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Abstract

This paper analyzes the connection between time preference heterogeneity and economic inequality. To achieve this, we extend the standard neoclassical growth model by introducing three additional features, namely (i) heterogeneity in consumers' discount rates, (ii) direct preferences for wealth, and (iii) human capital formation. The second feature prevents the wealth distribution from collapsing into a degenerate distribution. The third feature generates a strong positive correlation between earnings and capital income across consumers. A calibrated version of the model is able to generate patterns of wealth and income inequality that are very similar to those observed in the United States.

Keywords: Inequality, Heterogeneity, Time Preference, Human Capital

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1 Introduction

Empirical studies show that individuals do not discount future values at the same rate.[1] Since individuals' asset accumulation and schooling choices are strongly affected by the way they discount the future, this type of heterogeneity would naturally lead to cross-sectional differences in wealth and income. To examine the connection between time preference heterogeneity and economic inequality, this study develops a dynamic competitive equilibrium model in which consumers only differ in terms of their discount rates. It is shown that a calibrated version of the model can generate patterns of wealth and income inequality that are very similar to those observed in the United States.

The importance of time preference heterogeneity in explaining wealth inequality is well recognized in existing studies. There is now a vast literature in macroeconomics that uses the incomplete markets model of Huggett (1993, 1996) and Aiyagari (1994) to explain wealth and income inequality.[2] The standard incomplete markets model, however, has difficulty in explaining certain features of the wealth distribution in the United States. In particular, it fails to generate a high concentration of wealth at the top end of the wealth distribution.[3] Krusell and Smith (1998) show that introducing time preference heterogeneity can significantly improve the Aiyagari (1994) model in this regard. Similarly, Hendricks (2007) shows that introducing this type of heterogeneity into the life-cycle model of Huggett (1996) can improve the model's ability to account for wealth inequality.

In both Krusell and Smith (1998) and Hendricks (2007), cross-sectional variation in income is mainly driven by uninsurable idiosyncratic earnings risk, which is exogenous and independent of the heterogeneity in discount rates. These two sources of consumer heterogeneity are then used to account for the wide dispersion in wealth. This approach, however, ignores the effects of time preferences on lifetime earnings. Intuitively, more patient individuals are more willing to invest in financial assets as well as human capital than less patient ones. A higher level of human capital then leads to a higher level of lifetime earnings for those who are more patient. This intuition is consistent with empirical findings. Lawrance (1991) and Warner and Pleeter (2001) find that more-educated households and individuals tend to have lower discount rates than less-educated ones. This connection between patience and educational attainment suggests that human capital formation may provide an additional channel through which time preference heterogeneity can give rise to wealth and income inequality.

[1] A detailed review of these studies can be found in Frederick *et al.* (2002) Section 6.

[2] An excellent review of this literature can be found in Heathcote *et al.* (2009).

[3] See Castañeda *et al.* (2003) for a detailed discussion of this problem.

The main objective of this study is to explore the quantitative implications of this additional channel. To achieve this, we generalize the standard deterministic neoclassical growth model to allow for three important features, namely (i) heterogeneity in time preference, (ii) human capital formation, and (iii) consumers' direct preferences for wealth. The assumption of direct wealth preference has long been used in economic studies. In an early paper, Kurz (1968) introduces wealth preference into the optimal growth model and explores the long-run properties of the model. Zou (1994) interprets this type of preference as reflecting the "capitalist spirit," or the tendency to treat wealth acquisition as an end in itself rather than a means of satisfying material needs. Cole *et al.* (1992) suggest that the inclusion of financial wealth in consumers' preferences can be viewed as a reduced-form specification to capture people's concern for their wealth-induced status within society. Subsequent studies have followed these traditions and interpreted this type of preference as either capturing the spirit of capitalism or reflecting the demand for wealth-induced status. In this paper, we refer to this feature simply as wealth preference. There is now a rapidly growing literature that explores the implications of wealth preference on a wide range of issues, such as asset pricing, economic growth, expectations-driven business cycles, effects of fiscal policy and wealth inequality.[4]

The main purpose of introducing wealth preference in our model is as follows. It is now well known that the standard neoclassical growth model has difficulty in generating realistic wealth distribution based on differences in discount rates alone. Becker (1980) shows that when consumers have time-additive separable preferences and different constant discount rates, all the wealth in the neoclassical world will eventually be concentrated in the hands of the most patient consumers. In other words, the wealth distribution is degenerate and extremely unequal in the long run. Several existing studies have identified conditions under which the long-run wealth distribution is non-degenerate.[5] In this study, we show that a non-degenerate wealth distribution can be obtained by assuming that consumers

[4]Studies that explore the implications of wealth preference on asset pricing include Bakshi and Chen (1996), and Boileau and Braeu (2007) among others. Studies on economic growth include Zou (1994) and Smith (1999) among others. Karnizova (2010) introduces this type of preference into a neoclassical growth model with capital adjustment costs and shows that the model can generate expectations-driven business cycles. Gong and Zou (2002) and Nakamoto (2009) examine the welfare implications of fiscal policy when consumers value wealth directly. Finally, Luo and Young (2009) explore the implications of wealth preference on wealth inequality. This study will be discussed in greater detail later on.

[5]Boyd (1990) shows that Becker's result is no longer valid when consumers have recursive preferences. Sarte (1997) establishes the existence of a non-degenerate wealth distribution by introducing a progressive tax structure into Becker's model. Sorger (2002) shows that Becker's result cannot be extended to the case where consumers are strategic players, rather than price-takers, in the capital market. Espino (2005) establishes a non-degenerate wealth distribution by assuming that consumers have private information over an idiosyncratic preference shock. Except for Sarte (1997), none of these studies have explored the quantitative implications of their model. Sarte shows that a calibrated version of his model can replicate the income distribution in the United States. However, unlike the current study, he does not attempt to explain wealth and income inequality simultaneously.

have direct preferences for wealth. The intuition behind this result can be explained as follows. In the original Becker (1980) model where there is no direct wealth preference, a consumer will choose to hold a constant positive level of financial wealth only when the equilibrium interest rate is identical to his discount rate. Since there is only one interest rate in the neoclassical model, it is not possible for consumers with different discount rates to maintain constant positive level of wealth simultaneously. In the long-run equilibrium, interest rate is equated to the lowest discount rate in the population. Thus, only the most patient consumers would have positive asset holdings. All other consumers with discount rate greater than the equilibrium interest rate will deplete their wealth until it reaches zero. Thus, the long-run wealth distribution in the Becker (1980) model is extremely unequal. Introducing direct preferences for wealth changes this result by creating some additional benefits of holding financial assets. These additional benefits fundamentally change the consumers' saving behavior. In particular, consumers are now willing to maintain constant positive level of wealth even if the interest rate is lower than their discount rates. These additional benefits not only prevent the consumers from depleting their wealth to zero, they also induce different types of consumers to hold different levels of wealth. Thus, the equilibrium wealth distribution is non-degenerate.

To illustrate the theoretical and quantitative implications of wealth preference, we begin with a baseline model in which there is no human capital. In the baseline model, we adopt the same economic environment as in Becker (1980), which features a neoclassical production technology, a complete set of competitive markets, and consumers with different discount rates. The only modification we make to Becker's model is the inclusion of financial wealth in consumers' preferences. A calibrated version of the baseline model is able to replicate some key features of the wealth distribution in the United States. In particular, it is able to generate a large group of wealth-poor consumers and a very small group of extremely wealthy ones. The baseline model, however, cannot produce large variations in earnings across consumers. This type of variation is important in explaining income inequality because earnings is the most important source of income in the model economy. Consequently, a model with only time preference heterogeneity and wealth preference cannot explain the observed patterns of wealth and income inequality simultaneously. The same problem remains even if we allow for endogenous labor supply. Introducing human capital formation helps improve this result in two ways. First, consumers' earnings are now tied to their discount rates through the investment in human capital. This provides a channel via which time preference heterogeneity can lead to significant variations in earnings across consumers. Second, introducing human capital helps create a strong positive correlation between earnings and capital income. This happens because more patient consumers have higher earnings and more financial wealth than less patient ones. This in turn generates a substantial degree of income inequality in our model. A calibrated version of the model with all three features is able to replicate the observed patterns of wealth and income inequality in the United States.[6]

The current study differs from Krusell and Smith (1998) in three important ways: First, the current study aims to explain *both* wealth and income inequality using only one source of consumer heterogeneity, namely differences in discount rates. Second, the current study takes into account the endogenous components of labor income, namely endogenous labor hours and human capital formation. Third, instead of assuming that individuals' discount rates are stochastic and idiosyncratic in nature, the current study focuses on fixed, predetermined differences in discount rates across individuals.[7]

This study is also close in spirit to Luo and Young (2009) in the sense that both studies analyze wealth and income inequality in the presence of wealth preference. There are two major differences between the two studies. First, the source of consumer heterogeneity is different in the two models. In Luo and Young (2009), consumers share the same discount rate but face idiosyncratic uncertainty in labor productivity as in the Aiyagari (1994) model. Thus, this study does not consider the effects of time preference heterogeneity on wealth and income inequality. Second, the earnings distribution in the two models are determined by different factors. In Luo and Young (2009), earnings are jointly determined by labor productivity shock and consumers' labor-leisure choices. In particular, human capital formation is not considered in their model. Despite these differences in model specification, both studies find that wealth preference is a force that tends to *reduce* wealth inequality. In our model, this tendency is manifested in two ways. First, the equilibrium wealth distribution is no longer extremely unequal once we introduce wealth preference into Becker's model. Second, in the quantitative analysis, we find that the degree of wealth inequality decreases as we increase the coefficient that controls the strength of wealth preference. Similar results are also reported in Luo and Young (2009).

[6] We do not claim that other factors, such as life-cycle factors, income uncertainty, precautionary savings, redistributive taxation and transfer programs, are not important in understanding economic inequality. The main purpose of the calibration exercise is to illustrate the quantitative relevance of the mechanism captured by this model in explaining economic inequality.

[7] Existing studies show that predetermined factors (or ex ante heterogeneity) are at least as important as idiosyncratic shocks (or ex post heterogeneity) in explaining cross-sectional variation in lifetime utility. Keane and Wolpin (1997) argue that as much as 90 percent of the dispersion in lifetime utility can be attributed to predetermined, fixed factors. The remaining ten percent is attributed to exogenous idiosyncratic shocks. More recently, Huggett, Ventura and Yaron (2011) find that predetermined factors are more important in explaining the dispersion in lifetime earnings and lifetime wealth than idiosyncratic shocks.

The rest of this paper is organized as follows. Section 2 describes the baseline model environment, presents the main theoretical results, and evaluates the quantitative relevance of this model. Section 3 extends the baseline model by including endogenous labor supply. Section 4 extends the baseline model by introducing human capital formation. Section 5 discusses the main determinants of wealth and income inequality in the model with human capital. This is followed by some concluding remarks in Section 6.

2 The Baseline Model

2.1 Preferences

Consider an economy inhabited by $N > 1$ groups of infinitely-lived agents. Each group is indexed by a subjective discount factor β_i, for $i \in \{1, 2, ..., N\}$. The discount factors can be ranked according to $0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_N < 1$. Consumers within the same group are identical in all aspects. The share of type-i consumers in the population is given by $\lambda_i \in (0, 1)$. The size of total population is constant and is normalized to one, hence $\sum_{i=1}^{N} \lambda_i = 1$.

There is a single commodity in this economy which can be used for consumption and investment. The consumers' preferences are represented by

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{i,t}, k_{i,t}),$$

where $c_{i,t}$ is the consumption of a type-i consumer at time t and $k_{i,t}$ is the stock of physical capital owned by the consumer at the beginning of time t. The (period) utility function $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is identical for all consumers and have the following properties:

Assumption A1 The function $u(c, k)$ is twice continuously differentiable, strictly increasing and strictly concave in (c, k). It also satisfies the Inada condition for consumption, i.e., $\lim_{c \rightarrow 0} u_c(c, k) = \infty$, where $u_c(c, k)$ is the partial derivative with respect to c.

Assumption A2 The function $u(c, k)$ is homogeneous of degree $1 - \sigma$, with $\sigma > 0$.

Assumption A2 is imposed to ensure the existence of balanced growth equilibria. Under this assumption, the partial derivatives $u_c(c, k)$ and $u_k(c, k)$ are both homogeneous of degree $-\sigma$. We can then define a function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ according to

$$\Phi(z) \equiv \frac{u_k(z,1)}{u_c(z,1)}. \tag{1}$$

By Assumption A1, the function $\Phi(\cdot)$ is continuously differentiable and non-negative. We now impose some additional assumptions on this function.

Assumption A3 The function $\Phi(z)$ defined by (1) is strictly increasing, with $\Phi(0) = 0$, and satisfies $\lim_{z\rightarrow\infty} \Phi(z) = \infty$.

Assumption A3 serves two important roles in the theoretical analysis. First, it plays a role in ensuring the uniqueness of balanced-growth equilibrium. Second, it ensures that more patient consumers would have more asset holdings than less patient ones in this type of equilibrium. The details of these will become clear in Section 2.5. It is straightforward to check that $\Phi(\cdot)$ is strictly increasing if $u_{ck}(c,k) \geq 0$. The converse, however, is not necessarily true. In other words, Assumption A3 does not preclude the possibility of having a negative cross-derivative for some values of c and k.[8]

All three assumptions stated above are satisfied by the following functional forms which are commonly used in existing studies,

$$u(c,k) = \frac{1}{1-\sigma}\left(c^{1-\sigma} + \theta k^{1-\sigma}\right), \tag{2}$$

with $\sigma > 0$ and $\theta > 0$, and

$$u(c,k) = \frac{1}{1-\widetilde{\sigma}}\left[\zeta c^{\psi} + (1-\zeta) k^{\psi}\right]^{\frac{1-\widetilde{\sigma}}{\psi}}, \tag{3}$$

with $\widetilde{\sigma} > 0$, $\zeta \in (0,1)$ and $\psi < 1$.[9]

2.2 The Consumers' Problem

In each period, each consumer is endowed with one unit of time which is supplied inelastically to the market. The consumers receive labor income from work and capital income from their previous savings.

[8] Majumdar and Mitra (1994) show that, in a model with homogeneous consumers, the sign of the cross derivative $u_{ck}(c,k)$ plays an important role in determining the dynamic properties of the model. In the current study, we only focus on stationary equilibria.

[9] The additively separable specification is used in Zou (1994), Gong and Zou (2001), and Luo and Young (2009) among others. The non-separable specification is used in Boileau and Braeu (2007) and Karnizova (2010). The second study assumes that wealth effect is derived from the stock of physical capital owned by the consumer at the *end* of the current period, i.e., $k_{i,t+1}$.

All savings are held in the form of physical capital, which is the only asset in this economy. As in Becker (1980), the consumers are not allowed to borrow in every period.

Let w_t and r_t be the market wage rate and rental rate of physical capital at time t. Given a sequence of wage rates and rental rates, the consumers' problem is to choose a sequence of consumption and asset holdings so as to maximize their lifetime utility, subject to the sequential budget constraints and borrowing constraints. For each type-i consumer, this problem can be expressed as

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u\left(c_{i,t},k_{i,t}\right)$$

subject to

$$c_{i,t} + k_{i,t+1} - \left(1-\delta_k\right) k_{i,t} = w_t + r_t k_{i,t}, \tag{4}$$

$$k_{i,t+1} \geq 0,$$

and the initial condition $k_{i,0} > 0$. The parameter $\delta_k \in (0,1)$ is the depreciation rate of physical capital.

The consumer's optimal choices are completely characterized by the budget constraint in (4), and the Euler equation for consumption,

$$u_c\left(c_{i,t},k_{i,t}\right) \geq \beta_i \left[u_k\left(c_{i,t+1},k_{i,t+1}\right) + \left(1 + r_{t+1} - \delta_k\right) u_c\left(c_{i,t+1},k_{i,t+1}\right)\right], \tag{5}$$

which holds with equality if the borrowing constraint is not binding, i.e., $k_{i,t+1} > 0$. Introducing direct preferences for wealth essentially creates some additional benefits for holding wealth. These additional benefits are captured by the term $u_k\left(c_{i,t+1},k_{i,t+1}\right) > 0$ in the Euler equation. If consumers have no direct preference for wealth, i.e., $u_k(c,k) \equiv 0$, then the Euler equation in (5) is identical to the one in Becker (1980).

2.3 Production

Output is produced according to a standard neoclassical production function:

$$Y_t = F\left(K_t, X_t L_t\right),$$

where Y_t denote aggregate output at time t, K_t is aggregate capital, L_t is aggregate labor and X_t is the level of labor-augmenting technology. We will refer to $\widehat{L}_t \equiv X_t L_t$ as effective unit of aggregate labor. The technological factor is assumed to grow at a constant exogenous rate so that $X_t \equiv \gamma^t$ for all t, where $\gamma \geq 1$ is the exogenous growth factor and X_0 is normalized to one. The production function $F : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ is assumed to have all the usual properties which are summarized below.

Assumption A4 The production function $F\left(K, \widehat{L}\right)$ is twice continuously differentiable, strictly increasing and strictly concave in each argument. It exhibits constant returns to scale and satisfies the following conditions: $F\left(0, \widehat{L}\right) = 0$ for all $\widehat{L} \geq 0$, $F(K, 0) = 0$ for all $K \geq 0$, $\lim_{K \rightarrow 0} F_K\left(K, \widehat{L}\right) = \infty$ and $\lim_{K \rightarrow \infty} F_K\left(K, \widehat{L}\right) = 0$.

Since the production function exhibits constant returns to scale, we can focus on a representative firm whose problem is given by

$$\max_{K_t, L_t} \left\{F\left(K_t, X_t L_t\right) - w_t L_t - r_t K_t\right\},$$

for any $t \geq 0$. The solution of this problem is completely characterized by the first-order conditions:

$$w_t = X_t F_{\widehat{L}}\left(K_t, X_t L_t\right) = X_t F_{\widehat{L}}\left(\widehat{k}_t, 1\right), \tag{6}$$

$$r_t = F_K\left(K_t, X_t L_t\right) = F_K\left(\widehat{k}_t, 1\right), \tag{7}$$

where $\widehat{k}_t \equiv K_t / \left(X_t L_t\right)$ is the level of physical capital per effective unit of aggregate labor at time t.

2.4 Competitive Equilibrium

Let $\mathbf{c}_t = \left(c_{1,t}, c_{2,t}, ..., c_{N,t}\right)$ denote a distribution of consumption across groups at time t. Similarly, define $\mathbf{k}_t$ as the distribution of physical capital at time t. Given an initial distribution $\mathbf{k}_0$, a competitive equilibrium for this economy consists of a sequence of distributions, $\{\mathbf{c}_t, \mathbf{k}_t\}_{t=0}^{\infty}$, a sequence of aggregate inputs, $\{K_t, L_t\}_{t=0}^{\infty}$, and a sequence of prices, $\{w_t, r_t\}_{t=0}^{\infty}$, so that

(i) Given the prices $\{w_t, r_t\}_{t=0}^{\infty}$, the allocation $\{c_{i,t}, k_{i,t}\}_{t=0}^{\infty}$ solves a type-i consumer's problem.

(ii) Given the prices $\{w_t, r_t\}_{t=0}^{\infty}$, the aggregate inputs $\{K_t, L_t\}_{t=0}^{\infty}$ solve the representative firm's problem in every period, i.e., (6) and (7) are satisfied for all $t \geq 0$.

(iii) All markets clear in every period so that, for each $t \geq 0$,

$$K_t = \sum_{i=1}^{N} \lambda_i k_{i,t}, \qquad \text{and} \qquad \sum_{i=1}^{N} \lambda_i c_{i,t} + K_{t+1} - (1-\delta_k) K_t = F(K_t, X_t).$$

In both theoretical and quantitative analyses, we confine our attention to balanced-growth equilibria which are independent of the initial conditions. Thus, the initial distribution of physical capital is irrelevant to our analyses. A balanced-growth equilibrium is formally defined as a sequence $\mathcal{S} = \{\mathbf{c}_t, \mathbf{k}_t, K_t, L_t, w_t, r_t\}_{t=0}^{\infty}$ such that

(i) $\mathcal{S}$ is a competitive equilibrium as defined above.

(ii) The rental rate of physical capital is stationary over time, i.e., $r_t = r^*$ for all $t \geq 0$.

(iii) Individual consumption and asset holdings, aggregate capital and wage rate are all growing at the same constant rate. The common growth factor is given by $\gamma \geq 1$.

2.5 Theoretical Results

We now provide a set of conditions under which the baseline model possesses a unique balanced-growth equilibrium. We also show that the wealth distribution in the unique equilibrium is non-degenerate. These results are summarized in Theorem 1. The main ideas of the proof are as follows. A balanced-growth equilibrium is mainly characterized by a constant rental rate r^* which clears the market for physical capital. Once this variable is determined, all other variables in a balanced-growth equilibrium can be uniquely determined. Thus, it suffices to establish the existence and uniqueness of r^*. To achieve this, we first formulate the supply and demand for physical capital as a function of r.

Denote by $\widehat{k}^d(r)$ the amount of physical capital per effective unit of aggregate labor that the representative firm desires when the rental rate is r. The function $\widehat{k}^d(r)$ is implicitly defined by

$$r = F_K\left(\widehat{k}^d, 1\right). \tag{8}$$

Under Assumption A4, the function $\widehat{k}^d : \mathbb{R}_{++} \to \mathbb{R}_+$ is continuously differentiable and strictly decreasing. Moreover, $\widehat{k}^d(r)$ approaches infinity as r tends to zero from the right and approaches zero as r tends to infinity. If r is an equilibrium rental rate, then the equilibrium wage rate at time t is uniquely determined by $w_t = \gamma^t \widehat{w}(r)$, where

$$\widehat{w}(r) = F_{\widehat{L}}\left(\widehat{k}^d(r), 1\right). \tag{9}$$

Next, we consider the supply side of the physical capital market. Along any balanced-growth equilibrium path, individual consumption and asset can be expressed as $c_{i,t} = \gamma^t \widehat{c}_i$ and $k_{i,t} = \gamma^t \widehat{k}_i$, where $\widehat{c}_i$ and $\widehat{k}_i$ are stationary over time. The values of $\widehat{c}_i$ and $\widehat{k}_i$ are determined by the consumer's budget constraint and the Euler equation for consumption. Along a balanced growth path with rental rate r, the budget constraint becomes

$$\widehat{c}_i = \widehat{w}(r) + \left(r - \widehat{\delta}_k\right)\widehat{k}_i, \tag{10}$$

where $\widehat{\delta}_k \equiv \gamma - 1 + \delta_k \geq \delta_k$, and the Euler equation can be expressed as

$$\frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - r \geq \Phi\left(\frac{\widehat{c}_i}{\widehat{k}_i}\right), \tag{11}$$

which holds with equality if $\widehat{k}_i > 0$. By Assumption A3, we have $\Phi(z) \geq 0$ for all $z \geq 0$. In the above condition, z is the consumption-wealth ratio for a type-i consumer, which must be non-negative in equilibrium. Thus, the Euler equation is valid only for $r \leq \widehat{r}_i$, where $\widehat{r}_i \equiv \gamma^\sigma/\beta_i - (1 - \delta_k) > 0$. This essentially imposes an upper bound on the equilibrium rental rate, which is $\min_i \{\widehat{r}_i\} = \widehat{r}_N$.[10] For any $r \in (0, \widehat{r}_N)$, it is never optimal for any type of consumer to choose $\widehat{k}_i = 0$.[11] It follows that the Euler equation for consumption will always hold with equality in a balanced-growth equilibrium. Combining equations (10) and (11) gives

$$\frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - r = \Phi\left[\frac{\widehat{w}(r)}{\widehat{k}_i} + r - \widehat{\delta}_k\right]. \tag{12}$$

This implicitly defines a relationship between $\widehat{k}_i$ and r. Formally, this can be expressed as $\widehat{k}_i = g_i(r)$, where $g_i(\cdot)$ is a continuously differentiable function defined on $(0, \widehat{r}_i)$.

[10] If $r > \widehat{r}_N$, then the Euler equation will not be satisfied for some type of consumers and so r cannot be an equilibrium rental rate.

[11] To see this, suppose the contrary that a type-i consumer chooses to have $\widehat{k}_i = 0$ in a balanced-growth equilibrium with rental rate r. Then the right-hand side of (7) would become infinite as $\lim_{z \to \infty} \Phi(z) = \infty$ under Assumption A3. This clearly exceeds the left-hand side of the inequality for any $r \in (0, \widehat{r}_N)$ and hence gives rise to a contradiction. This also means that in order to have $\widehat{k}_i > 0$ in equilibrium, one can replace the assumption of $\lim_{z \to \infty} \Phi(z) = \infty$ by $\lim_{z \to \infty} \Phi(z) > \gamma^\sigma/\beta_1 - (1 - \delta_k)$ in Assumption A3.

Denote by $\widehat{k}^s(r)$ the aggregate supply of physical capital when the rental rate is $r \in (0, \widehat{r}_N)$. Formally, this is defined as

$$\widehat{k}^s(r) = \sum_{i=1}^{N} \lambda_i g_i(r). \tag{13}$$

Since each $g_i(r)$ is continuously differentiable on $(0, \widehat{r}_N)$, the function $\widehat{k}^s(r)$ is also continuously differentiable on this range. A balanced-growth equilibrium exists if there exists at least one value r^*, within the range $(0, \widehat{r}_N)$, that solves the physical capital market equilibrium condition:

$$\widehat{k}^d(r) = \widehat{k}^s(r).$$

Once r^* is determined, all other variables in the balanced-growth equilibrium can be uniquely determined. If there exists a unique value of r^*, then the balanced-growth equilibrium is also unique. Theorem 1 provides the conditions under which a unique value of r^* exists. The proof of this result can be found in Appendix A.

Theorem 1 *Suppose Assumptions A1-A4 are satisfied. Suppose* $\beta_i \gamma^{1-\sigma} < 1$ *for all* $i \in \{1, ..., N\}$, *and*

$$\widehat{k}^d\left(\widehat{\delta}_k\right) > \widehat{k}^s\left(\widehat{\delta}_k\right). \tag{14}$$

Then there exists a unique balanced-growth equilibrium. In the unique equilibrium, all types of consumers hold a strictly positive amount of capital. In addition, more patient consumers would have more consumption and hold more capital than less patient ones, i.e., $\beta_i > \beta_j$ *implies* $\widehat{c}_i > \widehat{c}_j$ *and* $\widehat{k}_i > \widehat{k}_j$.

We now explain the intuitions behind Theorem 1. Set $\gamma = 1$ for the moment. In the original Becker (1980) model, where consumers have no direct preference for wealth, the Euler equation is given by

$$\rho_i \equiv \frac{1}{\beta_i} - 1 \geq r^* - \delta_k, \tag{15}$$

with equality holds if $\widehat{k}_i > 0$. The parameter ρ_i is the discount rate or rate of time preference for a type-i consumer. This equation suggests that a consumer with no direct preference for wealth will invest according to the following rules: (i) accumulate assets indefinitely if the effective rate of return $(r^* - \delta_k)$ exceeds his rate of time preference, (ii) deplete the stock of assets until it reaches zero (the lower bound) if the effective rate of return is lower than his rate of time preference, and (iii) maintain a constant positive amount of assets if the two are equal. Since there is only one effective rate of return from savings, it is not possible for different types of consumers to maintain a constant amount of assets simultaneously. In addition, no one can accumulate assets indefinitely in a stationary equilibrium. Thus, the effective rate of return must be equated to the lowest rate of time preference in the population. In other words, only the most patient group of consumers will have positive asset holdings in any stationary equilibrium. All other groups of consumers will deplete their wealth until it reaches zero.

Introducing direct preferences for wealth breaks this spell by creating some additional benefits of holding wealth. These additional benefits fundamentally change the consumers' saving behavior. In particular, a consumer is now willing to maintain a constant positive level of assets even if the effective rate of return is lower than his rate of time preference. This is again evident from the Euler equation for consumption, which can be expressed as

$$\rho_i - (r^* - \delta_k) = \frac{u_k\left(\widehat{c}_i, \widehat{k}_i\right)}{u_c\left(\widehat{c}_i, \widehat{k}_i\right)}.$$

Since $u_k\left(\widehat{c}_i, \widehat{k}_i\right) > 0$, we have $\rho_i > (r^* - \delta_k)$ for all i. It is now possible to obtain a non-degenerate wealth distribution because consumers with different rates of time preference can choose a different value of $\widehat{k}_i$ based on the above equation. For impatient consumers, they are willing to hold a constant level of wealth only if they are compensated by large utility gains from wealth. Under the stated assumptions, these gains are diminishing in $\widehat{k}_i$. Thus, less patient consumers would choose a smaller value of $\widehat{k}_i$ than more patient ones.

To establish the results in Theorem 1, we have imposed two mild regularity conditions. The first condition requires $\beta_i \gamma^{1-\sigma} < 1$ for all $i \in \{1, ..., N\}$. This condition is both necessary and sufficient to ensure that the lifetime utility for all types of consumers is finite along the balanced growth path.[12] The second condition, stated in (14), ensures that the unique equilibrium rental rate r^* is greater than $\widehat{\delta}_k$. According to (10), $r^* > \widehat{\delta}_k$ is both necessary and sufficient to guarantee that individual consumption and asset holdings are positively correlated in the balanced-growth equilibrium. It is important to point out that condition (14) can be checked before solving for the equilibrium rental rate. More specifically,

[12] This condition is commonly used in models that allow for long-term growth in per-capita consumption. See, for instance, King, Plosser and Rebelo (1988) p.203.

$\widehat{k}^d\left(\widehat{\delta}_k\right)$ can be determined by substituting $r = \widehat{\delta}_k$ into (8). For each $i \in \{1, 2, ..., N\}$, define x_i by

$$\frac{1}{x_i} = \Phi^{-1}\left(\frac{\gamma^\sigma}{\beta_i} - \gamma\right).$$

Then, $g_i\left(\widehat{\delta}_k\right) = \widehat{w}\left(\widehat{\delta}_k\right) x_i$ and $\widehat{k}^s\left(\widehat{\delta}_k\right)$ is given by

$$\widehat{k}^s\left(\widehat{\delta}_k\right) = \widehat{w}\left(\widehat{\delta}_k\right) \sum_{i=1}^{N} \lambda_i x_i.$$

This shows that both $\widehat{k}^d\left(\widehat{\delta}_k\right)$ and $\widehat{k}^s\left(\widehat{\delta}_k\right)$ can be explicitly related to the fundamentals of the economy. To give a more concrete example, suppose the production function takes the Cobb-Douglas form,

$$F(K, XL) = K^\alpha (XL)^{1-\alpha}, \qquad \alpha \in (0,1), \tag{16}$$

and the utility function is given by (2). Then, condition (14) holds if and only if

$$\frac{\alpha}{\widehat{\delta}_k} > (1-\alpha)\,\theta^{\frac{1}{\sigma}} \left[\sum_{i=1}^{N} \lambda_i \left(\frac{\gamma^\sigma}{\beta_i} - \gamma\right)^{-\frac{1}{\sigma}}\right].$$

2.6 Numerical Results

We now examine the extent of economic inequality that can be generated by the baseline model. To achieve this, we have to specify the form of utility function and production function, and assign specific values to the model parameters. Some of these values are chosen based on empirical findings. Others are chosen to match some real-world targets. The details of this procedure are explained below.

2.6.1 Functional Forms and Parameters

In the numerical exercise, the production function is assumed to take the Cobb-Douglas form as in (16) and the utility function is additively separable as in (2). Under this specification, the parameter θ captures the importance of wealth preference in the utility function. In particular, a higher value of θ means that the same increase in wealth would generate a larger gain in utility. The original Becker model corresponds to the case in which $\theta = 0$.

Table 1 Benchmark Parameters in Baseline Model.

σ	Inverse of intertemporal elasticity of substitution	1
α	Share of capital income in total output	0.33
γ	Common growth factor	1.022
$\beta_{\min}$	Minimum value of subjective discount factor	0.966
$\beta_{\max}$	Maximum value of subjective discount factor	0.992
N	Number of distinct groups of agents	1,000

On period in the model is a year. The share of capital income in total output (α) is 0.33. The growth rate of per-capita variables ($\gamma - 1$) is 2.2 percent, which is the average annual growth rate of real per-capita GDP in the United States over the period 1950-2000. In the benchmark scenario, the parameter σ in the utility function is set to one. The range of subjective discount factors is chosen based on the estimates in Lawrance (1991). Using data from the Panel Study of Income Dynamics over the period 1974-1982, Lawrance (1991) estimates that the average rate of time preference for households in the bottom five percent of the income distribution is 3.5 percent, after controlling for differences in age, educational level and race. This implies an average discount factor of 1/(1+0.035)=0.966. The estimated rate of time preference for the richest five percent is 0.8 percent, which corresponds to a discount factor of 0.992.[13]

In the benchmark scenario, we consider a hypothetical population of one thousand groups of consumers and assume that the subjective discount factors are uniformly distributed between $\beta_{\min} = 0.966$ and $\beta_{\max} = 0.992$. In other words, we set $N = 1,000$ and $\lambda_i = 1/N$ for all i. The mean discount factor is 0.979. The choice of N is immaterial for our benchmark results. A uniform distribution is used for the following reason. Take wealth inequality as an example. In the stationary equilibrium, wealth inequality is driven by two types of variations: (i) variations in population shares across groups, captured by $\{\lambda_i\}_{i=1}^{N}$, and (ii) variations in the equilibrium level of asset holdings across groups, captured by $\left\{\widehat{k}_i\right\}_{i=1}^{N}$. By adopting a uniform distribution, we can rule out the first type of variation. Thus, wealth inequality in the benchmark results is entirely driven by the cross-sectional variations in asset holdings. The same argument applies to inequality in income. Our benchmark results then provide a

[13] To obtain these results, Lawrance (1991) estimate the Euler equation for a model without direct preferences for wealth. This range of values, however, encompasses the values of discount factors that are typically used in quantitative studies (with or without wealth preference). In Section 2.6.4, we will examine the effects of changing these endpoint values on the baseline results.

clear illustration of how much inequality can be generated by the key features of the model, namely wealth preference and heterogeneous discount factors. After presenting the benchmark results, we will examine the effects of relaxing the uniform distribution assumption and changing the values of β_{min} and β_{max}.

In the benchmark results, we focus on the relationship between θ and the degree of wealth and income inequality. To achieve this, we consider different values of θ ranging from 0.005 to 0.5. For each value of θ, the depreciation rate δ_k is recalibrated so that the capital-output ratio is maintained at 3.0. Table 1 summarizes the parameter values used in the benchmark economy.

2.6.2 Benchmark Results

Table 2 summarizes the main findings of this exercise. The reported results include the Gini coefficients for wealth and income, the coefficients of variation for wealth and income, and the shares of wealth held by the bottom and top percentiles of the wealth distribution. The data of these inequality measures are taken from Díaz-Giménez *et al.* (2011).

The results in Table 2 show a strong negative relationship between wealth inequality and the value of θ. This can also be seen from Figure 1, which shows the Lorenz curves for wealth under different values of θ. As θ approaches zero, both the Gini coefficient of wealth and the share of total wealth held by the wealthiest consumers increase towards unity. This means the wealth distribution becomes more and more concentrated when the importance of wealth preference diminishes. This result is consistent with theoretical predictions as $\theta = 0$ corresponds to the original Becker (1980) model. For small values of θ, the baseline model is able to generate a highly concentrated distribution of wealth with a large group of wealth-poor consumers and a small group of extremely wealthy ones. In particular, under certain value of θ, the model is able to replicate certain key measures of wealth inequality in the United States. For example, when $\theta = 0.01123$ the Gini coefficient of wealth generated by the model is 0.816, when $\theta = 0.01796$ the wealthiest one percent own 33.6 percent of total wealth in the model economy. These figures coincide with the actual data reported in Díaz-Giménez *et al.* (2011).

As the value of θ increases, the wealth distribution becomes more and more equal. The intuition behind this result is as follows. An increase in θ means that the same increase in asset holdings would now generate a larger gain in utility. This has two opposing effects on wealth inequality. First, a stronger preference for wealth encourages all types of consumers to accumulate more assets. This effect tends to be larger for the wealth-rich than for the wealth-poor. Thus, holding other things constant, an increase in θ would make the wealth distribution more unequal. Second, since aggregate savings increase as θ increases, the effective rate of return from savings $(r^* - \delta_k)$ needs to be adjusted downward in order to maintain the same capital-output ratio. Since more patient consumers are more responsive to interest rate changes than less patient ones, this would lower the share of total wealth owned by the wealthiest consumers and make the wealth distribution more equal. The overall effect of θ on wealth inequality then depends on the relative magnitude between these two forces. Our results show that the second effect dominates under the benchmark parameter values.

Table 2 also shows that the baseline model tends to generate a relatively low degree of income inequality. This happens because (i) earnings are identical for all consumers in this economy, and (ii) earnings represent a sizable portion of income for most of the consumers. Table 3 reports the share of total income from earnings for different wealth groups. When θ is less than 0.025, earnings accounts for more than 80 percent of total income for the majority of the consumers.

In sum, our quantitative results show that the baseline model is able to replicate some key features of the wealth distribution in the United States. However, it falls short of explaining income inequality. This is partly because earnings are identical for all consumers. The two extensions considered in Sections 3 and 4 are intended to change this feature of the baseline model.

2.6.3 Relaxing the Uniform Distribution Assumption

We now examine the effects of changing the shape of the distribution of discount factors. To achieve this, we assume that the size of each type is determined by

$$\lambda_i = \left(\frac{i}{N}\right)^{\frac{1}{\upsilon}} - \left(\frac{i-1}{N}\right)^{\frac{1}{\upsilon}}, \qquad \text{with } \upsilon > 0,$$

for $i \in \{1, 2, ..., N\}$. The endpoints of the distribution are fixed at their benchmark values, i.e., $\beta_{\min} = 0.966$ and $\beta_{\max} = 0.992$. This specification of λ_i is desirable for two reasons: (i) the skewness of the distribution is conveniently controlled by a single parameter υ, and (ii) it includes the benchmark uniform distribution as special case (i.e., $\upsilon = 1$). When $\upsilon > 1$, the size of the most patient group is less than $1/N$ and the distribution is more concentrated on low values of β. The opposite is true when $\upsilon \in (0, 1)$. Intuitively, a high value of υ represents an economy in which most of the consumers have similar values of discount factor clustered around $\beta_{\min}$, while a small groups of consumers are relatively more patient.

To better understand the effects of υ on wealth inequality, we consider two experiments. In the first experiment, we focus on the extent of wealth inequality under different values of υ. In each case, the depreciation rate δ_k is adjusted to maintain the capital-output ratio at 3.0. All other parameters (including θ) are fixed at their benchmark values. These results are shown in Panel (A) of Table 4. In the second experiment, both the Gini coefficient for wealth and the capital-output ratio are kept constant. This is achieved by adjusting both θ and δ_k for each value of υ. The results of the second experiment are summarized in Panel (B) of Table 4.

We begin by summarizing the effects of changing υ on the distribution of discount factors. These results are the same for both panels. Increasing υ from 1.0 to 2.0 raises the size of the least patient group (λ_1) from 0.0010 to 0.0316, and reduces the size of the most patient group (λ_N) by half. Because of the skewness of the distribution, the mean value of β is greater than the median value when $\upsilon > 1$.

Panel (A) of Table 4 shows that the Gini coefficients produced by the baseline model are rather robust to changes in the size of the most patient group. For instance, reducing λ_N by half only raises the Gini coefficients of wealth and income by 7.0 percent and 6.7 percent, respectively. The share of total wealth owned by the wealthiest consumers are more sensitive to this change. Panel (B) shows that once we maintain the Gini coefficient of wealth at the same level as in the benchmark scenario, changing υ would have only a mild impact on the wealth distribution. These results show that the main mechanism of the model is robust to changes in the shape of the distribution of discount factors.

2.6.4 Changing the Range of Discount Factors

We now examine the effects of changing the range of discount factors. We maintain the uniform distribution assumption as in the benchmark scenario, but consider five different combinations of endpoint values. In the first variation, the benchmark values are reduced by 0.01 so that $\beta_{\text{min}} = 0.956$ and $\beta_{\text{max}} = 0.982$. In the second variation, the benchmark values are reduced by 0.02. In these two experiments, the range $\triangle\beta \equiv |\beta_{\text{max}} - \beta_{\text{min}}|$ is the same as in the benchmark scenario. In the third and fourth experiments, this range is reduced by half. We consider the upper half in the third experiment, i.e., $\beta_{\text{min}} = 0.979$ and $\beta_{\text{max}} = 0.992$, and the lower half in the fourth one. In the final experiment, we extend the benchmark interval to the left by 50 percent, so that $\beta_{\text{min}} = 0.953$ and $\beta_{\text{max}} = 0.992$. Similar to the previous subsection, we report two sets of results for each experiment. Panel (A) of Table 5 reports the results obtained when the capital-output ratio is kept at 3.0 and θ is fixed at 0.01123. Panel (B) reports the results obtained when both the Gini coefficient of wealth and the capital-output ratio are kept constant.

Two observations can be made from Panel (A). First, shifting the distribution of discount factors while leaving the range $\triangle\beta$ unchanged has only a small impact on the Gini coefficients. The share of total wealth owned by the wealthiest consumers is also quite robust to this change. Second, wealth inequality is positively related to the size of $\triangle\beta$. This is evident from the results of the last three experiments.[14] However, Panel (B) shows that once we maintain the Gini coefficient of wealth at the same level, changing the range of discount factors has only a negligible impact on the wealth distribution. These results show that the main mechanism of the baseline model is robust to different values of $\beta_{\min}$ and $\beta_{\max}$. They also show that the model does not rely on large values of discount factors (i.e., very patient consumers) to generate a high concentration of wealth.

3 Endogenous Labor Supply

In this section, we extend the baseline model to include endogenous labor supply decisions. The consumers' period utility function is now given by

$$u(c,k,l) = \frac{c^{1-\sigma}}{1-\sigma} + \theta\frac{k^{1-\sigma}}{1-\sigma} - \mu\frac{l^{1+1/\eta}}{1+1/\eta}, \tag{17}$$

where l denote the amount of time spent on working, $\eta > 0$ is the intertemporal elasticity of substitution of labor, and μ is a positive-valued parameter. Consumers' earnings are now endogenously determined by their choice of working hours. The rest of the model is the same as in Section 2.

A balanced-growth equilibrium for this economy can be defined similarly as in Section 2.4. This type of equilibrium now includes, among other things, a stationary distribution of labor hours which is represented by $\mathbf{l} = (l_1, l_2, ..., l_N)$. Let $\widehat{k}^d(r)$ and $\widehat{w}(r)$ be the functions defined in (8) and (9). The equilibrium values of $\left\{\widehat{c}_i, \widehat{k}_i, l_i\right\}_{i=1}^N$ and the equilibrium rental rate r^* are determined by

$$\widehat{c}_i = \widehat{w}(r)\, l_i + \left(r - \widehat{\delta}_k\right)\widehat{k}_i, \tag{18}$$

[14] Similar results can be obtained under different values of θ. These results are not shown in the paper but are available upon request.

$$\frac{\gamma^{\sigma}}{\beta_i} - (1-\delta_k) - r = \theta \left(\frac{\widehat{c}_i}{\widehat{k}_i} \right)^{\sigma}, \tag{19}$$

$$\frac{\widehat{w}(r)}{\widehat{c}_i} = \mu (l_i)^{\frac{1}{\eta}}, \tag{20}$$

$$\sum_{i=1}^{N} \lambda_i \widehat{k}_i = \left(\sum_{i=1}^{N} \lambda_i l_i \right) \widehat{k}^d(r), \tag{21}$$

where $\widehat{\delta}_k \equiv \gamma - 1 + \delta_k$. Equations (18) and (19) can be obtained from the consumers' budget constraint and their Euler equation, respectively, after imposing the balanced-growth conditions. Equation (20) is the first-order condition with respect to labor. Equation (21) is the physical capital market equilibrium condition.

We now consider the same numerical exercise as in Section 2.6. The production function again takes the Cobb-Douglas form and the parameter values in Table 1 are used. In particular, the distribution of discount factors is assumed to be uniform, with $\beta_{\min} = 0.966$ and $\beta_{\max} = 0.992$. The intertemporal elasticity of substitution of labor is set to 0.4.[15] As in Section 2.6, we focus on the relationship between θ and the degree of economic inequality. We consider the same set of values for θ as in Table 2. In each case, the preference parameter μ is chosen so that the average amount of time spent on working is one-third and the depreciation rate δ_k is chosen so that the capital-output ratio is 3.0.

Table 6 shows the inequality measures obtained under $\eta = 0.4$. When comparing these to the baseline results in Table 2, it is immediate to see that they are very similar. Introducing endogenous labor supply decisions does not change the fundamental mechanism in the baseline model. In particular, the model continues to generate a high degree of wealth inequality when θ is small and a relatively low degree of income inequality in general. A comparison to the results in Table 2 also shows that allowing for endogenous labor supply actually *lowers* the Gini coefficient of income. This can be explained by Figure 2, which shows the relationship between discount factor and labor supply. Most of the consumers in this economy, except those who are very patient, choose to have the same amount of labor. Consequently, the distribution of labor hours is close to uniform. This explains why the extended model generates a similar degree of income inequality as the baseline model. Due to the wealth effect, wealth-rich consumers tend to work less than wealth-poor ones. This creates a negative correlation

[15] As a robustness check, we also consider two other values of this elasticity, namely 0.2 and 1.0. The results are almost identical to those obtained under $\eta = 0.4$. In particular, increasing this elasticity from 0.2 to 1.0 only marginally affects the Gini coefficients of wealth and income. These results are not shown in the paper but are available from the author upon request.

between earnings and capital income. This negative correlation in effect reduces income inequality in the model with endogenous labor supply.

4 Human Capital Formation

4.1 The Model

We now extend the baseline model to include human capital formation. Suppose in each period, each consumer is endowed with one unit of time which can be divided between market work and on-the-job training. Consider a type-i consumer with human capital $h_{i,t}$ at the beginning of time t. If he spends a fraction $l_{i,t} \in [0,1]$ of time on market work during the period, then his earnings are given by $w_t l_{i,t} h_{i,t}$. We refer to $l_{i,t} h_{i,t}$ as effective unit of labor hours. The variable w_t is now the market wage rate for an effective unit of labor hours. The consumer also receives $\varphi \left(1 - l_{i,t}\right)^{\epsilon} h_{i,t}^{\varsigma}$ units of newly produced human capital, where $\varphi > 0$, $\epsilon \in (0,1)$ and $\varsigma \in (0,1)$. His human capital at time $t+1$ is then given by

$$h_{i,t+1} = \varphi \left(1 - l_{i,t}\right)^{\epsilon} h_{i,t}^{\varsigma} + \left(1 - \delta_h\right) h_{i,t}, \tag{22}$$

where $\delta_h \in (0,1)$ is the depreciation rate of human capital.

The consumer's is now given by

$$\max_{\{c_{i,t}, l_{i,t}, k_{i,t+1}, h_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u\left(c_{i,t}, k_{i,t}\right)$$

subject to

$$c_{i,t} + k_{i,t+1} - \left(1 - \delta_k\right) k_{i,t} = w_t l_{i,t} h_{i,t} + r_t k_{i,t},$$

$$k_{i,t+1} \geq 0, \qquad l_{i,t} \in [0,1],$$

the human capital accumulation equation in (22), and the initial conditions: $k_{i,0} > 0$ and $h_{i,0} > 0$. The utility function is assumed to satisfy Assumptions A1-A3. The rest of the model economy remains the same as in Section 2. In particular, long-term growth in per-capita variables is again fueled by an exogenous improvement in labor-augmenting technology and the exogenous growth factor is $\gamma \geq 1$.[16]

[16] Unlike the endogenous growth model considered in Lucas (1988), human capital accumulation does not serve as the engine of growth in here. This is implied by the condition $\varsigma \in (0,1)$, which means there are diminishing returns of $h_{i,t}$ in the production function of human capital. The main idea of introducing human capital in this model is to increase the variation in earnings across consumers.

Let $\mathbf{h}_t = (h_{1,t}, ..., h_{N,t})$ denote a distribution of human capital at time t. Similarly, define $\mathbf{l}_t$ as a distribution of labor hours at time t. Given the initial distributions $\mathbf{k}_0$ and $\mathbf{h}_0$, a competitive equilibrium consists of a sequence of distributions, $\{\mathbf{c}_t, \mathbf{k}_t, \mathbf{l}_t, \mathbf{h}_t\}_{t=0}^{\infty}$, a sequence of aggregate inputs, $\{K_t, L_t\}_{t=0}^{\infty}$, and a sequence of prices, $\{w_t, r_t\}_{t=0}^{\infty}$, so that

(i) Given the prices, the allocation $\{c_{i,t}, k_{i,t}, l_{i,t}, h_{i,t}\}_{t=0}^{\infty}$ solves each type-i consumer's problem.

(ii) Given the prices, the aggregate inputs $\{K_t, L_t\}_{t=0}^{\infty}$ solve the representative firm's problem in each period.

(iii) All markets clear in every period, i.e.,

$$K_t = \sum_{i=1}^{N} \lambda_i k_{i,t} \quad \text{and} \quad L_t = \sum_{i=1}^{N} \lambda_i l_{i,t} h_{i,t}, \qquad \text{for each } t \geq 0.$$

A balanced-growth equilibrium can be defined similarly as in Section 2.4. Specifically, a balanced-growth equilibrium is a sequence $\mathcal{S} = \{\mathbf{c}_t, \mathbf{k}_t, \mathbf{l}_t, \mathbf{h}_t, K_t, L_t, w_t, r_t\}_{t=0}^{\infty}$ such that

(i) $\mathcal{S}$ is a competitive equilibrium as defined above.

(ii) The rental rate of physical capital is stationary over time, i.e., $r_t = r^*$ for all $t \geq 0$.

(iii) The distributions of labor hours and human capital are stationary over time.

(iv) Individual consumption and asset holdings, aggregate capital and wage rate are all growing at the same constant rate. In particular, the common growth factor is $\gamma \geq 1$.

Define the transformed variables $\widehat{c}_i \equiv c_{i,t}/\gamma^t$ and $\widehat{k}_i \equiv k_{i,t}/\gamma^t$. Along any balanced growth path, the equilibrium values of $\left\{\widehat{c}_i, \widehat{k}_i, l_i, h_i\right\}_{i=1}^{N}$ and the equilibrium rental rate r^* are determined by

$$\frac{\gamma^{\sigma}}{\beta_i} - (1 - \delta_k) - r = \Phi\left(\frac{\widehat{c}_i}{\widehat{k}_i}\right), \tag{23}$$

$$\widehat{c}_i = \widehat{w}(r)\, l_i h_i + \left(r - \widehat{\delta}_k\right) \widehat{k}_i, \tag{24}$$

$$\frac{l_i}{1 - l_i} = \frac{1}{\epsilon}\left\{\frac{1}{\delta_h}\left[\frac{1}{\beta_i} - (1 - \delta_h)\right] - \varsigma\right\}, \tag{25}$$

$$h_i = \left[\frac{\varphi}{\delta_h}(1 - l_i)^{\epsilon}\right]^{\frac{1}{1-\varsigma}}, \tag{26}$$

$$\sum_{i=1}^{N}\lambda_i\widehat{k}_i=\left(\sum_{i=1}^{N}\lambda_i l_i h_i\right)\widehat{k}^d\left(r\right), \tag{27}$$

where $\widehat{\delta}_k\equiv\gamma-1+\delta_k$. Similar to the baseline model, the functions $\widehat{k}^d(r)$ and $\widehat{w}(r)$ are defined by (8) and (9), respectively. Equations (23) and (24) can be obtained from the Euler equation for consumption and the consumers' budget constraint, after imposing the balanced-growth conditions. Equations (25) and (26) can be obtained from the first-order conditions with respect to $l_{i,t}$ and $h_{i,t+1}$, and the human capital accumulation equation. Equation (27) is the physical capital market equilibrium condition. The mathematical derivations of (23)-(27) are shown in Appendix B.

The main theoretical results in Section 2.5 can be extended to the current model. Specifically, under some mild regularity conditions, there exists a unique balanced-growth equilibrium for this economy. This unique equilibrium has two important properties. First, the borrowing constraint is not binding for all types of consumers. Thus, the Euler equation in (23) holds with equality for all i. Second, the wealth distribution in the unique equilibrium is non-degenerate. The formal proof of these results are shown in Appendix B.

Before concluding this section, we want to highlight several important features of the distributions of labor hours and human capital. In the unique balanced-growth equilibrium, the values of $\{l_i,h_i\}_{i=1}^N$ can be obtained by solving (25) and (26). These equations show that the distributions of labor hours and human capital are non-degenerate, and are completely determined by two factors: (i) the distribution of subjective discount factors and (ii) the parameters in the human capital accumulation process. This has two important implications. First, the values of $\{l_i,h_i\}_{i=1}^N$ are independent of the utility function $u(c,k)$. Thus, changing the parameters in the utility function would have no impact on the distributions of labor hours, human capital and earnings. Second, the values of $\{l_i,h_i\}_{i=1}^N$ are independent of the equilibrium rental rate r^* and the consumers' asset holdings $\left\{\widehat{k}_i\right\}_{i=1}^N$. Thus, in the stationary equilibrium, the distribution of earnings is not affected by the consumers' savings decisions.

4.2 Parameter Values

In the quantitative exercise, we use the same specification for production technology and utility function, and the same distribution of discount factors as in the benchmark scenario in Section 2.6. Specifically, the production function for output takes the Cobb-Douglas form with $\alpha=0.33$. The utility function is additively separable as in (2), with benchmark parameter value $\sigma=1$. In Section 5.5, we report

Table 7 Benchmark Parameters in Model with Human Capital.

α	Share of capital income in total output	0.33
γ	Common growth factor	1.022
δ_k	Depreciation rate of physical capital	0.08004*
σ	Inverse of intertemporal elasticity of substitution	1.0
θ	Strength of wealth preference	0.01202
β_{min}	Minimum value of subjective discount factor	0.966
β_{max}	Maximum value of subjective discount factor	0.992
N	Number of groups of consumers	1,000
φ	Parameter in human capital production	1.0
ϵ	Parameter in human capital production	0.939
ς	Parameter in human capital production	0.871
δ_h	Depreciation rate of human capital	0.037

* This figure has been rounded off to the fourth significant figure.

the results obtained under different values of σ. The population is divided into 1,000 groups and the discount factors are uniformly distributed between 0.966 and 0.992.[17]

As for the parameter values in the human capital production function, we normalize φ to unity and set the values of ϵ and ς according to the estimates reported in Heckman *et al.* (1998). Using data from the National Longitudinal Survey of Youth for the period 1979-1993, these authors find that the values of ϵ and ς for people who have completed at least one year of college education are 0.939 and 0.871, respectively. For those who do not have any college education, the corresponding values are 0.945 and 0.832. We use the first set of parameter values in the numerical analysis because workers with college education account for a larger share of U.S. labor force than those without college education.[18] As for the depreciation rate of human capital, Heckman *et al.* (1998) assume that it is zero. Other studies in the existing literature typically find that this rate is greater than zero.[19] In the benchmark scenario, we set $\delta_h = 0.037$ which is consistent with the estimate reported in Heckman (1976).

[17] The choice of $N = 1,000$ is again immaterial for our benchmark results. In particular, changing the number of groups to either 500 or 5,000 has virtually no impact on our benchmark results.

[18] Over the past twenty years, workers with at least some college education have accounted for an increasingly larger share of U.S. labor force. In 1992, this type of worker represented 51.8 percent of civilian labor force (over 25 years old). This increased to 62.1 percent by the year 2010. These figures are based on the data reported in the U.S. Statistical Abstract.

[19] See Browning *et al.* (1999) Table 2.3 for a summary of these studies.

The two remaining parameters, θ and δ_k, are calibrated so that the model can match two real-world statistics. In the benchmark scenario, we choose the value of θ so that the Gini coefficient of wealth predicted by the model is 0.816, which coincides with the value reported in Díaz-Giménez *et al.* (2011). The required value of θ is 0.01202. Similar calibration strategy is also used in Krusell and Smith (1998), Erosa and Koreshkova (2007), and Hendricks (2007) to determine the parameter values in the Markov process of the random discount factor.[20] The choice of θ, however, has no impact on the distribution of earnings. As explained earlier, the distributions of labor hours and human capital are independent of the utility function. Thus, the distribution of earnings in the model is not affected by the preference parameters σ and θ. The second parameter δ_k is calibrated so that the capital-output ratio generated by the model is 3.0. The parameter values used in the quantitative exercise are summarized in Table 7.

4.3 Benchmark Results

Table 8 summarizes the characteristics of the earnings, income and wealth distributions obtained under the benchmark parameter values. The first three columns show the Gini coefficients, the coefficients of variation and the mean-to-median ratios for the three variables. The mean-to-median ratio is intended to measure the degree of skewness in these distributions. The rest of Table 8 shows the share of earnings, income and wealth owned by consumers in different percentiles of the corresponding distribution.

Under the benchmark parameter values, the wealth distribution in the model economy is highly concentrated with a large group of wealth-poor consumers and a small group of extremely wealthy ones. For instance, the share of total wealth owned by consumers in the second quintile of the wealth distribution is merely 1.3 percent, whereas the share owned by the wealthiest five percent is 58.5 percent. These figures are very close to the actual values observed in the United States. As for the income distribution, the model is able to generate a Gini coefficient and a mean-to-median ratio that are similar to the observed values. It is also able to replicate reasonably well the share of aggregate income owned by different quintiles of the income distribution.

As for earnings, the model predicts a more equal distribution than that observed in the data. In the model economy, earnings-poor consumers own a larger share of total earnings than their real-world counterparts. Consequently, the Gini coefficient predicted by the model is much lower than the actual

[20] Conceptually, this strategy of choosing θ is also no different from choosing the preference parameter μ in (17) to match the average amount of time spent on work, a common practice in the real business cycle literature. In both cases, the unobserved, undetermined parameter is chosen so that certain prediction of the model can match its empirical counterpart.

value.[21] The big difference between the model's prediction and the actual value can be explained by two factors. First, in the actual data, a large number of households have reported negative earnings. According to Díaz-Giménez *et al.* (2011), the average earnings of households in the bottom quintile of the U.S. earnings distribution are negative due to sizable business losses. In the model economy, earnings must be strictly positive. This restriction reduces the range and dispersion of the earnings distribution, which in turn lowers earnings inequality in the model. Second, and more importantly, almost all the households in the bottom quintile of the U.S. earnings distribution are not workers. As shown in Díaz-Giménez *et al.* (2011) Table 4, retirees and nonworkers represent 96.9 percent of these households, and labor income only account for 0.2 percent of their total income. If we consider only households headed by employed worker, then the Gini coefficient for earnings in the United States is 0.47. This value is much closer to the one predicted by the model which assumes that all consumers are employed.[22]

5 Discussion

The benchmark results in Table 8 show that our model is able to generate realistic patterns of wealth and income inequality. To achieve this, we have extended the standard neoclassical growth model to allow for (i) direct preferences for wealth, (ii) human capital formation, and (iii) heterogeneity in subjective discount factors. In the above analysis, we assume that the utility function is logarithmic (i.e., $\sigma = 1$) and additively separable, and the distribution of discount factors is uniform. In this section, we examine the significance of each of these features in explaining wealth and income inequality. The main objective of this exercise is to better understand the determinants of wealth and income inequality in our model.

5.1 Strength of Wealth Preference

The purpose of this subsection is to illustrate the effects of wealth preference on wealth and income inequality in the extended model. To achieve this, we compute a series of balanced-growth equilibria

[21] Our results on earnings inequality, however, are comparable to those obtained by Pijoan-Mas (2006) and Erosa and Koreshkova (2007). In the benchmark model of Pijoan-Mas (2006), the Gini coefficient and the coefficient of variation for the earnings distribution are 0.33 and 0.65, respectively. In the benchmark model of Erosa and Koreshkova (2007), the Gini coefficient of earnings is 0.289.

[22] As shown in Díaz-Giménez *et al.* (2011), the Gini coefficients of income and wealth for employed workers in the United States are 0.48 and 0.78, respectively. These values are still quite close to the ones generated by the model.

using different values of θ ranging from 0.005 to 0.5. For each value of θ, the depreciation rate δ_k is recalibrated so that the capital-output ratio is maintained at 3.0. All other parameters values are the same as in Table 7.

The results of this exercise are shown in Table 9.[23] Similarly to the results shown in Table 2, inequality in wealth and income decrease as θ increases. But the decline in income inequality is much smaller than the decline in wealth inequality. This happens because (i) consumers' earnings are not affected by the parameter θ, and (ii) for most of the consumers in this economy, earnings account for a large fraction of their income.[24] Thus, changing θ has only a mild impact on the income distribution.

When comparing the results in Table 2 and Table 9, we can see that removing human capital formation from the extended model only lowers the Gini coefficient of wealth by 1.5 percent when $\theta = 0.01202$. In other words, wealth inequality in the extended model is mainly driven by wealth preference and the heterogeneity in discount factors.

5.2 Non-Separable Utility Function

In the existing literature, it is also common to use the following non-separable utility function,

$$u(c,k) = \begin{cases} \frac{1}{1-\widetilde{\sigma}} \left[\zeta c^{\psi} + (1-\zeta) k^{\psi}\right]^{\frac{1-\widetilde{\sigma}}{\psi}} & \text{for } \psi < 1 \text{ and } \psi \neq 0, \\ \frac{1}{1-\widetilde{\sigma}} \left(c^{\zeta} k^{1-\zeta}\right)^{1-\widetilde{\sigma}}, & \text{for } \psi = 0, \end{cases}$$

with $\widetilde{\sigma} > 0$ and $\zeta \in (0,1)$. Under the additively separable utility function, the Euler equation in the balanced-growth equilibrium is given by

$$\frac{\gamma^{\sigma}}{\beta_i} - (1-\delta_k) - r = \theta \left[\frac{\widehat{w}(r) l_i h_i}{\widehat{k}_i} + r - \widehat{\delta}_k\right]^{\sigma}.$$

Under the non-separable specification, the Euler equation becomes

$$\frac{\gamma^{\widetilde{\sigma}}}{\beta_i} - (1-\delta_k) - r = \frac{1-\zeta}{\zeta} \left[\frac{\widehat{w}(r) l_i h_i}{\widehat{k}_i} + r - \widehat{\delta}_k\right]^{1-\psi}.$$

[23] As explained earlier, the earnings distribution is independent of θ. Thus, for all the cases considered in Table 9, the earnings distribution is the same as in the benchmark scenario.

[24] When θ is 0.05 or less, earnings represent more than 70 percent of income for those in the bottom four quintiles (i.e., the bottom 80 percent) of the wealth distribution.

A direct comparison between these two equations suggests that they can be made identical by a suitable choice of parameter values. When this is imposed, the equilibrium wealth distribution and the equilibrium rental rate will be identical under these two specifications of utility function.[25] Formally, let $\widehat{\mathbf{k}} = \left(\widehat{k}_1, ..., \widehat{k}_N\right)$ be the distribution of physical capital obtained under the additively separable specification and a common growth factor γ. Then the same distribution can be obtained under a non-separable utility function with $\psi = 1 - \sigma$, $\zeta = 1/(1+\theta)$, and a common growth factor $\widetilde{\gamma} = \gamma^{\frac{1-\psi}{\widetilde{\sigma}}}$.[26] This observation suggests that these two forms of utility function are likely to yield quantitatively similar results *in the balanced-growth equilibrium.*[27] The additively separable form is preferred because it involves fewer parameters.

5.3 Relaxing the Uniform Distribution Assumption

We now perform the same sensitivity analysis as in Section 2.6.3. In particular, the share of each group in the population is now determined by

$$\lambda_i = \left(\frac{i}{N}\right)^{\frac{1}{\upsilon}} - \left(\frac{i-1}{N}\right)^{\frac{1}{\upsilon}}, \qquad \text{with } \upsilon > 0,$$

for $i \in \{1, 2, ..., N\}$. The endpoints of the distribution are fixed at their benchmark values, i.e., $\beta_{\text{min}} = 0.966$ and $\beta_{\text{max}} = 0.992$. The benchmark results in Table 8 then corresponds to the case when $\upsilon = 1$. We consider two calibration exercises. In the first exercise, we examine the extent of economic inequality under different values of υ. The results are shown in Panel (A) of Table 10. For each value of υ, the depreciation rate of physical capital is adjusted so as to maintain the capital-output ratio at 3.0. All other parameters (including θ) are fixed at their benchmark values. In the second exercise, both θ and δ_k are recalibrated in each case so that the two calibration targets (Gini coefficient of wealth and the capital-output ratio) are the same as in the benchmark scenario. The results of the second experiment are summarized in Panel (B) of Table 10.

Overall, the results of this exercise are similar to those obtained from the baseline model. Panel (A) of Table 10 shows that the Gini coefficients produced by the model are rather robust to changes in

[25] Since the values of $\{l_i, h_i\}_{i=1}^N$ are independent of the utility function, the distributions of labor hours, human capital and earnings are also identical under these two specifications of utility function.

[26] In particular, our benchmark results can be obtained from a non-separable utility function with $\psi = 0$, $\widetilde{\sigma} = 1$ and $\zeta = 1/(1 + 0.01202)$.

[27] We stress that the above argument is valid only in the balanced-growth equilibrium. The two specifications are likely to yield very different results along any transition path.

the size of the most patient group. More specifically, reducing λ_N by half raises the Gini coefficients of earnings, income and wealth by 13.6 percent, 10.8 percent and 8.5 percent, respectively. The share of total wealth and total income owned by the richest consumers are more sensitive to this change. The intuitions behind these results are as follows. First, consider the increase in earnings inequality. In the stationary equilibrium, this type of inequality is driven by (i) cross-sectional variations in the population share, $\{\lambda_i\}_{=1}^N$, and (ii) cross-sectional variations in human capital and labor hours, $\{h_i, l_i\}_{i=1}^N$. As shown in (25) and (26), the values of $\{h_i, l_i\}_{i=1}^N$ are independent of the effective rate of return $(r^* - \delta_k)$ and the population shares. This means changing υ has no impact on the values of $\{h_i, l_i\}_{i=1}^N$. Thus, the increase in earnings inequality that we observed in Panel (A) of Table 10 is completely driven by the changes in $\{\lambda_i\}_{=1}^N$. In particular, an increase in υ lowers the share of very patient consumers in the population. Since these consumers tend to have more human capital and higher earnings than the less patient ones, a large portion of total earnings is now concentrated in the hands of fewer consumers. Thus, the earnings distribution becomes more unequal as υ increases.

An increase in υ has a similar effect on wealth inequality. Specifically, such an increase means that a large portion of total wealth is now concentrated in the hands of fewer consumers. This makes the wealth distribution more unequal. However, an increase in υ would also induce changes in the effective rate of return from savings. This creates a second effect on wealth inequality. More specifically, an increase in the share of less patient consumers leads to a decline in aggregate savings. In order to maintain the same capital-output ratio, we need to adjust the effective rate of return upward as υ increases. Since more patient consumers are more responsive to interest rate changes than less patient ones, this widens the differences in asset holdings across groups and further increases wealth inequality. As for income, since it is just the sum of earnings and capital income, income inequality increases as earnings and wealth inequality increase.

Next, we turn to the results in Panel (B) of Table 10. Since adjusting θ has no effect on the earnings distribution, the Gini coefficients of earnings are the same as in Panel (A). When the Gini coefficient of wealth is held constant, increasing υ from 1.0 to 2.0 raises the Gini coefficient of income by 6.5 percent, which is smaller than the increase in Panel (A). The most significant difference between the two panels is that, when the Gini coefficient of wealth is held constant, an increase in υ would lower the share of total wealth and total income held by the richest consumers. This happens because we need to adjust θ upward as υ increases so as to maintain the Gini coefficient of wealth at the same level. As shown in

Table 9, this tends to lower the share of total wealth and total income held by the richest consumers. The results in Panel (B) thus show that the qualitative effects of θ on wealth and income inequality are robust to changes in υ.

5.4 Changing the Range of Discount Factors

We now examine the effects of changing the range of discount factors. Similar to Section 2.6.4, we maintain the uniform distribution assumption and consider five different combinations of endpoint values. We report two sets of results for each experiment. Panel (A) of Table 11 reports the results obtained when the capital-output ratio is kept at 3.0 and θ is fixed at 0.01202. Panel (B) reports the results obtained when the two calibration targets are kept constant.

We begin by summarizing the results in Panel (A). Similar to the model without human capital, shifting the distribution of discount factors while leaving the range $\triangle\beta$ unchanged only has a mild impact on the Gini coefficients. For instance, reducing the benchmark endpoint values by 0.01 would raise the Gini coefficients of earnings and income by 4.5 percent and 2.2 percent, respectively. The effect of this on the Gini coefficient of wealth is negligible. The share of total wealth and total income owned by the richest consumers is also quite robust to this change. The results of the last three experiments show that inequality in all three variables are positively related to the size of $\triangle\beta$. Overall, the results in Panel (A) suggest that the extended model does not rely on large values of discount factor (i.e., very patient consumers) to generate a high concentration of wealth and income. Instead, economic inequality in our model is largely determined by the relative magnitude between $\beta_{\min}$ and $\beta_{\max}$.

Next, consider the results in Panel (B) of Table 11. When the Gini coefficient of wealth is held constant, income inequality is less sensitive to changes in $\triangle\beta$. The main differences between the two panels are the effects of changing $\triangle\beta$ on the share of total wealth and total income held by the richest consumers. When the parameter θ is kept constant, these shares are positively related to the size of $\triangle\beta$. When the Gini coefficient of wealth is kept constant, these shares become negatively related to the size of $\triangle\beta$. This happens because, in order to maintain the same Gini coefficient of wealth, we need to adjust θ upward as the range of discount factors widens. This in turn lowers the share of total wealth and total income held by the richest consumers.

5.5 Changing the Intertemporal Elasticity of Substitution

Table 12 reports the results obtained under different values of σ. Panel (A) shows that, when θ is held constant, an increase in σ has only a mild impact on the Gini coefficients. In particular, increasing σ from 1.0 to 1.8 only raises the Gini coefficient of income and wealth by 1.7 percent and 3.6 percent, respectively. The share of total wealth owned by the wealthiest agents, however, is rather sensitive to this change. The intuitions of this result are as follows. Similarly to an increase in θ, an increase in σ would induce two opposing effects on wealth inequality. First, an increase in σ lowers the intertemporal elasticity of substitution (IES) for consumption. Holding other things constant, every consumer would now prefer to have a flatter consumption profile and less savings. In particular, the reduction in savings tends to be larger for the wealthy consumers than for the poor ones. Thus, holding other things constant, this effect would make the wealth distribution more equal. Second, since aggregate savings decline as σ increases, we need to adjust the effective rate of return from savings upward in order to maintain the same capital-output ratio. This induces a much larger increase in asset holdings for the wealthy consumers than for the poor ones, which in turn drives up the differences in wealth across groups. Hence, the second effect would make the wealth distribution more unequal and increase the share of wealth owned by the wealthiest consumers. The results in Panel (A) of Table 12 suggest that the second effect dominates under the benchmark parameter values.

When the Gini coefficient of wealth is kept constant, an increase in σ has no effect on the Gini coefficients. The same increase in σ now induces a smaller increase in the share of total wealth owned by the wealthiest agents than in Panel (A). This happens because we need to adjust θ upward as σ increases so as to maintain the same Gini coefficient of wealth. As shown in Table 9, this tends to reduce the share of wealth owned by the wealthiest agents, and thus partially offsets the effects of σ on the top end of the wealth distribution.

6 Concluding Remarks

This paper presents a highly tractable dynamic general equilibrium model that can generate patterns of wealth and income inequality that are very similar to those observed in the United States. To achieve this, we extend the standard deterministic neoclassical growth model to include three features: (i) consumer heterogeneity in time preference, (ii) direct preferences for wealth, and (iii) human capital formation. We show that a model with the first two features alone is able to replicate the patterns of wealth inequality observed in the United States. Such a model, however, cannot generate substantial degree of income inequality. Thus, we also need to introduce human capital in order to account for both wealth and income inequality.

Admittedly, the model considered in this study is rather stylized and has abstracted away a number of factors that are also relevant in explaining economic inequality. One possible extension is to introduce idiosyncratic uncertainty into the current framework. Such a model can then be used to evaluate the relative importance of predetermined factors and idiosyncratic shocks in explaining wealth and income inequality.

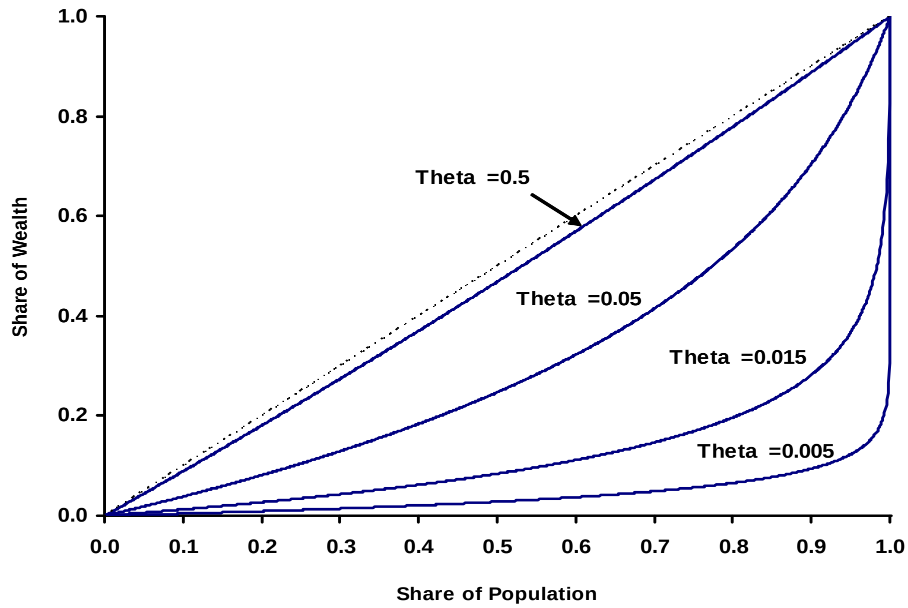

Figure 1: Lorenz Curves for the Wealth Distribution.

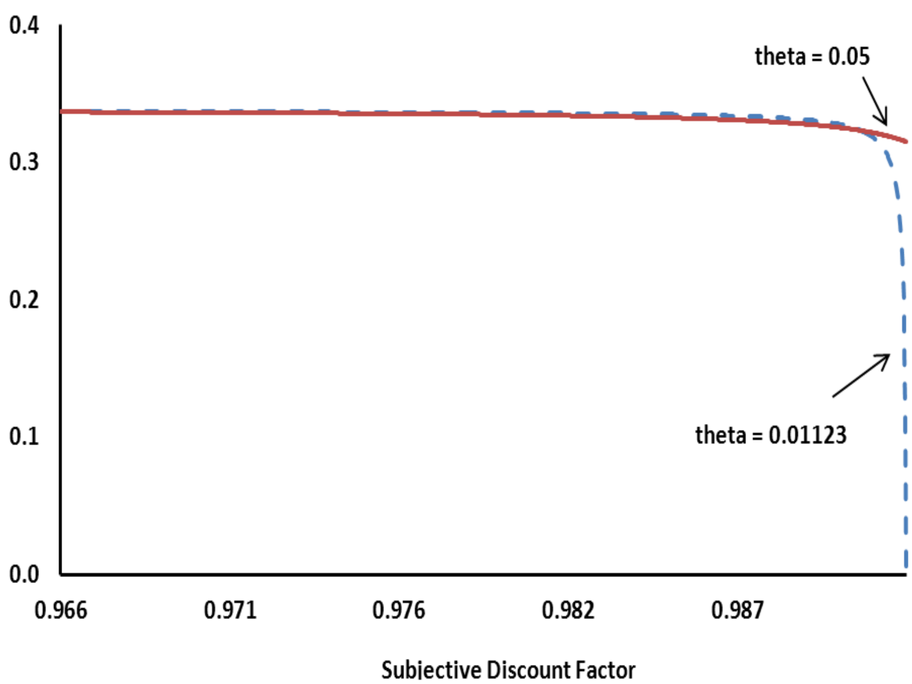

Figure 2: Relationship between Labor Supply and Discount Factor.

Table 2 Wealth and Income Inequality in Baseline Model.

	Gini Coeff.		C.V.		Share of Wealth (%) Held by			
					Bottom	Top		
θ	Wealth	Income	Wealth	Income	40%	10%	5%	1%
0.005	0.918	0.303	22.04	7.27	2.1	90.6	87.7	80.9
0.010	0.836	0.276	13.19	4.35	4.1	81.1	75.4	61.9
0.01123	**0.816**	0.269	11.25	3.71	4.6	78.8	72.4	57.3
0.01202	0.804	0.284	10.08	3.33	5.0	7.7	70.5	54.4
0.01796	0.709	0.234	4.19	1.38	7.4	66.3	56.2	**33.6**
0.025	0.608	0.201	2.11	0.70	10.2	54.3	41.4	17.1
0.050	0.375	0.124	0.78	0.26	18.4	29.5	17.3	4.1
0.100	0.201	0.066	0.37	0.12	26.9	17.6	9.3	1.9
0.500	0.041	0.014	0.07	0.02	37.0	11.2	5.6	1.1
Data	**0.816**	**0.575**	**6.02**	**4.32**	**0.9**	**71.4**	**60.3**	**33.6**

Data Source: Díaz-Giménez *et al.* (2011). Note: C.V. refers to the coefficient of variation.

Table 3 Share of Total Income from Earnings (%) in Each Wealth Group.

	Percentiles in Wealth Distribution						
θ	Bottom 1%	1-5%	5-10%	40-60%	90-95%	95-99%	Top 1%
0.005	98.0	98.0	97.9	96.1	78.1	56.8	17.2
0.010	96.2	96.1	95.9	92.5	64.2	40.6	10.0
0.01123	95.7	95.6	95.4	91.6	61.6	38.0	9.2
0.01202	95.4	95.3	95.1	91.1	60.0	36.5	8.7
0.01796	93.3	93.2	92.9	87.3	50.6	29.0	7.6
0.025	91.0	90.8	90.4	83.3	44.6	26.6	11.1
0.050	84.6	84.3	83.7	74.6	45.6	38.1	33.1
0.100	78.0	77.7	77.1	69.4	54.9	52.5	51.1
0.500	69.6	69.5	69.3	67.1	64.6	64.4	64.2
Data	**88.8**	**94.9**	**93.3**	**89.7**	**67.0**	**57.8**	**31.3**

Data Source: Díaz-Giménez *et al.* (2011) Table 6, excluding transfers from total income.

Table 4 Changing the Distribution of Discount Factors.

Panel (A) Holding capital-output ratio and $\theta = 0.01123$ constant.

	Discount Factors						Share of Wealth (%) Held by			
			Share of Agents with		Gini Coeff.		Bottom		Top	
υ	Mean	Median	$\beta = 0.966$	$\beta = 0.992$	Wealth	Income	40%	10%	5%	1%
1.0	**0.9790**	**0.9790**	**0.0010**	**0.0010**	**0.816**	**0.270**	**4.6**	**78.8**	**72.4**	**57.3**
1.2	0.9778	0.9773	0.0032	0.0008	0.836	0.276	5.3	80.1	74.7	62.0
1.5	0.9764	0.9752	0.0100	0.0007	0.855	0.282	6.0	81.6	77.2	67.0
2.0	0.9747	0.9725	0.0316	0.0005	0.873	0.288	6.8	83.3	80.0	72.3

Panel (B) Holding capital-output ratio and Gini coefficient of wealth constant.

	Discount Factors						Share of Wealth (%) Held by			
			Share of Agents with		Gini Coeff.		Bottom		Top	
υ	Mean	Median	$\beta = 0.966$	$\beta = 0.992$	Wealth	Income	40%	10%	5%	1%
1.0	**0.9790**	**0.9790**	**0.0010**	**0.0010**	**0.816**	**0.270**	**4.6**	**78.8**	**72.4**	**57.3**
1.2	0.9778	0.9773	0.0032	0.0008	0.816	0.269	5.9	77.7	71.6	57.4
1.5	0.9764	0.9752	0.0100	0.0007	0.816	0.269	7.6	76.6	71.0	58.0
2.0	0.9747	0.9725	0.0316	0.0005	0.816	0.269	9.8	75.7	70.9	59.7

Note: Figures in bold are the benchmark results for $\theta = 0.01123$ reported in Table 2.

Table 5 Changing the Range of Discount Factors.

Panel (A) Holding capital-output ratio and $\theta = 0.01123$ constant.

		Gini Coeff.		Share of Wealth (%)			
				Bottom	Top		
$\beta_{\min}$	$\beta_{\max}$	Wealth	Income	40%	10%	5%	1%
0.966	**0.992**	**0.816**	0.269	**4.6**	**78.8**	**72.4**	**57.3**
0.956	0.982	0.820	0.271	4.5	79.3	73.0	58.2
0.946	0.972	0.824	0.272	4.4	79.7	73.5	59.0
0.979	0.992	0.640	0.211	9.3	58.2	46.2	21.6
0.966	0.979	0.649	0.214	9.1	59.2	47.4	22.9
0.953	0.992	0.878	0.290	3.1	86.0	81.7	71.5

Panel (B) Holding capital-output ratio and Gini coefficient of wealth constant.

		Gini Coeff.		Share of Wealth (%)			
				Bottom	Top		
$\beta_{\min}$	$\beta_{\max}$	Wealth	Income	40%	10%	5%	1%
0.966	**0.992**	**0.816**	0.269	**4.6**	**78.8**	**72.4**	**57.3**
0.956	0.982	0.816	0.269	4.6	78.8	72.4	57.3
0.946	0.972	0.816	0.269	4.6	78.8	72.4	57.3
0.979	0.992	0.816	0.269	4.7	78.8	72.4	57.4
0.966	0.979	0.816	0.269	4.7	78.9	72.5	57.4
0.953	0.992	0.816	0.269	4.6	78.8	72.3	57.0

Note: Figures in bold are the benchmark results for $\theta = 0.01123$ reported in Table 2.

Table 6 Wealth and Income Inequality when $\eta = 0.4$.

	Gini Coeff.		C.V.		Share of Wealth (%) Held by			
					Bottom	Top		
θ	Wealth	Income	Wealth	Income	40%	10%	5%	1%
0.005	0.918	0.299	23.11	7.61	2.1	90.5	87.7	81.2
0.010	0.836	0.270	15.80	5.19	4.2	81.0	75.4	62.9
0.01123	0.816	0.262	14.10	4.63	4.7	78.7	72.4	58.5
0.01206	0.803	0.258	13.03	4.28	5.0	77.2	70.4	55.6
0.01796	0.706	0.224	5.54	1.80	7.5	65.9	55.9	34.8
0.025	0.600	0.188	2.04	0.64	10.4	53.4	40.4	16.4
0.050	0.369	0.118	0.77	0.24	18.6	29.1	17.0	4.0
0.100	0.203	0.068	0.37	0.12	26.8	17.7	9.3	2.0
0.500	0.052	0.025	0.09	0.04	36.2	11.5	5.8	1.2
Data	**0.816**	**0.575**	**6.02**	**4.32**	**0.9**	**71.4**	**60.3**	**33.6**

Data Source: Díaz-Giménez *et al.* (2011). Note: C.V. refers to the coefficient of variation.

Table 8 Benchmark Results of Model with Human Capital.

				Share (%) Held by Consumers in Each Group										
			Mean-to-	Bottom			Quintiles					Top		
	Gini	C.V.	Median	1%	1-5%	5-10%	1st	2nd	3rd	4th	5th	10%	5%	1%
Earnings														
Model	0.397	0.73	1.34	0.2	0.9	1.2	5.4	8.9	15.1	25.8	44.5	25.2	13.4	28.2
Data	0.636	3.60	1.72	-0.1	0.0	0.0	-0.1	4.2	11.7	20.8	63.5	47.0	35.3	18.7
Income														
Model	0.536	1.39	1.82	0.1	0.6	0.9	3.8	6.4	11.1	20.2	58.2	41.1	28.3	10.5
Data	0.575	4.32	1.77	-0.1	0.3	0.6	2.8	6.7	11.3	18.3	60.9	47.1	36.9	21.0
Wealth														
Model	0.816	3.16	6.92	0.0	0.1	0.1	0.6	1.3	3.0	8.8	86.2	73.4	58.5	25.9
Data	0.816	6.02	4.61	-0.1	-0.1	-0.0	-0.2	1.1	4.5	11.2	83.4	71.4	60.3	33.6

Data source: Díaz-Giménez *et al.* (2011).

Table 9 Changing the Strength of Wealth Preference.

			Share of Income (%)				Share of Wealth (%)			
	Gini Coeff.		Bottom	Top			Bottom	Top		
θ	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
0.005	0.569	0.918	9.9	46.1	36.0	22.9	0.8	88.5	81.8	63.8
0.010	0.544	0.842	10.2	42.4	30.3	13.0	1.5	77.5	64.5	33.6
0.01202	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
0.025	0.497	0.700	10.8	35.0	20.8	5.1	3.6	54.9	35.9	9.7
0.050	0.463	0.597	11.6	30.4	16.8	3.7	6.1	40.9	23.8	5.4
0.100	0.437	0.517	12.5	27.8	15.0	3.2	8.8	33.1	18.3	4.0
0.500	0.407	0.426	13.9	25.7	13.7	2.9	12.9	26.7	14.3	3.0
Data	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Note: Figures in bold are the benchmark results reported in Table 8. The source of data is Díaz-Giménez *et al.* (2011).

Table 10 Changing the Distribution of Discount Factors.

Panel (A) Holding capital-output ratio and $\theta = 0.01202$ *constant.*

	Discount Factors							Share of Income (%) Held by				Share of Wealth (%) Held by			
			Share of Consumers with		Gini Coeff.			Bottom	Top			Bottom	Top		
υ	Mean	Median	$\beta = 0.966$	$\beta = 0.992$	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
1.0	**0.9790**	**0.9790**	**0.0010**	**0.0010**	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
1.2	0.9778	0.9773	0.0032	0.0008	0.417	0.556	0.838	12.6	39.9	28.0	11.0	2.2	73.8	59.7	28.0
1.5	0.9764	0.9752	0.0100	0.0007	0.436	0.576	0.861	15.7	38.7	27.7	11.8	2.7	74.4	61.2	30.9
2.0	0.9747	0.9725	0.0316	0.0005	0.451	0.594	0.885	20.0	37.4	27.6	13.0	3.4	75.4	63.5	35.1
Data	–	–	–	–	0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Panel (B) Holding capital-output ratio and Gini coefficient of wealth constant.

	Discount Factors							Share of Income (%) Held by				Share of Wealth (%) Held by			
			Share of Consumers with		Gini Coeff.			Bottom	Top			Bottom	Top		
υ	Mean	Median	$\beta = 0.966$	$\beta = 0.992$	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
1.0	**0.9790**	**0.9790**	**0.0010**	**0.0010**	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
1.2	0.9778	0.9773	0.0032	0.0008	0.417	0.549	0.816	12.7	38.7	26.2	9.0	2.6	69.9	54.2	22.1
1.5	0.9764	0.9752	0.0100	0.0007	0.436	0.561	0.816	16.0	35.8	23.7	7.6	3.8	65.5	49.2	18.2
2.0	0.9747	0.9725	0.0316	0.0005	0.451	0.571	0.816	20.8	32.4	20.9	6.2	5.8	60.1	43.4	14.5
Data	–	–	–	–	0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Note: Figures in bold are the benchmark results reported in Table 8. The source of data is Díaz-Giménez *et al.* (2011).

Table 11 Changing the Range of Discount Factors.

Panel (A) Holding capital-output ratio and $\theta = 0.01202$ constant.

			Gini Coeff.		Share of Income (%)				Share of Wealth (%)			
					Bottom		Top		Bottom		Top	
$\beta_{\min}$	$\beta_{\max}$	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
0.966	**0.992**	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
0.956	0.982	0.379	0.524	0.818	11.1	40.8	28.5	10.9	2.0	74.0	59.7	27.5
0.946	0.972	0.353	0.507	0.819	12.1	40.2	28.4	11.6	2.1	74.6	61.0	29.7
0.979	0.992	0.218	0.363	0.657	18.8	29.1	18.2	5.0	6.2	53.8	37.3	11.5
0.966	0.979	0.206	0.356	0.662	19.4	29.2	18.5	5.2	6.3	54.8	38.5	12.3
0.953	0.992	0.527	0.642	0.876	5.7	49.4	34.9	14.3	7.0	81.8	68.6	35.3
Data		0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Panel (B) Holding capital-output ratio and Gini coefficient of wealth constant.

			Gini Coeff.		Share of Income (%)				Share of Wealth (%)			
					Bottom		Top		Bottom		Top	
$\beta_{\min}$	$\beta_{\max}$	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
0.966	**0.992**	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
0.956	0.982	0.379	0.523	0.816	11.1	40.7	28.3	10.8	2.0	73.7	59.3	27.0
0.946	0.972	0.353	0.506	0.816	12.1	40.0	28.2	11.3	2.1	74.1	60.3	28.8
0.979	0.992	0.218	0.415	0.816	17.7	36.4	27.4	14.5	3.0	75.9	65.3	40.2
0.966	0.979	0.206	0.407	0.816	18.3	36.3	27.5	14.7	3.1	76.1	65.8	41.0
0.953	0.992	0.527	0.623	0.816	5.8	45.9	29.8	8.7	1.2	71.4	53.0	18.2
Data		0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Note: Figures in bold are the benchmark results reported in Table 8. The source of data is Díaz-Giménez *et al.* (2011).

Table 12 Changing the Intertemporal Elasticity of Substitution.

Panel (A) Holding capital-output ratio and $\theta = 0.01202$ constant.

				Share of Income (%)				Share of Wealth (%)			
	Gini Coeff.			Bottom	Top			Bottom	Top		
σ	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
1.0	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
1.2	0.397	0.539	0.826	10.3	41.7	29.8	13.9	2.0	75.1	63.0	36.3
1.4	0.397	0.542	0.835	10.4	42.1	30.9	17.1	2.2	76.6	66.5	46.2
1.6	0.397	0.544	0.841	10.4	42.5	31.7	19.4	2.3	77.6	69.0	53.0
1.8	0.397	0.545	0.845	10.4	42.7	32.3	20.9	2.4	78.3	70.7	57.5
Data	0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Panel (B) Holding capital-output ratio and Gini coefficient of wealth constant.

				Share of Income (%)				Share of Wealth (%)			
	Gini Coeff.			Bottom	Top			Bottom	Top		
σ	Earnings	Income	Wealth	40%	10%	5%	1%	40%	10%	5%	1%
1.0	**0.397**	**0.536**	**0.816**	**10.3**	**41.1**	**28.3**	**10.5**	**1.9**	**73.4**	**58.5**	**25.9**
1.2	0.397	0.536	0.816	10.4	41.2	29.1	12.9	2.2	73.7	60.9	33.2
1.4	0.397	0.536	0.816	10.4	41.3	29.7	15.2	2.4	74.0	62.8	40.2
1.6	0.397	0.536	0.816	10.5	41.3	30.1	16.9	2.7	74.1	64.1	45.5
1.8	0.397	0.536	0.816	10.6	41.4	30.5	18.2	2.8	74.3	65.1	49.3
Data	0.636	0.575	0.816	9.5	47.1	36.9	21.0	0.9	71.4	60.3	33.6

Note: Figures in bold are the benchmark results reported in Table 8. The source of data is Díaz-Giménez *et al.* (2011).

Appendix A

Proof of Theorem 1

The proof of this theorem is divided into three main steps. First, it is shown that there exists a rental rate $\widetilde{r}_N > \widehat{\delta}_k$ such that $\widehat{k}^s(r) \to \infty$ as r approaches $\widetilde{r}_N$ from the left. Since $\widetilde{r}_N < \infty$, it follows from Assumption A4 that $\widehat{k}^d(\widetilde{r}_N) < \infty$. Hence, we have

$$\lim_{r \to \widetilde{r}_N} \widehat{k}^s(r) = \infty > \widehat{k}^d(\widetilde{r}_N) \qquad \text{and} \qquad \widehat{k}^d\left(\widehat{\delta}_k\right) > \widehat{k}^s\left(\widehat{\delta}_k\right).$$

Since both $\widehat{k}^s(r)$ and $\widehat{k}^d(r)$ are continuous on $\left(\widehat{\delta}_k, \widetilde{r}_N\right)$, it follows from the intermediate value theorem that there exists at least one $r^* \in \left(\widehat{\delta}_k, \widetilde{r}_N\right)$ such that $\widehat{k}^d(r^*) = \widehat{k}^s(r^*)$. The second step is to show that there exists at most one such solution in the interval $(0, \widetilde{r}_N)$. Together, these two steps establish the existence and uniqueness of r^*. Finally, it is shown that $\beta_i > \beta_j$ implies $\widehat{c}_i > \widehat{c}_j$ and $\widehat{k}_i > \widehat{k}_j$.

Step 1 First, it is shown that for each $i \in \{1, 2, ..., N\}$, there exists a unique value $\widetilde{r}_i > \widehat{\delta}_k$ that solves

$$\Gamma_i(r) \equiv \frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - r = \Phi\left(r - \widehat{\delta}_k\right).$$

Note that the function $\Gamma_i(r)$ is a strictly decreasing function that satisfies

$$\Gamma_i\left(\widehat{\delta}_k\right) \equiv \frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - \widehat{\delta}_k = \frac{\gamma^\sigma}{\beta_i} - \gamma > 0.$$

The last inequality follows from the assumption that $\beta_i \gamma^{1-\sigma} < 1$. Thus, we have $\Gamma_i\left(\widehat{\delta}_k\right) > \Phi(0) = 0$. By Assumption A3, $\Phi\left(r - \widehat{\delta}_k\right)$ is strictly increasing in r. Consequently, the two functions $\Gamma_i(r)$ and $\Phi\left(r - \widehat{\delta}_k\right)$ will cross only once. This establishes the claim.

The fact that $\Gamma_i\left(\widehat{\delta}_k\right) > \Phi(0) = 0$ implies that $\Gamma_i(\widetilde{r}_i) = \Phi\left(\widetilde{r}_i - \widehat{\delta}_k\right) > 0$. Hence, it must be the case that $\widetilde{r}_i > \widehat{\delta}_k$ and $\widetilde{r}_i < \widehat{r}_i \equiv \gamma^\sigma/\beta_i - (1 - \delta_k)$, for all i. Given the ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$, it is straightforward to see that $\widehat{\delta}_k < \widetilde{r}_N \leq \widetilde{r}_{N-1} \leq \ldots \leq \widetilde{r}_1$.

By the definition of $\widetilde{r}_i$, we have $g_i(r) \to \infty$ as r approaches $\widetilde{r}_i$ from the left. To see this, suppose the contrary that $g_i(\widetilde{r}_i)$ is a finite positive real number. Then a contradiction follows immediately from (12) and the definition of $\widetilde{r}_i$. Thus, when r approaches $\widetilde{r}_N$ from the left, we have $g_N(r) \to \infty$ and $g_i(r) > 0$ for all $i \in \{1, 2, ..., N-1\}$. Hence, $\widehat{k}^s(r) = \sum_{i=1}^N \lambda_i g_i(r) \to \infty$ as r approaches $\widetilde{r}_N$.

Step 2 To establish the uniqueness of r^*, we need to consider the derivative of $\widehat{k}^s(r)$. Using equation (12), one can derive the derivative of $g_i(r)$, which is given by

$$g_i'(r) = \frac{1}{\widehat{w}(r)}\left\{[g_i(r)]^2 + \widehat{w}'(r)\,g_i(r) + \frac{[g_i(r)]^2}{\Phi'(z_i(r))}\right\},$$

where $z_i(r) \equiv \widehat{w}(r)/g_i(r) + r - \widehat{\delta}_k$ and $\widehat{w}'(r) = -\widehat{k}^d(r) < 0$. Hence the derivative of $\widehat{k}^s(r)$ is

$$\frac{d}{dr}\left[\widehat{k}^s(r)\right] = \frac{1}{\widehat{w}(r)}\left\{\sum_{i=1}^{N}\lambda_i[g_i(r)]^2 - \widehat{k}^d(r)\,\widehat{k}^s(r) + \sum_{i=1}^{N}\lambda_i\frac{[g_i(r)]^2}{\Phi'(z_i(r))}\right\}.$$

Let r^* be any value that satisfies $\widehat{k}^d(r^*) = \widehat{k}^s(r^*)$. The derivative of $\widehat{k}^s(r)$ at $r = r^*$ is

$$\frac{1}{\widehat{w}(r^*)}\left\{\sum_{i=1}^{N}\lambda_i[g_i(r^*)]^2 - \left[\widehat{k}^s(r^*)\right]^2 + \sum_{i=1}^{N}\lambda_i\frac{[g_i(r^*)]^2}{\Phi'(z_i(r^*))}\right\},$$

after we imposed the condition $\widehat{k}^d(r^*) = \widehat{k}^s(r^*)$. The above expression is strictly positive as

$$\sum_{i=1}^{N}\lambda_i[g_i(r^*)]^2 \geq \left[\sum_{i=1}^{N}\lambda_i g_i(r^*)\right]^2 = \left[\widehat{k}^s(r^*)\right]^2,$$

and $\Phi'(z) > 0$. Since $\widehat{k}^d(r)$ is strictly decreasing, this result means that $\widehat{k}^s(r)$ must be cutting $\widehat{k}^d(r)$ from below *at every intersection point.* Since both $\widehat{k}^d(r)$ and $\widehat{k}^s(r)$ are continuous, if there exists more than one value that solves $\widehat{k}^d(r^*) = \widehat{k}^s(r^*)$, then at least of them must have $\widehat{k}^s(r)$ cutting $\widehat{k}^d(r)$ from above. This gives rise to a contradiction and hence establishes the uniqueness of r^*.

Step 3 Totally differentiate the equation

$$\frac{\gamma^\sigma}{\beta} - (1-\delta_k) - r = \Phi\left[\frac{\widehat{w}(r)}{\widehat{k}} + r - \widehat{\delta}_k\right]$$

with respect to β and $\widehat{k}$ yields

$$\frac{d\widehat{k}}{d\beta} = \frac{\gamma^\sigma}{\widehat{w}(r)}\left(\frac{\widehat{k}}{\beta}\right)^2\left[\Phi'\left(\frac{\widehat{w}(r)}{\widehat{k}} + r - \widehat{\delta}_k\right)\right]^{-1} > 0.$$

Hence $\beta_i > \beta_j$ implies $\widehat{k}_i > \widehat{k}_j$. Since the equilibrium rental rate r^* is strictly greater than $\widehat{\delta}_k$, $\widehat{c}_i$ is positively related to $\widehat{k}_i$ according to (10). This completes the proof of Theorem 1. ■

Appendix B

In this appendix, we first provide the mathematical derivations of equations (23)-(27) and then establish the existence and uniqueness of equilibrium in the extended model. Let $\xi_{i,t}$ and $\chi_{i,t}$ be the Lagrange multipliers for the budget constraint and the human capital accumulation equation, respectively. The first-order conditions for the agent's problem are given by

$$u_c\left(c_{i,t}, k_{i,t}\right) = \xi_{i,t}, \tag{28}$$

$$\xi_{i,t} w_t h_{i,t} = \chi_{i,t} \epsilon \varphi \left(1 - l_{i,t}\right)^{\epsilon - 1} h_{i,t}^{\varsigma}, \tag{29}$$

$$\xi_{i,t} = \beta_i \left[u_k\left(c_{i,t+1}, k_{i,t+1}\right) + \xi_{i,t+1}\left(1 + r_{t+1} - \delta_k\right)\right], \tag{30}$$

$$\chi_{i,t} = \beta_i \left\{\xi_{i,t+1} w_{t+1} l_{i,t+1} + \chi_{i,t+1}\left[\varsigma\varphi\left(1 - l_{i,t+1}\right)^{\epsilon} h_{i,t+1}^{\varsigma - 1} + \left(1 - \delta_h\right)\right]\right\}. \tag{31}$$

Combining (28) and (30) gives

$$\frac{u_c\left(c_{i,t}, k_{i,t}\right)}{u_c\left(c_{i,t+1}, k_{i,t+1}\right)} = \beta_i \left[\frac{u_k\left(c_{i,t+1}, k_{i,t+1}\right)}{u_c\left(c_{i,t+1}, k_{i,t+1}\right)} + 1 + r_{t+1} - \delta_k\right].$$

Equation (23) can be obtained from this after imposing the balanced-growth conditions: $c_{i,t} = \gamma^t \widehat{c}_i$ and $k_{i,t} = \gamma^t \widehat{k}_i$. The derivation of (24) is straightforward and is omitted. Along a balanced-growth equilibrium path, individual human capital is stationary. It follows from the human capital accumulation equation that

$$\delta_h h_i = \varphi\left(1 - l_i\right)^{\epsilon} h_i^{\varsigma}.$$

Equation (26) follows immediately from this expression. Finally, combining (29) and (31) gives

$$\chi_{i,t} = \beta_i \chi_{it+1} \left\{\varphi\left(1 - l_{i,t+1}\right)^{\epsilon - 1} h_{i,t+1}^{\varsigma - 1}\left[\varsigma\left(1 - l_{i,t+1}\right) + \epsilon l_{i,t+1}\right] + \left(1 - \delta_h\right)\right\}. \tag{32}$$

In a balanced-growth equilibrium, the multiplier $\chi_{i,t}$ is growing at a constant rate. To see this, combine (28) and (29) to get

$$u_c\left(c_{i,t}, k_{i,t}\right) w_t = \chi_{i,t} \epsilon \varphi \left(1 - l_{i,t}\right)^{\epsilon - 1} h_{i,t}^{\varsigma}.$$

After imposing the balanced-growth conditions, i.e., $c_{i,t} = \gamma^t \widehat{c}_i$, $k_{i,t} = \gamma^t \widehat{k}_i$, $w_t = \gamma^t \widehat{w}(r)$, $l_{i,t} = l_i$ and $h_{i,t} = h_i$ for all t, we have

$$u_c \left(\widehat{c}_i, \widehat{k}_i \right) \widehat{w}(r) \gamma^{(1-\sigma)t} = \chi_{i,t} \epsilon \varphi (1 - l_i)^{\epsilon - 1} h_i^{\varsigma}.$$

Hence, $\chi_{i,t+1} = \gamma^{1-\sigma} \chi_{i,t}$. Substituting this into (32) gives

$$1 = \beta_i \gamma^{1-\sigma} \left\{ \varphi (1 - l_i)^{\epsilon - 1} h_i^{\varsigma - 1} \left[\varsigma (1 - l_i) + \epsilon l_i \right] + (1 - \delta_h) \right\}.$$

Equation (25) can be obtained by substituting (26) into this.

We now turn to the existence and uniqueness of balanced-growth equilibrium. These results are summarized in the following theorem.

Theorem A1 *Suppose $\beta_i \gamma^{1-\sigma} < 1$ for all $i \in \{1, ..., N\}$, and*

$$\left(\sum_{i=1}^{N} \lambda_i l_i h_i \right) \widehat{k}^d \left(\widehat{\delta}_k \right) > \widehat{k}^s \left(\widehat{\delta}_k \right). \tag{33}$$

Then there exists a unique balanced-growth equilibrium. In the unique equilibrium, all consumers hold a strictly positive amount of physical capital.

Proof of Theorem A1

The proof of this theorem uses the same steps as in the proof of Theorem 1. First, notice that for each $i \in \{1, 2, ..., N\}$, the variables l_i and h_i are independent of r^*. This is true because these two variables are determined by (25) and (26), which are independent of r^*. Along a balanced-growth equilibrium with rental price r, the optimal savings of a type-i agent is completely determined by

$$\frac{\gamma^\sigma}{\beta_i} - (1 - \delta_k) - r = \Phi \left[\frac{\widehat{w}(r) l_i h_i}{\widehat{k}_i} + r - \widehat{\delta}_k \right].$$

This equation is a direct analogue of (12) from the baseline model. In particular, this implicitly defines a continuous differentiable function $g_i : (0, \widehat{r}_i) \to \mathbb{R}_+$, that describes the relationship between $\widehat{k}_i$ and r. In the current setting, $\widehat{r}_i$ is defined as $\widehat{r}_i \equiv \gamma^\sigma / \beta_i - (1 - \delta_k) > 0$. Using the same argument as in the proof of Theorem 1, one can show that for each $i \in \{1, 2, ..., N\}$ there exists a unique value $\widetilde{r}_i \in \left(\widehat{\delta}_k, \widehat{r}_i \right)$

that solves

$$\Gamma_i(r) \equiv \frac{\gamma^\sigma}{\beta_i} - (1-\delta_k) - r = \Phi\left(r - \widehat{\delta}_k\right).$$

This again implies $g_i(r) \to \infty$ as r approaches $\widetilde{r}_i$ from the left. Hence, $\widehat{k}^s(r) = \sum_{i=1}^N g_i(r) \to \infty$ as r approaches $\min_i \{\widetilde{r}_i\} = \widetilde{r}_N$ from the left. This, together with condition (33), implies that there exists at least one $r^* \in \left(\widehat{\delta}_k, \widetilde{r}_N\right)$ that clears the capital market. To establish the uniqueness of r^*, we again consider the derivative of $\widehat{k}^s(r)$, which is now given by

$$\frac{d}{dr}\left[\widehat{k}^s(r)\right] = \frac{1}{\widehat{w}(r)}\left\{\sum_{i=1}^N \lambda_i \frac{[g_i(r)]^2}{l_i h_i} - \widehat{k}^d(r)\,\widehat{k}^s(r) + \sum_{i=1}^N \lambda_i \frac{[g_i(r)]^2}{l_i h_i \Phi'(z_i(r))}\right\},$$

where $z_i(r) \equiv \frac{\widehat{w}(r) l_i h_i}{g_i(r)} + r - \widehat{\delta}_k$. Let r^* be any value that satisfies

$$\widehat{k}^s(r) \equiv \sum_{i=1}^N \lambda_i g_i(r) = \left(\sum_{i=1}^N \lambda_i l_i h_i\right)\widehat{k}^d(r)$$

The derivative of $\widehat{k}^s(r)$ at $r = r^*$ is

$$\frac{1}{\widehat{w}(r^*)}\left\{\sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i} - \frac{\left[\sum_{i=1}^N \lambda_i g_i(r^*)\right]^2}{\sum_{i=1}^N \lambda_i l_i h_i} + \sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i \Phi'(z_i(r^*))}\right\}.$$

This above expression is strictly positive because, by the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i}\right)\left(\sum_{i=1}^N \lambda_i l_i h_i\right) \geq \left(\sum_{i=1}^N \sqrt{\frac{\lambda_i}{l_i h_i}} g_i(r^*) \cdot \sqrt{\lambda_i l_i h_i}\right)^2 = \left[\sum_{i=1}^N \lambda_i g_i(r)\right]^2,$$

which implies

$$\sum_{i=1}^N \lambda_i \frac{[g_i(r^*)]^2}{l_i h_i} - \frac{\left[\sum_{i=1}^N \lambda_i g_i(r^*)\right]^2}{\sum_{i=1}^N \lambda_i l_i h_i} \geq 0.$$

This completes the proof of the theorem. ■

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