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MIXED FRACTIONAL BROWNIAN MOTION, SHORT AND LONG-TERM

DEPENDENCE AND ECONOMIC CONDITIONS: The Case of the S&P-500 Index

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ABSTRACT:

The Kolmogorov-Mandelbrot-van Ness Process is a zero mean Gaussian process indexed by the Hurst Parameter (H). When it models financial data, a controversy arises as to whether or not financial data exhibit short or long-range dependence. This paper argues that the Mixed Fractional Brownian is a more suitable tool for the purpose as it leaves no room for controversy. It is used here to model the S&P-500 Index, sampled daily over the period 1950-2011. The main results are as follows: The S&P-500 Index is characterized by both short and long-term dependence. More explicitly, it is characterized by at least 12 distinct scaling parameters that are, *ex hypothesis*, determined by investors' approach to the market. When the market is dominated by "blue-chippers" or 'long-termists', or when bubbles are ongoing, the index is persistent; and when the market is dominated by "contrarians", the index jumps to anti-persistence that is a far-from-equilibrium state in which market crashes are likely to occur.

KEY WORDS: Gaussian Processes, Mixed Fractional Brownian Motion, Hurst Exponent, Local Self-similarity, Persistence, Anti-persistence. Definiteness of covariance Functions, Dissipative dynamic systems.

I- INTRODUCTION

Long-memory or long-term dependence (LTD) exists when past events influence the present and possibly future events. In various fields of science, ranging from astrophysics to biology, from psychology to language and economics, LTD is a characteristic of phenomena whose autocorrelation functions (acf) decay rather slowly. In economics, it is commonly assumed that when the dependence is short-ranged (STD), the acf decays to zero after a few lags or decays to zero exponentially. Whereas, for a process characterized by LTD, the acf decays as a power law. However, as is shown below, that understanding implicitly alludes to mono-fractality.

The presence and the extent of LTD in economics and finance is usually measured by the Hurst exponent, $H \varepsilon (0, 1)$, which is an asymptotic descriptive statistic that can only be estimated experimentally. An average value of $H < 1/2$ indicates STD; investors interpret it to mean that the immediate future will most likely contradict the recent past. And an average value of $H > 1/2$ characterizes LTD that investors interpret to mean that the future will most likely look like the past. But, how representative are these average values knowing that financial and economic time series are not stationary and, more importantly, are invariant to scale only over consecutive segments [1], [3], [7], [17]? Indeed, a review of some 50 studies of that matter carried out over the last 30 years or so has failed to determine whether or not these series are characterized by STD or LTD. Typically, LTD has found support in, say, [8], [4], [16], [13], [6], [5] and is rejected in, say, [2], [10], [11] and [12], among others. Moreover, the attempt to come up with an average value of H has produced a plethora of values that seem to vary with series' length, sampling intervals, and appraisal methods. Because of such variance in estimation and consequent claims and counter claims, economic and financial theorists remain divided over the issue.

In that connection, a few theorists have proposed the multi-fractal formalism [12] as an alternative to account for the changes in the scale of the fluctuations which generate heteroskedasticity of the overall dynamics. More recently, it has been proposed that a superposition of independent mono-fractals can easily account for the multiplicity of scales as well. This later approach was motivated by the recognition that global self-similarity is a strong idealization in the real world. There is ample evidence that man-made processes, if they are self-similar at all, are so over consecutive segments of length r_i, i ε n, even though determining the range (r) of a given scale is a difficult experimental problem when the inputs to these processes are unobservable. Perhaps, it is the reason why researchers rely on the Wavelet formalism (WT) or De-trended Fluctuation Analysis (DFA) based on the assumption that these formalisms may compensate for the absence scale invariance.

If the absence of global self similarity is a problem, the variability of scales can be well analyzed by the simple use of a multi-scalable fractional Brownian motion termed "Mixed fractional Brownian motion" (here after MfBm). These are super-positions of various self-similar and stationary segments, each with its own H index. MfBm are studied in [15], [19], [18], and [21] but, to our knowledge, has not been put to the test. The purpose of this paper is to study the S&P-500 Index as an MfBm, assuming that that process captures more accurately the reality of market outcomes and, therefore, may shed light on the debate.

The paper is divided into five parts. The first states the problem. The second shows that in MfBm, LTD and STD may alternate depending on market conditions. Part III discusses the data. Part IV presents our results which are further discussed in Part V.

II- PRELIMINARIES

Definition 1. In general, $X^H_t = \{X^H(t, \omega), t \,\varepsilon\, (0,T), \omega \,\varepsilon\, \Omega\}$ *is a real-valued Gaussian process, defined on a probability space* $(\Omega, \Gamma, \mathcal{P})$*, indexed by* $H \,\varepsilon\, (0,1)$*, satisfying* $X^H_t = E(X^H_t) = 0$*,* $\forall t \,\varepsilon\, \Re$*, and H is constant over segment* r_i*, i ε n. Here E denotes the expectation with respect to the probability law* $\mathcal{P}$ for X^H_t, $t \,\varepsilon\, \Re$, $X^H(t, \omega)$*, and* $[\Omega, \Gamma]$ *is a measurable space.*

The process X^H_t is termed Gaussian if $\forall t > 0$, because the probability distribution of the random vector $(X_{t1}, \ldots, X_{tn}) \,\varepsilon\, \Re$ is normal or Gaussian. The probability distribution of such a process is entirely determined by its zero mean and its covariance function R (t, s).

Definition 2. The MfBm $Z_t = \sum_{r=1}^{n} (b_r X^{Hr}_t)$*, where* $r \varepsilon n$, $H_r \varepsilon n$, $\forall H_r \varepsilon (0, 1)$*, is a linear combination of Gaussian processes of Def. 1. Therefore,* Z_t *is a centered Gaussian process or a superposition of n independent input streams, each with its own H.*

The X^{Hr}_t are Kolmogorov- Mandelbrot- van Ness processes [14] or ordinary fBm's taken as inputs into a dynamic input/output system, where Z_t is the output. Using input storage and Teletraffic terminologies (see [31]), unobservable inputs arrive either as "cars" or "trains", and Z_t is an observable output. The linear combination of Gaussian processes Z_t has mean and variance as follows:

$$E(Z_t) = 0$$

$$E(Z_t^2) = \sum_{r=1}^{n} b_r^2 (X^H_t)^2 = \sum_{r=1}^{n} (b_r)^2 |t|^{2Hr}.$$

Then Z_t is also completely characterized by its covariance function R (t, s):

$$R(t, s) = 2^{-1} \sum_{r=1}^{n} (b_r)^2 [t^{2Hr} + s^{2Hr} - |t - s|^{2Hr}], \forall t, s \varepsilon \Re_+, r \varepsilon n \text{ for } \forall H_r \varepsilon (0,1).$$

Z_t has the following essential properties:

Property 1. (Scale invariance). $X^{Hr}_t, t \geq 0$ *and* $m^H [X^{Hr}_t, t \geq 0]$ $(m \varepsilon \Re_+, r \varepsilon n)$ *have the same probability distribution. This property is a consequence of the covariance function R (t, s), which is homogeneous of order 2H.*

Property 2. (Stationary Increments). Over the interval (t, s), X^H_t *has a normal distribution with zero mean and variance given by* $E[X^{Hr}_t, X^{Hr}_s)] = |t - s|^{2Hr}$.

Property 3. (Dependence). Defining $S_1 = \{H_r \varepsilon \Re_+ | 0 < H_r < 1/2\}$; $S_2 = \{H_r \varepsilon \Re_+ | H_r = 1/2\}$, *and* $S_3 = \{H_r \varepsilon \Re_+ | 1/2 < H_r < 1\}$. *If* $H_r \varepsilon S_1$, Z_t *is anti-persistent or STD exists; if, on the other hand,* $H_r \varepsilon S_3$, Z_t *is persistent or LTD exists.*

As Z_t is a linear combination of X^{Hr}_t, we show in Appendix 1 that it has all the *sine qua non* properties of the process of Definition 1but over segment r_i, $\forall i \varepsilon n$. It can then be seen that Z_t collapses to an ordinary Brownian motion if $H_r = 1/2$, $\forall r \varepsilon n$; Z_t becomes an ordinary fractional Brownian motion (fBm) if r = 1; and it is an MfBm if $r \varepsilon n$. Then it follows that confusing MfBm and fBm would present additional estimation problems and may in fact be the root cause of the debate.

For completeness and clarity, consider the following auxiliary definitions:

Definition 4. The scaling exponent, β_r*, of a process is the slope of the log-log plot of the power law describing the behavior of the process under scaling transformation. If the process of Def. 1 has Property 1, then* $H_r = (\beta_r - 1)/2$, $\forall r \varepsilon n$.

Definition 5. A Uni-scaling process (or mono-fractal) is one for which a single β *suffices to characterize entirely the process. Mono-fractality is generated by additive random processes.*

Definition 6. A Multi-scaling process (or Multi-fractal) is one for which more than one β is needed to characterize the process. That is, $Z_t = X^H{}_t\,[\varphi\,(t)]$, where $\varphi\,(t)$ is the distribution function playing the role of a time-deformation; thus multi-fractality is generated by a multiplicative cascade of random processes.

Definition 7. The Mixed fractional Brownian motion is a superposition of various mono-fractals of Definition 5.

Simply put: If r = 1, $X^{Hr}{}_t$ is the Kolmogorov-Mandelbrot-van Ness Process [14] of Def. 5; if r ε n, Z_t is an MfBm with $X^{Hr}{}_t$ as inputs. The motivation for preferring Z_t over the multi-fractal formalism rests on the following: Multi-fractals are generalizations of the mono-fractal concept, which seem to model well phenomena in physics such as the energy dissipation in turbulence, fractal resistor networks, non-linear dynamic systems, some games of chance, etc. One approach to the analysis of multi-fractals involves a bisecting process starting with a characteristic probability and ends up, at the limit, with a self-affine distribution of the various scales that characterizes the process. That approach models, say, turbulence or the repartition of photons in an equally split beam of light, very well, but man-made processes such as markets are reflexive constructs in which the MfBm presents an advantage; that is, the MfBm is a fragmented mono-fractal where each fragment satisfies properties 1 and 2. In addition, the fragments are dated, thus making it somewhat easier to account for the reflexivity criterion by connecting scale variations to observable agents' behavior.

III- THE DATA

The concept of fragmented mono-fractality will now be tested against the S&P-500 Index which, it is recalled, is a market-value-weighted index of stock prices times the number of shares outstanding in which each stock weight is proportional to its market value. We will use the Microsoft Excel data set of closing prices, sampled daily from January 3rd, 1950 to February 28th, 2011, giving 15,389 data points. The analysis will be done with the Excel SS and the Benoit$_{TM}$ Wavelet system. The advantage of the latter is that different methods can be compared as the package includes Fourier and Wavelet transforms, Rescaled-range analysis, and various methods of filtration for white noise. But beforehand, we will de-trend the data set using logarithmic differences of whatever order necessary to achieve strict stationarity even at the risk of attenuating some low frequencies.

Figure 1 shows the time profile of the raw data. As it can be seen, the market became quite turbulent in the latter part of the period under study. Figure 2 is a magnified view of the first difference, while Figure 3 is the third difference used for the analysis.

The structure of Z_t points to a system with many critical points where bifurcations occur. To locate these points, we will first partition the data into smaller segments over which property 1 obtains. These segments will next be filtered to remove extra noise introduced by altering the segmentation or by extrinsic variations in H due to the finiteness of the segments.

We strongly suspect that changes in the H parameter are determined by agents' attitude which in turn determines their approach to the market. In that case, agents' types would be gauged either by their enduring characteristics or by their assessment of economic conditions.

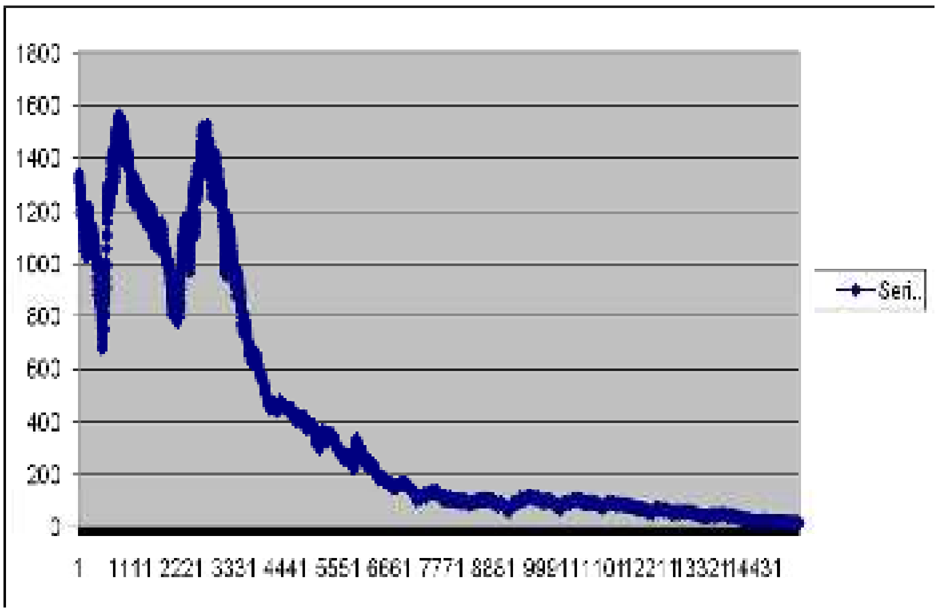

Figure 1: The Raw S&P-500 Index, 2011- 1950, showing Huge Fluctuations between 1999 and 2011.

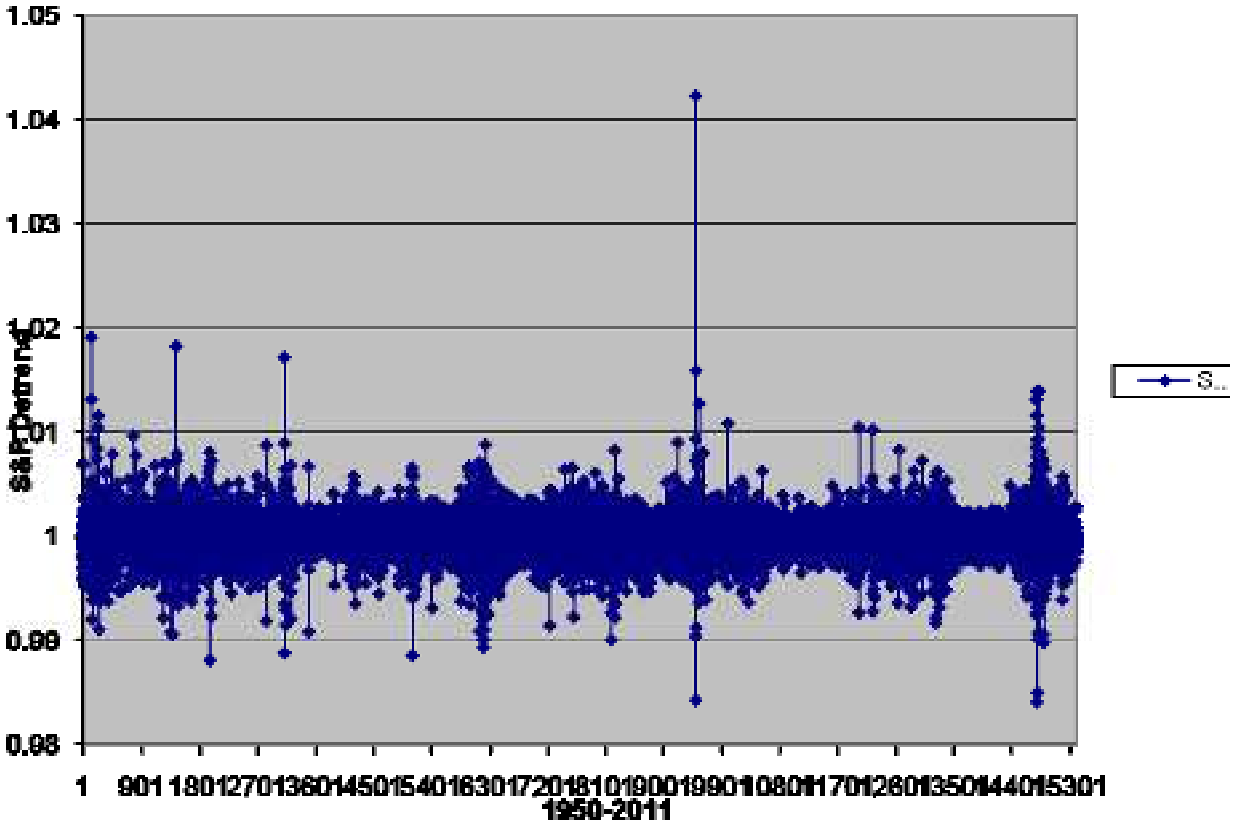

Figure 2: A magnified view of the first difference of the S&P-500 Index rotated clockwise 180° about the x-axis and counter-clockwise 180° about the y axis.

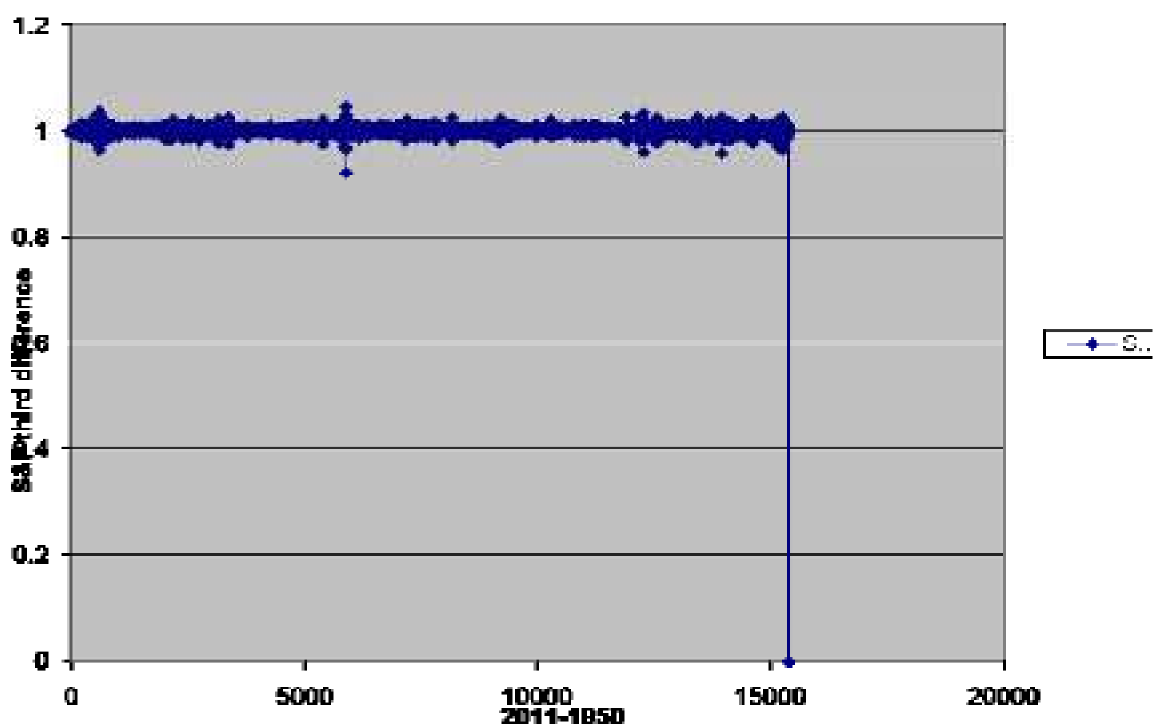

Figure 3: The third difference time profile of the S&P-500 from 2011 to 1950. The biggest spike occurred at 5891 or October 1987..

IV- THE RESULTS

The whole series were first divided into segments corresponding to variations in observable macroeconomic conditions, including changes in policy. But, subsequently and by trial and error, we were able to identify 12 segments of self-similarity. These segments were next filtered for noise. The reader should be aware, however, that the Microsoft Excel data set increases backward in time, hence in Figures 1 and 3 (the time profile of Z_t), the t-axis flows backward in time as well. As it can be seen in the figure, turbulence dominated the recent years. Figure 2 (the first difference) is another representation where t = 4 corresponds to January 3rd, 1950 and t = 15.389 represents February 24th, 2011due to the rotation about the axes. In Figure 3, the event of October 1987 is shown as the longest spike at 5891 corresponding to October 20th, 1987. In first and second differences, trend-free behavior was not completely achieved. For our inference we use the third difference shown in Figure 3.

As it can be seen in Table 1, in some instances there are differences between ranges and wavelet capacity. However, the property of scale invariance over r_i led us to suppose a constant H_r and constant Hausdorff dimensions over r_i ε n whenever r_i was not a power of 2. Our results are presented in Table 1 below.

Period	r_i	Wavelet Points	H_r	$d^{(1)}$
1950-58	2126	2^{11}	0.4760 ± 0.0482	1.5240
1958-61	673	2^{9}	0.5890 ± 0.0410	1.4110
1961-72	2591	2^{11}	0.5220 ± 0.0321	1.4780
1972-80	2053	2^{11}	0.2209 ± 0.0359	1.7791
1980-83	1076	2^{10}	0.2870 ± 0.0319	1.7130
1983-87$^{(2)}$	1074	2^{10}	0.5590 ± 0.0501	1.4410
1988-92	1127	2^{10}	0.5310 ± 0.0610	1.4690
1992-97	1291	2^{10}	0.4630 ± 0.0559	1.5370
1998-02	1149	2^{10}	0.6100 ± 0.0612	1.3900
2003-07$^{(2)}$	1024	2^{10}	0.1101 ± 0.0310	1.8899

2007-08	512	2^9	0.2811 ± 0.0326	1.7189
2009-11	535	2^9	0.1430 ± 0.0339	1.8570

Table 1: The Multi-fractality of the S&P-500 Index from 1950-2011. The market alternates between anti-persistence and persistence. (1) The Hausdorff dimension; (2) Values before crashes.

According to these results, the market alternated between anti-persistence and persistence. Up to 1958, it was moderately anti-persistent, but it jumped into persistence up to 1972. The period 1972-80 was characterized by many unexpected events such as the demise of the Bretton Woods system. There were also two significant oil shocks, a market crash during 1973-74, and political trouble in the oil producing regions of the Middle-East, etc. The latter part of the decade 1970-80 was frequently characterized by "stagflation". Table 1 shows that the market swung to anti-persistence until the end of the 1981-82 Recession. It is also interesting to observe that the long stretch of persistence during 1983-92 occurred during an also long stretch of what was at first thought to be real economic growth. After 1992, the market jumped back to anti-persistence until the end, except for the brief period 1998-02 when the dot.com bubble was ongoing. The alternation of persistence and anti-persistence reveals two important points: The first is the inadequacy of the unique scaling exponent to characterize the local variability of financial time series. The second is the very power of the fragmented mono-fractal formalism to do so.

Our results are in good agreement with those in [20] [1] and [3]; [3] and [20] use Wavelet multi-resolution analysis, while [1] uses the DFA. The next logical question at this juncture is: What is causing changes in persistence? Even though this is a question for further research, it is reasonable to conjecture at this juncture that periods of growth and recession may be the driving factors causing investors to approach the market as "cars" or as "trains". If that assessment is correct, it would then suggest that there exist two types of investors. Type I may be a blue-chipper who (after assessing the past) remains optimistic about the long-term. This type arrives in the market as "train". If this type dominates the field, the market would be characterized by growth and persistence. There is a caveat however. Type I may need some time to recognize bubble growth. But when Type I finally does, his or her abrupt conversion to Type II leads to market crashes as was the case during the periods 1972-73, 2001, and 2007. In all, we see whether growth is real or bubbling, the market tends to be persistent.

Type II is a contrarian who is uncertain about the future or who is naturally risk-averse; consequently, Type II focuses on mean reversal. When this type dominates the field, the market tends to be anti-persistent as was the case during 1972-83 and since 2003.

Again, if that assessment is correct, it would be possible to infer that the market is persistent during times of real economic growth. However, it swings to anti-persistence when a bubble bursting is imminent and remains so during times of uncertainty. Imbalances may accumulate during real growth or when a bubble is ongoing, but a market crash is a dissipation of such imbalances. However, regardless of where imbalances occur, the dissipative process takes place during anti-persistence in a manner similar to what is described by the Fluctuation Dissipation Theorem.

Fear, political turmoil, and recessions seem to exercise a downward pull on H, and real growth seems to have the opposite effect. Deregulation and high frequency trading (HFT) both seem to have the capacity to change input arrivals to “cars”; in that connection, the Gramm-Leach Act of 1999 and the Commodity Modernization Act of 2000, for example, had surely flooded the market (after the 2001 Recession) with huge contrarian participants that are still dominating the market at the present time. Moreover, HFT that accounted for 73 percent of all equity trading volume in 2009 may very well have the same effect on investors’ arrival. But again, these conjectures are left for further research.

According to Table 1, there were 7 periods of anti-persistence and the 5 periods of persistence; they can be seen in Figure 3, and better still, in the magnified Figure 2. Taken in its entirety, the evidence suggests that some investors (Type I) approach the market with a long-term view. Another interesting topic for further research would be to see first if Type I consists mainly of institutional investors, endowment funds, and sovereign funds, etc. that are attracted to private equity capital. And, second, if Type II consists of Hedge funds and High Frequency traders. In the meantime, what this paper demonstrates is that the market is a dynamic input/output system that is reflexive, and that reflexivity is the natural driver of both short and long-range dependence.

V- DISCUSSION

We have argued that modeling the MfBm or the mixed fractional Brownian motion could shed much light on the debate as to whether the S&P-500 Index is long-range dependent or not. The motivation for the suggested shift to MfBm is the observed fact that the Hurst exponent fluctuates over time. As the MfBm is a Gaussian process indexed by more than one Hurst exponent, it seems more suitable to account for a varying H,

We next examined the S&P-500 Index and found it to be neither globally self-similar nor stationary. Then the use of an MfBm allowed us to study the market over segments of self-similarity. Over the period 1950-2011, there were 5 episodes of persistence and 7 episodes of anti-persistence. The 1958-72 period was persistent but was also accumulating imbalances due to wars, the Cuban missile crisis, political turmoil, and increased volatility resulting from the demise of the Bretton Woods Agreements. All of these lead to the crash of 1973-74. The later part of the 1970s was a time of oil shocks, post market crash, and stagflation; until 1983, the market was then anti-persistent. After the 1981-82 Recession, optimism rose and the market became persistent. At the same time, however, Congress deregulated the thrift industry, leading to imprudent mortgage financing and outright malfeasance. Additionally, the Congress passed the Tax Reform Act of 1986 which reduced the value of real estate investments by limiting the extent to which losses could be deduced from gross income. The holders of these investments panicked and attempted to unload them, causing the collapse of the thrift industry and the housing crisis of the early 1990s. The rise in persistence during 1998-02 was fueled by a bubble. That is, after 1997, the rise in persistence was fueled by the dot.com bubble, leading to the collapse of 2000-02. Imbalances began to accumulate again during the housing bubble that began in 2002 and which is still unraveling in 2011. It can safely be argued that since 2003, the market is pinned down in anti-persistence. After 2009, high unemployment, inflation and the absence of real growth make uncertainty pervasive. Put more succinctly, as in the later part of the 1970s, the US economy is presently in stagflation. If our interpretation is correct as shown in Appendix 2 and by past experience, the market is presently in such disequilibrium that investors, whether type I or II, are scared of it. In the meantime, the field is presently dominated by big contrarian participants operating in a time frame of milliseconds who, beside their market power, possess the advantage of "front running". Market power and front running combine to send false market signals. It is not far-fetched then to suppose that these participants are not interested in the longer term; consequently, the market has been living in profound dis-equilibrium since 2003.

These results also vindicate John Maynard Keynes who assigned a role to confidence in the economic assessment of investors. "Business investment" he said "depends significantly on the state of confidence or on the animal spirit…. When confidence is high, the economy thrives; when it is low, it sickens."

Our results can be summarized as follows: 1) Financial data are neither globally self-similar nor stationary; 2) the Kolmogorov- Mandelbrot-van Ness process cannot adequately capture the essence of financial data, because these data are not scale invariant throughout; 3) the MfBm process offers a more realistic view of financial data due to its ability to capture the local variability of the process; 4) the S&P-500 Index is characterized by both short and long-term dependence; 5) the Hurst parameter varies with investors' behavior, and; 6) market crashes are more likely to occur when the market is far from equilibrium in S_1.

Appendix 1

1) For $Z_t = \sum_{r=1}^{n} (b_r X^{Hr}_t)$, $\forall$ H_r, r ε n; $E(Z_t) = 0$; $E(Z_t)^2 = \sum^{n}_{r=1} b_t^2 (X_r^{Hr})^2 = \sum^{n}_{r=1} (b_r)^2 |t|^{2Hr}$, for $\forall$ H_r ε (0, 1). Then for $t < s < u < v \geq 0$:

$$\text{(A1)} \qquad [(Z_s - Z_t)(Z_v - Z_u)] = E(Z_v Z_s) + E(Z_u Z_t) - E(Z_v Z_t) - E(Z_u Z_s)$$

$$= 2^{-1} \sum^{n}_{r=1} (b_r)^2 [-(|v - s|^{2Hr} + |u - t|^{2Hr}) + |v - t|^{2Hr} + |u - s|^{2Hr}] = 0. \text{ (Property 2).}$$

2) For $u > 0$, $t > 0$, the correlation coefficient ρ is:

$$\text{(A2)} \qquad \rho (Z_{t+u} - Z_t)(Z_{s+u} - Z_s) = [2 \sum^{n}_{r=1} (b_r)^2 u^{2Hr}]^{-1} \{ \Sigma^{n}_{r=1} (b_r)^2 [(t - s + u)^{2Hr} + (t - s - u)^{2Hr}$$

$$- 2 (t - s)^{2Hr}]\} \neq 0 \text{ for } H \neq 1/2;$$

further: $\sum_{r=1}^{n} \rho^{Hr} \to \infty$ for $H > 1/2$ and $\sum_{r=1}^{n} |\rho|^{Hr} \to -\infty$ for $H_r < 1/2$ (Property 3) .

3) For any $s < u < v \geq 0$, if Z_t is a Markov process, then:

$$\text{Cov}(Z_v, Z_s)\,\text{cov}(Z_u, Z_u) = \text{cov}(Z_u, Z_s)\,\text{cov}(Z_v, Z_u);$$

using (1), collecting like terms and simplifying, we have:

$$\text{(A3)} \qquad 2 (v^{2Hr} + s^{2Hr} - |v - s|^{2Hr}) = (u^{2Hr} + s^{2Hr} - |u - s|^{2Hr})(v^{2Hr} + u^{2Hr} - |v - u|^{2Hr}).$$

The unique solution to (A3) is $H_r = 1/2$, $\forall$r ε n. In other words, if $H_r \neq 1/2$, the equality in (A3) does not obtain. Then Z_t is not a Markov process.

Appendix 2

Let $\delta R(t, s)/\delta t| s = R_t$ be a variation of R (t, s) with respect to t while holding s constant. Similarly, $\delta^2(R(t, s)/\delta t^2 = R_{tt}$,

δIt follows at $H_r = 1/2$, $R_{t.} = R_s = 0$. Then:

(A2.1) $$R_{tt} = \sum^n_{r=1} b_r^2 H_r (2H_r - 1) [t^{2Hr-2} - |t - s|^{2Hr-2}] > (<) 0 \text{ for } H_r > (<) 1/2.$$

$$R_{ss} = \sum^n_{r=1} b_r^2 H_r (2H_r - 1) [s^{2Hr-2} + |t - s|^{2Hr-2}] > (<) 0 \text{ for } H_r > (<) 1/2.$$

$$R_{ts} = R_{st} = \sum^n_{r=1} b_r^2 H_r (2H_r - 1) |t - s|^{2Hr-2} > (<) 0 \text{ for } H_r > (<) 1/2.$$

Since $(R_{tt})(R_{ss}) > (R_{ts})^2$ for $\forall H_r \varepsilon (S_1, S_2)$, R (t, s) is negative definite for $H_r < 1/2$, positive definite for $H_r > 1/2$, and is undefined at the inflection point at $H_r = 1/2$. Hence, at the upper boundary of S_1, STD approaches a maximum, and LTD approaches a minimum at the lower boundary of S_3.

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