complete foresight, the risk averse producer cooperative firm produces more under perfect competition than under imperfect competition. »
>
> (Note that this proposition is also valid for the entrepreneurial firm.)

Thus the introduction of price uncertainty and risk aversion do not modify the conclusion reached by WARD (1958) that the producer cooperative firm under perfect competition has a higher level of output than the producer cooperative firm under imperfect competition.

Of more interest is the difference between levels of employment and production under certainty and uncertainty. Referring to figure 2, observe that $L_A > L_N$ and $L_B > L_M$.

The risk averse producer cooperative firm employs and produces more under price uncertainty, regardless of the structure of the output market. This result may appear paradoxical since the risk averse entrepreneurial firm produces less under price uncertainty, a result that reflects the firm's aversion to risky activities. Recalling equation (13) and the ensuing discussion, we have shown that for the risk averse producer cooperative firm operating under diminishing marginal returns, the variability of income drops as the levels of employment and production increase. The producer cooperative firm being risk averse can therefore reduce risk by producing more. This explains our result. Thus

> **PROPOSITION II**: « Risk aversion and diminishing marginal returns are necessary and sufficient conditions for the producer cooperative firm to produce more under price uncertainty than under certainty, regardless of the structure of the output market. »

(4) Note that $dE(\tilde{S})/d\sigma(\tilde{S}) = E[dS/d\sigma(\tilde{S})] = E\,[(dS/dL)\,(dL/d\sigma(\tilde{S})]$ and since $dL/d\sigma(\tilde{S})$ is different from zero $dE(\tilde{S})/d\sigma(\tilde{S}) = O$ implies $dS/dL = O$.

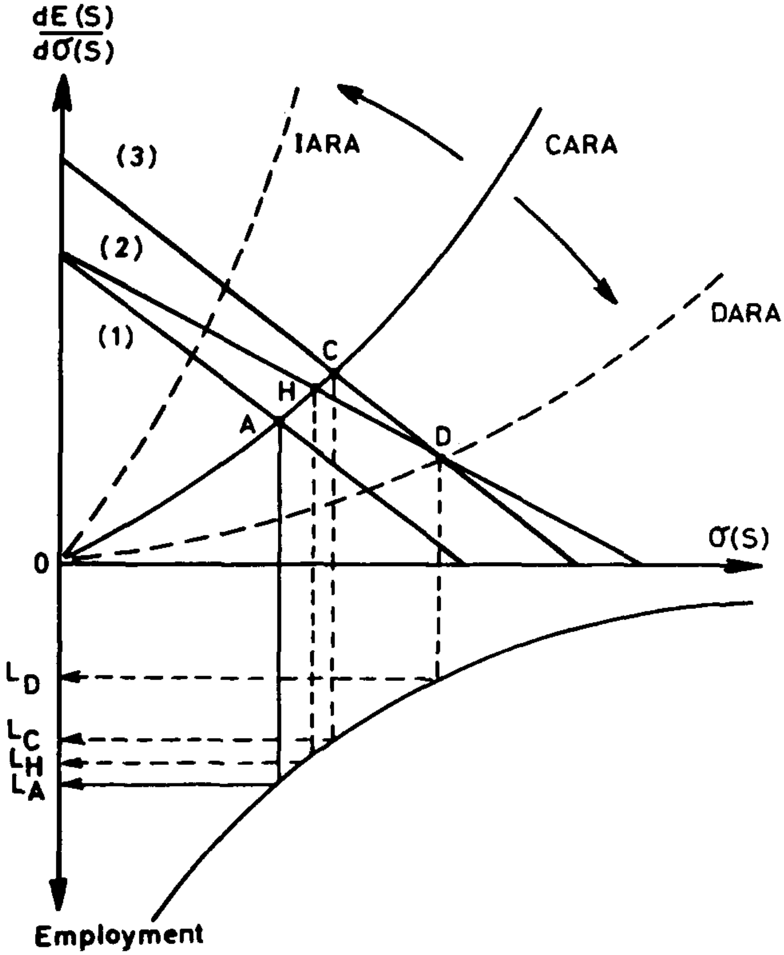


FIGURE 3

It is worth noting that under certainty the entrepreneurial firm produces more than the producer cooperative firm. This result is considerably weakened under price uncertainty when firms are risk averse since the entrepreneurial firm will reduce output and the producer cooperative firm will increase output, thus narrowing the production gap prevailing under certainty.

Finally, note that equilibrium under risk aversion requires the existence of positive income since the indifference curves and MRS curves are not defined when expected income is negative. Therefore

PROPOSITION III: «The risk averse producer cooperative firm will not produce at a loss under price uncertainty.»

THE COOPERATIVE RESPONSE TO CHANGES IN MARKET CONDITIONS

In this section we are concerned with the firm's response to a change in market conditions. Specifically, we would like to examine the effect of a change in any one of the model's parameters on the firm's optimal levels of employment and production. These are (i) the fixed costs F, (ii) the expected price $\bar{p}$ prevailing under perfect competition and (iii) $\sigma(\tilde{p})$, the degree of price uncertainty perceived by the firm.

The firm's response will depend on its attitude toward risk. However, the concept of risk aversion introduced in the previous section is not sufficient to describe the firm's response to changes in the model's parameters. Both ARROW (1965) and PRATT (1964) have shown that an index of Absolute Risk Aversion must be used. The firm is said to display increasing, constant or decreasing absolute risk aversion according as the index $R_A(\tilde{S}) = -U''(\tilde{S})/U'(\tilde{S})$ increases, remains constant or decreases with income $\tilde{S}$. In the mathematical appendix we demonstrate that Constant Absolute Risk Aversion (CARA) implies that the MRS curve remains fixed when the firm moves across its family of indifference curves. In other words, under CARA the MRS curve is the same for any of the firm's indifference curves. The complete family of indifference curves is summarized in a unique MRS curve under CARA. This is not the case under either Increasing Absolute Risk Aversion (IARA) or Decreasing Absolute Risk Aversion (DARA). Under IARA the MRS curve rotates counterclockwise as the firm moves to a higher indifference curve as illustrated in figure 3. Under DARA the MRS curve rotates clockwise as the firm moves to a higher indifference curve. Thus, contrary to the case of CARA, under both IARA and DARA to each indifference curve corresponds a unique MRS curve with higher indifference curves represented by MRS curves swinging to the left under IARA and to the right under DARA. There exists ample theoretical and empirical evidence showing that individuals display DARA as their wealth increases. In what follows we assume that the firm exhibits non-increasing absolute risk aversion.

The above results can now be applied to investigate the firm's behavior in response to changes in the model's parameters. First assume a decrease in the firm's fixed costs. Referring to equations (14) and (15), observe that the slopes of the MEI curves will increase but their intercepts will not change. This is illustrated in figure 3 by a shift of the MEI curve from position (1) to position (2). As a result of the shift in the MEI curve, the original equilibrium point moves from A to H under CARA and from A to D under DARA. Note that a reduction in fixed costs increases expected income and the firm is now on a higher indifference curve. The optimal level of employment will move from L_A to L_H under CARA and from L_A to L_D under DARA with $L_H < L_A$ and $L_D < L_A$. In both cases employment decreases and so does production. Under IARA the direction of the change in employment is ambiguous. It depends on the relative steepness of the slopes of the MRS and MEI curves. We can state the following proposition:

PROPOSITION IV: « Non-increasing absolute risk aversion is a sufficient condition for the output of the producer cooperative firm (under diminishing marginal returns) to vary directly with its fixed costs under price uncertainty. »

This result is similar to the case of certainty where the cooperative firm increases its employment and output in response to an increase in fixed costs so as to spread out the additional costs over a larger number of members. However, the firm must display either CARA or DARA for this result to hold under price uncertainty.

It is worth comparing the behavior of the risk averse producer cooperative firm to that of the risk averse entrepreneurial firm. The risk averse entrepreneurial firm does not alter its output in response to a change in fixed costs under CARA. Under DARA its output varies inversely with changes in fixed costs [5]. This result is due to the difference between the objective functions of the producer cooperative and entrepreneurial firms.

The impact of a change in the expected price faced by the risk averse producer cooperative firm under perfect competition can be examined in order to determine if a supply curve exists under price uncertainty.

(5) This standard results can be found in LELAND (1972).

An increase in the expected price increases the intercept of the MEI * curve without changing its slope as indicated by equation (15). The firm is now on a higher indifference curve and its MEI * curve shifts in a parallel fashion from position (1) to position (3) as illustrated in figure 3. The original equilibrium point shifts from A to C under CARA and from A to D under DARA. Employment decreases from L_A to L_C and from L_A to L_D and so does production. Again, under IARA the direction of the change is ambiguous. We can therefore state that

PROPOSITION V : « Nonincreasing absolute risk aversion is a sufficient condition for the output of the competitive producer cooperative firm (under diminishing marginal returns) to vary inversely with expected price. »

Thus the risk averse producer cooperative firm under perfect competition which exhibits either CARA or DARA has a downward sloping supply curve under price uncertainty. This result is similar to the case of the producer cooperative firm operating under certainty.

Finally, assume that the producer cooperative firm revises its expectation about the distribution of future prices by reducing its estimate of the standard deviation of price. This will increase its expected income (see equation (8)) and put the firm on a higher indifference curve. Referring to equations (14) and (15) note that the intercepts of the MEI curves will increase and their slopes decrease. The MEI curves will shift from position (1) to position (2) as illustrated in figure 4. The reduction in $\sigma(\tilde{p})$ will also affect the Transformation Curve in the lower quadrant in figure 4, shifting it inward toward the origin. The original equilibrium point will move from A to C under CARA and from A to D under DARA. Employment will decrease from L_A to L_C under CARA and L_A to L_D under DARA; and so does production. Again, the direction of the change in employment is ambiguous under IARA. Thus our last proposition is

PROPOSITION VI : « Non-increasing absolute risk aversion is a sufficient condition for the output of the producer cooperative firm (under diminishing marginal returns) to change directly with its perceived degree of uncertainty. »

Again, this result may appear paradoxical. However, it is consistent with proposition II. As $\sigma(\tilde{p})$ is reduced the cooperative firm operates in a less and less uncertain environment and therefore its production will decline according to proposition II. Note that the risk averse entrepreneurial firm which displays either CARA or DARA will vary its production inversely with the perceived degree of uncertainty.

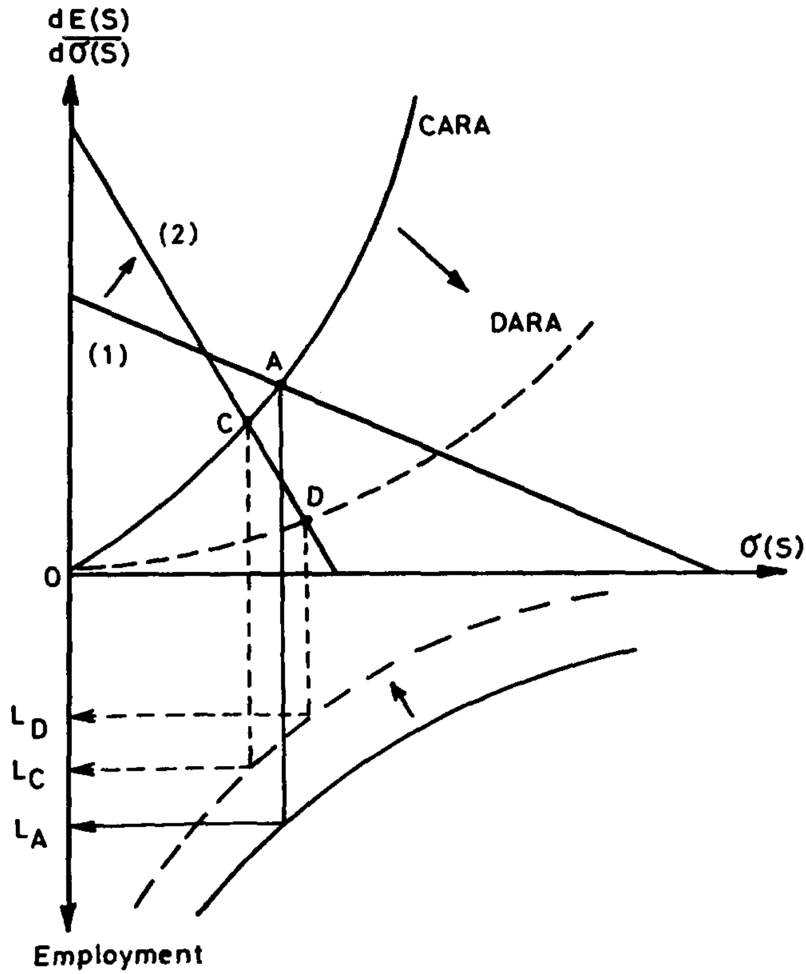


FIGURE 4

CONCLUSION

In this article we were concerned with the behavior of the risk averse producer cooperative firm facing an uncertain demand for its product. We have shown that the introduction of price uncertainty into the standard model of the producer cooperative firm affects the firm's optimal levels of employment and production. The risk averse producer cooperative firm produces more under uncertainty. We have also shown that if the cooperative firm displays nonincreasing absolute risk aversion, a change in its fixed costs or its perceived degree of uncertainty leads to a change in output in the same direction. As in the case of certainty, the competitive producer cooperative firm displaying nonincreasing absolute risk aversion has a downward sloping supply curve.

Comparing these results to those obtained for the entrepreneurial firm facing an uncertain demand, we have shown that the difference in the production levels of the two types of firm is considerably smaller under price uncertainty than it is under certainty. In general, when facing an uncertain demand, the producer cooperative and entrepreneurial firms respond in opposite directions to changes in similar market conditions.

MATHEMATICAL APPENDIX

1. — The Shape of the Indifference Curves and the MRS Curves

Along an indifference curve $E[U(\tilde{S})]$ is a constant. Differentiating with respect to $\sigma(\tilde{S})$ we get:

$$dE[U(\tilde{S})]/d\sigma(\tilde{S}) = E[U'(\tilde{S}) \,.\, d\tilde{S}/d\sigma(\tilde{S})] = 0 \qquad (A.1)$$

Since $\tilde{S}$ is a normally distributed random variable we can write $\tilde{S} = E(\tilde{S}) + \sigma(\tilde{S}) \,.\, \tilde{Z}$ where $\tilde{Z}$ is a normally distributed random variable with zero mean and unit standard deviation. It follows that

$$\begin{aligned} d\tilde{S}/d\sigma(\tilde{S}) &= dE(\tilde{S})/d\sigma(\tilde{S}) + \tilde{Z} \\ d\tilde{S}/d\sigma(\tilde{S}) &= MRS + \tilde{Z} \end{aligned} \qquad (A.2)$$

since the Marginal Rate of Substitution (MRS) is the slope of an indifference curve. Substituting equation (A.2) in equation (A.1) we get

$E[U'(\tilde{S})(MRS + \tilde{Z})] = 0$. Solving for MRS we obtain

$$MRS = -E[U'(\tilde{S}) \,.\, \tilde{Z}]/E[U'(\tilde{S})] < 0 \qquad (A.3)$$

Under risk aversion $U'(\tilde{S}) > 0$ and $U''(\tilde{S}) < 0$. The expression $U'(\tilde{S}) \,.\, \tilde{Z}$ is negative since the random variable $\tilde{Z}$ can take either a positive or a negative value with equal probability and zero mean and that the

« weights » $U'(\tilde{S})$ are larger when $\tilde{Z}$ is negative because $U'(\tilde{S})$ is a declining function of $\tilde{S}$, that is, $U''(\tilde{S}) < 0$. It follows that the MRS is positive and therefore, given risk aversion, indifference curves are upward sloping in the $E(\tilde{S}) - \sigma(\tilde{S})$ plane. Differentiating equation (A.3) with respect to $\sigma(\tilde{S})$ we get

$$dMRS/d\sigma(\tilde{S}) = -E[U''(\tilde{S})(MRS + \tilde{Z})^2]/E[U'(\tilde{S})] > 0 \qquad (A.4)$$

Again, under risk aversion, $U'(\tilde{S}) > 0$ and $U''(\tilde{S}) < 0$ and $dMRS/d\sigma(\tilde{S})$ is positive. It follows that, given risk aversion, indifference curves are upward sloping and convex to the origin as drawn in figure 1.

Note that equation (A.4) implies that the MRS are an increasing function of risk $\sigma(\tilde{S})$ under risk aversion. Finally, evaluating the MRS at the origin where $\sigma(\tilde{S}) = 0$ we have $\tilde{S} = E(\tilde{S})$ since $\tilde{S} = E(\tilde{S}) + \sigma(\tilde{S}) \,.\, \tilde{Z}$.

Using equation (A.3) we get

$$MRS(0) = -E[U'(E(\tilde{S})) \,.\, \tilde{Z}]/E[U'(E(\tilde{S}))] = E(\tilde{Z}) = 0$$

by observing that $U'(E(\tilde{S}))$ is not a random variable. It follows that MRS curves pass through the origin and are an increasing function of risk $\sigma(\tilde{S})$ as drawn in figure 2.

2. — The Shape of the Choice Set and the MEI Curves

Taking the derivatives of equation (6) with respect to L we get

$$dE(\tilde{S})/dL = qL^{-1}D'(q)q' + D(q)L^{-1}q' - D(q)L^{-2}q + FL^{-2} \qquad (A.5)$$

Introducing the price elasticity of demand $\varepsilon_D = -(D(q)/q)/(D'(q))$ equation (A.5) can be rewritten as

$$dE(\tilde{S})/dL = L^{-1}\left\{D(q)(1 - 1/\varepsilon_D)q' - D(q)qL^{-1} + FL^{-1}\right\}$$

$$= L^{-1}\left\{D(q)[qL^{-1}(1 - 1/\varepsilon_D)(q'/qL^{-1} - \frac{1}{1 - 1/\varepsilon_D})] + FL^{-1}\right\} \qquad (A.6)$$

Introducing the elasticity coefficient $\varepsilon_L = (q')/(qL^{-1}) < 1$ equation (A.6) can be rewritten as

$$dE(\tilde{S})/dL = L^{-2}[D(q)q(\varepsilon_L - 1 - \varepsilon_L/\varepsilon_D) + F] \qquad (A.7)$$

The derivative of equation (7) with respect to L is

$$d\sigma(\tilde{S})/dL = \sigma(\tilde{p})L^{-1}(q' - qL^{-1}) = \sigma(\tilde{S})(\varepsilon_L - 1)L^{-1} \qquad (A.8)$$

and the MEI can be expressed as

$$\text{MEI} = \frac{dE(\tilde{S})}{d\sigma(\tilde{S})} = \frac{dE(\tilde{S})}{dL} \Big/ \frac{d\sigma(\tilde{S})}{dL}$$

$$\text{MEI} = \frac{dE(\tilde{S})}{d\sigma(\tilde{S})} = \frac{D(q)}{\sigma(\tilde{p})}\left[1 - \frac{\varepsilon_L}{\varepsilon_D(\varepsilon_L - 1)}\right] + F(L\sigma(\tilde{S})\ (\varepsilon_L - 1))^{-1} \tag{A.9}$$

Under perfect competition D(q) is equal to expected price $\bar{p}$, the demand is perfectly elastic, that is, $\varepsilon_D = \infty$, and therefore equation (A.9) collapses to

$$\text{MEI}^* = \bar{p}/\sigma(\tilde{p}) + F\ (L\sigma(\tilde{S})\ (\varepsilon_L - 1))^{-1} \tag{A.10}$$

Referring to equation (A.9) note that the first term is always positive since $\varepsilon_D > 0$ and $0 < \varepsilon_L < 1$ and the second always negative. Thus there is a value of $\sigma(\tilde{S})$ for which MEI is zero, meaning the choice set has an extremum. We will show below that it is a maximum. Likewise referring to equation (A.10) note that the first term is always positive and the second always negative and therefore an extremum exists for the value $\sigma(\tilde{S})$ that makes MEI $^* = 0$.

To show that the extremum is actually a maximum, take the derivative of equation (A.10) with respect to $\sigma(\tilde{S})$. We have

$$\frac{d\text{MEI}^*}{d\sigma(\tilde{S})} = -\frac{F}{(L\sigma(\tilde{S})\ (\varepsilon_L - 1))^2}\left[L(\varepsilon_L - 1) + \sigma(\tilde{S})\ (\varepsilon_L - 1)\frac{dL}{d\sigma(\tilde{S})} + L\sigma(\tilde{S})\ \frac{d\varepsilon_L}{d\sigma(\tilde{S})}\right]$$

$$\frac{d\text{MEI}^*}{d\sigma(\tilde{S})} = -\frac{LF}{(L\sigma(\tilde{S})\ (\varepsilon_L - 1))^2}\left(\varepsilon_L + \sigma(\tilde{S})\ \frac{d\varepsilon_L}{d\sigma(\tilde{S})}\right) < 0 \tag{A.11}$$

Since $\varepsilon_L > 0$ and $d\varepsilon_L/d\sigma(\tilde{S}) = (d\varepsilon_L/dL)\ (dL/d\sigma(\tilde{S})) > 0$ it follows that dMEI $^*/d\sigma(\tilde{S})$ is negative and thus the extremum is a maximum. The same result can be obtained by taking the derivative of MEI with respect to $\sigma(\tilde{S})$ but the mathematics is much more complicated.

3. – Derivation of Equation (11)

Using equations (9) and (10), equations (6) and (7) become, respectively

$$E(\tilde{S}) = (a\text{-}bL^{1/2})L^{-1/2} - FL^{-1} = aL^{-1/2} - b\text{-}FL^{-1} \tag{A.12}$$

$$\sigma(\tilde{S}) = \sigma(\tilde{p})L^{-1/2} \tag{A.13}$$

From equation (A.13) we get $L^{-1/2} = \sigma(\tilde{S})/\sigma(\tilde{p})$ and $L^{-1} = \sigma^2(\tilde{S})/\sigma^2(\tilde{p})$. Substituting these expressions in equation (A.12) we get equation (11).

4. — The Absolute Risk Aversion Index and the MRS Curves

Evaluating the slope of the MRS curves at the origin, using equation (A.4) and recalling that $\tilde{S} = E(\tilde{S})$ at the origin, it follows that

$$[dMRS/d\sigma(\tilde{S})]_{\sigma(\tilde{S})=0} = -E[U''(E(\tilde{S}))\ (0 + \tilde{Z})^2]/E[U'\ (E(\tilde{S}))]$$
$$= U''\ (E(\tilde{S}))E(\tilde{Z}^2)/U'\ (E(\tilde{S}))$$

but $E(\tilde{Z}^2)$ is the variance of $\tilde{Z}$ which is equal to one and thus the slope of the MRS curves at the origin are equal to

$$[dMRS/d\sigma(\tilde{S})]_{\sigma(\tilde{S})=0} = -U''(E(\tilde{S}))/U'(E(\tilde{S}))$$

which is the ARROW-PRATT Index of Absolute Risk Aversion.

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