Asymmetric Bertrand duopoly: game complete analysis by algebra system Maxima

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ASYMMETRIC BERTRAND DUOPOLY: GAME COMPLETE ANALYSIS BY ALGEBRA SYSTEM MAXIMA

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Abstract:

In this paper we apply the *Complete Analysis of Differentiable Games* (introduced by D. Carfi in [3], [6], [8] and [9]) and already employed by himself and others in [4], [5], [7]) and some new algorithms employing the software *wxMaxima 11.04.0* in order to reach a **total knowledge** of the classic Bertrand Duopoly (1883), viewed as a complex interaction between two competitive subjects, in a particularly difficult **asymmetric case**. The software wxMaxima is an interface for the computer algebra system *Maxima*. Maxima is a system for the manipulation of symbolic and numerical expressions, including differentiation, systems of linear equations, polynomials, and sets, vectors, matrices. Maxima yields high precision numeric results by using exact fractions, arbitrary precision integers, and variable precision floating point numbers. Maxima can plot functions and data in two and three dimensions. The Bertrand Duopoly is a classic oligopolistic market in which there are two enterprises producing the same commodity and selling it in the same market. In this classic model, in a competitive background, the two enterprises employ as possible strategies the unit prices of their products, contrary to the Cournot duopoly, in which the enterprises decide to use the quantities of the commodity produced as strategies. The main solutions proposed in literature for this kind of duopoly (as in the case of Cournot duopoly) are the *Nash equilibrium* and the *Collusive Optimum*, without any subsequent critical exam about these two kinds of solutions. The absence of any critical quantitative analysis is due to the relevant lack of knowledge regarding the set of all possible outcomes of this strategic interaction. On the contrary, by considering the Bertrand Duopoly as a *differentiable game* (games with differentiable payoff functions) and studying it by the new topological methodologies introduced by D. Carfi, we obtain an exhaustive and complete vision of the entire payoff space of the Bertrand game (this also in asymmetric cases with the help of wxMaxima 11.04.0) and this total view allows us to analyze critically the classic solutions and to find other ways of action to select Pareto strategies, in the **asymmetric cases too**. In order to illustrate the application of this topological methodology to the considered infinite game, several compromise pricing-decisions are considered, and we show how the complete study gives a real extremely extended comprehension of the classic model.

Keywords: Asymmetric Bertrand Duopoly, Normal-form Games, Software algorithms in Microeconomic Policy, Complete Analysis of a normal-form complex interaction, Pareto optima, valuation of Nash equilibriums, Bargaining solutions.

JEL Classification: D7, C71, C72, C78

1. Paper introduction

This paper is organized into 8 sections:

- section 2 contains the **classic way to introduce and solve the duopoly**;
- section 3 contains the **definition of the functions of our asymmetric Bertrand's duopoly**;
- In section 4 we start **the study of the asymmetric game and find the critical zone**;
- In section 5 we have **the transformation of the strategic space** in both cases, proper and improper;
- In section 6 we study **the non-cooperative friendly phase** and find the extremes and Pareto boundaries of payoff space;
- In section 7 we study **the properly non-cooperative phase**, and obtain Nash equilibrium and the best replies correspondences;
- section 8 contains **the defensive and offensive phases**, obtained through the worst gain functions;
- In section 9, we study **the cooperative phase**, through the analysis of the best cooperative strategies.

2. General setting of Bertrand duopoly

We consider a duopoly (1, 2) with *production fixed cost* f and *production variable cost* a function v of the produced quantity, for both the producers; we shall assume the function v equal 0.

***The demand for enterprise* 1** is the affine *reaction function* Q_1, from the Euclidean plane of price bi-strategies $\mathbf{R}^2$ into the real line of quantities to be produced $\mathbf{R}$ – the demand Q_1 (p) is the aggregate reaction of consumers in the market to the pair p of prices imposed by the two enterprises (see for a general theory of reactions [11], [12], [13], [14]) - defined by

$$Q_1(p) = b + a_1p_1 + a_2p_2 , \quad (2.\ 1)$$

that is by the equality

$$Q_1(p) = b + (a|p), \quad (2.\ 2)$$

for every pair of prices p, where a is a pair of real numbers whose first component is negative and whose second component is positive, where b is a non-negative real and where (a|p) is the standard Euclidean scalar product of the two vectors a and p.

The components of the pair p are determined by the two enterprises 1 and 2, respectively.

The *demand for the enterprise 2* is the function Q_2 defined, in a perfectly analogous way as the first one, by

$$Q_2(p) = b + a_1p_2 + a_2p_1, \quad (2.\ 3)$$

that is

$$Q_2(p) = b + (a^-|p), \quad (2.\ 4)$$

for every pair of prices p, where a^- is the symmetric pair of a.

The ***classic way to solve the duopoly*** (see for instance: Davide Vannoni and Massimiliano Piacenza, University of Torino, Faculty of Economics, *Appunti di Microeconomia - Corso C - Lezione 10*, A. A. 2009/2010) is to determine the curves of best price reaction, for example, for enterprise 1, we consider the profit function P_1 defined by

$$P_1 (p_1,q_1) = p_1q_1 – f \quad (2.\ 5)$$

that, on the reaction demand function Q_1, assumes the form

$$g_1(p) = P_1 (p_1 ,Q_1 (p)) = p_1(b + a_1p_1 + a_2p_2) – f, \quad (2.\ 6)$$

for every price p_2, fixed by the enterprise 2, the price of maximum profit for enterprise 1 must satisfy the following stationary condition

$$D_1(g_1)(p) = b + 2a_1p_1 + a_2p_2 = 0. \quad (2.\ 7)$$

We note that the second derivative of the function $g_1(. ,p_2)$ is negative ($2a_1 < 0$), hence the above stationary condition is not only necessary but also sufficient in order to obtain maxima, we so determine the classic reaction curve of enterprise 1, the line of equation

$$p_1 = b/(2a_1) + p_2a_2/(2a_1). \quad (2.\ 8)$$

Symmetrically, the reaction curve of enterprise 2 is the line

$$p_2 = b/(2a_1) + p_1a_2/(2a_1). \quad (2.\ 9)$$

Now, by the intersection of the two reaction curves, we obtain the fixed-point equation

$$p_1 = b/(2a_1) + (b/(2a_1) + p_1a_2/(2a_1))a_2/(2a_1), \quad (2.\ 10)$$

and so finally we obtain the equilibrium price of the enterprise 1, and the same of enterprise 2:

$$p_2 = p_1 = - b/(2a_1 + a_2). \quad (2.\ 11)$$

Another classic solution is the symmetric collusive point C = (c,c) determined by maximization of the function H defined by

$$H(c) = P_1 (c ,Q_1 (c,c)) + P_2 (c ,Q_2 (c,c)) = 2c(b + (a_1 + a_2)c) – 2f, \quad (2.\ 12)$$

for every c.

But also in this case, an accurate analysis of this point is impossible since we do not know the geometry of the payoff space.

3. Formal description of Bertrand's normal form game

It will be a non-linear two-players gain game (*f*, >) (see also [6], [8] and [9]). The two players/enterprises shall be called *Emil* and *Frances* (following Aubin's books [1] and [2]).

Assumption 1 (strategy sets). The two players produce and offer the same commodity at the following prices: $x \in \mathbb{R}_\geq$ for Emil and $y \in \mathbb{R}_\geq$ for Frances. In more precise terms: the payoff function *f* of the game is defined on a subset of the positive cone of the Cartesian plane $\mathbb{R}^2$, interpreted as a space of bi-prices. We assume (by simplicity) that the set of all strategies (of each player) is the interval E = [0, +∞[.

Assumption 2 (asymmetry of the game). The game will be assumed asymmetric with respect to the players. In other terms, the payoff pair f(x,y) is not the symmetric of the pair f(y,x).

Assumption 3 (form of demand functions). Let the demand function Q_1 (defined on E^2) of the first player be given by

$$Q_1(x,y) = u - 2x + y, \quad (3.\ 1)$$

for every positive price pair (x, y) and let analogously the demand function of the second enterprise be given by

$$Q_2(x,y) = u - 4y + x, \quad (3.\ 2)$$

for every positive bi-strategy (x, y), where u is a positive constant (representing, obviously, the quantity $Q_i(0,0)$ demanded of good i, by the market, when both prices are fixed to be 0).

Remark (about elasticity). The demand's elasticity of the two functions with respect to the corresponding price is

$$e_1(Q_1)(x,y) = \partial_1Q_1(x,y)(x/Q_1(x,y)) = -2x/(u - 2x + y), \quad (3.\ 3)$$

and

$$e_2(Q_2)(x,y) = \partial_2Q_2(x,y)(y/Q_2(x,y)) = -4y/(u - 4y + x), \quad (3.\ 4)$$

for every positive bi-strategy (x, y).

Their values are negative, according to the economic law: *produced quantities are decreasing with respect to their prices*. So, if Emil (or Frances) increases his price, the consumers' demand will diminish.

Assumption 4 (payoff functions). First player's ***profit function*** is defined, classically, by the revenue

$$f_1(x,y) = x\, Q_1(x,y) - c = x(u - 2x + y) - c, \tag{3. 5}$$

for every positive bistrategy (x, y). Second player's ***profit function*** is defined by

$$f_2(x,y) = y\, Q_2(x,y) - c = y(u - 4y + x) - c, \tag{3. 6}$$

for every positive bistrategy (x, y), where the positive constant c is the fixed cost. So we assume the variable cost to be 0 (this is not a great limitation for our example, since our interest is the interaction between the two players and the presence of the variable cost does not change our approach).

The bi-gain function is so defined by

$$f(x,y) = (x(u - 2x + y), y(u - 4y + x)) - (c, c), \tag{3. 7}$$

for every bistrategy (x, y) of the game in the unbounded square E^2.

4. Study of the Bertrand's normal form game

When the fixed cost is zero, we can assume that Emil and Frances have the compact strategy sets

$$E = F = [0, u], \tag{4. 1}$$

indeed, we have the following property.

Property. *A necessary condition in order to obtain both the quantities $Q_i(x,y)$ positive is that the pair of prices (x,y) lies in the square E^2.*

Proof. The reader can easily prove the above interesting property, by imposing the positivity conditions for the affine functions Q_i. ♦

The improper Bertrand game. Besides, we will consider an extension of the Bertrand game with strategy spaces $E = F = [-u, u]$, in order to obtain a wider vision of the game itself by enlarging the bistrategy space.

Payoff function to examine. When the fixed cost c is zero (this assumption determines only a "reversible" translation of the gain space), the bi-gain function f from the compact square $[0, u]^2$ into the bi-gain plane $\mathbb{R}^2$ (respectively the function f from the square $[-u, u]^2$ into the same plane $\mathbb{R}^2$) is defined by

$$f(x,y) = (x(u - 2x + y), y(u - 4y + x)), \tag{4. 2}$$

for every bi-strategy (x, y) in the square $S = [0, u]^2$ (respectively, in the square $S = [-u, u]^2$), which is the convex envelope of its vertices, denoted by A, B, C, D starting from the origin (or from $(- u, u)$) and going anticlockwise.

When the characteristic price u is 1, we will obtain the payoff vector function defined by

$$f(x,y) = (x(1 - 2x + y), y(1 - 4y + x)), \tag{4. 3}$$

on the strategy square $S = [0, 1]^2$ (or $S = [-1, 1]^2$).

Now, we must find ***the critical space of the game*** and its image by the function f, before representing $f(S)$.

For, we determine (as explained in [3], [6], [8] and [9]) firstly the *Jacobian matrix* of the function f at any point (x, y) - denoted by $J_f(x,y)$. We will have, in vector form, the pair of gradients

$$J_f(x,y) = ((y - 4x + 1, x), (y, - 8y + x + 1)), \tag{4. 4}$$

and concerning the determinant of the above pair of vectors

$$\det J_f(x,y) = (-8y + x + 1)(y - 4x + 1) - xy =$$

$$= -8y^2 + 32xy - 7y - 4x^2 - 3x + 1. \qquad (4.5)$$

The *Jacobian determinant* is zero at those points (x_1, y_1) and (x_2, y_2) of the strategy square such that

$$y_1 = -\text{sqrt}(896\, x_1^2 - 544\, x_1 + 81)/16 + 2\, x_1 - 7/16, \qquad (4.6)$$

and

$$y_2 = \text{sqrt}(896\, x_1^2 - 544\, x_1 + 81)/16 + 2\, x_1 - 7/16. \qquad (4.7)$$

From a geometrical point of view, we will obtain two curves (Figure 4.1 with $S = [0, 1]^2$ and Figure 4.2 with $S = [-1, 1]^2$); they represent *the critical zone of Bertrand Game*.

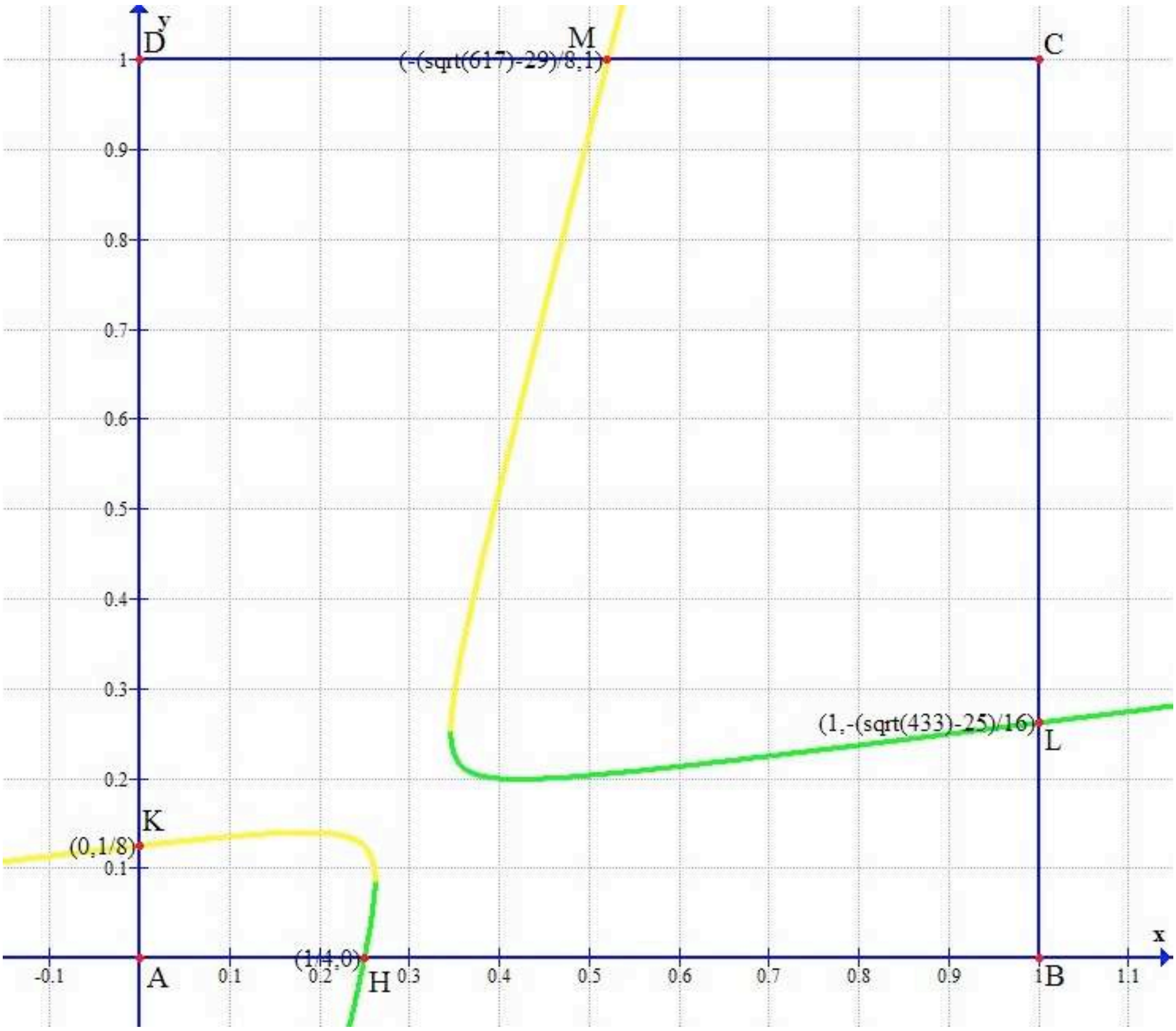

Figure 4. 1. Critical zone of Bertrand game

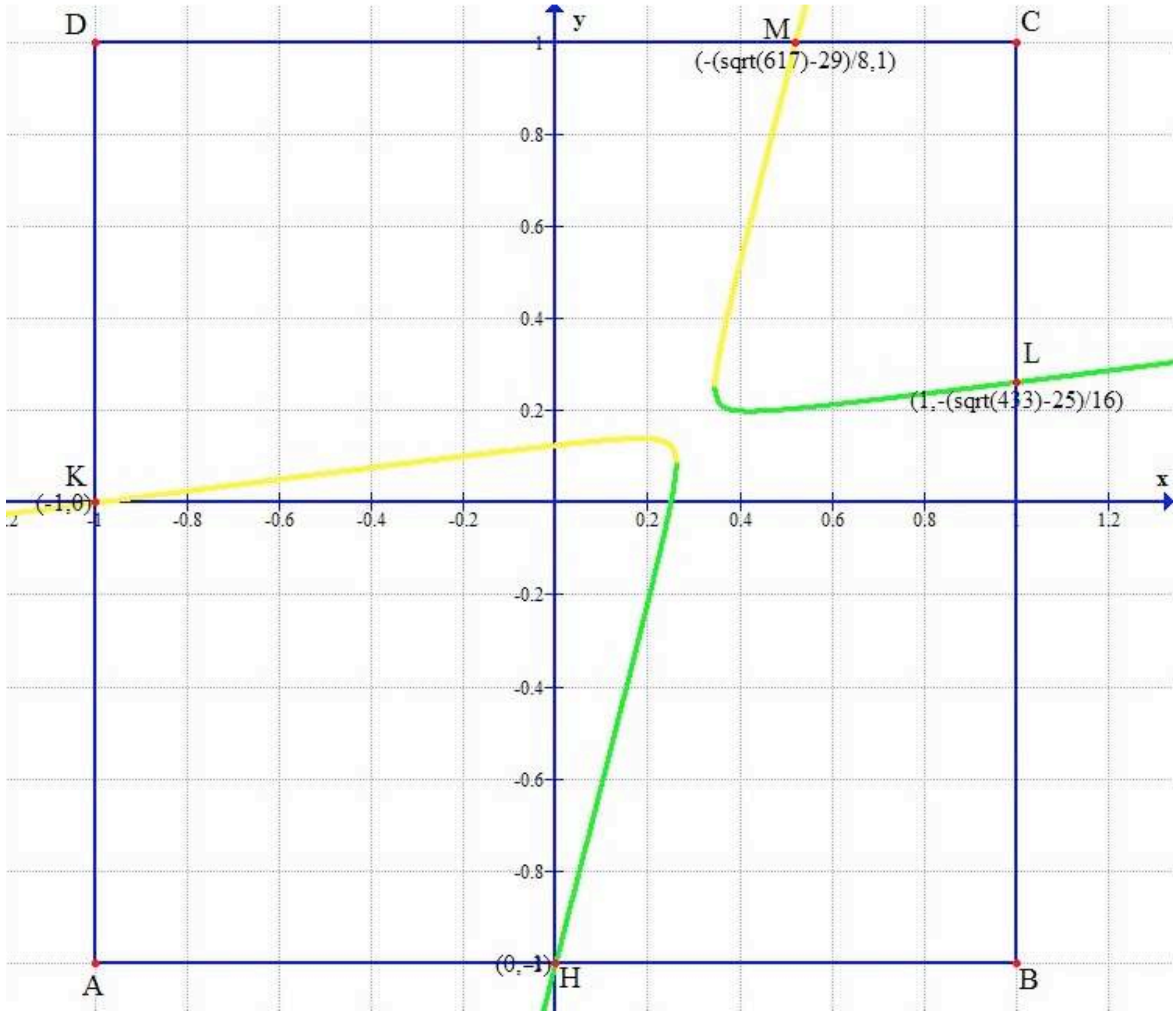

Figure 4. 2. Critical zone of improper Bertrand game

5. Transformation of the strategy space

It is readily seen that the intersection points of the yellow curve with the boundary of the strategic square are the two points

M = (-(sqrt(617) - 29)/8, 1) , K = (0, 1/8).

Remark. The point M is the intersection point of the yellow curve with the segment [C, D], its abscissa μ verifies the non-negative condition and the following equation

$$16 = \mathrm{sqrt}(896\mu_1^2 - 544\mu_1 + 81) + 32\mu_1 - 7, \tag{5. 1}$$

this abscissa is so

μ_1 = -(sqrt(617) - 29)/8

(approximately equal to 0,520064).

The point K is the intersection point of the **ordinate axis** and the ***best reply correspondence*** $y = (1/8)(x + 1)$, see paragraph 7 (7. 12).

It is, also, readily seen that the intersection points of the green curve with the boundary of the strategic square are the two points

L = (1, -(sqrt(433) - 25)/16) , H = (1/4, 0).

Remark. The point L is the intersection point of the green curve with the segment [B, C]. To find the coordinate simply solve the following system:

$$(x = 1,\ \mu_2 = \text{-sqrt}(896\, x^2 - 544\, x + 81)/16 + 2\, x - 7/16), \tag{5. 2}$$

this ordinate is so

μ_2 = -(sqrt(433) - 25)/16

(approximately equal to 0,261959).

The point H is the intersection point of the **abscissa axis** and the ***best reply correspondence*** $x = (1/4)(y + 1)$, see paragraph 7 (7. 12).

We start from Figure 4.1, with $S = [0, 1]^2$.

The transformations of the bi-strategy square vertices and of the points H, K, M, L are the following:

- A' = f(A) = f(0, 0) = (0, 0);
- B' = f(B) = f(1, 0) = (-1, 0);
- C' = f(C) = f(1, 1) = (0, -2);
- D' = f(D) = f(0, 1) = (0, -3);
- H' = f(H) = f(1/4, 0) = (1/8, 0);
- K' = f(K) = f(0, 1/8) = (0, 1/16);
- M' = f(M) = $f(\mu_1, 1) = (\mu_1 - \mu_1^2, \mu_1 - 3)$ approximately equal to (0,499, -2,479);
- L' = f(L) = $f(1, \mu_2) = (\mu_2 - 1, \mu - 2\mu_2^2)$ approximately equal to (-0,738, 0,249).

Starting from Figure 4.1, with $S = [0, 1]^2$, we can do the transformation of its sides.

Side [A, B]. Its parameterization is the function sending any point $x \in [0, 1]$ into the point $(x, 0)$; the transformation of this side can be obtained by transformation of its generic point $(x, 0)$, we have

$$f(x, 0) = (x - 2x^2, 0). \tag{5. 3}$$

We obtain the segment with end points B' and H', with parametric equations

$$X = x - 2x^2 \text{ and } Y = 0, \tag{5. 4}$$

with x in the unit interval.

Side [B, C]. It is parameterized by

$(x = 1, y \in [0, 1])$;

the figure of the generic point is

$$f(1, y) = (y - 1, -4y^2 + 2y). \tag{5. 5}$$

We can obtain the parabola passing through the points B', L', C' with parametric equations

$$X = y - 1 \text{ and } Y = -4y^2 + 2y. \tag{5. 6}$$

Side [C, D]. Its parameterization is

$(x \in [0, 1], y = 1)$;

the transformation of its generic point is

$$f(x, 1) = (-2x^2 + 2x, x - 3). \quad (5.7)$$

We can obtain the parabola passing through the points C', M', D' with parametric equations

$$X = -2x^2 + 2x \text{ and } Y = x - 3. \quad (5.8)$$

Side [D, A]. Its parameterization is

$(x = 0, y \in [0, 1])$;

the transformation of its generic point is

$$f(0, y) = (0, -4y^2 + y). \quad (5.9)$$

We obtain the segment [D', K'] with parametric equations

$$X = 0 \text{ and } Y = -4y^2 + y, \quad (5.10)$$

with y in the unit interval.

Now, we find ***the transformation of the critical zone***. The parameterization of the critical zone is defined by the equations

$$y_1 = -\,\mathrm{sqrt}(896\, x_1^2 - 544\, x_1 + 81)/16 + 2\, x_1 - 7/16 \quad \text{(see (4. 6))}$$

and

$$y_2 = \mathrm{sqrt}(896\, x_1^2 - 544\, x_1 + 81)/16 + 2\, x_1 - 7/16. \quad \text{(see (4. 7))}$$

The parameterization of the GREEN ZONE is

$(x \in [0, 1], y = y_1)$;

the transformation of its generic point is

$$f(x,y_1) = (x - 2x^2 + xy_1, y_1 - 4y^2{}_1 + xy_1), \quad (5.11)$$

a parameterization of the YELLOW ZONE is

$(x \in [0, 1], y = y_2)$;

the transformation of its generic point is

$$f(x,y_2) = (x - 2x^2 + xy_2, y_2 - 4y^2{}_2 + xy_2). \quad (5.12)$$

The transformation of the Green Zone has parametric equations

$$X = x - 2x^2 + xy_1 \text{ and } Y = y_1 - 4y^2{}_1 + xy_1, \quad (5.13)$$

and the transformation of the Yellow Zone has parametric equations

$$X = x - 2x^2 + xy_2 \text{ and } Y = y_2 - 4y^2_2 + xy_2. \quad (5.\ 14)$$

We have two colored curves in *green* and *black* (Figure 5.1), breaking by two points of discontinuity T and U obtained by resolving the following equation

$$896\, x^2 - 544\, x + 81 = 0; \quad (5.\ 15)$$

the solutions of the above equation are

$$x_1 = -\,(\mathrm{sqrt}(22) - 34)/112,\ x_2 = (\mathrm{sqrt}(22) + 34)/112, \quad (5.\ 16)$$

and then, replacing them in the parametrical equations of the critical zone, and putting

$t = \mathrm{sqrt}(22)+34$ with $s = -\,\mathrm{sqrt}(t^2/14- (34t)/7 +81)/16$

we obtain

$T_1 = (t(s + t/56- 7/16))/112 - t^2/6272 + t/112$

from

T$_1$ = ((sqrt(22)+34)*(-sqrt((sqrt(22)+34)^2/14-(34*(sqrt(22)+34))/7+81)/16+(sqrt(22)+34)/56-7/16))/112 +
- (sqrt(22)+34)^2/6272+(sqrt(22)+34)/112;

moreover

$T_2 = -\,4(s + t/56 - 7/16)^2 + s + (t(s + t/56 - 7/16))/112 + t/56 - 7/16,$

from

T$_2$ = - 4(-sqrt((sqrt(22)+34)^2/14-(34*(sqrt(22)+34))/7+81)/16+(sqrt(22)+34)/56-7/16)^2-sqrt((sqrt(22)+34)^2/14+
- (34*(sqrt(22)+34))/7+81)/16+((sqrt(22)+34)*(-sqrt((sqrt(22)+34)^2/14+
- (34*(sqrt(22)+34))/7+81)/16+(sqrt(22)+34)/56-7/16))/112+(sqrt(22)+34)/56-7/16;

and, putting

$u = 9 - \mathrm{sqrt}(6)\ 34-\mathrm{sqrt}(22)$ with $v = -\,\mathrm{sqrt}(-(34u)/7 + u^2/14 + 81)/16$

we obtain

$U_1 = ((u/56 + v - 7/16)u)/112 + u/112 - u^2/6272$

from

U$_1$ = (((34-sqrt(22))/56 - sqrt(-(34*(34-sqrt(22)))/7 + (34-sqrt(22))^2/14+81)/16-7/16)*(34-sqrt(22)))/112 +
+ (34-sqrt(22))/112 - (34-sqrt(22))^2/6272;

and

$U_2 = ((u/56 + v - 7/16)u)/112 + u/56 – 4(u/56 + v - 7/16)^2 + v - 7/16$

from

U2=(((34-sqrt(22))/56-sqrt(-(34*(34-sqrt(22)))/7+(34-sqrt(22))^2/14+81)/16-7/16)*(34-sqrt(22)))/112+(34-sqrt(22))/56-4*((34-sqrt(22))/56-sqrt(-(34*(34-sqrt(22)))/7+(34-sqrt(22))^2/14+81)/16-7/16)^2-sqrt(-(34*(34-sqrt(22)))/7+(34-sqrt(22))^2/14+81)/16-7/16.

So, we obtain - approximately - the point

$T = (0{,}194,\ 0{,}084)$,

and the point

$U = (0{,}147,\ 0{,}078)$.

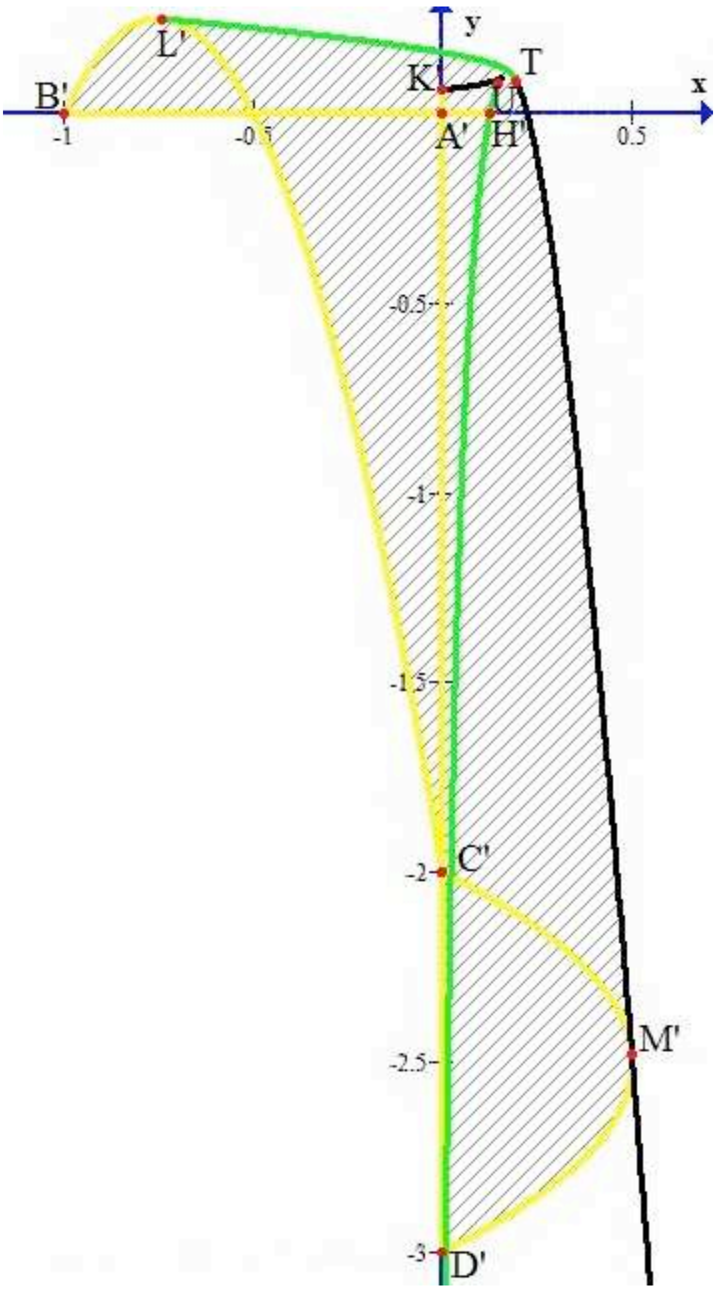

Figure 5. 1. Payoff space of asymmetric Bertrand game

Payoff space of the improper Bertrand's game

Starting from the Figure 4.2 with $S = [-1, 1]^2$; the projections of bistrategy square's points are the following:

- $A' = f(A) = f(-1, -1) = (-2, -4)$;
- $B' = f(B) = f(1, -1) = (-2, -6)$;
- $C' = f(C) = f(1, 1) = (0, -2)$;
- $D' = f(D) = f(-1, 1) = (-4, -4)$;
- $H' = f(H) = f(0, -1) = (0, -5)$;
- $K' = f(K) = f(-1, 0) = (-3, 0)$;
- $M' = f(M) = f(\mu_1, 1) = (\mu_1 - \mu_1^2, \mu_1 - 3)$ approximately equal to (0,499, -2,479);
- $L' = f(L) = f(1, \mu_2) = (\mu_2 - 1, \mu - 2\mu_2^2)$ approximately equal to (-0,738, 0,249).

Starting from Figure 4.2 with $S = [-1, 1]^2$, we can do the transformation of its sides.

Side [A, B]. Its parametric form is

$(x \in [-1, 1], y = -1)$;

the transformation of its generic point is

$$f(x, -1) = (-2x^2, -x - 5). \tag{5. 17}$$

We can obtain the parabola passing through the points A', H', B' with

$$X = -2x^2 \text{ and } Y = -x - 5. \tag{5. 18}$$

Side [B, C]. Its parameterization is

$(x = 1, y \in [-1, 1])$;

the transformation of its generic point is

$$f(1, y) = (y - 1, -4y^2 + 2y). \tag{5. 19}$$

We can obtain the parabola passing through the points B', L', C' with

$$X = y - 1 \text{ and } Y = -4y^2 + 2y. \tag{5. 20}$$

Side [C, D]. Its parameterization is

$(x \in [-1, 1], y = 1)$;

the transformation of its generic point is

$$f(x, 1) = (-2x^2 + 2x, x - 3). \tag{5. 21}$$

We can obtain the parabola passing through the points C', M', D' with parametric equations

$$X = -2x^2 + 2x \text{ and } Y = x - 3. \tag{5. 22}$$

Side [D, A]. Its parameterization is

$(x = -1, y \in [-1, 1])$;

the transformation of its generic point is

$$f(-1, y) = (-y - 3, -4y^2). \tag{5. 23}$$

We can obtain the parabola passing through the points D', K', A' with parametric equations

$$X = -y - 3 \text{ and } Y = -4y^2. \tag{5. 24}$$

For the transformation of the critical zone and the coordinates of the points of discontinuity please refer to the case $S = [0, 1]^2$, we must remember only to widen the interval considered from $x, y \in [0, 1]$ to $x, y \in [-1, 1]$.

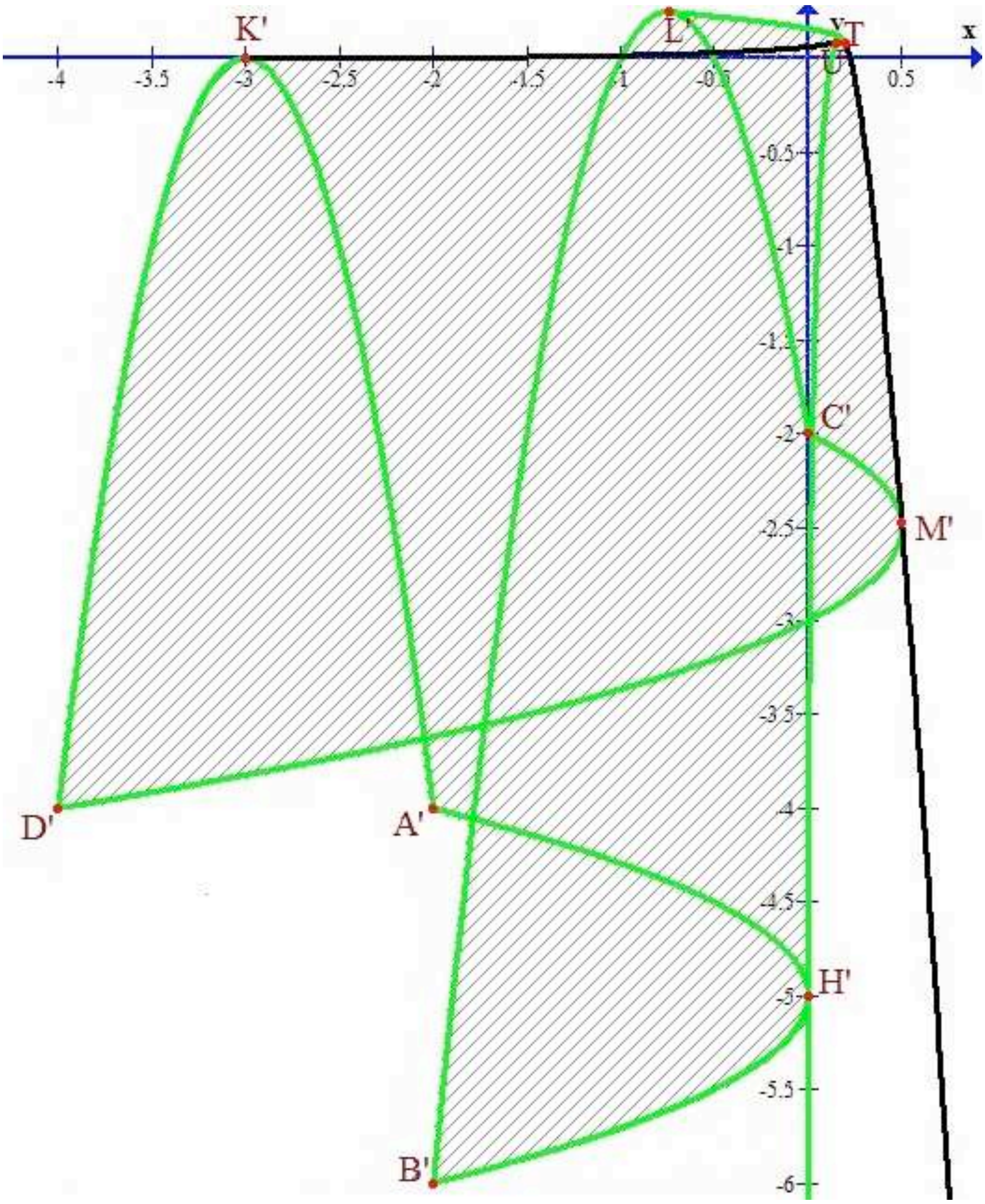

Figure 5. 2. Payoff space of the improper asymmetric Bertrand game

6. Non-cooperative friendly phase

Examining the Figure 5.1, in which $S = [0, 1]^2$, we will note that the game has two shadow extremes, that is the *shadow minimum*

α = (-1, -3)

and the *shadow maximum*

β = (0,49919, 0.24942).

The *Pareto minimal boundary of the payoff space* $f(S)$ is showed in the Figure 6.1 by the union of the point B' and the curves passing through the points C' and D'.

The *Pareto maximal boundary of the payoff space* $f(S)$ is the union of the two curve segments, on the transformations of the critical zone, with extreme points the pair of points

(L', T) and (T, M').

They are colored in *green* and in *black*.

Both Emil and Frances do not control the Pareto maximal boundary; they could reach the point L' and M' of the boundary, but the solution is not many satisfactory for them. In fact, a player will suffer a loss and the other one will have a small win.

Examining the Figure 5.2, in which

$S = [-1, 1]^2$,

we will note that the game has two shadow extremes, that is the *shadow minimum*

α = (-4, -6)

and the *shadow maximum*

β = (0,49919, 0.24942).

The *Pareto minimal boundary* $f(S)$ is showed in the Figure 6.2, it has only two points, the points D' and B'.

For *Pareto maximal boundary* $f(S)$, in the case $S = [0, 1]^2$, is the union of the boundary curves with end-points the pairs

(L', T) and (T, M'),

respectively.

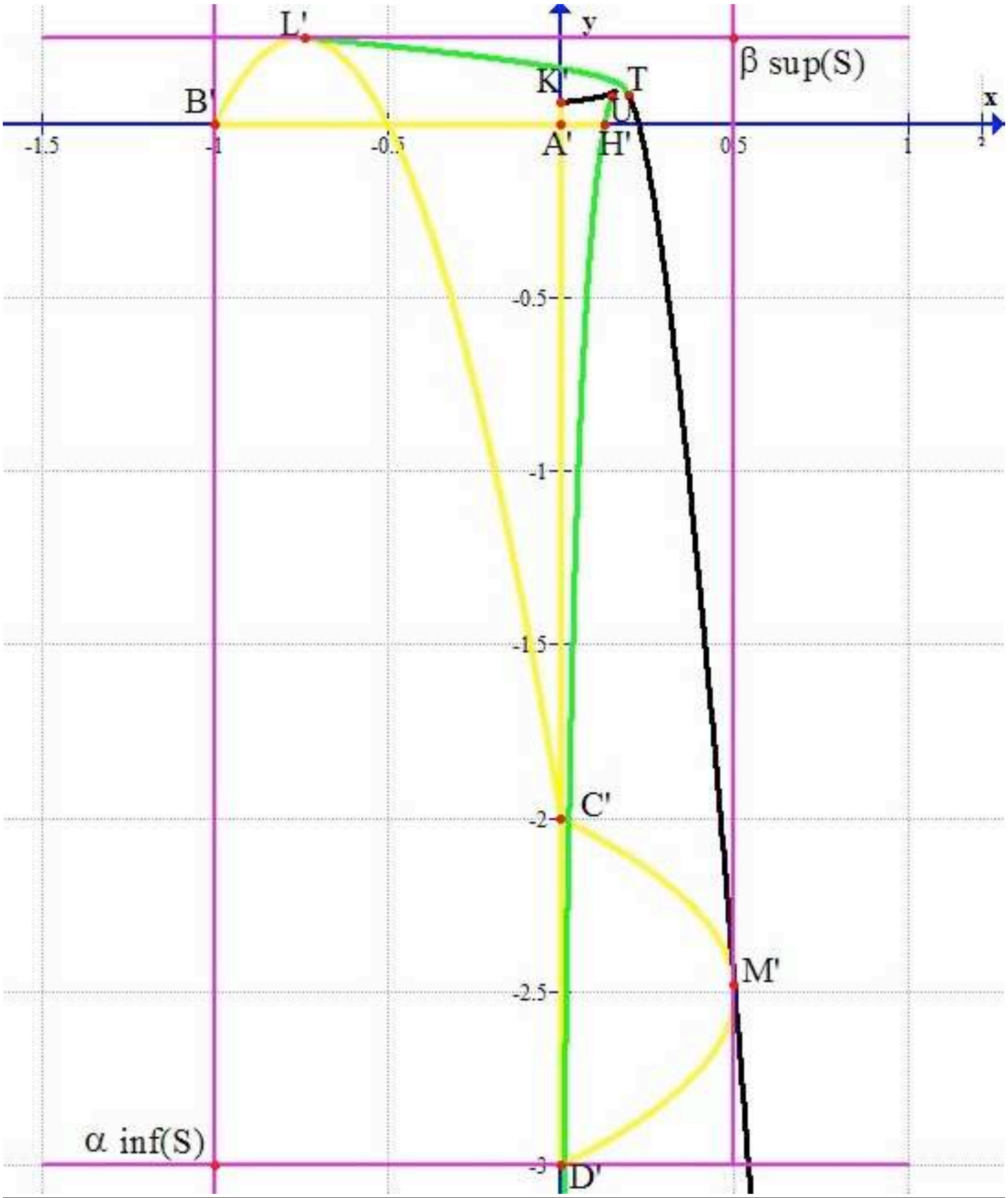

Figure 6. 1. Extrema of the Bertrand game

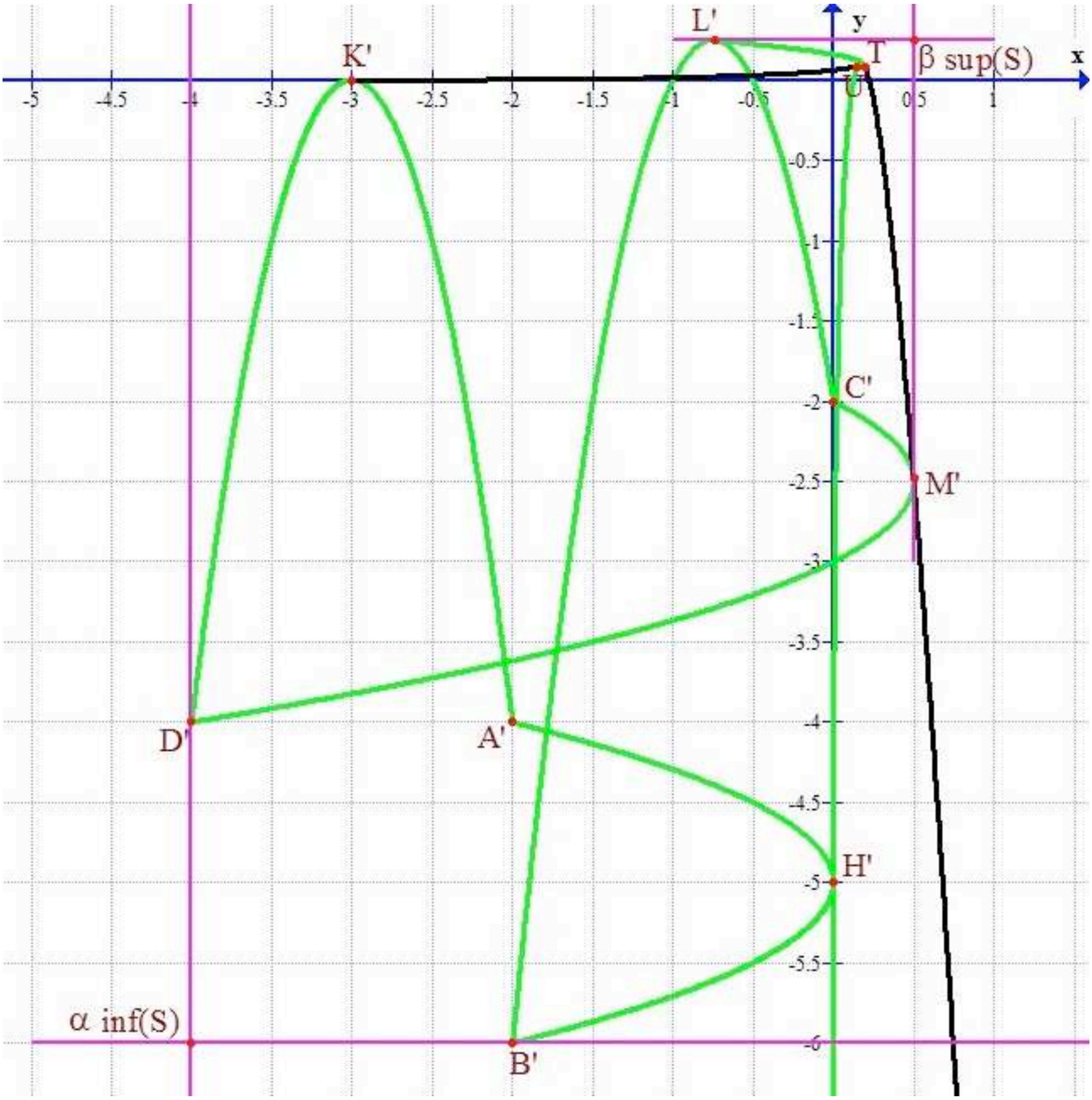

Figure 6. 2. Extrema of the improper Bertrand game

7. Properly non-cooperative (egoistic) phase

Now, we will consider ***the best reply correspondences*** between the two players Emil and Frances, in the cases $S = [0, 1]^2$ and $S = [-1, 1]^2$.

If Frances sells the commodity at the price y, Emil, in order to reply rationally, should maximize his *partial profit function*

$$f_1(\cdot, y) : x \mapsto x(1 - 2x + y), \quad (7.\ 1)$$

on the compact interval [0, 1] or [-1, 1].

According to the Weierstrass theorem, there is at least one Emil's strategy maximizing that partial profit function and, by Fermàt theorem, the ***Emil's best reply strategy to Frances' strategy*** y is the only price

$$B_1(y) = x^* := (1/4)(y + 1). \quad (7.\ 2)$$

Indeed, the partial derivative

$$f_1(\cdot, y)'(x) = -4x + 1 + y, \quad (7.\ 3)$$

is positive for $x < x^*$ and negative for $x > x^*$.

So, the Emil's best reply correspondence is the function B_1 from the interval [0, 1] into the interval [0,1], defined by

$$y \mapsto (1/4)(y + 1), \quad (7.\ 4)$$

in the proper case, and B_1 from [-1, 1] into [-1,1], defined by

$$y \mapsto (1/4)(y + 1), \quad (7.\ 5)$$

in the improper one.

If Emil sells the commodity at the price x, Frances, in order to reply rationally, should maximize his *partial profit function*

$$f_2(x, \cdot) : y \mapsto y\,(1 - 4\,y + x), \quad (7.\ 6)$$

on the compact interval [0, 1] or [-1, 1].

According to the Weierstrass theorem, there is at least one Frances' strategy maximizing that partial profit function and, by Fermàt theorem, the ***Frances' best reply strategy to Emil's strategy*** x is the only price

$$B_2(x) = y^* := (1/8)(x + 1). \quad (7.\ 7)$$

Indeed, the partial derivative

$$f_2(x, \cdot)'(\,y) = -8y + 1 + x, \quad (7.\ 8)$$

is positive for $y < y^*$ and negative for $y > y^*$.

So, the Frances' best reply correspondence is the function B_2 from the interval [0, 1] into the interval [0,1], defined by

$$x \mapsto (1/8)(\,x + 1), \quad (7.\ 9)$$

in the proper case, and B_2 from [-1, 1] into [-1,1], defined by

$$x \mapsto (1/8)(x + 1), \tag{7. 10}$$

in the improper one.

The *Nash equilibrium* is the fixed point of the symmetric Cartesian product function B of the pair of two reaction functions (B_2,B_1) defined (canonically) from the Cartesian product of the domains into the Cartesian product of the codomains (in inverse order), by

$$B : (x, y) \mapsto (B_1(y), B_2(x)), \tag{7. 11}$$

that is the only bi-strategy (x,y) satisfying the below system of linear equations

$$x = (1/4)(y + 1),\ y = (1/8)(x + 1), \tag{7. 12}$$

that is the point N = (9/31, 5/31) - as we can see also from the two Figures 7.1 and 7.2 - which determines a bi-gain

$$N' = (162/961, 100/961),$$

as Figures 6.3 and 6.4 will show.

The Nash equilibrium is not completely satisfactory, because it is not a Pareto optimal bi-strategy, but it represents the only properly non-cooperative game solution.

Remark (demand elasticity at Nash equilibrium). Concerning the Nash equilibrium, we can also calculate the demand's elasticity with respect to the corresponding price. At first, we must remember the two demand functions, and then we obtain

$$e_1(Q_1)(x,y) = \partial_1 Q_1(x,y)(x/Q_1(x,y)) = (-2x/(u - 2x + y)), \qquad \text{(see (3. 3))}$$

and

$$e_2(Q_2)(x,y) = \partial_2 Q_2(x,y)(y/Q_2(x,y)) = (-4y/(u - 4y + x)). \qquad \text{(see (3. 4))}$$

Then, we have

$$e_1(Q_1)(N) = \partial_1 Q_1(N)((1/3)/Q_1(x,y)) = (-(18/31)/(1 - (18/31) + (5/31))) = -1, \tag{7. 13}$$

and

$$e_2(Q_2)(N) = \partial_2 Q_2(N)((1/3)/Q_2(x,y)) = (-(20/31)/(1 - (20/31) + (9/31))) = -1. \tag{7. 14}$$

So we can deduce that:

at the non-cooperative equilibrium N, since

$$|e_1(Q_1)(N)| = 1 \text{ and } |e_2(Q_2)(N)| = 1, \tag{7. 15}$$

the demands will be elastic with respect to the prices; therefore if the price increases of one unit, demand will reduce of one unit too.

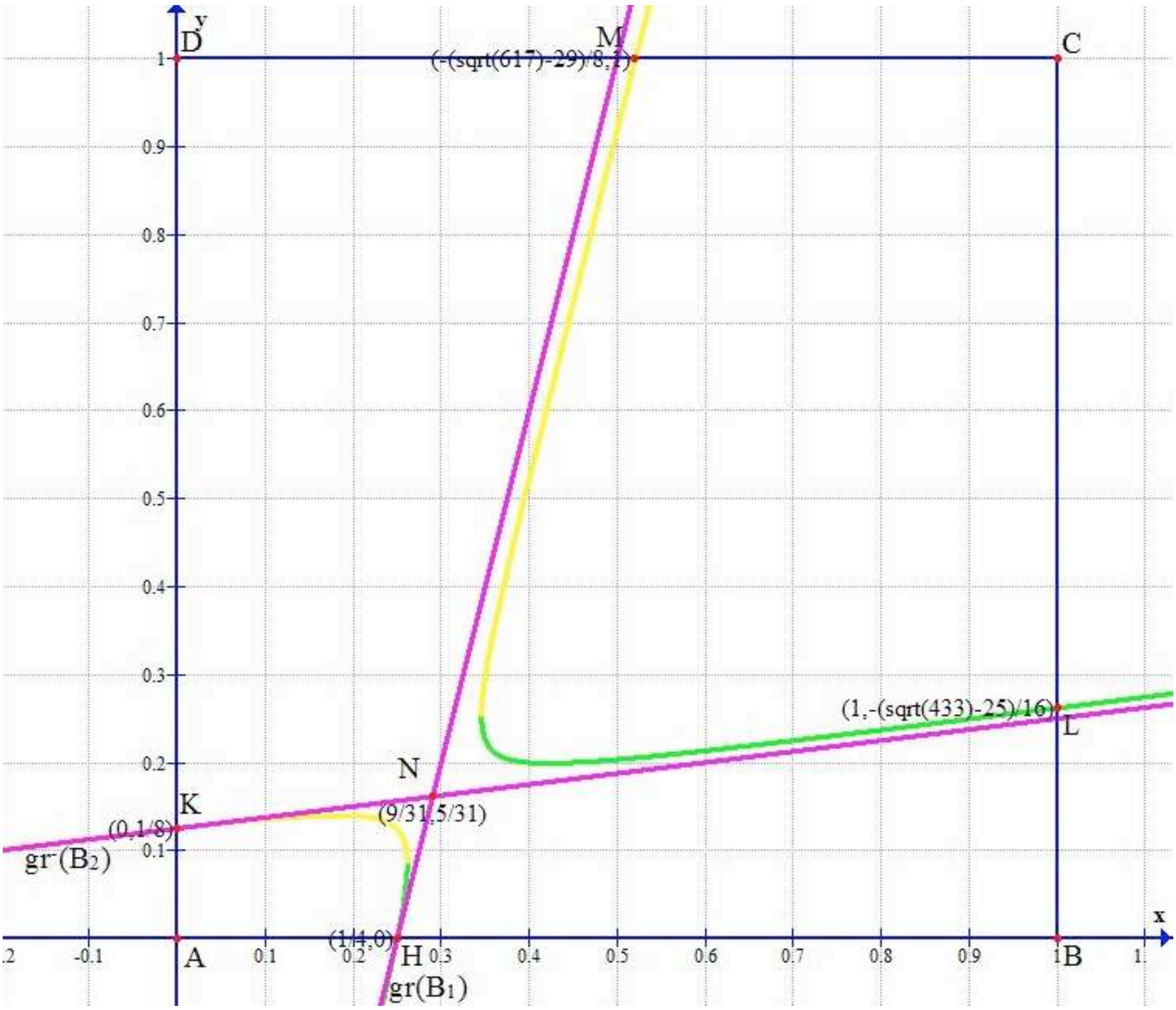

Figure 7. 1. Nash Equilibrium of the proper game

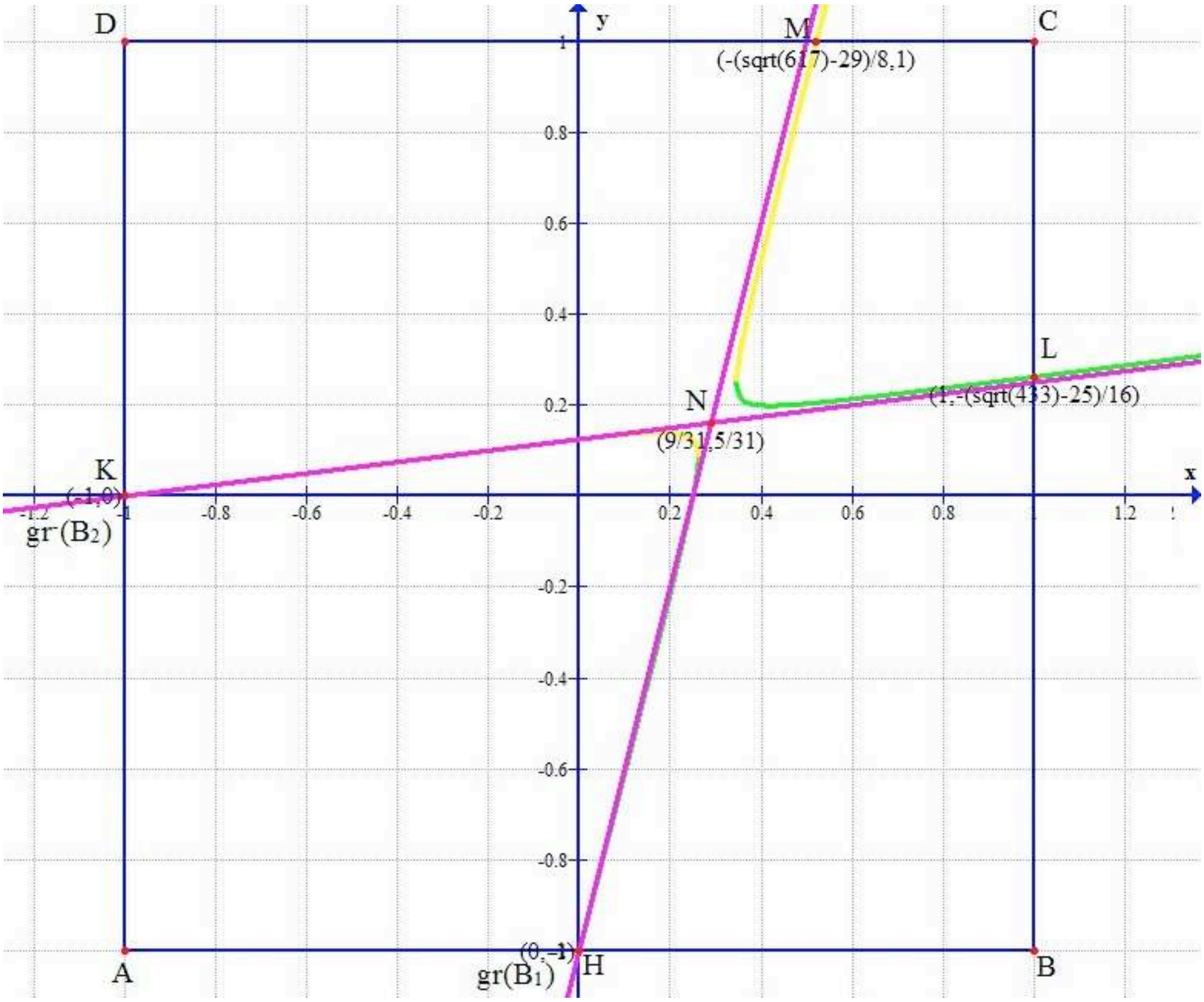

Figure 7. 2. Nash equilibrium of the improper game

Figure 7. 3. Payoff at Nash equilibrium of the proper Bertrand game

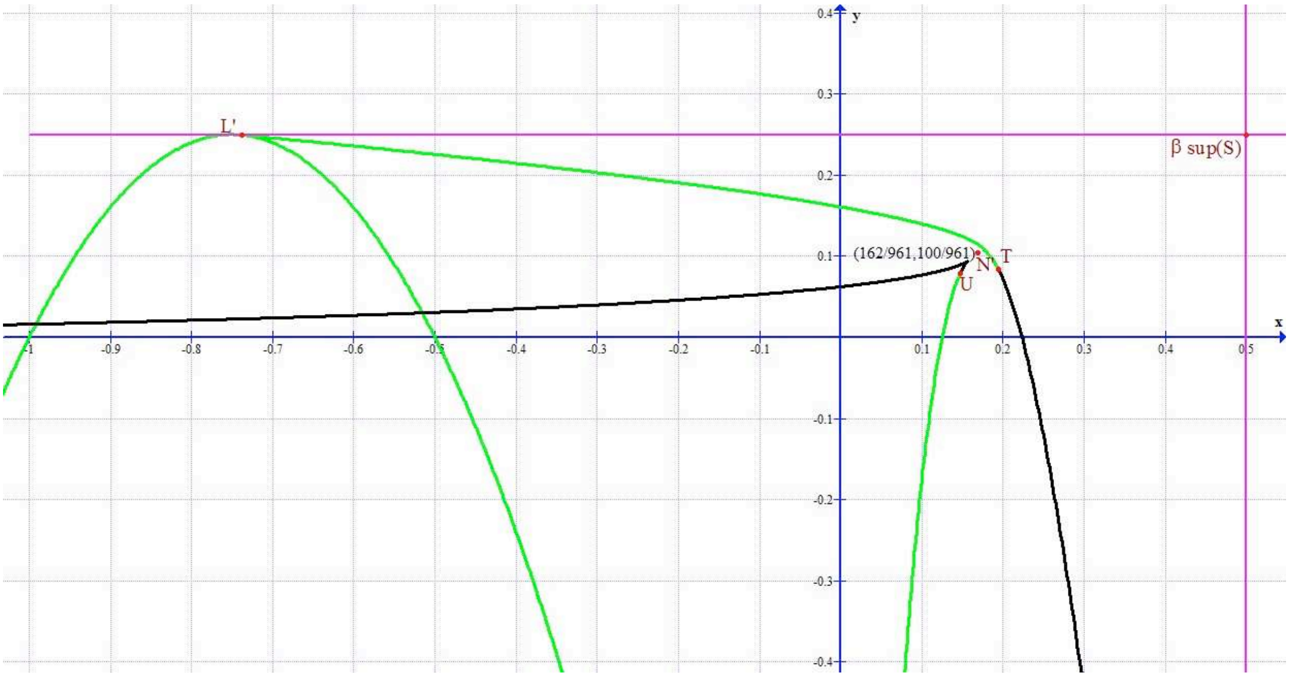

Figure 7. 4. Payoff at Nash equilibrium of the improper game

8. Defensive and offensive phase

Players' conservative values are obtained through their ***worst gain functions***.

Worst gain functions. On the square $S = [0, 1]^2$, ***Emil's worst gain function*** is defined by

$$f^{\#}_1(x) = \inf_{y \in F} x(1 - 2x + y) = x - 2x^2, \tag{8. 1}$$

its maximum will be

$$v^{\#}_1 = \sup_{x \in E} (f^{\#}_1(x)) = \sup_{x \in E} (x - 2x^2) = 1/8, \tag{8. 2}$$

attained at the conservative strategy $x^{\#} = 1/4$.

Frances' worst gain function is defined by

$$f^{\#}_2(y) = \inf_{x \in E} y(1 - 4y + x) = y - 4y^2, \tag{8. 3}$$

its maximum will be

$$v^{\#}_2 = \sup_{y \in F} (f^{\#}_2(y)) = \sup_{y \in F} (y - 4y^2) = 1/16 \tag{8. 4}$$

attained at the unique conservative strategy $y^{\#} = 1/8$.

Conservative bivalue. The *conservative bivalue* is

$v^{\#} = (v^{\#}_1, v^{\#}_2) = (1/8, 1/16)$.

The worst offensive multifunctions are determined by the study of the *worst* gain functions.

The Frances' worst offensive reaction multifunction O_2 is defined by $O_2(x) = 0$, for every Emil's strategy x; indeed, fixed an Emil's strategy x the Frances' strategy 0 minimizes the partial profit function $f_1(x, .)$. The Emil's worst offensive correspondence versus Frances is defined by $O_1(y) = 0$, for every Frances' strategy y.

The dominant offensive strategy is 0 for both players, indeed the offensive correspondences are constant.

The offensive equilibrium A = (0,0) bring to the payoff *A* ' = (0, 0), in which the profit is zero for both players.

The core of the payoff space (in the sense introduced by J.P. Aubin) is the part of the Pareto maximal boundary contained in the cone of upper bounds of the conservative bi-value $v^{\#}$; the conservative bi-value gives us a bound for the choice of cooperative bistrategies.

Conservative phase of the Extended Bertrand game

If the strategy space is the extended square $S = [-1, 1]^2$, ***Emil's worst gain function*** is defined by

$$f^{\#}_1(x) = \inf_{y \in F} x(1 - 2x + y) = -2x^2, \tag{8. 5}$$

its maximum will be

$$v^{\#}_1 = \sup_{x \in E} (f^{\#}_1(x)) = \sup_{x \in E} (-2x^2) = 0, \tag{8. 6}$$

attained at the ***conservative strategy*** $x^{\#} = 0$.

Frances' worst gain function is defined by

$$f^{\#}_2(y) = \inf_{x \in E}\, y(1 - 4y + x) = -4y^2, \tag{8.7}$$

its maximum will be

$$v^{\#}_2 = \sup_{y \in F} (f^{\#}_2(y)) = \sup_{y \in F} (-4y^2) = 0, \tag{8.8}$$

attained at the ***conservative point*** $y^{\#} = 0$.

The conservative bivalue in the improper case is

$v^{\#} = (v^{\#}_1, v^{\#}_2) = (0, 0)$.

The worst offensive multifunctions can be determined by the study of the *worst* gain functions.

For every strategy Emil could choose, he has the minimum gain when Frances sells his commodity at the price -1. This result is unusual from an economic point of view, but it can make sense in a *short period deep competition*.

Then ***Frances' worst offensive multifunction*** is defined by $O_2(x) = -1$, for every Emil's strategy x; we obtain, also, $O_1(y) = -1$, for every Frances' strategy y.

The ***dominant offensive strategy*** is -1 for both players, and the offensive (dominant) equilibrium $A = (-1, -1)$ brings to the point $A' = (-2, -4)$, in which a severe loss is registered for both players.

The ***conservative knot*** of the game is the point (0, 0), whose image is the point (0, 0).

The core of the payoff space is the part of Pareto maximal boundary contained into the cone of upper bounds of the conservative bi-value $v^{\#}$; this bounds the choice of cooperative bistrategies.

9. Cooperative phase

The best compromise solution (in the sense introduced by J.P. Aubin), obtained by the intersection of the straight line passing through the points $v^{\#}$ and β, and the green curve of critical zone (showed graphically in the Figures 9.1 and 9. 3), is not satisfactory; therefore we have chosen **the Core Best Compromise** (showed graphically in the Figures 9. 2 and 9. 4).

This point is obtained drawing the parallel and the perpendicular of the points C1 and C2, which are the two extremes of the core, and joining the $v^{\#}$ with sup_CORE(S).

We can note that Frances will have a greater bi-gain and it will be near the Nash equilibrium in the proper case (figure 9. 2). On the contrary, in the improper one (Figure 9. 4), the Frances bi-gain will be better than Nash equilibrium.

Besides, *the core best compromise* solution coincides with the *Kalai-Smorodinsky bargaining solution*.

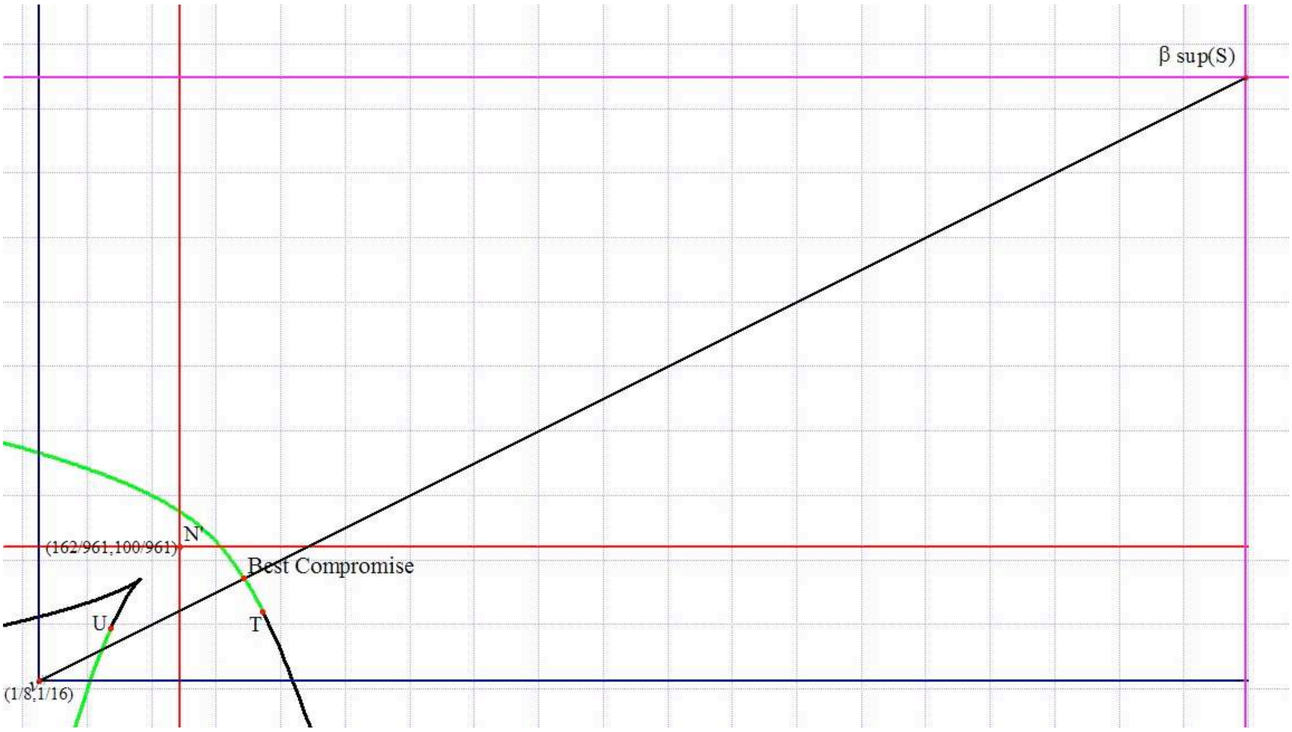

Figure 9. 1. Conservative Analysis

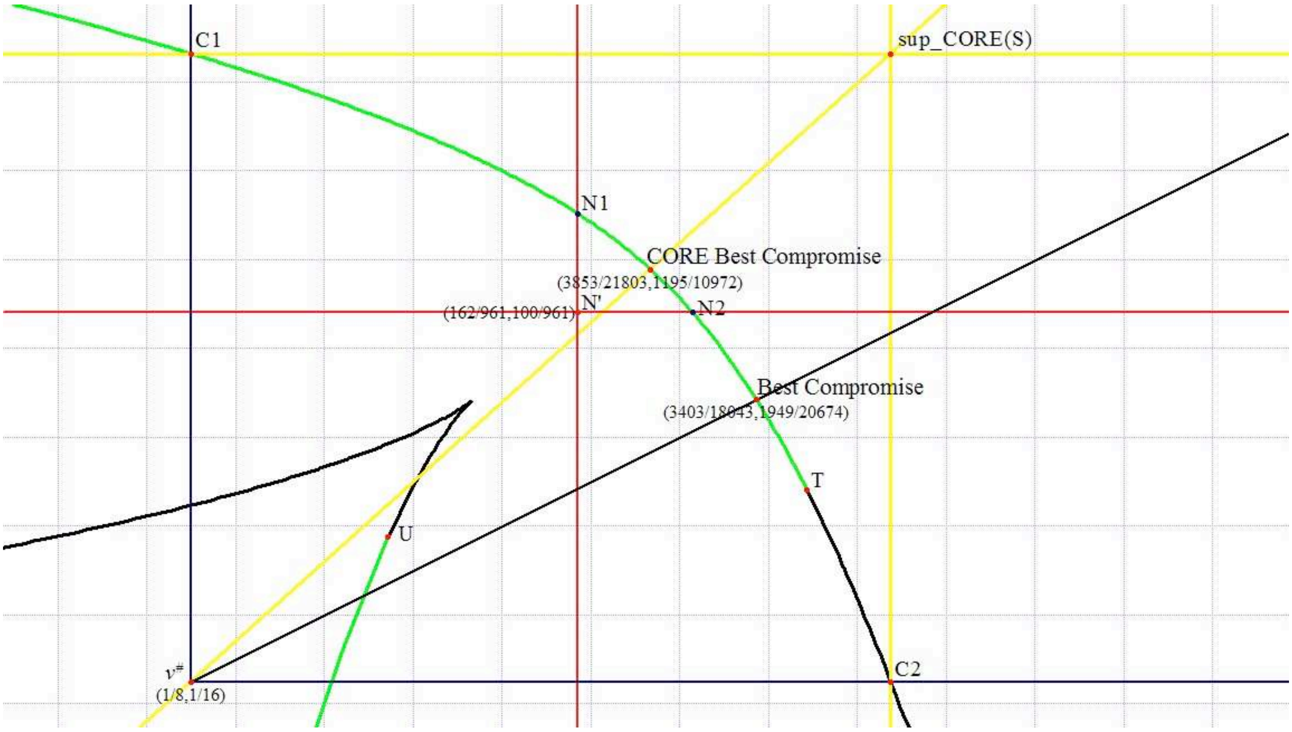

Figure 9. 2. The core and the Kalai-Smorodinsky payoff of the proper game

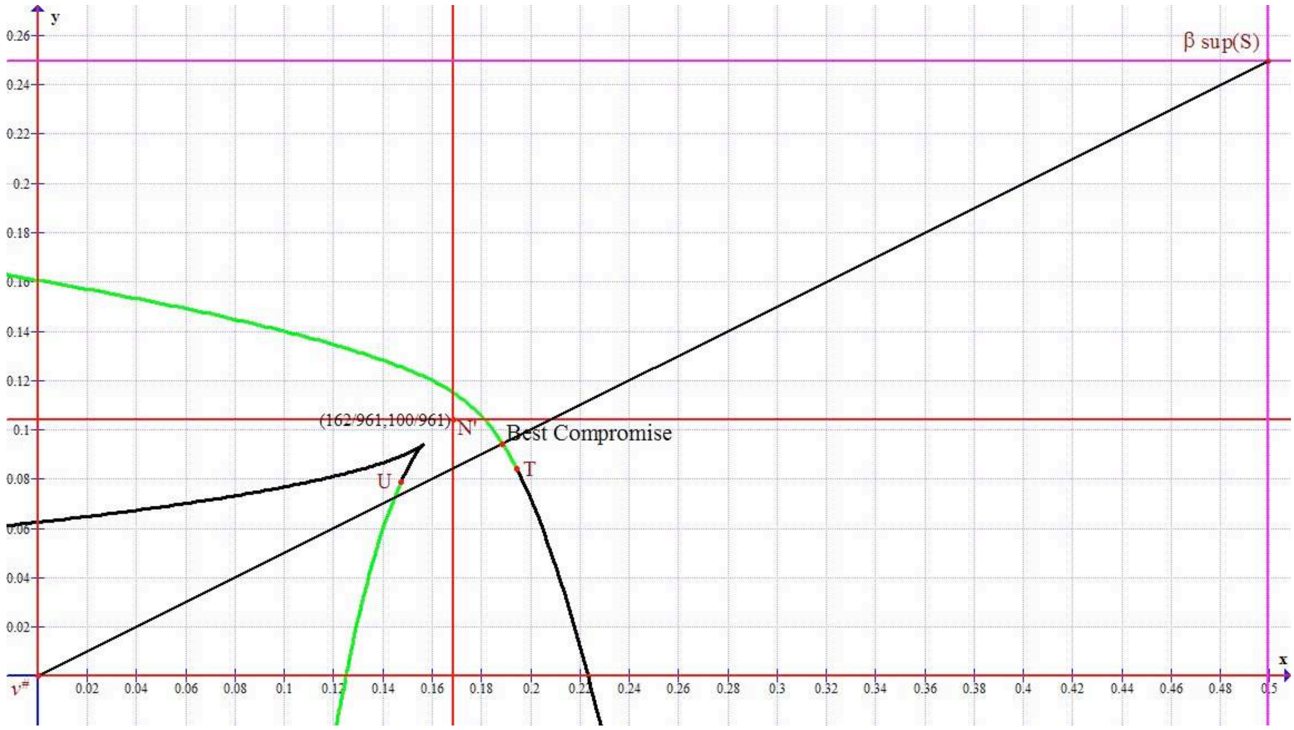

Figure 9. 3. Conservative exam of the improper Bertrand game

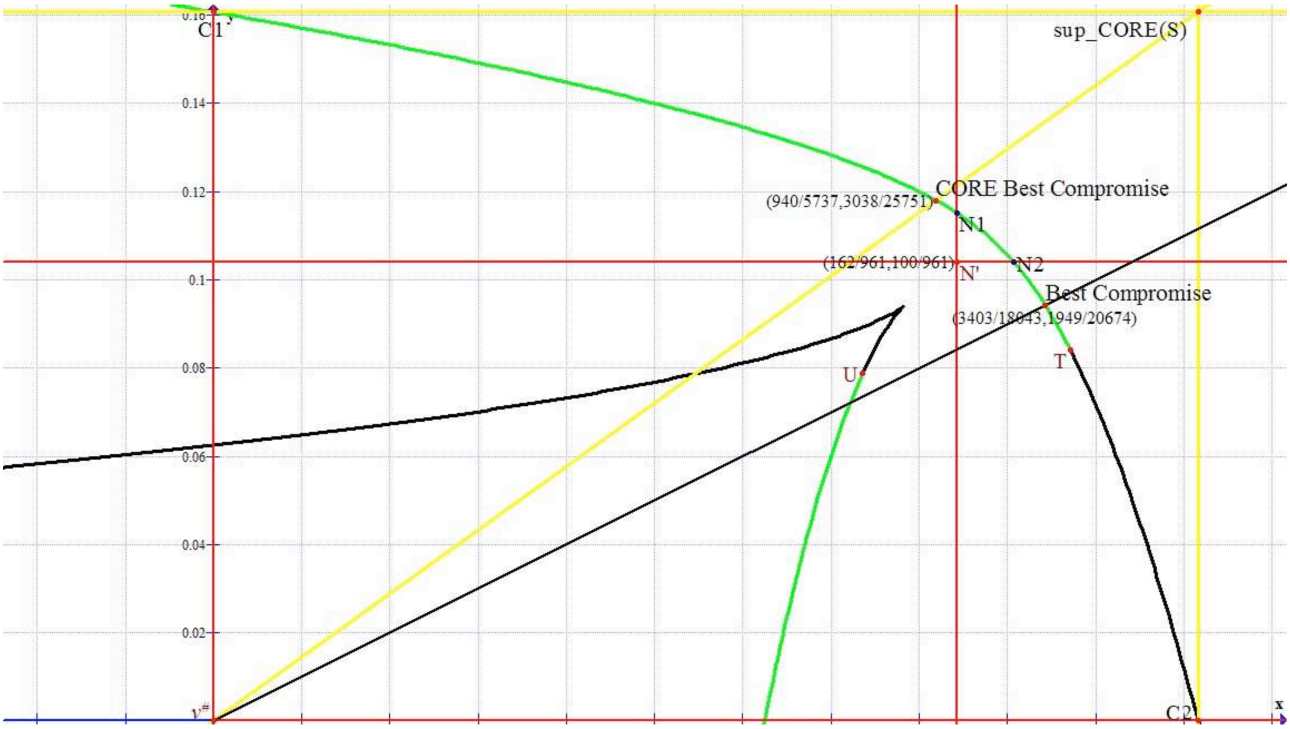

Figure 9. 4. The Core of the improper Bertrand game

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