The theorem of existence of the ruptures in probability scale and the basic question of insurance

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Abstract

The theorem of existence of the ruptures in the probability scale was proved in 2010. The ruptures can exist near the borders of finite intervals and of the probability scale. The theorem is used to analyze and to partially answer to the basic questions of insurance. The question is “To insure or not”.

The aim of the research is to provide insurance by new variations of mathematical methods. Its importance consists in the better understanding of possible origins of insurance processes and factors, which influence them. Such understanding will help to manage these insurance processes. Its methodology is to reveal pure mathematical aspects of insurance processes and to analyze these aspects by pure mathematical methods, including application of the theorem. Its most significant result: when uncertainty increases, then taking the theorem into account may reverse insurant’s and insurer’s decisions to the opposite ones.

The sketch of the theorem is given. It includes: the general lemma and the general theorem for finite intervals, the lemma and the theorem for the probability evaluation, the theorem for the probability.

An example of the ruptures in the probability scale is presented.

The question is analyzed from the points of view of insurant and insurer. The analysis is made purely mathematically for the uniform case of medium insurance value as in the automobile insurance. The analysis may be also relevant for time intervals between profitable and unprofitable periods of insurance cycle.

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Introduction

The theorem of existence of the ruptures in the probability scale was proved in (Harin, 2010-1). The theorem was applied to the modeling (Harin, 2010-2) and to the theory of complex systems (Harin, 2010-3). The theorem is being detailed and refined (see, e.g., Harin, 2010-4).

In this article the theorem is applied to the insurance.

The methodology of the research is to reveal pure mathematical aspects of insurance processes and to analyze these aspects by pure mathematical methods, including application of the theorem. The conditions of the considered insurance process are narrowed to exclude all non-mathematical conditions and (as far as possible) boundary effects. Then a mathematical model of the considered processes is revealed. Then the mathematical model is analyzed by pure mathematical methods including application of the theorem.

The article is organized as follows:

In Section 1, the conditions of the questions are formulated, the simple calculations are performed and the answers of an insurant and insurer are given without taking the theorem into account.

In Section 2, the theorem of existence of ruptures is sketched. The sketch includes: the general lemma and the general theorem for finite intervals, the lemma and the theorem for the probability evaluation, the theorem for the probability.

In Section 3, an example of ruptures in probability scale is given.

In Section 4, an application of the theorem to the question is given. Simple calculations show: at the considered conditions and at high level of uncertainty, taking the theorem into account may change the answers to the opposite ones.

1. The basic questions of insurance

1.1. Formulation of the question. Two points of view. The basic questions of insurance is “To insure or not”. This question may be analyzed from two points of view.

One of them is the point of view of insurant. He or she may take a standard form contract or leave it.

The another point of view is that of insurer. The insurer may offer the standard form contract with common prices or raise prices in questionable line of insurance or walk away from this line when prices fall below a prudent premium.

The question may be typical for time intervals between profitable and unprofitable (‘soft’ and ‘hard’) periods of insurance cycle.

1.2. Conditions. Let us consider following conditions of the discussed insurance process:

1) Let us eliminate all the conditions except the mathematical ones. Let us eliminate psychological conditions and aspects such as in (Cutler et al, 2008). Let us eliminate conditions such as comfort, heightened level of service etc.

So, the first aim is to obtain pure mathematical conditions and pure mathematical problem.

2) Let us eliminate boundary value conditions and problems as far as possible. Namely, suppose the insurance value is essentially less than the value of insurant’s spare cash.

So, the second aim is to obtain uniform conditions (as far as possible).

3) Let us consider an ideal case, when the profit and deductions in reserves are equal to zero. In this case, the difference between the insurance payments, fees and the insurance values, moneys includes only unavoidable insurance expences such as wage, amortisation, lease of office rooms, etc.

As the result, we obtain only four conditions:

- insurance money, value V,
- probability p of insurance event, insured accident
- insurance payment, fee F,
- insurance expenses E,

1.3. Calculation. In the ideal case the insurance fee F includes

$$F = Vp + E\ .$$

In real cases it transforms to the inequality

$$F \geq Vp + E\ .$$

It is obvious that an average insurant pays the additional insurance payment which equals to E and is additional to the pure product of the insurance value V and the probability p of the insurance event.

1.4. Answers. So, at pure mathematical, uniform conditions, average insurant's answer should be negative. At the same conditions in the ideal case, insurer's answer should be more positive than negative.

In the real case, when the profit and deductions in reserves are not equal to zero and are sufficiently high, insurer's answer should be positive.

Note.

For some insurants, the individual probability $p_{individ}$ of insurance event may be more than p.

In the case, when

$$Vp_{individ} > F$$

insurant's answer should be positive. In the case, when

$$Vp_{individ} + E > F$$

insurer's answer should be negative. However such cases are not the item of this article.

2. Theorem of existence of ruptures in probability scale

2.1. General lemma and theorem

2.1.1. General preliminary notes. Suppose an interval $X=[A, B]$: $0<(B-A)<\infty$. Suppose a quantity $f(x)$:

1) for $x<A$ and $x>B$, the statement $f(x)\equiv 0$ is true and for $A\leq x\leq B$ the statement $f(x)\geq 0$ is true, and

$$\int_{-\infty}^{+\infty} f(x)dx = C_f\ ,\ \text{where}\ \ 0 < C_f < \infty\ ;$$

2) the initial moment of the first order, the mathematical expectation exists

$$EX = \frac{1}{C_f}\int_{-\infty}^{+\infty} xf(x)dx \equiv M\ ;$$

3) for n : $1<n<\infty$, at least one central moment exists

$$E(X-M)^n = \frac{1}{C_f}\int_{-\infty}^{+\infty}(x-M)^n f(x)dx\ .$$

The maximal possible value of a central moment may be estimated from its definition

$$|E(X-M)^n| \equiv |\frac{1}{C_f}\int_{-\infty}^{+\infty}(x-M)^n f(x)dx| \le$$
$$\le \frac{1}{C_f}\int_{-\infty}^{+\infty}|(x-M)^n| f(x)dx \le \quad .$$
$$\le \frac{1}{C_f}(B-A)^n\int_{-\infty}^{+\infty} f(x)dx = (B-A)^n$$

More precise estimation of this value is provided (see, e.g., Harin, 2010-4) by the sum of modules of the central moments of the functions that are concentrated at the borders of the interval: $\delta(x-A)\times(B-M)/(B-A)$ and $\delta(x-B)\times(M-A)/(B-A)$

$$Max(E(X-M)^n) \le$$
$$\le |(A-M)^n\frac{B-M}{B-A}| + |(B-M)^n\frac{M-A}{B-A}| .$$

It leads to the well-known maximum for $n=2$ and $M_{max}=(B-A)/2$

$$Max(E(X-M)^2) = (\frac{B-A}{2})^2$$

and, for $n=2k>>1$, - to the maximums at $M_{max}\approx A+(B-A)/2n$ and $M_{max}\approx B-(B-A)/2n$ (see, e.g., Harin, 2010-4)

$$Max(E(X-M)^n) \approx \frac{1}{\sqrt{e}}\frac{(B-A)^n}{2n} .$$

2.1.2. General lemma about tendency to zero for central moments. If, for $f(x)$, defined in the section 2.1.1, $M\equiv E(X)$ tends to A or to B, then, for $1<n<\infty$, $E(X-M)^n$ tends to zero.

The proof: For $M\to A$,

$$|E(X-M)^n| \le$$
$$\le |(A-M)^n\frac{B-M}{B-A}| + |(B-M)^n\frac{M-A}{B-A}| \le$$
$$\le [(B-A)^{n-1}+(B-A)^{n-1}]\frac{(M-A)(B-M)}{B-A} \le \quad .$$
$$\le 2(B-A)^{n-1}(M-A)\xrightarrow[M\to A]{} 0$$

So, if $(B-A)$ and n are finite and $M\to A$ (that is $(M-A)\to 0$), then $E(X-M)^n\to 0$.

For $M\to B$, the proof is similar.

The lemma has been proved.

Note. More precise (see, e.g., Harin, 2010-4) estimation may be obtained for central moments' tendency to zero, e.g. for $M\to A$

$$|E(X-M)^n| \le (B-A)^{n-1}(M-A)\xrightarrow[M\to A]{} 0 .$$

2.1.3. General theorem of existence of ruptures for expectation. If there are: $f(x)$ defined in the section 2.1.1, n : $1<n<\infty$, and $r_{dispers}$: $|E(X-M)^n| \ge r_{dispers}>0$, then $r_{expect}>0$ exists : $A<(A+r_{expect})\le E(X)\le(B-r_{expect})<B$.

The proof: From the lemma, for $M\to A$,

$$0 < r_{dispers} \le |E(X-M)^n| \le 2(B-A)^{n-1}(M-A) ,$$

$$0 < \frac{r_{dispers}}{2(B-A)^{n-1}} \le (M-A) ,$$

$$r_{expect} \equiv \frac{r_{dispers}}{2(B-A)^{n-1}} .$$

For $M\to B$, the proof is similar.

As long as $(B-A)$, n and $r_{dispers}$ are finite and $r_{dispers}>0$, then r_{expect} is finite, $r_{expect}>0$, both $(M-A)\ge r_{expect}>0$ and $(B-M)\ge r_{expect}>0$.

The theorem has been proved.

Note. More precisely (see, e.g., Harin 2010-4)

$$r_{expect} \equiv \frac{r_{dispers}}{(B-A)^{n-1}} .$$

So, if a finite $(n<\infty)$ central moment of a quantity, which is defined for a finite interval, cannot approach 0 closer, than by a nonzero value $r_{dispers}>0$, then the expectation of the quantity also cannot approach a border of this interval closer, than by the nonzero value $r_{expect}>0$.

More general: If a quantity is defined for a finite interval and a nonzero rupture $r_{dispers}>0$ exists between zero and the zone of possible values of a finite $(n<\infty)$ central moment of the quantity, then the nonzero ruptures $r_{expect}>0$ also exist between a border of the interval and the zone of possible values of the expectation of this quantity.

2.2. Theorem of existence of ruptures in probability scale

2.2.1. Preliminary notes. For a series of tests of number K, including $K\rightarrow\infty$, let the density $f(x)$ of a probability estimation, frequency F : $F\equiv M\equiv E(X)$, has the characteristics defined in the section 2.1, in particular $f(x)$ is defined for $[0, 1]$ and $C_f=1$.

2.2.2. Lemma about tendency to zero for central moments of density of probability evaluation. If a density $f(x)$ is defined in the section 2.2.1, and either $E(X)\rightarrow 0$ or $E(X)\rightarrow 1$, then, for $1<n<\infty$, $E(X-M)^n\rightarrow 0$.

The proof: As long as the conditions of this lemma satisfy the conditions of the lemma of the section 2.1.2, then the statement of this lemma is as true as the statement of the lemma of the section 2.1.2.

The lemma has been proved.

2.2.3. Theorem of existence of ruptures for probability estimation. If: a density $f(x)$ is defined in the section 2.2.1, there are n : $1<n<\infty$, and $r_{dispers}>0$: $E(X-M)^n\geq r_{dispers}>0$, then, for the probability estimation, frequency $F\equiv M\equiv E(X)$, r_{expect} exists such as $0<r_{expect}\leq F\equiv M\equiv E(X)\leq(1-r_{expect})<1$.

The proof: As long as the conditions of this theorem satisfy the conditions of the theorem of the section 2.1.3, then the statement of this theorem is as true as the statement of the theorem of the section 2.1.3.

The theorem has been proved.

2.2.4. Theorem of existence of ruptures in probability scale. If, for the interval $[0,1]$, P is defined such as, when the number K of tests tends to infinity, the probability estimation, frequency F tends at that to P, that is $P=LimF$, nonzero ruptures $0<r_{expect}\leq F\leq(1-r_{expect})<1$ exist between the probability estimation and every border of the interval, then the same nonzero ruptures $0<r_{expect}\leq P\leq(1-r_{expect})<1$ exist between P and every border of the interval.

The proof: Consider the left boundary 0 of the segment $[0; 1]$. The frequency F_K is not less then r_{mean}. Hence we obtain for P

$$P = \lim_{K\to\infty} F_K \geq \lim_{K\to\infty} r_{mean} = r_{mean}$$

So $P\geq r_{mean}$.

Note this is true both for a monotonous convergence and a dominated convergence. The reason is the fixation of the minimal value by the conditions of the theorem.

For the right boundary 1 the proof is similar to above one. So, $r_{expect}\leq P\leq(1-r_{expect})$.

The theorem has been proved.

To what an extent a probability satisfies the conditions applied on P, to such an extent the theorem is true for the probability as well.

The theorem may be formulated also for needs of practical applications:

If, for the series of tests, when the number K of tests tends to infinity and a probability estimation, frequency F tends at that to a probability P, a rupture $r_{dispers}>0$ exists between 0 and the zone of possible values of dispersion D of the density f of the probability estimation F, then the ruptures $r_{expect}>0$ also exist near the borders of the probability scale. The ruptures $r_{expect}>0$ exist between the borders and both the zone of possible values of the probability estimation, frequency F, and the zone of possible values of the probability P.

3. An example of ruptures in probability scale

3.1. Conditions. The simplest example of such ruptures is the aiming firing at a target in the one-dimensional approach:

Let, at the precise aiming, some scattering of hits takes place due to, e.g., scattering of bullet dimensions (if the diameter of bullet is less than the diameter of barrel of gun, then the bullet will fly out the barrel not through the optical axis of the barrel, but through some beam of trajectories which are distributed around this axis).

Let the dimension of the target is equal to $2L>0$ and, at the precise aiming, the uncertainty, the scattering of hits obeys the normal law with the dispersion σ^2. Then the maximal probability P_{in_Max} of hit in the target and the minimal probability $P_{out\_min}=1-P_{in\_Max}$ of miss are equal to (see, e.g., Abramowitz and Stegun, 1972):

3.2. Results. For $\sigma=0$:

$P_{in\_Max}=1$ and $P_{out\_min}=0$, that is, there are no ruptures in the probability scale for hits and misses, that is $r_{expect}=1-P_{in\_Max}=P_{out\_min}=0$.

For $L=3\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,997<1$ and, for P_{out}, $0<0,003=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits in the target and misses are equal to $r_{expect}=0,003>0$.

For $L=2\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,95<1$ and, for P_{out}, $0<0,05=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits in the target and misses are equal to $r_{expect}=0,05>0$.

For $L=\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,68<1$ and, for P_{out}, $0<0,32=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits in the target and misses are equal to $r_{expect}=0,32>0$.

3.3. Conclusion. Thus:

For zero $\sigma=0$ - there are no ruptures ($r_{expect}=0$).

For nonzero $\sigma>0$:

- the nonzero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of hit in the target $0\le P_{in}\le P_{in\_Max}=1-r_{expect}<1$ and 1;
- the same nonzero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of miss $0<r_{expect}=P_{out\_min}\le P_{out}\le 1$ and 0.

Note, the dispersion of scattering of hits σ^2 may determine the dispersion D of the probability estimation of hits in the target and misses, but the dispersion σ^2 is not the same as the dispersion D.

Analogously, an uncertainty in a parameter or in some parameters may lead to nonzero dispersion of the density of a probability estimation in the insurance. This nonzero dispersion may lead to the ruptures in probability scale for insurance processes.

4. An application of the theorem to the question

So, in real circumstances, when a nonzero dispersion of the density of a probability estimation exists, the ruptures r_{expect} can exist in the probability scale near the borders of the scale, including the rupture r_{expect} near zero. This shifts the probability evaluation and the probability p from zero to the middle of the probability scale. In any case, the probability p can not be less than the rupture r_{expect}

$$p \geq r_{\exp ect} .$$

Two cases may be of interest: the first

$$F < Vr_{\exp ect} \leq Vp ,$$

and the second (which includes the first)

$$F < Vr_{\exp ect} + E \leq Vp + E .$$

4.1. Insurant's point of view. If

$$F < Vr_{\exp ect} ,$$

then insurant's answer should be positive.

So, taking the theorem into account may change insurant's answer to the opposite one.

In other words, when uncertainty increases, then insurant's answer becomes more positive.

4.2. Insurer's point of view. If

$$F < Vr_{\exp ect} + E ,$$

then insurer's answer should be negative.

So, taking the theorem into account may change insurer's answer to the opposite one.

In other words, when uncertainty increases, then insurer's answer becomes more negative. The insurer should raise prices in the questionable, highly uncertain line of insurance or walk away from this line (when prices fall below a prudent premium).

4.3. Insurance cycles. The theorem may be also applied to insurance cycles. Uncertainties, that may play role, may have various types of nature. For example, they may be noises in the spectral analysis of insurance cycle (Venezian, 2006). They may be responses to fluctuations in the supply of property-liability insurance (Winter, 1991) etc. Note, the theorem (as it is) cannot be applied to unpredictable events such as the 11 September 2001 attack.

Conclusions

The general conclusion of this article is to pay attention to uncertainties.

Due to the theorem, when uncertainty increases, then the probability of insurance event may increase and the insurer should raise prices in the questionable, highly uncertain line of insurance or walk away from this line. This may be also relevant when one deal with the problems of insurance cycles.

Further researches of the item, including both fundamental and applied researches, should be carried out to develop practical recommendations for the insurance industry.

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