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When Leisure Becomes Excessive: a Bifurcation Result in Endogenous Growth Theory

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Abstract: The traditional assumption concerning endogenous labor supply in models of economic growth is that utility increases with leisure, independently of the specific time allocation of the representative agent observed at a given moment. In this note, we explore the consequences, over dynamic stability, of assuming that the agent dislikes having free time in excess, i.e., of considering that the marginal utility of leisure is not necessarily positive for every value of the leisure share (in particular, for high values of this share). By including this assumption in a typical AK endogenous growth model, we find that the system will rest, independently of parameter values, on a bifurcation line.

Keywords: Labor-leisure choice, Leisure utility, Endogenous growth, Dynamic analysis.

JEL classification: O41, J22, C61.

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1. Introduction

In modern societies it is legitimate to ask whether there is always a positive relation between leisure and the utility leisure brings. Unemployed people or people working at partial time often dislike having too much free time and most of us find personal realization and social recognition in work. This idea is not taken into account by macroeconomic models, namely the models of growth and business cycles in which the endogenous determination of the labor-leisure choice is central to the analysis [among many others, these models include Stokey and Rebelo (1995), Ortigueira (2000) and Duranton (2001)].

In this note, we explore the consequences of assuming that the representative agent prefers an intermediate share of leisure time than extreme values (no leisure or too much leisure). These consequences are addressed under an endogenous growth setup. Interesting dynamic results are obtained in what concerns, both, transitional dynamics and the long term balanced growth path.

The argument we propose is not uncontroversial and it should be understood in relative terms. Some societies value leisure more than others and, thus, the point in which individuals begin to withdraw less utility from leisure as leisure rises varies from one social context to another. For instance, Glaeser, Sacerdote and Scheinkman (2003) and Blanchard (2004) highlight the difference in hours worked in Europe and North-America; the difference can be explained, in the view of these authors, by a cultural predilection for leisure that has gained weight in Europe since the 1960s. Americans work more hours not only because this allows them to increase their income but also because there is a cultural context that inhibits individuals from getting utility from leisure when they do not participate or participate scarcely on the effort of creating value to the economy they belong to. Although this seems a more pronounced tendency in North-America then in Europe, this is indeed a trend that we can identify all over the developed world.

The note is organized as follows. Section 2 describes the model's features, section 3 addresses the properties of the steady-state, section 4 analyzes local stability and section 5 concludes.

2. A Model of Excess of Leisure

Assume a representative agent that maximizes the following sequence of utility functions over an infinite horizon,

$$U_0 = \sum_{t=0}^{+\infty} \beta^t \cdot U(c_t, 1-\ell_t) \tag{1}$$

In expression (1), $c_t \geq 0$ stands for the real level of consumption and $\ell_t \in [0,1]$ is the share of the agent's time allocated to work; obviously, $1-\ell_t$ is the share of time allocated to leisure. Parameter $\beta \in [0,1]$ is the discount factor. The utility function takes the following functional form,

$$U(c_t, 1-\ell_t) = \ln\left[\frac{c_t}{(1-\ell_t)^{m\cdot(1-\ell_t)}}\right],\ m>0 \tag{2}$$

Under function (2), consumption and leisure produce utility separately. Concerning consumption, marginal utility is positive and diminishing, as conventionally assumed. In what respects leisure, we consider that utility rises with leisure when the time allocated to working hours is relatively high, but as the unoccupied time increases the utility withdrawn from leisure falls – the representative agent dislikes having too much free time. Figure 1 represents the relation between leisure and utility for a constant level of consumption $\bar{c}$. In the limit circumstance in which the agent does not work at all, she will not withdraw any utility from the free time she possesses.

*** Figure 1 ***

The resource constraint is a trivial capital accumulation equation,

$$k_{t+1} = \tilde{y}_t - c_t - (1-\delta)\cdot k_t,\ k_0 \text{ given} \tag{3}$$

Variable $k_t \geq 0$ respects to the stock of physical capital, $\delta \geq 0$ is the rate of capital depreciation and $\tilde{y}_t$ stands for the effective level of income. This last variable is considered in contrast with the potential level of income, which is given by a constant returns production function, i.e., $y_t = A \cdot k_t$, with $A>0$ the level of technology. The potential level of income is defined as the output that is generated when the available working hours are integrally used in production. Normalizing the amount of the agent's time to 1, the amount of effective working hours is ℓ_t. Therefore, assuming that production is proportional to the number of hours worked, we should consider $\tilde{y}_t = \ell_t \cdot k_t$.

3. Steady-State Existence and Uniqueness

Let p_t be the co-state variable of k_t. The current-value Hamiltonian function of the proposed problem is:

$$\aleph(k_t, p_t, c_t) = U(c_t, 1-\ell_t) + \beta \cdot p_{t+1} \cdot (\tilde{y}_t - c_t - \delta \cdot k_t) \tag{4}$$

First-order conditions are,

$$\aleph_c = 0 \Rightarrow \beta \cdot p_{t+1} = c_t^{-1} \tag{5}$$

$$\aleph_\ell = 0 \Rightarrow m \cdot [1 + \ln(1-\ell_t)] + \beta \cdot p_{t+1} \cdot A \cdot k_t = 0 \tag{6}$$

$$\beta \cdot p_{t+1} - p_t = -\aleph_k \Rightarrow p_t = (1 + A \cdot \ell_t - \delta) \cdot \beta \cdot p_{t+1} \tag{7}$$

$$\lim_{t \to +\infty} k_t \cdot \beta^t \cdot p_t = 0 \quad \text{(transversality condition)} \tag{8}$$

Under an endogenous growth setup, we define the steady state as the long run locus in which: *i*) the labor share is constant, $\ell^* \equiv \ell_{t+1} = \ell_t$, and *ii*) consumption and capital grow at a same rate, $\frac{c^*}{k^*} \equiv \frac{c_{t+1}}{k_{t+1}} = \frac{c_t}{k_t}$; the second point is a straightforward consequence of the first, given the shape of constraint (3).

Proposition 1. *The steady state exists and it is unique.*

Proof: Let γ>-1 be the growth rate of k_t and c_t in the steady state. Hence, we can define variables that do not grow in the balanced growth path, $\hat{k}_t \equiv \frac{k_t}{(1+\gamma)^t}$ and

$\hat{c}_t \equiv \dfrac{c_t}{(1+\gamma)^t}$. From the optimality conditions, after replacing the original variables by $\hat{k}_t$ and $\hat{c}_t$, we get the following system,

$$\hat{k}_{t+1} = \frac{1 + A \cdot \ell_t - \delta}{1+\gamma} \cdot \hat{k}_t - \frac{1}{1+\gamma} \cdot \hat{c}_t \qquad (9)$$

$$\hat{c}_{t+1} = \frac{\beta}{1+\gamma} \cdot (1 + A \cdot \ell_{t+1} - \delta) \cdot \hat{c}_t \qquad (10)$$

with

$$\ell_t = \frac{\exp\left(\dfrac{A}{m} \cdot \dfrac{\hat{k}_t}{\hat{c}_t} + 1\right) - 1}{\exp\left(\dfrac{A}{m} \cdot \dfrac{\hat{k}_t}{\hat{c}_t} + 1\right)} \qquad (11)$$

Computing steady state relations, one obtains

$$\frac{\hat{c}^*}{\hat{k}^*} = \frac{1-\beta}{\beta} \cdot (1+\gamma) \qquad (12)$$

$$\ell^* = \frac{1 + \gamma - \beta \cdot (1-\delta)}{\beta \cdot A} = \frac{\exp\left(\dfrac{A}{m} \cdot \dfrac{\beta}{1-\beta} \cdot \dfrac{1}{1+\gamma} + 1\right) - 1}{\exp\left(\dfrac{A}{m} \cdot \dfrac{\beta}{1-\beta} \cdot \dfrac{1}{1+\gamma} + 1\right)} \qquad (13)$$

The steady-state exists in the form we have defined it (a constant consumption-capital ratio and a constant labor share). To confirm that the steady state is unique, one just has to prove that the growth rate γ is, under (13), a unique value. We have two expressions involving γ. The first one is a linear function with a positive slope, that starts at $-(1-\delta)/A$ (this is the value for which γ=-1); note that the constraint $\beta \cdot (1-\delta) \leq 1+\gamma \leq \beta \cdot (1+A-\delta)$ must hold in order for ℓ^* to be an admissible value. The second expression corresponds to a decreasing function of γ, with $\lim_{\gamma \to -1} \dfrac{\exp(\cdot) - 1}{\exp(\cdot)} = 1$

and $\lim_{\gamma \to +\infty} \frac{\exp(\cdot)-1}{\exp(\cdot)} = 1 - 1/\exp(-1) \approx 0.6321$. These two lines intersect in one and only one point and, thus, the steady state growth rate is unique; also unique is the share of time allocated to labor and, according to (12), the consumption-capital share. Figure 2 draws the intersection between the two expressions in (13).

*** Figure 2 ***

■

4. Local Dynamics

We are concerned with understanding if the steady state as defined in the previous section is achievable independently of initial values (k_0,c_0) in the vicinity of the steady state, i.e., if there is local stability. The evaluation of the dynamics in the neighbourhood of (k^*, c^*) produces a bifurcation result, as stated in proposition 2.

Proposition 2. *The system rests on a bifurcation line, i.e., one of the eigenvalues of the Jacobian matrix of the system is equal to* 1. *The other eigenvalue locates outside the unit circle.*

Proof: Linearizing the system in the steady state vicinity, one obtains:

$$\begin{bmatrix} \hat{k}_{t+1} - \hat{k}^* \\ \hat{c}_{t+1} - \hat{c}^* \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} + \sigma & -\frac{1}{1+\gamma} \cdot \left(1 + \frac{\beta}{1-\beta} \cdot \sigma\right) \\ \frac{1-\beta}{\beta} \cdot (1+\gamma) \cdot \sigma & \frac{1 - \sigma \cdot [(1-\beta) + \beta \cdot \sigma]}{1 + \beta \cdot \sigma} \end{bmatrix} \cdot \begin{bmatrix} \hat{k}_t - \hat{k}^* \\ \hat{c}_t - \hat{c}^* \end{bmatrix} \tag{14}$$

with $\sigma \equiv \frac{A \cdot [\beta \cdot (1 + A - \delta) - (1+\gamma)]}{m \cdot (1-\beta) \cdot (1+\gamma)^2} \geq 0$.

System (14) is derived in appendix.

The trace and the determinant of the Jacobian matrix in (14) are $Tr(J) = (1+\beta)/\beta$ and $Det(J) = 1/\beta$. Thus, the system rests over the bifurcation line $1 - Tr(J) + Det(J) = 0$ and the eigenvalues of the Jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = Det(J) = 1/\beta > 1$ ■

Figure 3 sketches the phase diagram of this system. The represented line corresponds to the isoclines of system (14) (i.e., $\hat{k}_{t+1} - \hat{k}_t = 0$ and $\hat{c}_{t+1} - \hat{c}_t = 0$), which are, in the present case, coincidental and equal to $\hat{c}_t - \hat{c}^* = \frac{1-\beta}{\beta} \cdot (1+\gamma) \cdot (\hat{k}_t - \hat{k}^*)$. As one observes, the steady state will not be reached, unless the initial point is already the steady-state or if any disturbance on the value of consumption is provoked by the representative agent.

*** Figure 3 ***

5. Discussion

We have developed an endogenous growth model with endogenous labor supply. Differently from the conventional assumption that utility increases with leisure independently of its amount, we have assumed that leisure in excess is less valued by the representative agent than a relatively intermediate level of leisure. The individual does not withdraw too much utility from too much spare time. In the limit, if the agent does not work at all, no utility comes from leisure, exactly as if all the available time was allocated to work.

The imposed assumption allows to find a unique steady state characterized by the existence of a unique growth rate, which cannot be presented explicitly, and a consumption-capital ratio, that is as much higher as the higher is the economy's growth rate; thus, the balanced growth path evidences the idea that the more the economy grows, the more the representative agent is able to consume per unit of physical capital. Another steady state result concerns the labor share: the faster is the pace of growth of the economy, the less the representative agent allocates time to working hours; this can be confirmed by looking at figure 2.

The central result is that the unconventional form of the utility function concerning leisure produces a bifurcation-instability outcome. Through the linearization of the system around the steady state, we compute a dimension 2 Jacobian matrix with an eigenvalue equal to 1 and the other eigenvalue higher than 1. Any form of stability (a stable node, a stable focus or a saddle-path stable equilibrium) is ruled out.

The results should be compared with the ones of a similar model with conventional positive marginal utility of leisure. With an utility function

$U(c_t, 1-\ell_t) = \ln c_t + n \cdot (1-\ell_t)$, n>0, solving the same maximization problem, one obtains a constant over time share of labor, $\ell_t = \frac{A/n - (1-\beta)\cdot(1-\delta)}{(1-\beta)\cdot A}$ and a constant consumption-capital ratio $\frac{c_t}{k_t} = \frac{A}{n}$; consumption and capital grow at rate $g = \frac{\beta}{1-\beta}\cdot\frac{A}{n} - 1$. Therefore, basically, the assumption of leisure in excess introduces transitional dynamics over an endogenous growth model that under a trivial framework can be described as being permanently on a balanced growth path.

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Appendix – Derivation of the Linearized System

The linearization of equation (9), in the neighbourhood of (k^*, c^*), yields,

$$\hat{k}_{t+1} - \hat{k}^* = \left(\frac{1}{\beta} + \sigma\right)\cdot(\hat{k}_t - \hat{k}^*) - \frac{1}{1+\gamma}\cdot\left(1 + \frac{\beta}{1-\beta}\cdot\sigma\right)\cdot(\hat{c}_t - \hat{c}^*) \qquad (a1)$$

Relatively to equation (10), this can be rewritten as,

$$\hat{c}_t = \frac{1+\gamma}{\beta} \cdot \frac{1}{1 + A \cdot \ell_{t+1} - \delta} \cdot \hat{c}_{t+1} \qquad (a2)$$

The linearization of (a2) around (k^*, c^*) allows to write

$$\hat{c}_t - \hat{c}^* = -(1-\beta) \cdot (1+\gamma) \cdot \sigma \cdot (\hat{k}_{t+1} - \hat{k}^*) + (1 + \beta \cdot \sigma) \cdot (\hat{c}_{t+1} - \hat{c}^*) \qquad (a3)$$

Having (a1) in consideration, we rearrange (a3) to present it as follows,

$$\hat{c}_{t+1} - \hat{c}^* = \left(\frac{1-\beta}{\beta} \cdot (1+\gamma) \cdot \sigma \right) \cdot (\hat{k}_t - \hat{k}^*) + \left(\frac{1 - \sigma \cdot \left[(1-\beta) + \beta \cdot \sigma \right]}{1 + \beta \cdot \sigma} \right) \cdot (\hat{c}_t - \hat{c}^*) \qquad (a4)$$

Equations (a1) and (a4) form system (14).

Figures

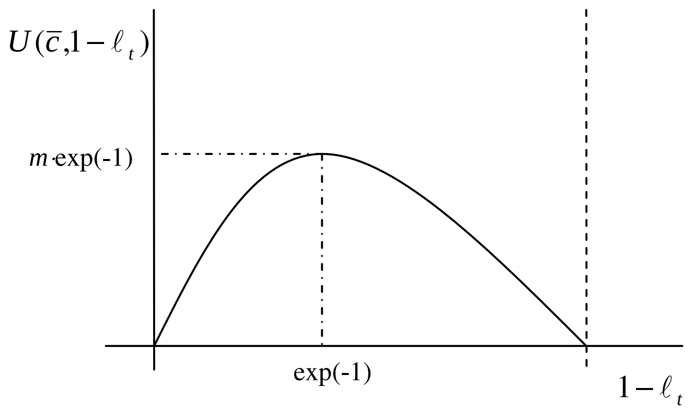

Figure 1 – The utility of leisure.

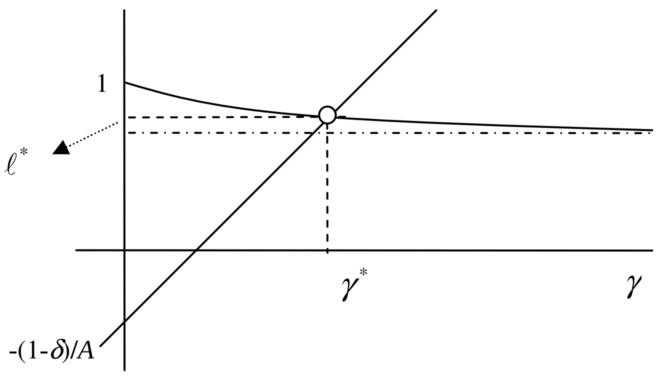

Figure 2 – Growth rate uniqueness.

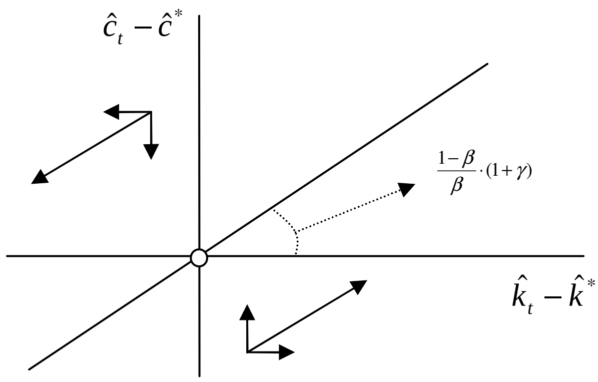

Figure 3 – Phase diagram.