Socially determined time preference in discrete time

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Abstract: The aim of the paper is to develop a discrete time version of a one-sector optimal growth model with endogenous time preference. The intertemporal discount rate is determined by social factors (i.e., factors that are external to the individual agent), namely the economy wide levels of consumption and income. In continuous time, the combined effect of the previous factors is known to eventually produce local indeterminacy, instead of the well known saddle-path equilibrium of the standard Ramsey model. In discrete time, the possibility of local indeterminacy is explored under several types of Ramsey models with endogenous time preference: neo-classical and endogenous growth models, and models with production externalities and endogenous labor supply. Besides finding various possibilities regarding local dynamics, we also find that one of the models can give place to endogenous fluctuations, although this occurs only under rather exceptional circumstances.

Keywords: Endogenous time preference, Growth models, Stability analysis, Technological externalities, Endogenous labor supply.

JEL classification: O41, C61.

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1. Introduction

The Ramsey growth model constitutes a fundamental tool regarding the analysis of material accumulation and of patterns of consumption over time. It is sufficiently flexible to explain different patterns of growth, arising from distinct technological conditions, different patterns of preferences or different assumptions regarding the shape of the production function, the economic properties attached to its inputs or the way one understands capital depreciation and obsolescence. One of the factors that exerts influence over the outcome of the growth model is the kind of rate of time preference one considers. Typically, economists have almost always adopted a constant rate of time preference. This is more the result of an analytical convenience than an assumption with strong empirical support.

For instance, Boyarchenko and Levendorskii (2005) identify a set of anomalies attached to the notion of a constant discount rate; these problems are the following: first, there is evidence that, in reality, the discounting is hyperbolic, i.e., the instantaneous discount rate decreases with time or, in other words, individuals discount over short horizons at a higher rate than over long horizons [some models that explore dynamic choices under hyperbolic discounting include Laibson (1997), Barro (1999) and O'Donoghue and Rabin (1999), just to cite a few]. Second, a sign effect is plausible to occur, i.e., it is likely to have gains discounted more strongly than losses; arguments in favour of this evidence are found in work concerning the psychology of decisions as it is the case of Kahneman and Tversky (1979) and Kahneman (2003). Third, a delay-speedup asymmetry, relating to the idea that if a change of the delivery time of an outcome is perceived as a acceleration from some reference point, then the discount rate is larger than if the change is perceived as a delay, relatively to that reference point. Fourth, it is possible to imagine a negative discounting for losses, since many times the agent prefers to expedite payments or other losses. Fifth, there is clearly, in practice, a magnitude effect, in the sense that small outcomes are more strongly discounted when compared with large outcomes. Finally, there is a stronger preference, or a higher impatience, regarding improving sequences. A detailed discussion concerning the previous effects over the representative agent discounting of future outcomes can be found in Frederick, Loewenstein and Donoghue (2002).

The above paragraph elucidates about the great variety of forces involving the subjective choice of a discount rate or rate of time preference. It also allows to perceive that, certainly, many of the forces that influence such choice are endogenous both to the representative individual relatively to which some economic problem is stated and solved, but also to the economic system as a whole. In the model to develop below, some of these influences are assumed; specifically, we will focus on the economy wide determinants of the individual rate of intertemporal preference.

Basically, in the literature, two candidates for explaining the rate of time preference as endogenous are normally taken. In the tradition of Uzawa (1968) and Epstein (1987), these are the level of consumption and the level of income. The main assumptions tend to be synchronized with the empirical evidence, that is, rising consumption levels tend to imply rising impatience (a higher discount rate), while rising income tends to generate more patient individuals (a lower discount rate).

In this paper, we will follow closely the work by Meng (2006), who develops a model of endogenous time preference where the factors affecting the individual discount rate are the economy wide aggregate levels of consumption and income. The influence of individual levels of consumption and income is overlooked, and only aggregate levels are considered. The influence of such factors has been also thoroughly documented in the literature [see the references in Meng (2006)] and it comes from logical arguments: a jealousy effect explains the presence of aggregate consumption as a determinant of the time preference (higher consumption levels in society imply increasing impatience), while the economy's income positive impact over patience is meaningful under the idea that a wealthier society produces less impatient individuals (in what concerns the timing of consumption).

As stated, we develop the same model as Meng (2006), including the two variations he considers (technological externalities and endogenous labor supply), in discrete time. This analysis is relevant if one wants to confirm if the local indeterminacy result of the continuous time setup continues to hold when one changes the assumed notion of time. We find that, under neo-classical growth, local indeterminacy is a possible stability outcome, but we cannot exclude the presence of saddle-path stability or instability, depending on the specific values of parameters. We also analyze simple one sector endogenous growth models with endogenous time preference, and conclude that, under the chosen specification, the system will rest always over a bifurcation line and, hence, local indeterminacy is ruled out.

Relatively to the two extensions, an endogenous labor supply – endogenous growth specification does not allow for finding local indeterminacy for reasonable parameter values, while the model with production externalities, besides reintroducing the possibility of indeterminacy, is the only one capable of displaying endogenous fluctuations. These fluctuations are triggered by a flip bifurcation that generates a period doubling process that culminates in the presence of chaotic motion. Such outcome is rare, occurring only for extreme values of parameters; nevertheless, we cannot exclude it, and thus we add a new candidate explanation for the possibility of endogenous business cycles, alongside with the ones already explored in the literature: increasing returns / production externalities with a constant discount rate [Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002)], learning [Cellarier (2006)] or financial development [Caballé, Jarque and Michetti (2006)], just to cite some of the most meaningful.

To be precise, the eventual presence of endogenous business cycles under an endogenously determined rate of time preference is not a completely new result; Drugeon (1998) assumes an endogenous time preference rate that depends both on the individual level of consumption and the aggregate level of consumption. His findings point to the presence of 'sustained oscillation motion', that is, endogenous fluctuations. Our model adds the result of endogenous cycles in a setup where time preference is exclusively determined by economy wide factors.

The remainder of the paper is organized in the following sequence. Section 2 describes the model. Section 3 explores its main local stability conditions under neo-classical growth. Section 4 studies stability conditions under endogenous growth. In sections 5 and 6, two variations of the model are analyzed: a production externalities / increasing returns framework and an endogenous leisure - endogenous growth setup. Section 7 is destined to a brief remark about global dynamics and the presence of endogenous cycles in one of the models. Section 8 concludes.

2. Meng's Model in Discrete Time

The benchmark model to consider is a discrete time version of the continuous time growth model with socially determined time preference developed in Meng (2006). Consider an economy where a representative agent intends to maximize a sequence of utility functions from the present time moment, t=0, to infinity, $t\to\infty$. This sequence of utility functions is presented as follows,

$$U_0 = \sum_{t=0}^{+\infty}\left\{U(c_t)\cdot\left[\prod_{v=0}^{t}\beta(C_v,Y_v)\right]\right\} \tag{1}$$

In equation (1), $c_t \geq 0$ represents the agent's real consumption at moment t, $U(c_t)$ is the consumption utility function, $\beta(C_v, Y_v)$ respects to the discount factor, and C_v and Y_v are, respectively, the economy wide average levels of consumption and income at time v. These two variables are standard (or average) values determined by the whole society and that serve as a reference for the individual agent in setting up her degree of impatience regarding consumption (i.e., her intertemporal discount rate). We will use interchangeably the terms discount rate and rate of time preference; these are not necessarily the same when endogenous discounting is assumed, but according to Meng (2006) they coincide when the discount rate is determined solely by economy wide factors.

A conventional CIES utility function is assumed, i.e., $U(c_t) = (c_t^{1-\theta} - 1)/(1-\theta)$, with $\theta \in (0, +\infty)/\{1\}$ the inverse of the elasticity of intertemporal substitution. This function fulfils the main requirements concerning consumption utility stylized facts, i.e., marginal utility is positive ($U'>0$) but diminishing ($U''<0$).

The resource constraint is a trivial one. We just consider a capital accumulation equation, where $k_t \geq 0$ represents the stock of capital and $\delta \geq 0$ is the depreciation rate:

$$k_{t+1} = y_t - c_t + (1-\delta) \cdot k_t \text{, } k_0 \text{ given} \tag{2}$$

Variable $y_t \geq 0$ corresponds to the representative agent's level of income. Income respects to output as given by a conventional neo-classical production function, $y_t = f(k_t)$. Function f exhibits positive and decreasing marginal returns ($f'>0$, $f''<0$), and, for the analytical treatment of the model, we just take a Cobb-Douglas functional form, $y_t = A \cdot k_t^{\alpha}$, with $A>0$ the technological level and $\alpha \in (0,1)$ the output-capital elasticity. The economy wide level of income at time t can be presented as $Y_t = f(K_t)$.

To solve the problem of maximization of (1) subject to (2), we build up the Hamiltonian function,

$$\aleph(k_t, c_t, q_t) = U(c_t) + \beta(C, Y) \cdot q_{t+1} \cdot \left[f(k_t) - c_t - \delta \cdot k_t \right] \tag{3}$$

In expression (3), q_t respects to the current-value co-state variable (shadow-price) of k_t. First-order conditions are:

$$\aleph_c = 0 \Rightarrow \beta(C,Y)\cdot q_{t+1} = c_t^{-\theta} \tag{4}$$

$$\beta(C,Y)\cdot q_{t+1} - q_t = -\aleph_k \Rightarrow q_t = \left[1+\alpha\cdot A\cdot k_t^{-(1-\alpha)} - \delta\right]\cdot \beta(C,Y)\cdot q_{t+1} \tag{5}$$

$$\lim_{t\to+\infty} k_t \cdot \beta(C,Y)^t \cdot q_t = 0 \quad \text{(transversality condition)} \tag{6}$$

In equilibrium, we have $C_t=c_t$ and $Y_t=y_t$; thus, $\beta(C,Y)=\beta(c_t,y_t)$ in optimality conditions (4) to (6). Combining (4) and (5), the following equation of motion for the representative agent's level of consumption holds,

$$c_{t+1} = \left[\beta(c_t,y_t)\cdot\left(1+\alpha\cdot A\cdot k_{t+1}^{-(1-\alpha)} - \delta\right)\right]^{1/\theta}\cdot c_t \tag{7}$$

The dynamic system relatively to which stability conditions will be discussed is composed by equations (2) and (7). Once again, we call the attention for the similarities between our model and the continuous time version of Meng (2006). As he says, "*Note that compared with the standard Ramsey-Cass-Koopmans model with a constant discount rate, the only difference is that here in Eq. (9) the discount rate depends on consumption and capital, whereas the resource equation (10) remains unchanged.*" page 2676. Equation (9) in Meng's presentation corresponds to a continuous-time version of (7), while his equation (10) has correspondence on our equation (2). The time preference is exogenous to the agent and, hence, it does not disturb the way optimality conditions are derived. Nevertheless, optimality implies a coincidence between aggregates' values from the economy wide point of view and from the point of view of the individual agent.

The signs of the derivatives of the discount factor function in equation (7) are the following: $\beta_c<0$ and $\beta_y>0$. These conditions intend to make the model close to the empirical evidence. They state that the individual rate of time preference increases with the economy's level of consumption, that is, individual impatience rises when the agent observes higher levels of consumption in society. This is a jealousy effect; average consumption matters to the isolated individual in the sense that the willingness to defer consumption in time falls as one sees the overall consumption level rising. This jealousy effect arises in contrast to a wealth effect. When the income of the whole society increases, the isolated agent will be more willing to defer consumption, that is, patience rises. Thus, the rate of time preference falls with an increase in the economy's living standard. This offsetting effect of two countervailing forces leads, in the continuous time version of the model, to local indeterminacy. In the following sections, we ask if this result continues to hold in discrete time.

3. Endogenous Impatience and Neo-Classical Growth

3.1 Linearity in the Discount Rate

In a first version of the model, we consider that the discount rate is linear in its arguments (as does Meng). This implies writing the discount factor as $\beta(c_t, y_t) = 1/(1+\rho(c_t, y_t))$, with $\rho(c_t, y_t) = \rho_0 + \rho_1 \cdot c_t - \rho_2 \cdot y_t$; parameters ρ_0, ρ_1, ρ_2 are all positive values, given the reasoning developed in the last paragraph of the previous section.

The following assumption is central on the development of the model, and it allows for obtaining tractable local stability results,

Assumption 1. The steady state discount rate is $\rho^* = \rho_0$.

The above assumption states that the jealousy effect and the wealth effect are such that they offset each other in the steady state. Under assumption 1, the steady state may be characterized as in proposition 1.

Proposition 1. *Defining a balanced growth path / steady state as the set of constant values (k^*, c^*) that is obtained for $k^* \equiv k_{t+1} = k_t$ and $c^* \equiv c_{t+1} = c_t$, such balanced growth path exists and it is unique.*

Proof: If, in the steady state, assumption 1 holds, then the following relation also holds: $c^* = \frac{\rho_2}{\rho_1} \cdot A \cdot (k^*)^\alpha$. From constraint (2), we find a second relation between the steady state levels of consumption and capital, which is $c^* = \frac{\rho_2 \cdot \delta}{\rho_1 - \rho_2} \cdot k^*$. By solving a system with these two relations, a unique pair (k^*, c^*) is found; this is $k^* = \left[\frac{A\cdot(\rho_1-\rho_2)}{\rho_1\cdot\delta}\right]^{1/(1-\alpha)}$, $c^* = \rho_2\cdot\left(\frac{A}{\rho_1}\right)^{1/(1-\alpha)}\cdot\left(\frac{\rho_1-\rho_2}{\delta}\right)^{\alpha/(1-\alpha)}$ ■

The steady state result derived in the proof of proposition 1 deserves a few comments. First, as one should expect, the higher is the technology level and the lower is the depreciation rate, the higher are the steady state levels of capital and consumption. Second, the steady state as presented implies some constraints over parameters,

i) $\rho_1 \geq \rho_2$. This condition guarantees non-negative k^* and c^*;

ii) $\rho_0 = \frac{\rho_2-(1-\alpha)\cdot\rho_1}{\rho_1-\rho_2}\cdot\delta$. This constraint comes from the steady state evaluation of (7). Observe that the steady state rate of time preference is proportional to the rate of capital depreciation. If capital does not depreciate, the representative agent will not discount future consumption. This is the result of considering that the jealousy effect and the economy's output effect offset each other over the balanced growth path;

iii) $\rho_2 \geq (1-\alpha)\cdot\rho_1$. This boundary constraint avoids the existence of a negative discount rate.

One of our main purposes consists on understanding if the indeterminacy result found in continuous time for $\rho_0, \rho_1, \rho_2 > 0$ continues to hold under the discrete time version of the model. To undertake such an evaluation, we linearize (2) and (7) in the steady state vicinity to obtain the following matricial system,

$$\begin{bmatrix} k_{t+1}-k^* \\ c_{t+1}-c^* \end{bmatrix} = \begin{bmatrix} 1+\rho_0 & -1 \\ \left[\frac{\alpha\cdot c^*}{1+\rho_0}-(1-\alpha)\cdot\delta\right]\cdot\frac{\rho_1\cdot\rho_2\cdot\delta}{(\rho_1-\rho_2)\cdot\theta} & 1+\frac{(1-\alpha)\cdot\rho_1\cdot\rho_2\cdot\delta^2}{\theta\cdot(\rho_1-\rho_2)\cdot(1+\rho_0)}-\frac{\rho_1\cdot c^*}{\theta\cdot(1+\rho_0)} \end{bmatrix}\cdot\begin{bmatrix} k_t-k^* \\ c_t-c^* \end{bmatrix} \quad (8)$$

Stability conditions are presented in proposition 2.

Proposition 2. *Local indeterminacy [i.e., fixed-point stability or the presence of two eigenvalues of the Jacobian matrix in (8) inside the unit circle] requires the validity of the following conditions,*

$$2\cdot(2+\rho_0)+\frac{(1-\alpha)\cdot\rho_1\cdot\rho_2\cdot\delta^2}{\theta\cdot(\rho_1-\rho_2)\cdot(1+\rho_0)}-\frac{[2-(1-\alpha)\cdot\delta]\cdot\rho_1\cdot c^*}{\theta\cdot(1+\rho_0)}>0$$

$$\frac{(1-\alpha)\cdot\rho_1\cdot\delta}{\theta\cdot(1+\rho_0)}\cdot\left[c^*-\frac{\rho_2\cdot\delta}{\rho_1-\rho_2}\right]>0$$

$$\frac{[1-(1-\alpha)\cdot\delta]\cdot\rho_1\cdot c^*}{\theta\cdot(1+\rho_0)}-\rho_0>0.$$

Proof: The trace and the determinant of the Jacobian matrix in system (8) are

$$Tr(J)=2+\rho_0+\frac{(1-\alpha)\cdot\rho_1\cdot\rho_2\cdot\delta^2}{\theta\cdot(\rho_1-\rho_2)\cdot(1+\rho_0)}-\frac{\rho_1\cdot c^*}{\theta\cdot(1+\rho_0)}$$

$$Det(J)=1+\rho_0-\frac{[1-(1-\alpha)\cdot\delta]\cdot\rho_1\cdot c^*}{\theta\cdot(1+\rho_0)}$$

The stability conditions in the proposition are just the trivial conditions that guarantee that the two eigenvalues of J lie inside the unit circle, $1+Tr(J)+Det(J)>0$, $1-Tr(J)+Det(J)>0$ and $1-Det(J)>0$ ■

Stability conditions in proposition 2 are not very informative. One understands that local indeterminacy is guaranteed for some combinations of parameter values but that this is not surely a universal result. The following corollary narrows the possibility of local indeterminacy to values of the technology parameter above a given combination of parameters.

Corollary 1. $A>\frac{\rho_1\cdot\delta}{\rho_1-\rho_2}$ is a necessary condition for local indeterminacy.

The above condition is obtained from the second inequality of proposition 2; the condition presented in the corollary is equivalent to $c^*>\frac{\rho_2\cdot\delta}{\rho_1-\rho_2}$.

A numerical example allows for a better understanding of the model's dynamics. We will attribute concrete values to every parameter except θ; this is our bifurcation parameter. We adopt α=0.3 [as in Meng (2006)]; the depreciation rate is the one assumed in the calibration of a growth model by Guo and Lansing (2002), δ=0.067 [Meng's model ignores capital depreciation; in our model we have stated before the importance of this parameter, since if no depreciation exists, then there is not, as well, steady state discounting]; we take ρ_1=0.2, and given the constraint on ρ_2 (in this case, 0.14≤ρ_2≤0.2), we choose to work with ρ_2=0.16. Under these parameter values, the steady state discount rate is ρ_0=0.0335.

To highlight the result in corollary 1, we take, separately, two values for the technology index; first, we consider A=0.2 and, subsequently, we take A=0.4. Under the selected parameter values, local indeterminacy imposes A>0.335; therefore, the first case implies that indeterminacy is not an admissible stability result, while the second case, with A above the computed threshold value, means that we may find indeterminacy if the first and third conditions of proposition 2 are satisfied as well.

Let us begin by addressing the case A=0.2. Our numerical example involves the following steady state values for consumption and capital: $c^* = 0.1823$ and $k^* = 0.4786$. The Jacobian matrix is presentable as $J = \begin{bmatrix} 1.0335 & -1 \\ 0.0003/\theta & 1-0.0328/\theta \end{bmatrix}$.

Stability conditions are $1+Tr(J)+Det(J)>0 \Rightarrow \theta > 0.0163$, $1-Tr(J)+Det(J)>0 \Rightarrow -0.0008/\theta > 0$ and $1-Det(J)>0 \Rightarrow \theta < 1.003$. We confirm that for a positive value of the utility function parameter, local indeterminacy never holds (second condition is false). First and third conditions imply that for a given interval of values of θ, saddle-path stability is observable (to be precise, saddle-path stability holds for $0.0163 < \theta < 1.003$); outside such interval, instability will prevail. This stability result may be depicted graphically, on a trace-determinant diagram. Regard that $Tr(J) = 2.0335 - 0.0328/\theta$ and $Det(J) = 1.0335 - 0.0336/\theta$; thus, a relation between trace and determinant is easily computed: $Det(J) = 1.0244 \cdot Tr(J) - 1.0488$. Figure 1 draws this line in a trace-determinant diagram, where one observes that the local indeterminacy area (inside the inverted triangle formed by the bifurcation lines) is never crossed. The represented line is limited from above, since as the trace of matrix J reaches 2.0355, parameter θ becomes an infinite value.

*** Figure 1 ***

Consider now the second example, A=0.4. The steady state values of the endogenous variables are $c^* = 0.4907$ and $k^* = 1.2883$. The Jacobian matrix will be $J = \begin{bmatrix} 1.0335 & -1 \\ 0.0051/\theta & 1-0.0925/\theta \end{bmatrix}$. As before, one computes stability conditions:

$1+Tr(J)+Det(J)>0 \Rightarrow \theta > 0.0442$, $1-Tr(J)+Det(J)>0 \Rightarrow 0.0051/\theta > 0$ and $1-Det(J)>0 \Rightarrow \theta < 2.6089$. Thus, in this case local indeterminacy exists as long as $0.0442 < \theta < 2.6089$. Once again, the result becomes more clear if presented graphically; note that $Tr(J)=2.0335-0.0925/\theta$ and $Det(J)=1.0335-0.0874/\theta$, and, therefore, the trace-determinant relation will be: $Det(J)=0.9449 \cdot Tr(J)-0.8876$. Comparing the expression of this line with the one in the previous example, we observe that both lines have to end in the point *Tr*(*J*)=2.0335, since this is the point where the utility function parameter becomes infinite (the elasticity of intertemporal substitution becomes zero); furthermore, this second line is less sloped, and it is precisely this characteristic that will make the line enter the stability / indeterminacy inverted triangle. Stability / indeterminacy will hold for $0.0442 < \theta < 2.6089$, instability prevails if $\theta > 2.6089$ and saddle-path stability requires $\theta < 0.0442$. See figure 2.

*** Figure 2 ***

3.2 Linearity in the Discount Factor

The possibility of an indeterminacy result explored above has taken a specific functional form for the socially determined discount function. In this subsection, we investigate if such result is similar for an alternative specification of such function. Specifically, we now assume, instead of linearity in the discount rate, linearity in the discount factor, i.e., we take $\beta = \beta_0 - \beta_1 \cdot C + \beta_2 \cdot Y$. The signals in the expression have changed relatively to the linear discount rate function, but they intend to express the same as before: a high discount factor is synonymous of more patience (low discount rate), while a low discount factor means less patience (high discount rate). Therefore, $\beta_0, \beta_1, \beta_2 \in [0,1]$.

Assumption 2 is similar to assumption 1.

<u>**Assumption 2**</u>. The steady state discount factor is $\beta^* = \beta_0$.

Given assumption 2, the balanced growth path result is close to the one in proposition 1. Once again, the steady state point (k^*, c^*) exists and it is unique. Under assumption 2, condition $c^* = \frac{\beta_2}{\beta_1} \cdot A \cdot (k^*)^\alpha$ holds, and, therefore, given resource constraint (2), the steady state values of capital and consumption are $k^* = \left[\frac{A\cdot(\beta_1-\beta_2)}{\beta_1\cdot\delta}\right]^{1/(1-\alpha)}$ and $c^* = \beta_2\cdot\left(\frac{A}{\beta_1}\right)^{1/(1-\alpha)}\cdot\left(\frac{\beta_1-\beta_2}{\delta}\right)^{\alpha/(1-\alpha)}$. In this case, the inequality $\beta_1 \geq \beta_2$ must be satisfied. Replacing the steady state values in the difference equation concerning the motion of the consumption variable, the following value for β_0 is obtained: $\beta_0 = \frac{\beta_1-\beta_2}{(1-\delta)\cdot(\beta_1-\beta_2)+\alpha\cdot\delta\cdot\beta_1}$. Because $\beta_0 \leq 1$ must be verified, a new constraint emerges: $\beta_2 \geq (1-\alpha)\cdot\beta_1$.

Similarly to the discount rate linear function, we proceed with the local analysis of the dynamics of the model. The linearized system in the steady state vicinity is now

$$\begin{bmatrix} k_{t+1}-k^* \\ c_{t+1}-c^* \end{bmatrix} = \begin{bmatrix} 1/\beta_0 & -1 \\ \left[\beta_2 - \frac{1-\alpha}{k^*}\cdot\beta_0\right]\cdot\frac{\alpha\cdot\beta_1\cdot\delta\cdot c^*}{\beta_0\cdot(\beta_1-\beta_2)\cdot\theta} & 1+\left[(1-\alpha)\cdot\frac{\beta_0}{k^*}\cdot\frac{\alpha\cdot\beta_1\cdot\delta}{\beta_1-\beta_2}-\frac{\beta_1}{\beta_0}\right]\cdot\frac{c^*}{\theta} \end{bmatrix}\cdot\begin{bmatrix} k_t-k^* \\ c_t-c^* \end{bmatrix} \quad (9)$$

Proposition 3 presents the stability conditions,

Proposition 3. In the linear discount factor function case, local indeterminacy requires the following inequalities to be satisfied,

$$2\cdot\left(\frac{1+\beta_0}{\beta_0}\right)+\left[(1-\alpha)\cdot\frac{\beta_0}{k^*}\cdot\frac{\alpha\cdot\beta_1\cdot\delta}{\beta_1-\beta_2}-\frac{[2-(1-\alpha)\cdot\delta]\cdot\beta_1}{\beta_0}\right]\cdot\frac{c^*}{\theta}>0$$

$$\left[\frac{(1-\alpha)\cdot\delta\cdot\beta_1}{\beta_0}-(1-\alpha)\cdot\frac{\beta_0}{k^*}\cdot\frac{\alpha\cdot\beta_1\cdot\delta}{\beta_1-\beta_2}\right]\cdot\frac{c^*}{\theta}>0$$

$$[1-(1-\alpha)\cdot\delta]\cdot\frac{\beta_1\cdot c^*}{\theta}-(1-\beta_0)>0$$

Proof: The trace and the determinant of the Jacobian matrix in system (9) are:

$Tr(J)=\frac{1+\beta_0}{\beta_0}+\left[(1-\alpha)\cdot\frac{\beta_0}{k^*}\cdot\frac{\alpha\cdot\beta_1\cdot\delta}{\beta_1-\beta_2}-\frac{\beta_1}{\beta_0}\right]\cdot\frac{c^*}{\theta}$ and $Det(J)=\frac{1}{\beta_0}-[1-(1-\alpha)\cdot\delta]\cdot\frac{\beta_1\cdot c^*}{\beta_0\cdot\theta}$. Applying the same set of stability conditions as in proposition 2, we arrive to the obtained inequalities■

Corollary 2. A necessary condition for stability is $A>\frac{\alpha^{1-\alpha}\cdot\beta_1\cdot\delta}{[(\beta_1-\beta_2)\cdot(1-\delta)+\alpha\cdot\beta_1\cdot\delta]^{2\cdot(1-\alpha)}\cdot(\beta_1-\beta_2)^{\alpha}}$. This condition is obtained directly from the second stability expression in proposition 3.

To further discuss local dynamics recover the numerical example used before. Namely, assume, once again, α=0.3 and δ=0.067. Take also β_1=0.2 and β_2=0.16 (these values have correspondence in ρ_1 and ρ_2 of the previous analysis only for analytical convenience; note that given the other parameter values we had to select a value of β_2 obeying $0.14\leq\beta_2\leq 0.2$). In this case, β_0=0.9676 (which corresponds to a discount rate of ρ_0=0.0335, which is precisely the same found in our first case). The indeterminacy necessary condition in the above corollary is, under our example, $A>1.3108$. Thus, this second case requires a higher technology level for local indeterminacy to prevail.

Consider two examples: first, A=0.2 and, second, A=2. For A=0.2, the Jacobian matrix of system (9) is $J=\begin{bmatrix}1.0335 & -1\\ -0.0238/\theta & 1-0.0117/\theta\end{bmatrix}$. The corresponding trace and determinant are $Tr(J)=2.0335-0.0117/\theta$ and $Det(J)=1.0335-0.0359/\theta$. Stability conditions, as presented in proposition 3, indicate that indeterminacy is absent (according to the condition in corollary 2, the value of the technology index is lower than the one necessary to find such outcome); stability conditions applied to our example also say that saddle-path stability exists for $0.0117<\theta<1.0716$, while for any other value of the utility function parameter, instability will be evidenced.

The absence of local indeterminacy can also be verified by looking at the line that relates trace and determinant; in this case, this is $Det(J)=3.0684\cdot Tr(J)-5.2051$. We refrain from representing this line graphically, since it is located qualitatively in the same position as the line in figure 1, i.e., given that its slope is above unity, the line will be below (to the right) of bifurcation line $1-Tr(J)+Det(J)=0$; thus, the region inside the unit circle is never crossed by the computed line.

The other example, with A=2, will be similar to the one characterized in figure 2. The slope of the trace-determinant line will be below one, and therefore this line will be located, partially, inside the unit circle. In this particular case, $c^* = 3.441$ and $k^* = 12.8396$ are the steady state values of variables. The Jacobian matrix comes $J = \begin{bmatrix} 1.0335 & -1 \\ 0.0383/\theta & 1-0.693/\theta \end{bmatrix}$. Trace and determinant are $Tr(J) = 2.0335 - 0.693/\theta$ and $Det(J) = 1.0335 - 0.6779/\theta$. Stability conditions are all satisfied under $0.161<\theta<20.236$. Above the upper bound for θ, the system becomes unstable, and for $\theta<0.161$, saddle-path stability prevails. This result is identical (in qualitative terms) to the one found for the discount rate linearity case (for the level of technology above a given threshold value). Once more, the trace-determinant relation is straightforwardly computed: $Det(J) = 0.9782 \cdot Tr(J) - 0.9557$. Because the slope of this line is below one, we guarantee that the indeterminacy area is crossed, on a way very similar to the one discussed with figure 2.

Our main conclusion is that indeterminacy results will not defer significantly if one considers a linear discount rate function or a linear discount factor function. In both cases, relatively high technology levels guarantee local indeterminacy, as long as the utility function parameter stays within a given interval.

4. Endogenous Impatience and Endogenous Growth

The previous model is now adapted to a scenario of endogenous growth. Basically, two different assumptions are considered relatively to the benchmark setup. First, the neo-classical production function gives place to an AK production function; second, instead of assuming a discount factor function $\beta(c_t, y_t)$ (linear in ρ or in β), we consider a function $\beta(\hat{c}_t, \hat{y}_t)$, where $\hat{c}_t$ and $\hat{y}_t$ represent detrended consumption and income variables, i.e., considering that the original variables grow, under a balanced growth path, at rate γ, we have $\hat{c}_t \equiv \frac{c_t}{(1+\gamma)^t}$ and $\hat{y}_t \equiv \frac{y_t}{(1+\gamma)^t}$; likewise, we define $\hat{k}_t \equiv \frac{k_t}{(1+\gamma)^t}$. The optimal control problem of utility maximization is solved as the original model, and, considering the detrended variables, the system we want to analyze is composed by the following two equations,

$$\hat{k}_{t+1} = \frac{1+A-\delta}{1+\gamma} \cdot \hat{k}_t - \frac{1}{1+\gamma} \cdot \hat{c}_t \tag{10}$$

$$\hat{c}_{t+1} = \left[\beta(\hat{c}_t, \hat{k}_t) \cdot (1+A-\delta)\right]^{1/\theta} \cdot \frac{1}{1+\gamma} \cdot \hat{c}_t \tag{11}$$

We will study the dynamics as before, first by assuming linearity in the discount rate and, on a second moment, by taking linearity in the discount factor.

Consider first $\beta(\hat{c}_t, \hat{y}_t) = 1/(1+\hat{\rho}_t)$, with $\hat{\rho}_t = \rho_0 + \rho_1 \cdot \hat{C}_t - \rho_2 \cdot \hat{Y}_t$. Variables $\hat{C}_t$ and $\hat{Y}_t$ correspond to consumption and income average social levels (detrended), which, in equilibrium, enter in the decision process of the individual agent as endogenous variables. In this case, and reconsidering that $\hat{\rho}^* = \rho_0$, we obtain a steady-state consumption-capital ratio: $\frac{\hat{c}^*}{\hat{k}^*} = \frac{\rho_2}{\rho_1} \cdot A$; since consumption and capital grow at a same steady-state rate, this ratio is also equal to $\frac{c^*}{k^*}$. Using the ratio to evaluate (10) in the steady-state, the growth rate of the considered aggregates is obtained; the result is $\gamma = \frac{\rho_1 - \rho_2}{\rho_1} \cdot A - \delta$. As before, we assume $\rho_1 \geq \rho_2$. Finally, the evaluation of (11) in the steady state requires $\rho_0 = \frac{1+A-\delta}{(1+\gamma)^\theta} - 1$. To guarantee a positive ρ_0, we must have $\theta < \frac{\ln(1+A-\delta)}{\ln(1+\gamma)}$.

The study of local dynamics is undertaken through the linearization of system (10), (11) in the steady state vicinity. Note that the balanced growth path is characterized by a unique consumption-capital ratio and an equilibrium growth rate that can be positive, zero or negative. The linearized system is

$$\begin{bmatrix} \hat{k}_{t+1} - \hat{k}^* \\ \hat{c}_{t+1} - \hat{c}^* \end{bmatrix} = \begin{bmatrix} \dfrac{1+A-\delta}{1+\gamma} & -\dfrac{1}{1+\gamma} \\ \dfrac{\rho_2 \cdot A \cdot \hat{c}^*}{(1+\rho_0)\cdot\theta} & 1 - \dfrac{\rho_1 \cdot \hat{c}^*}{(1+\rho_0)\cdot\theta} \end{bmatrix} \cdot \begin{bmatrix} \hat{k}_t - \hat{k}^* \\ \hat{c}_t - \hat{c}^* \end{bmatrix} \tag{12}$$

Proposition 4 refers to the indeterminacy / stability result.

Proposition 4. *In the endogenous growth model with a linear discount rate function, local indeterminacy cannot hold.*

Proof: The result is easy to achieve once the trace and the determinant are computed:

$$Tr(J) = 1 + \frac{1 + A - \delta}{1 + \gamma} - \frac{\rho_1 \cdot \hat{c}^*}{(1 + \rho_0) \cdot \theta}$$

$$Det(J) = \frac{1 + A - \delta}{1 + \gamma} - \frac{\rho_1 \cdot \hat{c}^*}{(1 + \rho_0) \cdot \theta}$$

One observes that $Det(J) = Tr(J) - 1$ and, thus, the system will rest over the bifurcation line $1 - Tr(J) + Det(J) = 0$, the same is to say that one of the eigenvalues of the Jacobian matrix is equal to 1 independently of the values of parameters■

Let us re-examine the model with a linear discount factor function: $\hat{\beta}_t = \beta_0 - \beta_1 \cdot \hat{C}_t + \beta_2 \cdot \hat{Y}_t$. Steady state results are: $\frac{\hat{c}^*}{\hat{k}^*} = \frac{\beta_2}{\beta_1} \cdot A$, $\gamma = \frac{\beta_1 - \beta_2}{\beta_1} \cdot A - \delta$ and $\beta_0 = \frac{(1+\gamma)^\theta}{1 + A - \delta}$. We must guarantee $\beta_1 \geq \beta_2$ and, as before, $\theta < \frac{\ln(1 + A - \delta)}{\ln(1+\gamma)}$.

Linearization yields,

$$\begin{bmatrix} \hat{k}_{t+1} - \hat{k}^* \\ \hat{c}_{t+1} - \hat{c}^* \end{bmatrix} = \begin{bmatrix} \frac{1 + A - \delta}{1 + \gamma} & -\frac{1}{1+\gamma} \\ \frac{\beta_2 \cdot A \cdot \hat{c}^*}{\beta_0 \cdot \theta} & 1 - \frac{\beta_1 \cdot \hat{c}^*}{\beta_0 \cdot \theta} \end{bmatrix} \cdot \begin{bmatrix} \hat{k}_t - \hat{k}^* \\ \hat{c}_t - \hat{c}^* \end{bmatrix} \quad (13)$$

Trace and determinant of the Jacobian matrix are:

$$Tr(J) = 1 + \frac{1 + A - \delta}{1 + \gamma} - \frac{\beta_1 \cdot \hat{c}^*}{\beta_0 \cdot \theta}$$

$$Det(J) = \frac{1 + A - \delta}{1 + \gamma} - \frac{\beta_1 \cdot \hat{c}^*}{\beta_0 \cdot \theta}$$

A same type of result as the one in the previous specification is obtained, i.e., $1 - Tr(J) + Det(J) = 0$ holds, and therefore local indeterminacy is never found since one of the eigenvalues of J stays over the unit circle.

5. Production Externalities

We consider in this section a variation of the benchmark neo-classical model that is also able to generate indeterminacy. This variation modifies two of the fundamental hypothesis of the model. First, we introduce an externality in production, and thus production may be subject to increasing returns to scale; second, we let the discounting function depend only on the aggregate level of consumption [as in the externalities version of Meng's model]. Analytically, these two assumptions are translated as follows,

i) $y_t = f(k_t, K_t)$. We take a Cobb-Douglas production function $y_t = A \cdot k_t^{\alpha} \cdot K_t^{\eta}$, with $\eta \in (0,1)$;

ii) $\beta(C_t) = 1/(1+\rho(C_t))$, with $\rho(C_t) = \rho_0 + \rho_1 \cdot C_t$. We consider only the linear discount rate function case, since, as in the benchmark neo-classical model, the linear discount factor case produces very similar results.

Solving the model for the representative agent, one will have

$$k_{t+1} = A \cdot k_t^{\alpha+\eta} - c_t + (1-\delta) \cdot k_t \tag{14}$$

$$c_{t+1} = \left[\beta(c_t) \cdot \left(1 + (\alpha+\eta) \cdot A \cdot k_{t+1}^{-[1-(\alpha+\eta)]} - \delta\right)\right]^{1/\theta} \cdot c_t \tag{15}$$

A new assumption regarding the steady state level of the discount rate is needed,

Assumption 3. The steady state discount rate is some constant $\rho^* > \rho_0$.

Under assumption 3, the following result is straightforward,

Proposition 5. *In the socially determined time preference model with externalities in the production of final goods and as long as assumption 3 holds, the steady state exists and it is unique.*

Proof: Under assumption 3, the discount function implies the following balanced growth path value for consumption: $c^* = \dfrac{\rho^* - \rho_0}{\rho_1}$. From (15), a unique steady state stock of capital is determined: $k^{*}=\left[\frac{(\alpha+\eta)\cdot A}{\rho^{*}+\delta}\right]^{1/[1-(\alpha+\eta)]}$. Finally, the steady state is also characterized by the existence of some ρ^{*} equilibrium value; from equation (14), one understands that this value is the solution of the equation $\frac{\rho^{*}-\rho_0}{\rho_1}=A\cdot\left[\frac{(\alpha+\eta)\cdot A}{\rho^{*}+\delta}\right]^{(\alpha+\eta)/[1-(\alpha+\eta)]}-\delta\cdot\left[\frac{(\alpha+\eta)\cdot A}{\rho^{*}+\delta}\right]^{1/[1-(\alpha+\eta)]}$ ■

Corollary 3. Increasing marginal returns to capital must hold, once the externality effect is considered. Analytically, $\alpha+\eta>1$.

The above condition guarantees a positive steady state discount rate. To understand why this is so, take the last equation in the proof of proposition 5. Note that such equality requires $\left(\frac{\alpha+\eta}{\rho^{*}+\delta}\right)^{-1}-\delta>0$ if one wants ρ^{*} to be positive; the presented inequality is equivalent to $\rho^{*}>[(\alpha+\eta)-1]\cdot\delta$, which, in turn, requires $\alpha+\eta>1$.

The linearization of the model around the steady state point leads to:

$$\begin{bmatrix} k_{t+1}-k^{*} \\ c_{t+1}-c^{*} \end{bmatrix}=$$
$$\begin{bmatrix} 1+\rho^{*} & -1 \\ \frac{\rho^{*}-\rho_0}{\theta\cdot\rho_1}\cdot\left[((\alpha+\eta)-1)\cdot\frac{\rho^{*}+\delta}{k^{*}}\right] & 1-\frac{\rho^{*}-\rho_0}{\theta\cdot\rho_1\cdot(1+\rho^{*})}\cdot\left[((\alpha+\eta)-1)\cdot\frac{\rho^{*}+\delta}{k^{*}}+\rho_1\right] \end{bmatrix}\cdot\begin{bmatrix} k_t-k^{*} \\ c_t-c^{*} \end{bmatrix} \quad (16)$$

Stability conditions are given by proposition 6.

Proposition 6. *The model with socially determined time preference and technological externalities is locally indeterminate if the following conditions hold:*

$$2\cdot(2+\rho^{*})-\frac{\rho^{*}-\rho_0}{\theta\cdot\rho_1\cdot(1+\rho^{*})}\cdot\left[((\alpha+\eta)-1)\cdot\frac{\rho^{*}+\delta}{k^{*}}+\rho_1+\rho_1\cdot(1+\rho^{*})\right]>0;$$

$$\frac{\rho^{*}-\rho_0}{\theta}\cdot\left[((\alpha+\eta)-1)\cdot\frac{\rho^{*}+\delta}{k^{*}}-\rho_1\cdot\rho^{*}\right]>0;$$

$$(\theta-1)\cdot\rho^{*}+\rho_0<0.$$

Proof: Trace and determinant of the Jacobian matrix in (16) are, respectively, $Tr(J) = 2 + \rho^* - \frac{\rho^* - \rho_0}{\theta \cdot \rho_1 \cdot (1+\rho^*)} \cdot \left[((\alpha+\eta)-1) \cdot \frac{\rho^* + \delta}{k^*} + \rho_1 \right]$ and $Det(J) = 1 + \frac{\theta - 1}{\theta} \cdot \rho^* + \frac{\rho_0}{\theta}$. The conditions in the proposition are the direct result of considering stability relations $1 + Tr(J) + Det(J) > 0$, $1 - Tr(J) + Det(J) > 0$ and $1 - Det(J) > 0$, into the discussed setup■

Corollary 4. Two of the necessary conditions for indeterminacy are:

i) $((\alpha+\eta)-1) \cdot \frac{(\rho^* + \delta) - (\alpha+\eta)\cdot\delta}{(\alpha+\eta)\cdot\left(\rho^* - \rho_0\right)} \cdot (\rho^* + \delta) > \rho^*$;

ii) $\rho^* > \frac{\rho_0}{1-\theta}$.

The first inequality is equivalent to the second condition in proposition 6, while the second is obtained directly from the third condition in proposition 6, i.e., $1 - Det(J) > 0$. Observe that this last inequality requires $\theta<1$.

Let us consider a numerical example to confirm the possibility of local indeterminacy. We assume α=0.3, η=0.8, $\rho^* = 0.04$, $\rho_1 = 0.05$ and δ=0.067. To obey to the first condition of corollary 4, one should have $\rho_0 > 0.0319$; we consider $\rho_0 = 0.035$. Relatively to the value of *A*, the last relation in the proof of proposition 5 is equivalent to:

$$A = \left[\frac{\frac{\rho^* - \rho_0}{\rho_1}}{\left[\frac{\alpha+\eta}{\rho^*+\delta}\right]^{(\alpha+\eta)/[1-(\alpha+\eta)]} - \delta \cdot \left[\frac{\alpha+\eta}{\rho^*+\delta}\right]^{1/[1-(\alpha+\eta)]}} \right]^{1-(\alpha+\eta)}$$

, which, for the selected parameter values will be *A*=0.0863.

Under this numerical example, steady state values of variables are $c^* = 0.1$ and $k^* = 3.3098$ and the Jacobian matrix is $J = \begin{bmatrix} 1.04 & -1 \\ 0.0003/\theta & 1-0.0051/\theta \end{bmatrix}$. The respective trace and determinant come: $Tr(J) = 2.04 - 0.0051/\theta$ and $Det(J) = 1.04 - 0.005/\theta$. The computation of stability conditions lead to the result of local indeterminacy for 0.0025<θ<0.125. The system is saddle-path stable for θ<0.0025 and unstable if θ>0.125. This result is represented graphically in figure 3; the line in this graphic is $Det(J) = 0.9804 \cdot Tr(J) - 0.9608$.

*** Figure 3 ***

Local indeterminacy is found for extremely low values of θ (extremely high values of the elasticity of intertemporal substitution) and for values of ρ_0 near ρ^*.

6. Endogenous Labor Supply and Endogenous Growth

A last exercise consists in assuming leisure as an argument of the utility function, that is, we assume a leisure-labor trade-off and an optimal selection of the allocation of time by the representative agent. This variation from our benchmark model follows Meng's specification in the sense that it takes the economy wide level of income as the only argument of the discount function, but it departs from such specification by considering endogenous growth, i.e., an AK production function. Therefore, the stability result can be explored in a hybrid framework: we have seen that endogenous discounting and endogenous growth did not produce indeterminacy under the conventional Ramsey optimal growth model; here, we may investigate if this result continues to hold if workload optimization is considered along with consumption optimization.

Assume that the representative agent solves the following maximization problem:

$$MaxU_0 = \sum_{t=0}^{+\infty} \left\{ U(c_t, \ell_t) \cdot \left[\prod_{v=0}^{t} \beta(Y_v) \right] \right\} \tag{17}$$

In problem (17), variable $\ell_t \in (0,1)$ is the share of the representative agent's time associated to labor, and thus $1-\ell_t$ will be the share of time allocated to leisure (we assume that the representative agent is endowed with one unit of time, and thus the referred shares coincide with the amount of time that the agent spends working and resting). We consider a utility function that is concave regarding consumption but linear in terms of leisure. Taking $m>0$, the adopted functional form is: $U(c_t, \ell_t) = \left(c_t^{1-\theta} - 1\right)/(1-\theta) + m \cdot (1-\ell_t)$.

The resource constraint is, again, (2), but now the production function is $y_t = f(k_t, \ell_t) = A \cdot k_t \cdot \ell_t$. This production function reveals that there are constant marginal returns of capital (it is an AK function) and that only a part of the available working hours are effectively used to generate wealth, i.e., if all the agent's available time was allocated to work, then $y_t = A \cdot k_t$; in reality, only a fraction of the available time is allocated to the production of final goods and therefore only a fraction of the potential output is effectively produced.

The discount factor is, in this case, $\beta(Y) = 1/(1+\rho_t)$, with $\rho_t = \rho_0 - \rho_2 \cdot Y_t$.

The Hamiltonian function of this problem is (q_t is a co-state variable),

$$\aleph(k_t, c_t, \ell_t, q_t) = U(c_t, \ell_t) + \beta(Y) \cdot q_{t+1} \cdot \left[f(k_t, \ell_t) - c_t - \delta \cdot k_t \right] \tag{18}$$

First-order conditions come,

$$\aleph_c = 0 \Rightarrow \beta(Y) \cdot q_{t+1} = c_t^{-\theta} \tag{19}$$

$$\aleph_\ell = 0 \Rightarrow \beta(Y) \cdot q_{t+1} = \frac{m}{A \cdot k_t} \tag{20}$$

$$\beta(Y) \cdot q_{t+1} - q_t = -\aleph_k \Rightarrow q_t = \left[1 + A \cdot \ell_t - \delta\right] \cdot \beta(Y) \cdot q_{t+1} \tag{21}$$

$$\lim_{t \to +\infty} k_t \cdot \beta(Y)^t \cdot q_t = 0 \quad \text{(transversality condition)} \tag{22}$$

As in previous cases, $Y_t = y_t$ in equilibrium, i.e., $\beta(Y) = \beta(y_t)$. From the optimality conditions, one withdraws the following system of difference equations,

$$k_{t+1} = A \cdot k_t \cdot \ell_t - \left(\frac{A \cdot k_t}{m} \right)^{1/\theta} + (1-\delta) \cdot k_t \tag{23}$$

$$\ell_{t+1} = \frac{1}{A} \cdot \left[\frac{k_{t+1}}{\beta(y_t) \cdot k_t} - (1-\delta) \right] \tag{24}$$

with $c_t = \left(\frac{A \cdot k_t}{m} \right)^{1/\theta}$. System (23)-(24) has some relevant differences relatively to the models one has analyzed before. There is a contemporaneous relation between

consumption and stock of capital and therefore we need to analyze the dynamic behavior of only one of these variables, but another endogenous variable with attached dynamic motion arises: the share of labor time.
To study the dynamics of the model, we begin by stating assumption 4.

Assumption 4. The steady state discount rate is some constant $\rho^* < \rho_0$.

With assumption 4, proposition 7 comes

Proposition 7. *In the socially determined time preference endogenous growth model with endogenous labor supply, under assumption 4 the steady state exists and it is unique.*

Proof: Defining the steady state as the long run locus for which $k^* \equiv k_{t+1} = k_t$, $c^* \equiv c_{t+1} = c_t$ and $\ell^* \equiv \ell_{t+1} = \ell_t$, we make use of the discount function, of equations (23) and (24) and of the relation between capital and consumption withdrawn from optimality conditions, to compute the following unique values: $\ell^* = \frac{1}{A}\cdot(\rho^* + \delta)$, $k^* = \frac{\rho_0 - \rho^*}{\rho_2 \cdot (\rho^* + \delta)}$ and $c^* = \left[\frac{A}{m}\cdot\frac{\rho_0 - \rho^*}{\rho_2 \cdot (\rho^* + \delta)}\right]^{\theta}$ ■

The steady state result imposes a specific value for the balanced growth path of the equilibrium discount rate. This is such that the technology level has to be given by $A = m \cdot (\rho^*)^{\theta} \cdot \left[\frac{\rho_0 - \rho^*}{\rho_2 \cdot (\rho^* + \delta)}\right]^{\theta - 1}$.

Linearizing in the steady state vicinity,

$$\begin{bmatrix} k_{t+1} - k^* \\ \ell_{t+1} - \ell^* \end{bmatrix} = \begin{bmatrix} 1 + \frac{\theta - 1}{\theta}\cdot\rho^* & A\cdot\frac{\rho_0 - \rho^*}{\rho_2\cdot(\rho^* + \delta)} \\ \frac{\rho_2\cdot(\rho^* + \delta)}{A}\cdot\left[\frac{(\theta - 1)\cdot(1 + \rho^*)\cdot\rho^*}{\theta\cdot(\rho_0 - \rho^*)} - 1\right] & 1 + \rho^* - \frac{\rho_0 - \rho^*}{\rho^* + \delta} \end{bmatrix} \cdot \begin{bmatrix} k_t - k^* \\ \ell_t - \ell^* \end{bmatrix} \tag{25}$$

Proposition 8. *The indeterminacy conditions of the endogenous growth / endogenous time preference / endogenous labor supply model are the following:*

i) $\theta > \dfrac{2\cdot\rho^*}{3+\rho_0+3\cdot\rho^*-2\cdot\dfrac{\rho_0-\rho^*}{\rho^*+\delta}}$;

ii) $\theta < \dfrac{(\rho_0+\delta)\cdot\rho^*}{(\rho^*+\delta)\cdot(2\cdot\rho^*-\rho_0)+(\rho_0-\rho^*)}$;

iii) $\theta > \dfrac{(\rho_0-\rho^*)\cdot\rho^*}{(\rho_0-\rho^*)\cdot(1+\rho^*)-(\rho^*+\delta)\cdot\rho_0}$;

iv) $\rho_0 > \dfrac{\rho^*\cdot(1+\rho^*)}{1-\delta}$.

Proof: Trace and determinant of the matrix in system (25) are $Tr(J)=2+\dfrac{2\cdot\theta-1}{\theta}\cdot\rho^*-\dfrac{\rho_0-\rho^*}{\rho^*+\delta}$ and $Det(J)=1+\rho_0-\dfrac{\rho_0-\rho^*}{\rho^*+\delta}\cdot\left(1+\dfrac{\theta-1}{\theta}\cdot\rho^*\right)$. The first stability condition in the proposition is directly computed from $1+Tr(J)+Det(J)>0$, the second from $1-Tr(J)+Det(J)>0$ and the third from $1-Det(J)>0$; the fourth condition in the proposition is a necessary condition for $1-Det(J)>0$ to hold■

Corollary 5. Parameters *A*, *m* and ρ_2 are irrelevant for the analysis of stability. This is a straightforward conclusion that one reaches by looking at the expressions of the trace and determinant of the Jacobian matrix in (25).

Under reasonable parameter values, local indeterminacy is absent. To confirm this, take, as usual, δ=0.067, and consider $\rho^*=0.03$ (other numerical examples for other reasonable values of these two parameters produce a similar result of no indeterminacy; given the practical impossibility of presenting meaningful general results, we just explore this example). For the chosen parameter values, condition iv) in proposition 8 implies that $\rho_0>0.0331$; for conditions ii) and iii) to be simultaneously satisfied, we must have $\rho_0<0.0327$. The two constraints on the value of ρ_0 are incompatible, and thus the requirements for local indeterminacy are not fulfilled.

Take, for instance, $\rho_0=0.032$. This value satisfies one of the boundary conditions on the parameter but not the other. In this example, $Tr(J)=2.0394-0.03/\theta$ and $Det(J)=1.0108+0.0006/\theta$; the trace-determinant line is

$Det(J) = 1.05 - 0.02 \cdot Tr(J)$. Now, assume $\rho_0 = 0.034$; in this case, the first constraint on ρ_0 is satisfied but the second is not. The trace is $Tr(J) = 2.0188 - 0.03/\theta$ and the determinant is $Det(J) = 0.9915 + 0.0012/\theta$; now, one has the following relation: $Det(J) = 1.072 - 0.04 \cdot Tr(J)$. By violating two different stability conditions, the computed trace-determinant relations are incompatible with the existence of local indeterminacy. Figure 4 presents these two lines, revealing that the inverted triangle of stability is not crossed by any of them.

*** Figure 4 ***

Figure 4 represents solely the quadrant of the trace-determinant relation where these are both positive. The relevant point is that the two presented lines are bounded for small intervals of values of trace and determinant, in order to allow for a positive and finite value for θ. We observe that for admissible values of this parameter the system is, in the first case ($\rho_0 = 0.032$) unstable, and in the second case ($\rho_0 = 0.034$) saddle-path stable. Thus, saddle-path stability is admissible for values of ρ_0 relatively far (and above) ρ^*.

7. Global Dynamics

Two dimensional dynamic systems in discrete time are known to eventually produce nonlinear long term motion. Cycles of various periodicities, quasi-periodicity and chaos may arise after the transition from fixed-point stability to instability or saddle-path stability, through a bifurcation process. Global dynamics can only be addressed resorting to numerical examples (i.e., with concrete values attributed to the various parameters). Recovering the examples of previous sections, it is possible to investigate if the found bifurcation points mean the occurrence of cycles or if, as the local analysis shows, the transition from stability to instability is the only dynamic feature that is encountered. By exploring the different examples, one finds that cycles arise solely on the production externalities model.[1] The endogenous cycles appear below the lower bound of the interval of values of θ that allow for stability. The flip

[1] This analysis was made resorting to IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of M. Lines and A. Medio. The figures in this section were drawn using this software.

bifurcation, occurring at θ=0.0025, triggers a process of period doubling bifurcations that leads to chaotic motion for extremely small values of the parameter of the utility function.

Figure 5 displays the respective bifurcation diagram (confirm that the bifurcation point is, in fact, θ=0.0025). Figures 6, 7 and 8 complement the graphical presentation by representing an attracting set (the set of long term values to which the system converges) and the long term time series of consumption and capital. These last three figures are presented for a value of θ for which chaos exists – in this illustration, consumption and capital time series will never converge to the steady state and they will not, as well, diverge to infinity.

As a result, we might say that endogenous time preference can generate long term endogenous business cycles but only under some extreme circumstances (externalities in the production of final goods and an extremely high elasticity of intertemporal substitution).

8. Conclusions

We have explored a standard discrete time optimal control growth model, where the rate of time preference is endogenous and socially determined. The representative agent intertemporal preference is influenced by the aggregate level of consumption (more economy wide consumption increases individual impatience) and by the aggregate level of income (an economy with a higher capacity to generate wealth exerts a positive effect over individual patience). Several versions of the model were addressed, namely Ramsey-like neo-classical and endogenous growth setups (where endogenous discounting was modelled through, both, a linear discount rate function and a linear discount factor function), a framework where externalities in the production of final goods were assumed and, finally, a scenario with leisure as an argument of the utility function.

We have confirmed the continuous time result of local indeterminacy as a stability result frequently obtained. In terms of local dynamics, the conventional neo-classical model and the externalities model allow for a variety of stability results (indeterminacy / fixed-point stability, saddle-path stability and indeterminacy), depending on values of parameters. Endogenous growth models with endogenous time preference lead to a bifurcation result independently of parameter values, and thus local indeterminacy never holds. The endogenous leisure model can present unstable or saddle-path stable dynamic outcomes, however indeterminacy was not encountered.

The only model where bifurcations lead to cycles and chaotic motion is the one with increasing returns due to production externalities. In this model, we regard that an extremely high elasticity of intertemporal substitution implies a flip bifurcation that leads to a period doubling route to chaos. Therefore, the socially determined time preference framework is capable of generating long term endogenous fluctuations, but these are, in fact, a rare phenomenon under the discussed type of modelling specification.

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Figures

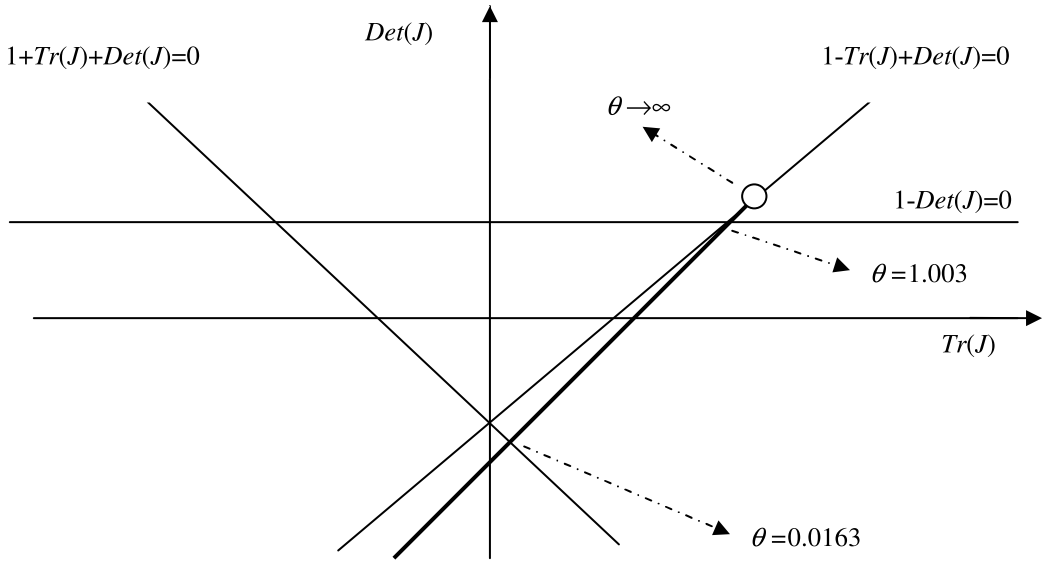

Figure 1 – Trace-determinant relation in the neo-classical growth model with a linear discount rate function (*A*=0.2).

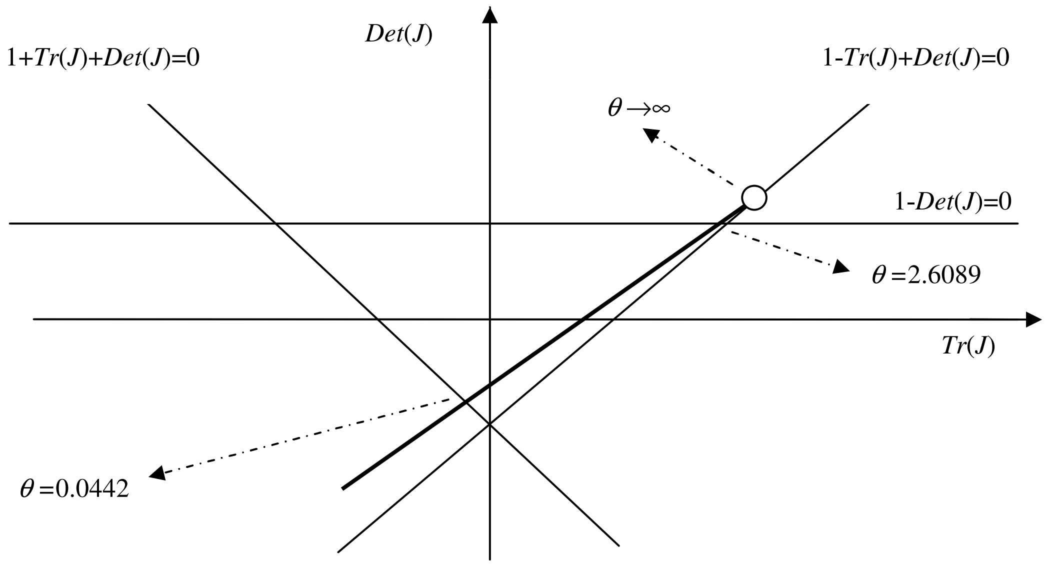

Figure 2 – Trace-determinant relation in the neo-classical growth model with a linear discount rate function (*A*=0.4).

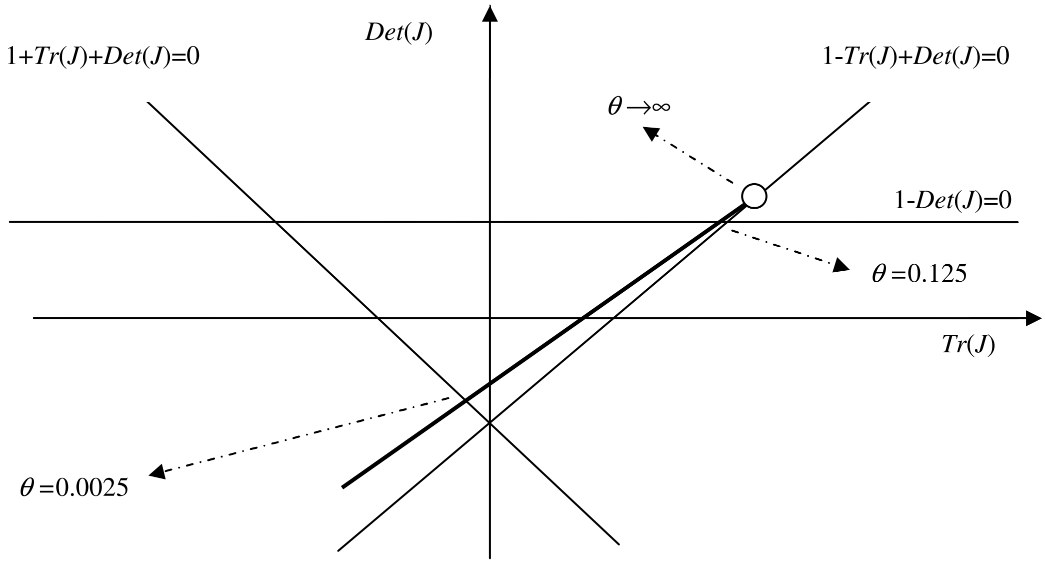

Figure 3 – Trace-determinant relation in the model with productive externalities.

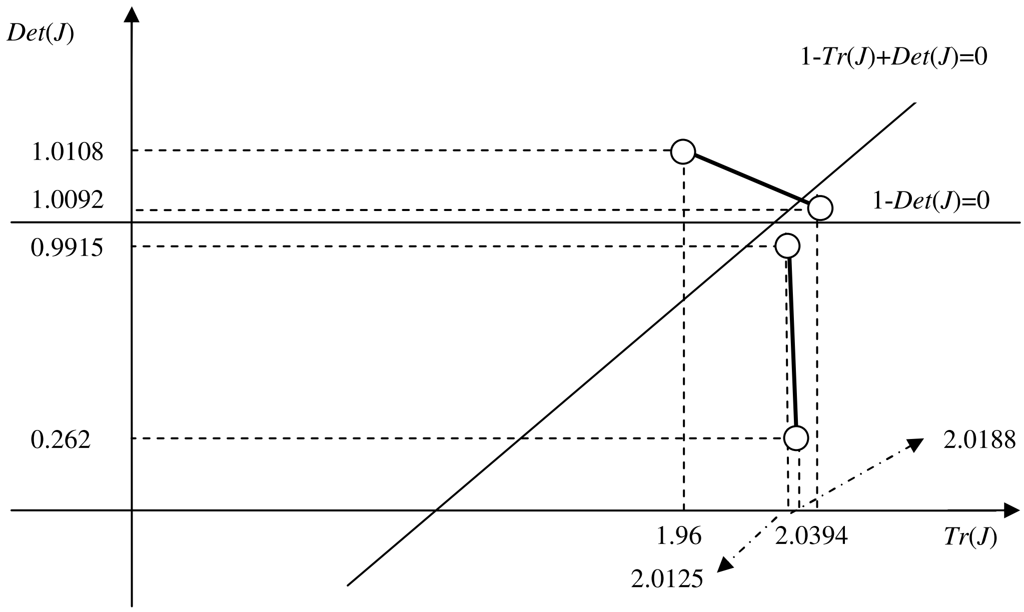

Figure 4 – Trace-determinant relation in the model with endogenous labor supply.

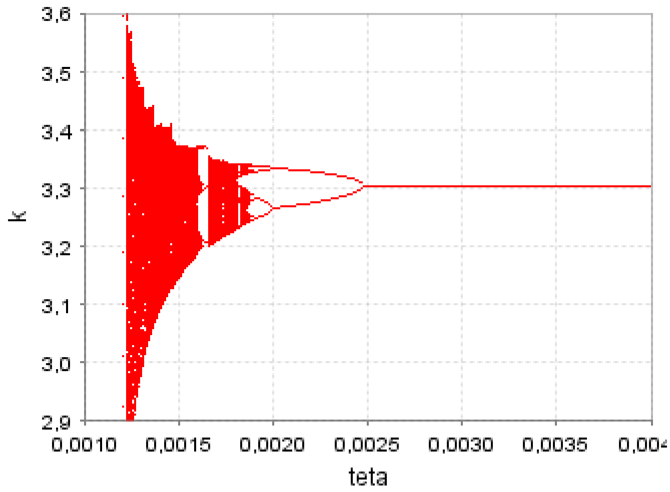

Figure 5 – Bifurcation diagram (externalities model).

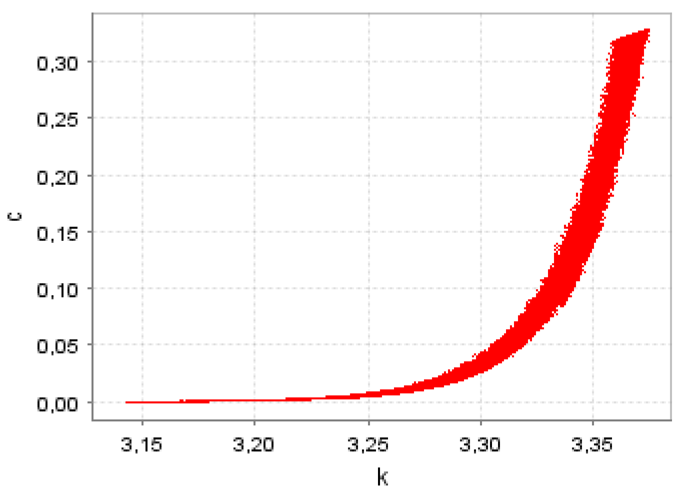

Figure 6 – Attractor, θ=0.0015 (externalities model).

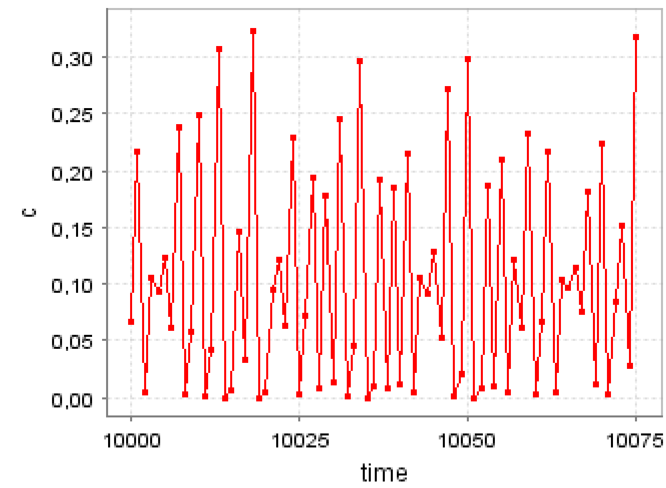

Figure 7 – Consumption long term time series, θ=0.0015 (externalities model).

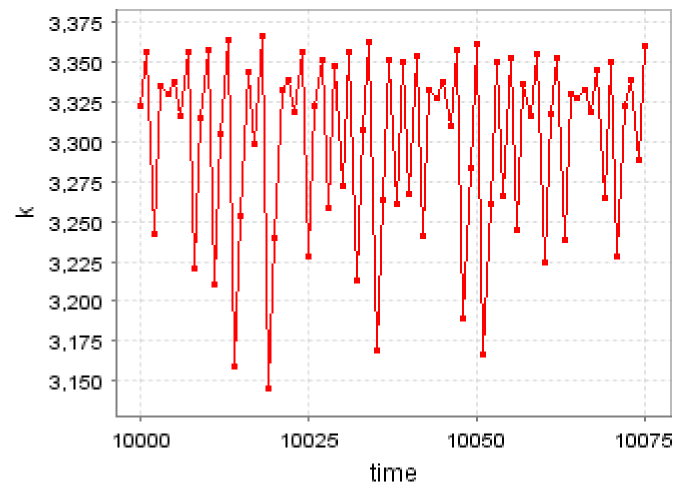

Figure 8 – Capital long term time series, θ=0.0015 (externalities model).