Target variation in a loss avoiding pension fund problem

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Target Variation in a Loss Avoiding Pension Fund Problem*

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Abstract

This study builds on the findings in Krawczyk (2008), where a 'cautious relaxed' utility measure is introduced in the solving of a dynamic portfolio management problem. The new measure provides distributions that are left skewed in contrast to the right skewed distributions previously found. This paper builds on these findings by testing the effect of increasing the client's target and introducing the manager's preferences. It is found that increasing the target causes the distribution to become less left skewed, causing higher probabilities of loss. The pension fund manager considering his own payoff does not significantly affect the results and in some cases improves them.

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1 Introduction

In a recent paper Krawczyk (2008) a 'cautious-relaxed' performance measure is introduced to replace the classical risk-averse utility function usually utilised in a portfolio management problem. The problem solved in Krawczyk (2008) and this paper concerns a pension fund manager deciding how to allocate funds provided to him by a client. Using this new 'cautious-relaxed' performance measure generates left skewed payoff distributions as opposed to the right skewed payoffs that had previously been found. The new utility function involves setting a target for the fund to reach and causes the manager to take on a strategy where more is allocated to the risky asset when the fund starts to perform poorly. The aims of this paper is to see how the dynamics of this problem change when the size of the target is varied and when the manager considers his own preferences, with the goal of analysing the effect that this has on fund distributions.

The hypothetical scenario involves an initial payment x_0 made by a client to a pension fund manager. The manager has the choice between two assets, one being risk-free and the other a more volatile risky asset with a higher expected payoff. The money is then allocated between the two assets as decided on by the manager. Allocation of these assets is decided on through the maximisation of a given utility function. The classical measure to be utilised is one that is mentioned in papers such as Merton (1971) where a concave risk averse utility function is maximised. In Krawczyk (2008) a function that is punishing of any value below a set target and rewarding of any amount above this target is introduced. The level of punishment for under-performance is greater than the utility gained by managers who surpass the target. This means that with certain parameters, if the target is within reach the manager will allocate all funds to the risk-free asset. In Krawczyk (2008) only one target is dealt with and no research is performed on the effect of increasing or decreasing the target, a shortfall that is to be dealt with in this paper.

This paper continues by introducing a model that incorporates the preferences of the pension fund manager. Prior to this, the assumption is made that the manager is willing to allocate the portfolio subject only to the customer's preferences. This essentially introduces a principal-agent dynamic to the problem. In this section the portfolio is allocated subject to a weighted average of both the manager's and the client's utility measure. It is found that the effect this has on fund distributions is not overly significant and in some cases causes an improvement.

The decision of how to allocate a portfolio is one that has been solved on countless occasions with the most prominent paper being Merton (1971). Here Merton built a continuous version of a model proposed in Samuelson (1969). These were the first pioneering papers in terms of dynamic portfolio management. This is different to papers such as Markowitz (1952) which focus on a static version of the problem as opposed to a dynamic one. These papers focused on utility functions that are HARA[1] as opposed to the target seeking type seen in Krawczyk (2008). Utility measures that are unsymmetrical are also seen in prospect theory, in papers such as Tversky and Kahneman (1992) and Berkelaar et al. (2004), where a target seeking agent also has a kinked function.

In Pliska (1986) an optimal portfolio is chosen by modeling security prices as semi-martingales. In Boda et al. (2004) Markov decision processes are used to maximise the probability that wealth exceeds a certain target. In Cairns (2000) a contribution rate is incorporated into the problem meaning rather than allocating a set amount paid at the beginning of the investment period the agent continuously contributes to the fund. This is an idea that in the future could be incorporated into the setup used in this paper. There have been countless other methods used at solving such dynamic problems. These include using VaR and CVaR as constraints as in Yiu (2004) and Bogentoft et al. (2001), but the strategies found in these models still generate right skewed distributions, which are deemed not preferable by pension fund optimisers.

The paper proceeds as follows. Section 2 and 3 introduce the set up for the problem and the methods used to solve it. In Section 4 the effect of varying of the client's target is first analysed. Section 5 deals with the manager's incentives and assesses the resulting distributions when the manager considers his or her own payoff. Section 6 finishes with some concluding remarks.

2 Setting up the Problem

2.1 The model

Consider an agent with savings of x_0 that will not be required for a period of T years(for example he/she may be retiring in T years). The agent hires a pension fund manager to invest this money for the T year period, which is then used to fund a portfolio decided on by the manager. The investment strategy is chosen in order to maximise a certain objective function that could be decided on by the manager after consultation with the agent. The manager's portfolio consists of two assets one being risky and the other risk-free, as is commonly seen in the literature. It is assumed that there are no transaction fees, meaning that rebalancing the portfolio has zero cost. The price of the risky asset $p(t)$ follows a geometric Brownian motion

[1]Hyperbolic absolute risk aversion

as follows:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dw \tag{1}$$

where dw is a standard Brownian motion. The α term is a constant and represents the drift of the asset, higher levels of this represent higher expected returns for the asset. The σ term is also a constant and represents the volatility of the asset. The risk-free asset's price $q(t)$ follows the process:

$$dq(t) = rq(t)dt \tag{2}$$

where $r < \alpha$ and $r, \alpha, \sigma > 0$. At any point in time $u(t)$ is the proportion of wealth invested in the risky asset, and wealth $x(t)$ follows the process:

$$dx(t) = (1 - u(t))rx(t)dt + u(t)x(t)(\alpha dt + \sigma dw) - U(t)dt \tag{3}$$

where U(t) is the agent's consumption rate.

The manager needs to choose a strategy such that he/she maximises discounted expected utility[2]. This could mean that (4) is maximised:

$$J(x_0, u) = \mathbb{E}\left[\int_0^T e^{-\rho t} g(U(t))dt | x(0) = x_0\right] \tag{4}$$

where $g(U(t))$ is the client's instantaneous utility function, ρ is the discount rate and $\mathbb{E}$ is a mathematical expectation. Now (4) can be augmented as shown in Krawczyk (2008) to become:

$$J(x_0, u) = \mathbb{E}\left[\int_0^T e^{-\rho t} g(U(t))dt + e^{-\rho T} s(x(T)) | x(0) = x_0\right] \tag{5}$$

where $s(x(T))$ is the final payoff function. For this study, $U(t) = 0$ meaning is is assumed that the agent will not withdraw any money from the fund until time T [3]. Therefore (5) becomes:

$$J(x_0, u) = \mathbb{E}[e^{-\rho T} s(x(T)) | x(0) = x_0] \tag{6}$$

Therefore the problem to be solved is to maximise (6) subject to (3) and $x(t) > 0$ and $0 \leq u(t) \leq 1$ [4].

[2]Note that for the sake of this study, as is common in the literature, transaction costs are ignored

[3]$U(t)$ can also be made positive without any effect on the manager's strategy

[4]This inequality forbids the act of short selling.

2.2 A New Measure

The choice of $s(x(T))$ is now the issue of concern. Note that as $e^{-\rho T}$ is a constant it can be omitted without affecting the result. The utility measure used in Merton (1971) is concave as follows:

$$s(x(T)) = \frac{1}{\delta}[x(T)]^{\delta}, 0 < \delta < 1 \tag{7}$$

Analytically the solution to this problem can easily be found as in Fleming and Rishel (1975) to be

$$u(t) = \frac{(\alpha - r)}{(1-\delta)\sigma^2}. \tag{8}$$

This implies that the optimal strategy for the fund manager is static, it does not change with time and wealth. The same proportion of wealth is always invested into the risky and risk-free asset. This seems to be quite unintuitive in that you would expect a manager to change his strategy in order to adjust to the performance of the portfolio. For the remainder of this paper this manager will be referred to as the Merton manager.

In Krawczyk (2008) a different utility measure is introduced in which the manager sets a target level for the wealth at the end of the time period. If the wealth is below the target it is heavily punished and any performance above it is moderately rewarded. This is given using the following function:

$$s(x(T)) = \begin{cases} (x(T) - x_T)^{\kappa} & \text{if } x(T) \geq x_T, \\ -(x_T - x(t))^{a} & \text{otherwise} \end{cases} \quad 0 < \kappa < 1, a > 1. \tag{9}$$

Here x_T is a target set prior to the investment period, and the values of κ and α are set according to the level of reward/punishment for under/over performance. Note that the measure is concave both above and below the target. This is different to the prospect theoretic utility measure mentioned in Krawczyk et al. (2011) in that this is convex below the target. The target, x_T is selected so that the following inequalities are satisfied:

$$0 \leq x_0 < x_0 e^{(r-\epsilon)T} < x_T \tag{10}$$

where ϵ is the management fee. This fee is assumed to be charged at a continuous rate by the manager. Note that the inclusion of this fee changes the dynamics of the problem, in that (3) is now altered to become

$$dx(t) = (1 - u(t))(r - \epsilon)x(t)dt + u(t)x(t)((\alpha - \epsilon)dt + \sigma dw). \tag{11}$$

The inequalities are currently set as a matter of common sense, it seems unrealistic that a manager would set himself a target that is below the guaranteed level of wealth. The effect of relaxing these inequalities will be investigated later in this paper.

In Krawczyk (2008) the parameters set for the study are such that the target will never be exceeded, as all wealth is allocated to the risk-free asset when the target is guaranteed without the risky asset. This means once $x(t) = x_T e^{-(T-t)(r-\epsilon)}$, implying that the target can be reached with certainty, all wealth is allocated to the risk-free asset. This idea is supported by findings in Samuelson (1974), with regards to the favourable bet theorem, in which when dealing with a kinked utility function a bet has to be considerably favourable for it not to be avoided. In this context the bet is the risky asset, and with the assumed parameters the return may not be sufficient enough to draw an investor guaranteed to make the target, away from the risk-free asset. A sensitivity analysis is performed in Krawczyk (2008) that shows varying the level of κ, a and σ can cause $u(t)$ to never reach zero. In other words if the risky asset were to be less volatile or the measure to be less punishing of loss, the manager is willing to risk not achieving the target for the chance of surpassing it.

2.3 Comparing Fund Distributions

The final fund values for 5000 runs of each investment strategy is simulated using Monte-Carlo simulation making it possible to make a comparison between investment methods. Given two alternative distributions, assessing which one is the stronger performer can be quite difficult and depends on preferences. It is tempting to only look at a distribution's mean when comparing it with others but this can be very deceiving especially when discussing one's savings for retirement. In Krawczyk (2008) it is shown that simulating the strategies for the Merton manager provides payoffs that are right skewed, but with a relatively high mean. However, simulating the strategy provided by the cautious-relaxed utility measure gives a left skewed payoff distribution, but with a lower mean.

These results imply that the Merton manager's mean could be driven by very high outliars and not incorporate a high probability of loss. Because of this, characteristics such as the median, the standard deviation and the probability of loss are also important to consider. These statistics will give a better idea of what is causing the mean to be at the level that it is. This is where the cautious-relaxed manager in Krawczyk (2008) shows some favourable characteristics as the median is higher, the standard deviation lower and probability of loss is smaller.

The Value at Risk (VaR) and Conditional Value at Risk (CVaR) will also be taken into account as measures of success for each distribution. $_\beta$VaR for a given probability level β is defined to be the lowest level of loss such that the probability of making a larger loss is β. In this scenario loss will be defined as any pension value below the initial outlay. $_\beta$CVaR is the the mean level of loss above the $_\beta$VaR. See Krawczyk (2008) for more information on these measures. After looking at the characteristics for each distribution, a discussion of a pension fund manager's motivations can take place with the goal of analysing the effects of varying the target.

2.4 Parameters

The distributions are very dependent on parameters chosen for the model. To begin with, this paper will be similar to Krawczyk et al. (2011) in its choice of parameters. Table 1 shows the parameters that are to be utilised for this study. The period considered is ten years and the original amount provided to the manager by the customer is \$40,000.

r	α	σ	ϵ	x_T	κ	a
0.05	0.085	0.20	.005	100,000	0.88	1.5

Table 1: Chosen Parameters

With these parameters the fund will accumulate to $40000e^{(0.05-.005)10} = 62,732.49$ if all money is invested in the risk-free asset. From this point on this figure will be referred to as the guaranteed amount. It should be expected that a pension fund would regularly surpass this value. The target of 100,000 was decided on because if the volatility from the risky asset is eliminated and only the drift remained then \$40,000 would become $e^{(.085-.005)10} = 89,021.63$, which is relatively close to \$100,000.

3 Computation of Strategies and distributions

3.1 SOCSol

The investment strategies are generated using a program called SOCSol, developed in Matlab and introduced in Azzato and Krawczyk (2008a). The program discretises optimal control problems and solves them using Markov Chains. See Azzato and Krawczyk (2008a), Windsor and Krawczyk (1997) and Azzato and Krawczyk (2008b) for more details on SOCSol.

3.2 The Merton Manager

Plugging in the values given in the above table to the Merton solution given in Fleming and Rishel (1975) shown in (8) and using a δ of 0.05 gives an optimal $u(t)$ of 0.921. This means that 92.1% of wealth at any time is invested in the risky asset. This investment strategy provides the distribution of payoffs shown in Figure 1. The distribution is generated using Monte-Carlo simulation. The payoff is right skewed with a relatively high probability of either being below the original investment or below the guaranteed amount.

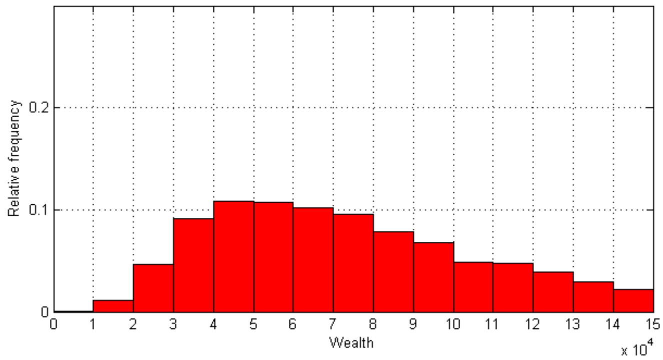

Figure 1: Payoff distribution for Merton's Manager

Figure 1 shows that there is a good probability of having a high level of wealth at the end of the 10 years. However, there is also a high probability of making a loss, the probability of wealth being less than the guaranteed amount is about 40%. Many agents saving for their pension would find this to be an unsatisfactory result.

3.3 The Cautious-Relaxed Manager

Figure 2 shows investment strategies[5] for a manager using a cautious-relaxed utility measure with the parameters given above. Two lines are drawn representing the

[5]These are obtained in SOCSol using 50 as the discretisation of wealth and a time step of .01 of a year. The upper bound for wealth is set at 1,000,000.

investment strategies at time 0 and time 6. This could be done for any time on the discrete grid. The y-axis is $u(t)$, the proportion to be invested into the risky asset and the x-axis is the level of wealth or $x(t)$.

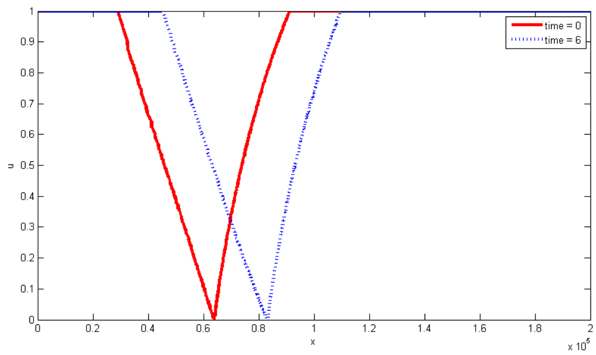

Figure 2: Control rule for target of 100,000

A strategy for any level of wealth is provided by SOCSol but it is important to note that some levels of $x(t)$ will never be reached. For example at time 0 if $x(0) = 63,762.82$, $u(t) = 0$ and this would be maintained for the entire investment period. Therefore at any time period when x_T is obtainable without the risky asset all funds would be allocated to the risk-free asset, meaning any $x(t)$ to the right of the x-intercept will never be reached at the corresponding time. Investment strategies above this level are arbitrarily given by the solution method and illustrate the manager's strategy if somehow the fund value was over this level. The strategies given by this are dynamic in that they change with time and wealth, this is different to the static strategy seen in the Merton model.

Despite this not being the norm in the literature, it seems intuitive that a manager would adjust his allocations with performance. For example if a manager's fund is performing poorly, one would expect that a manager would be well advised to shift his investment towards more risky investments in order to at least have the chance of recouping losses. Figure 3 shows the payoff distributions for a manager using a cautious-relaxed utility function with a target of \$100,000.

This distribution could be considered favourable to the Merton manager's in that it is right skewed and the probability of loss seems to be much lower, which many investors would prefer. Further discussion of these properties will take place in Section 4.2.

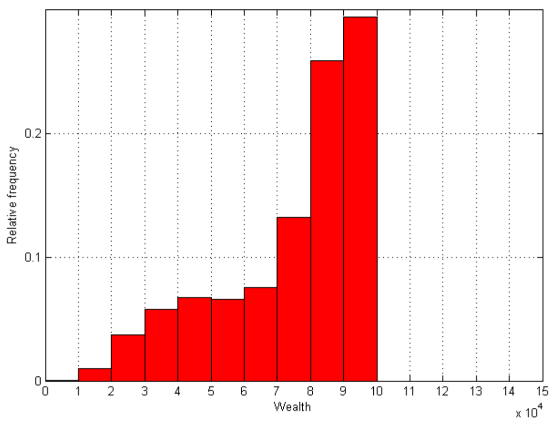

Figure 3: Payoff Distribution for Target of 100,000

4 Varying the target

4.1 Strategies

The question remains now what is the best target for the manager to set, in order to get the most preferable payoff distribution. Firstly, a definition of what pension optimisers regard as 'optimal' is required and obviously will depend on preferences, which makes this a very difficult definition to make. Some investors will be willing to take the risk of receiving a low payoff for the chance of receiving a high one as shown in the Merton payoff distributions. Potentially, setting a high target would give similar distributions to that given in the classical model and could provide a better strategy for fund manager's investments.

In Figure 4 the investment strategies for managers setting targets of \$80,000, \$120,000, \$140,000 and \$160,000 are shown respectively. These appear to be similar to those shown in Figure 2 but are shifted left for lower and right for higher targets. This being due to the fact that a higher target requires more risk and therefore at any point in time a higher fund value is required in order to reach the target. Varying the target does not only shift the manager's strategy, the slope of the rules appear to become flatter with higher targets and steepens with lower targets. This means that an increase in wealth has a more significant impact on the investment strategy for a lower target than one that is greater. This is due to the fact that obviously more investment is required in the risky asset to guarantee a higher target.

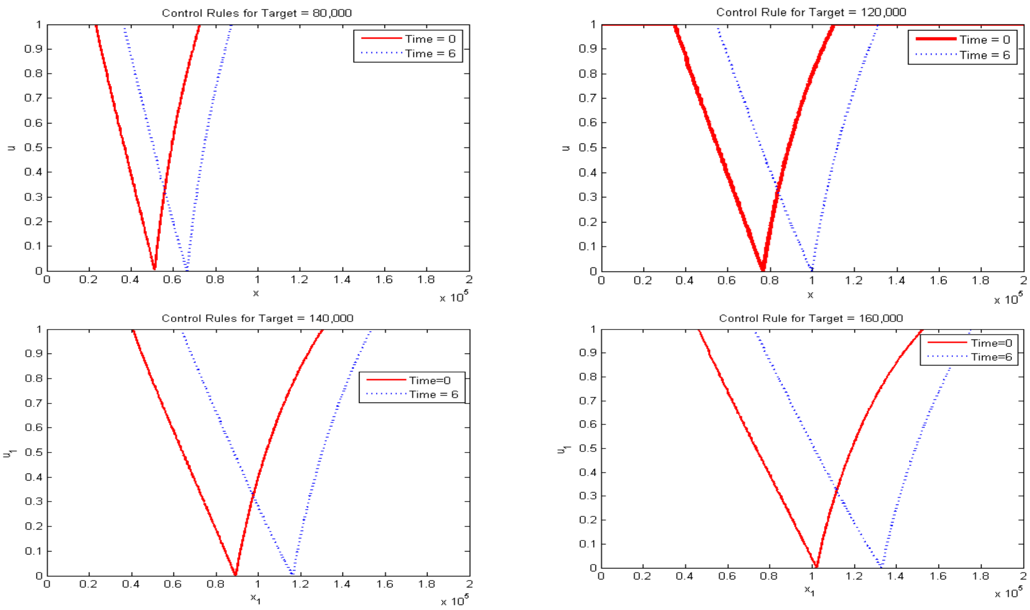

Figure 4: Portfolio management strategies for times 0 and 6.

4.2 Distributions

Again with Monte-Carlo simulation the distribution of payoffs generated by these strategies can be produced, giving the distributions shown in Figure 5.

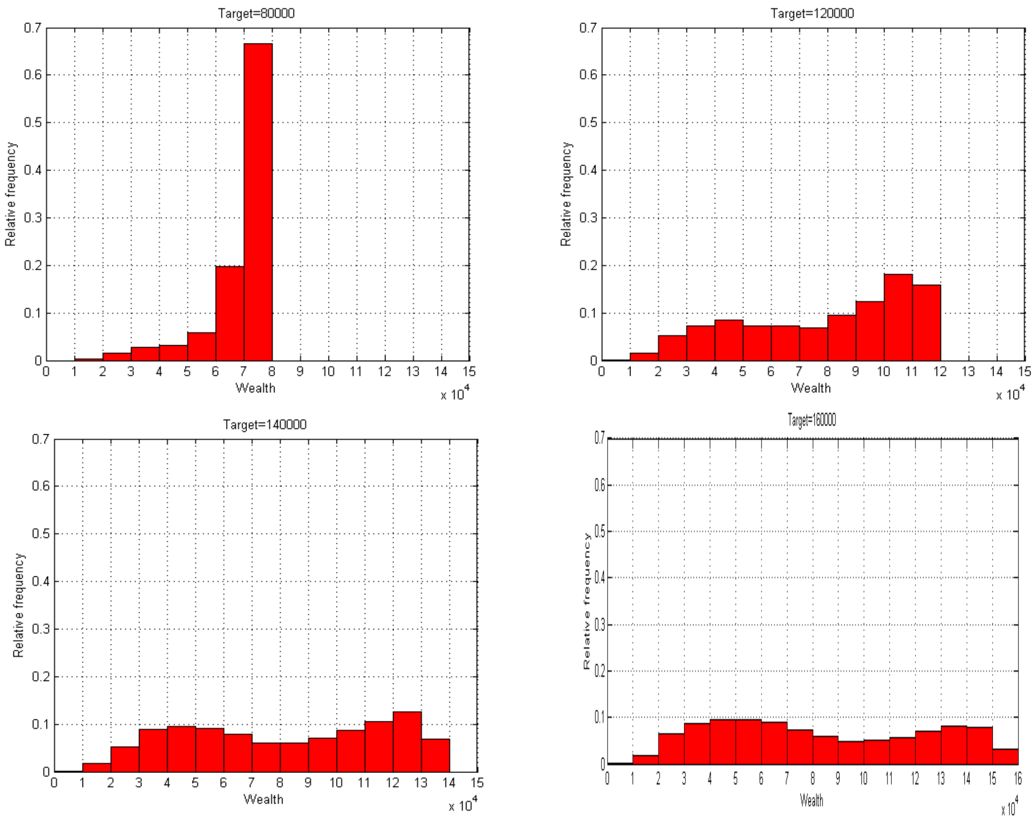

Figure 5: Portfolio payoff distributions at time 10

Table 2 shows different measures of 'success' for the different strategies. The best value under each criteria is in bold font.

This table shows some interesting results. Firstly the mean always increases with the target but at a decreasing rate. The mean for a target of \$160,000 is \$83,310 which is getting relatively close to the mean of the Merton manager \$86,308. The median is highest for a target of \$120,000 and it begins to decrease after this, all of the targets have higher medians than the Merton manager, which indicates left skewness. The 40th percentile [6] peaks with the \$100,000 manager and begins to drop as the target gets higher. The standard deviation of payoffs

[6] The 40th percentile is the value at which 40% of fund values fall below.

Target	Merton	80,000	100,000	120,000	140,000	160,000
Mean of $x(10)$	**86308**	69434	75228	78777	82081	83310
Median of $x(10)$	73207	73618	83343	**87306**	82389	77190
40-th percentile of $x(10)$	63092	71700	**78119**	74131	68088	64065
Std. dev. of $x(10)$	53737	**11231**	21331	29408	35517	40539
Coeff. of skew. of $x(10)$	1.8789	**-2.1697**	-1.0518	-0.4942	-0.1180	0.1696
5% VaR	11304.78	**-2525.47**	9760.09	13121.55	13806.04	14248.73
5% CVaR	17074.97	**8768.84**	16146.53	18819.93	19205.84	19851.18
10% VaR	5101.34	**-15305.95**	-328.77	6067.88	6062.49	8302.43
10% CVaR	12531.69	**-3040.65**	10496.73	10742.56	14462.86	15538.68
P($x(10) > .95x_T$)	N.A.	**0.3072**	0.1240	0.0572	0.0296	0.0186
P($x(10) >$ \$75000)	0.4848	0.3968	**0.6436**	0.595	0.5460	0.5182
P($x(10) <$ \$62,732)	0.3964	**0.1742**	0.2482	0.3242	0.3556	0.3876
P($x(10) <$ \$40,000)	0.1494	**0.0414**	0.0984	0.1456	0.1530	0.1706
P($x(10) <$ \$20,000)	0.0116	**0.0034**	0.0108	0.0164	0.0170	0.0194

Table 2: Final fund return distribution statistics. Highest ranked statistics are in bold font.

increases with the target implying that the higher the goal the more volatile the distribution is. This is very much an expected result because the higher the target the higher the proportion allocated to the more volatile risky asset. However, the highest standard deviation of \$40,539 pales in comparison to the standard deviation of \$53,737 belonging to the Merton manager.

The skewness coefficient[7] is highly negative for lower targets, and increases with higher targets, eventually becoming positive for a target of \$160,000. This means the favourable property of left skewness begins to fade and eventually the distribution becomes right skewed, similar to the Merton manager. The VaR and CVaR for both 0.05 and 0.10 at first tends to increase significantly with the target but then appear to plateau for targets above \$120,000. It is interesting to see that both the VaR and CVaR for targets above \$100,000 are higher than that of the Merton manager, and that the 10% VaR for the \$140,000 target is lower that the \$120,000 target. In fact, it seems as if VaR and CVaR are measures in which the cautious-relaxed measure does not perform that well, especially with higher targets. This could be due to the fact that when these levels of loss are being made, the cautious-relaxed manager is investing all funds in the risky asset which

[7]The skewness coefficient is calculated as $E[\frac{(X-\mu)}{\sigma}^3]$ where μ is the mean and σ is standard deviation. It provides a measure of asymmetry in the distribution.

is putting more money at risk.

The probability of being within 5% of the target decreases as the target increases as one would expect. An interesting thing to note is that the Merton manager has a higher probability of having a fund value below \$62,732 but he has a better probability of providing a fund value of above \$40,000 than both the \$140,000 and \$160,000 targets. This could be due to the fact that the target seeking managers, upon reaching the point where they are near \$40,000 are so far away from their target that they allocate all funds to the risky asset, which can be detrimental to their fund value.

In many ways the manager targeting \$160,000 is very similar to the Merton manager in the characteristics of their distributions. The preferable properties of the manager with a \$100,000 target's distribution begin to get lost with these higher targets and the line between them and the Merton manager begins to blur. However it is important to consider the fact that the target-seeking managers statistics are affected a lot less by outliars. A strong performing risky asset can cause the value of the Merton fund to tend towards extremely high values causing big increases in the mean. This is not the case with the target seeking managers as once these funds are within their target they are not allowed to increase in value to the same extent.

4.3 Which to Choose?

In order to make a fair comparison of these strategies it is important to look at the different motivations of clients when deciding on the manager's strategy. When trying to advertise for new customers, the manager would have to decide which of these strategies would appeal most to a prospective customer and would depend on preferences. If the manager could show all the different distributions to the client, the question is whether or not the client would prefer to see the left-skewed distributions of the cautious-relaxed manager or the right-skewed distributions produced by the Merton manager. A client saving for his/her retirement would most likely find the high probability of making a loss in Merton's method unacceptable and tend towards the more right-skewed distributions of the lower targeting managers. An agent willing to take on a more risky investment could also tend towards the higher targets over the Merton manager's strategy because the volatility of these managers is still considerably lower. This is despite the fact that the means are still lower than those of the Merton manager, the medians are much more favourable.

Looking at Table 2, \$80,000 has the highest number of criteria in bold font.

This is mainly due to the fact that the target is relatively unambitious, and when it comes to probability of loss this target performs very well. However, in terms of mean and median this target performs the least favourably. This could potentially lead one to select \$100,000 as a target, where the 40th percentile and the probability of being greater than \$75,000 are in bold. This has come at the cost of slightly higher probabilities of loss, but with a higher mean and median than the \$80,000 target. Further increases in the targets cause smaller changes in these statistics, therefore \$100,000 seems to be the most suitable target.

Figure 6 shows the PDFs and CDFs of the different managers. These are generated using the Parzen–Rosenblatt window method, which is a convenient way of smoothing the data and is often considered an improvement on the jaggedness of the histograms shown in Figures 5 and 3 .

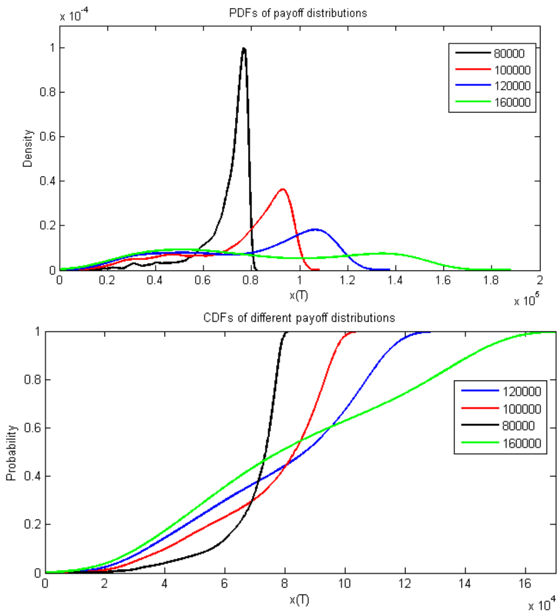

Figure 6: PDFs and CDFs for final distributions.

As the target increases the PDFs begin to flatten out and become less peaked. An interesting feature in the PDFs is the significant drop in the peak from the 80,000 target line to the 100,000 line, showing how much more likely achieving a target close to 80,000 is than the other targets. The CDFs act as would be expected, also becoming less and less steep as the target increases.

4.4 A Lower Target

An interesting extension would be loosen the inequalities set in (10) to check the effect of setting the target below the guaranteed amount. This would imply that the manager starts investing to the right of the x-intercept at the beginning of the period. This is because the target is guaranteed at this point, meaning that some of the wealth can be risked by investing in the risky asset. If the risky asset performs poorly, wealth is allocated back towards the risk-free asset. However, if the risky asset performs a larger proportion of funds are allocated to it. This means that the client is effectively setting a minimum amount that they will allow the value of their investment to fall too. For this example the target will be set to \$60,000 and \$50,000 which is below the guaranteed amount of \$62,732. The investment strategies and fund distributions are shown in Figure 7.

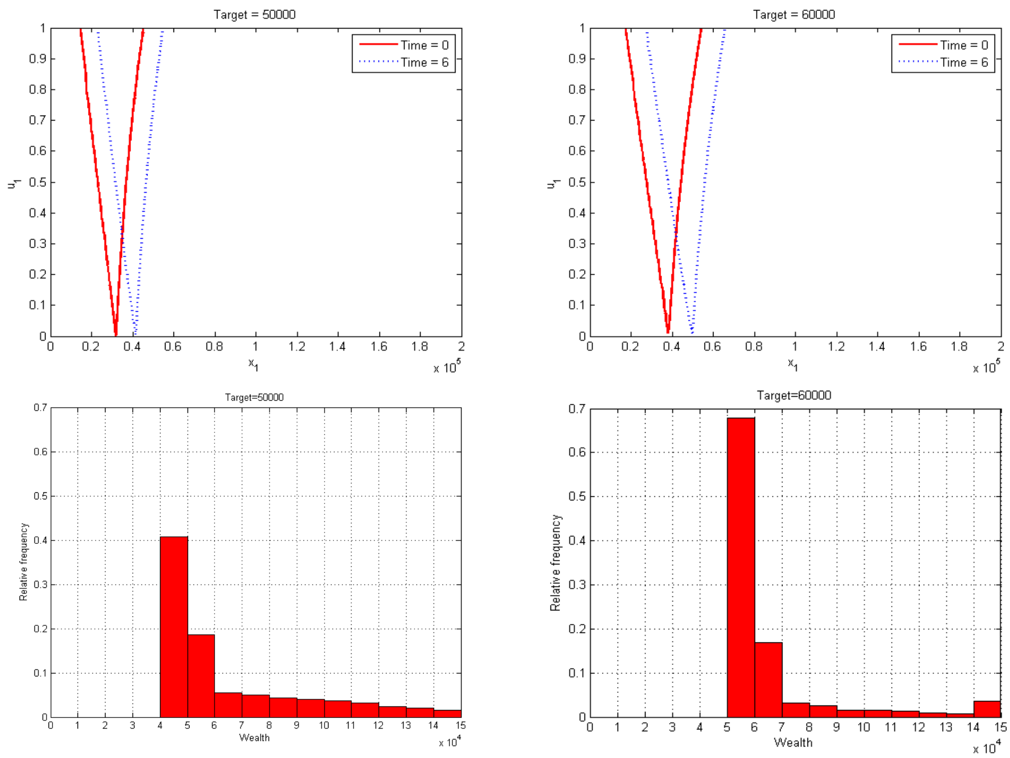

Figure 7: Histogram and control rules for targets of 50,000 and 60,000

This again gives right skewed distributions, but unlike Merton's model the probability of being below the original outlay of \$40,000 is zero. Some pension fund customers could be drawn to this distribution, because of the fact there is a probability of some higher payoffs without the risk of significant loss shown in the other distributions. Note that this changes the dynamics of the manager's behaviour from one that is 'loss avoiding' to one in which he is setting a minimum value for the fund to fall too. The performance of this manager is still not competitive with the others in that the moment the portfolio begins to perform poorly all wealth is allocated to the risk-free asset and kept there for the remainder of the investment period. This is shown in the histograms by high frequencies of values at or close to the target. Like the Merton manager the performance of this fund is driven by outliars, which can provide misleading statistics.

5 The Manager's Incentives

In the previous sections it is assumed that the manager will be willing to follow the choices of the loss avoiding client, with no concern for the effect that this could have on their own payoff. It is possible that the manager will not be satisfied with the customer's decision and be tempted to deviate from it.

5.1 Introducing the Manager's Utility

In preceding chapters the expected value of the final payoff function has been maximised without consideration of the manager's motivations. The effect of the manager deciding to shy away from the preferences of the customer towards his or her own payoffs is to be tested. This will be completed with the intention of testing the consequences that this may have on the investment strategy and distribution of the fund. It will be assumed that the manager will now maximise a weighted average of his own and the customer's utility function. This would mean that the following equation would be maximised instead of (5):

$$J(x_0, u) = \mathbb{E}\left[(1-\nu)\int_0^T e^{-\rho t} f(x_m(t))dt + \nu e^{-\rho t} s(x(T)) | x(0) = x_0\right] \tag{12}$$

where ν is the amount of weight the manager places on the customer's preferences and $f(x_m(t))$ is the manager's instantaneous utility function with regards to the manager's revenue $x_m(t)$. This implies that the closer ν is to 0 the more the manager is thinking about his own payoff in the present. It will be assumed that the manager is risk neutral. This choice is made because the management fee is relatively low in comparison with the actual fund value, having the manager risk averse will cause the value of the manager's utility function to be dwarfed by the measure of the customer. Therefore it will be assumed that the the manager's instantaneous utility function is

$$f(x_m(t)) = x_m(t) = 0.005x(t). \tag{13}$$

Note that the discounting term has been added back into the customers utility measure in (12) because now the manager's utility is being discounted, so too must the customers in order to be consistent. The problem to be solved now is to maximise (12) subject to (3) and other constraints mentioned in Section 2.1. The target is to be fixed at \$100,000. Values utilised for ν start at 0.5 and are iteratively decreased from there. The first value for ν that has any visible effect on the investment strategy of the manager is 0.0001. This result is caused by the fact that the management fee is so small in comparison with the fund value, but this level of weighting is not as infeasible as it first sounds. The reason for this is that it is unlikely the manager will be investing the funds of one customer, the fund will be managing the money of multiple clients. This means that while it is difficult to envision the manager placing a weight so low on one customer's payoffs, if the fund had 100 clients this weighting would become $100\nu = 0.01$. The investment strategies for $\nu = 0.0001, 0.00001$ is shown in Figure 8.

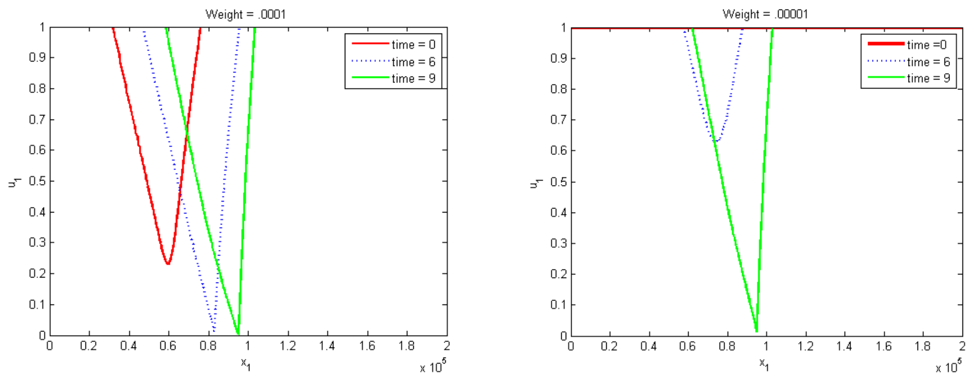

Figure 8: Control rules for managers with $\nu = .0001, .00001$

The manager's strategy when ν is set to 0 is to allocate all money to the risky asset, i.e. set $u(t) = 1$, this is due to the assumption of risk neutrality. These control rules show that the manager's strategy tends away from those shown in Section 3 towards this more aggressive one, the higher the level of ν is set. The change to the strategy at time 0 occurs first as at this point the manager's payoff is discounted the least, and therefore the reward is higher to the manager. A strategy for time 9 is included in Figure 8 because it better illustrates the change in strategy over time. In this sense the problem has changed in that the timing of wealth gains is now important to the manager and his tendency is to be more aggressive at times closer to 0 than at times further away. A good example of this is that the shape of the strategy at time 9 does not change at all with the increase in ν. This is in contradiction to what is seen in earlier sections in that the client's preference is to be more aggressive in later time periods than earlier ones. Here, these two are effectively counteracting each other. The manager with a weighting of $\nu = .00001$'s strategy appears to be very similar to that of a prospect theoretic manager in Krawczyk et al. (2011) in that at time 0, $u(t) = 0$ and at time 6 the strategy loops down in a similar fashion. The measure used for that manager is one that is also target seeking, but where the penalty for failing to reach the target is not as harsh as the cautious relaxed one used in this paper.

5.2 A Better Distribution?

The next question to ask is what effect do these new strategies have on the distributions of the fund? Figure 9 shows the histograms of the customer's final payoffs for the two strategies alongside the distribution obtained previously in Figure 3. The summary statistics for these managers as well as a manager with $\nu = 1$ are shown in Table 3.

The distribution when ν is .0001 appears to be superior to the unselfish manager's in that the probability of being within the 90,000-100,000 bin is higher without significantly affecting the probabilities in the lower bins. When ν is equal to .00001 the histogram has the familiar left skewed shape but also has reached values above the target.

Table 3 shows two of the better average fund values obtained so far. The mean of the manager with a $\nu = .00001$ is \$86,252, which is very similar to that of the Merton manager but with a higher median and a 5% smaller chance of making a loss. However, when compared with the unselfish \$100,000 targeting manager the standard deviation is significantly higher as too is the probability of loss. The manager with $\nu = .0001$ has a mean better than the \$100,000 targeting manager but this is counteracted with a slightly more volatile distribution and higher chance of loss. The probability of being within 5% of the target of \$100,000 is 5% better than previously. Note that the probability of making over \$75,000 has decreased and the probability of making under the secure amount has increased. This shows as expected a risk neutral manager who starts to think of his own payoffs over that of his client will cause the mean of distribution of the fund to to increase at the cost of a higher level of volatility. These distributions still show some strong characteristics, however, and most likely the customer will not be able to spot the manager's deviation unless he or she is able to see the manager's investment

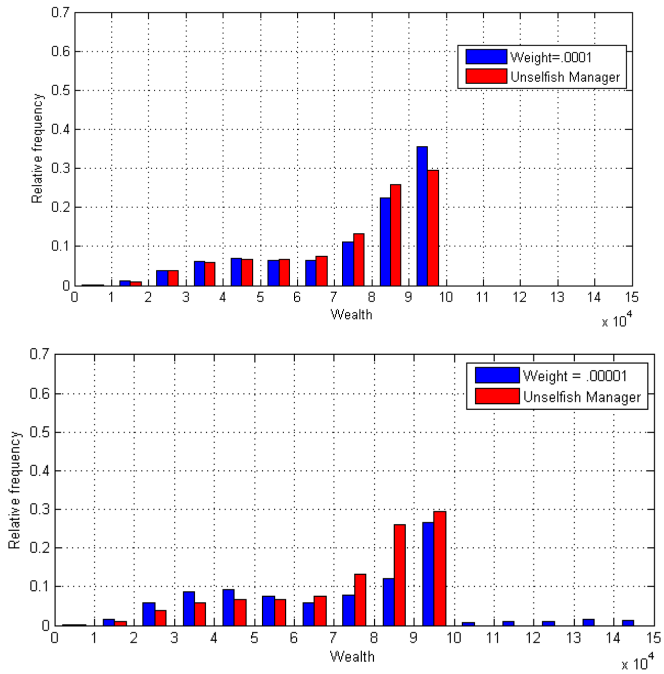

Figure 9: Wealth distributions for managers with $\nu = 0.0001, 0.00001$

strategy.

5.3 Sensitivity Analysis

According to some performance measures and Figure 9 it seems as if the manager with a $\nu = .0001$ (from here on referred to as the selfish manager) has actually improved the client's final payoff distribution by considering his own payoff. The volatility has increased slightly but the mean and probability of achieving the target have also increased. However when considering the probability of making over \$75,000 the selfish manager does not perform as well, but only to a very small extent. To investigate this matter further, in this section a sensitivity analysis will take place. The goal being to vary the volatility parameters and testing the effect that this has on the performance of the two managers.

Firstly to ensure that the manager acting selfishly is in fact not in the clients best interest the mean value of the client's utility will be calculated using $s(x(T))$ from (9). This is important because the original problem maximises the client's expected utility, so the manager acting selfishly should cause a decrease in the

ν	0.0001	0.00001	0	1
Mean of $x(10)$	75449	86252	**88532**	75228
Median of $x(10)$	**84860**	83183	74044	83343
40-th percentile of $x(10)$	**78458**	72003	62475	78119
Std. dev. of $x(10)$	22322	56190	59330	**21331**
Coeff. of skew. of $x(10)$	-1.0000	2.7986	2.1023	**-1.0518**
5% VaR	10094.47	14146.19	13853.81	**9760.09**
5% CVaR	16794.07	19484.15	19583.19	**16146.53**
10% VaR	1559.01	6623.27	7239.17	**-328.77**
10% CVaR	11213.83	14918.14	15099.21	**10496.73**
P($x(10) > .95x_T$)	0.1722	0.3174	**0.345**	0.1240
P($x(10) > 75000$)	0.6426	0.5782	0.4904	**0.6436**
P($x(10) < \$62,732$)	0.2594	0.3450	0.4024	**0.2482**
P($x(10) < \$40,000$)	0.1112	0.1618	0.1690	**0.0984**
P($x(10) < \$20,000$)	0.0118	0.0178	0.0190	**0.0108**

Table 3: Summary statistics for fund values where manager places weighting of $1-\nu$ on his own payoff. Highest ranked statistics are in bold font.

expected value of the measure $s(x(T))$. Theoretically, it should be that the increase in volatility is a price that the customer is not willing to pay for the extra chance of making the target. The unselfish manager provides a mean value for $s(x(T))$ of -4915298 and the manager with $\nu = .0001$ provides a value of -4970002. This result is comforting in that with the utility measure in (9) the unselfish manager does in fact provide a higher mean utility level at time 10.

This could possibly bring into question the quality of the utility measure $s(x(T))$ in that the red bars in the top histogram in Figure 9 are preferred to the blue bars. The selfish manager has a significantly larger mean and probability of making the target, at the cost of only a slightly higher probabilities of loss. Remember that $s(x(T))$ is a function arbitrarily set with the intention of providing better fund distributions. Figure 9 could potentially show that this function is not an ideal measure of a client's preferences.

In order to make a fairer comparison of the two managers, the volatility of the risky asset will be varied. This will show how each manager fares with both a more and a less volatile risky asset. One would expect to see the selfish manager provide a higher probability of loss with a more volatile asset as the strategy is more aggressive. On the other hand with a less volatile asset this manager could potentially provide a better payoff distribution. Figure 10 shows the investment strategies and final payoff distributions for the unselfish manager when that volatility of the risky asset σ is decreased to 0.05. This means that the problem becomes a close approximation to one that is deterministic rather than stochastic. The strategy of the selfish manager is omitted because $u(t) = 1$ for all values of wealth and time.

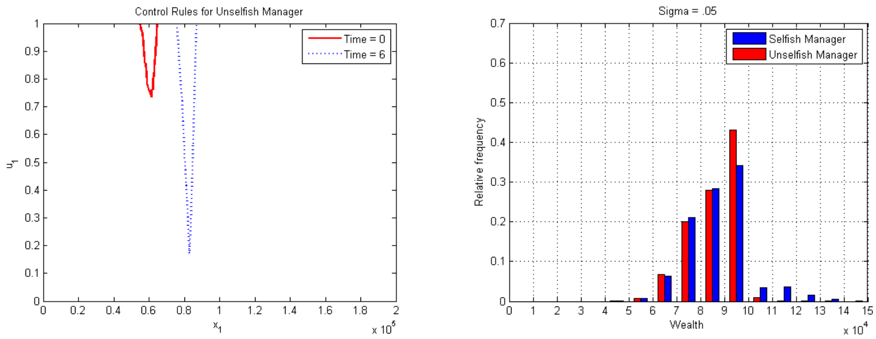

Figure 10: Control rule for unselfish manager and final distributions when $\sigma = 0.05$

Looking at the histogram one could potentially come to the conclusion that the selfish manager does perform better than the unselfish manager in this case. From the graph the only drawback is a slightly higher chance of obtaining a payoff between \$70,000 and \$80,000. Other than this, the graphs look very similar until the fund values are above \$90,000, at this point the selfish manager easily outperforms the unselfish manager. The probability that the client will receive a fund in excess of the target is much higher with a selfish manager. The median and VaRs of the selfish manager are also an improvement on the unselfish manager. The only drawback is that the probability of obtaining over \$75,000 is slightly lower. These deficiencies are quite minimal however, demonstrating that the cautious-relaxed utility measure may not be ideal when dealing with problems that are or close to deterministic. Graphs demonstrating the difference between the two managers at different volatilities are shown in Figure 14.

Due to the discrete nature of SOCSol, it is possible to calculate the fund value at discrete time steps during the investment period. From this an approximate value of each manager's payoff throughout the period can be calculated by summing the revenues at each time step. This will give an idea of the improvement in the manager's payoff from the new strategy. The average selfish manager's payoff for a sigma of .05 is \$2,681 compared to \$2,670 for an unselfish manager. This shows that there has been an increase in revenue from the management fee, but a very minimal one.

A possible method for identifying the differences between two investment strate-

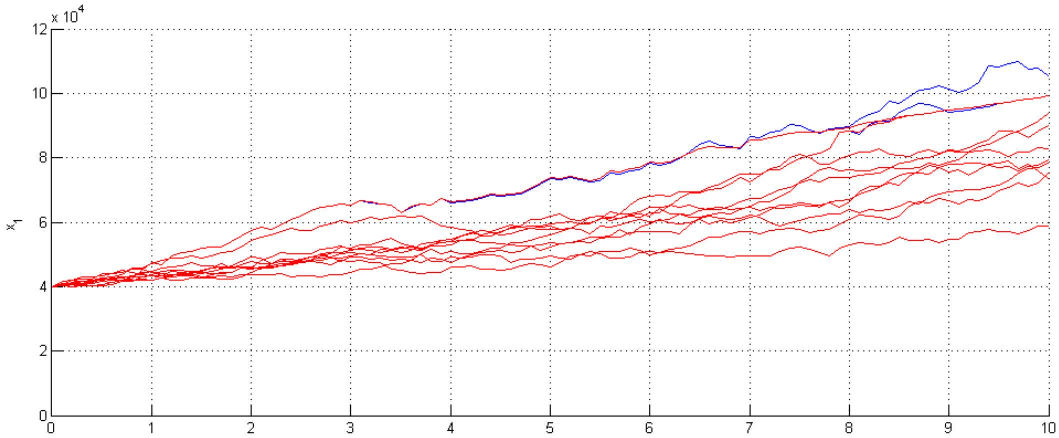

Figure 11: Time profiles for selfish and unselfish manager when $\sigma = 0.05$

gies is plotting the paths or time profiles of both managers' investments. The graphs show how ten selfish and unselfish funds evolve over time, given the investment strategies shown in Figure 10. The results for both managers are simulated using the same random numbers, meaning the behaviour of the risky asset is identical for both of their portfolios. The time profiles when σ is 0.05 are shown in Figure 11. To help illustrate the difference between the two managers the simulations for both selfish and unselfish are shown on the same plot. The selfish manager's are shown in blue and the unselfish manager's are shown in red. Figure 11 shows the similarities between the strategies of the two managers at this level of volatility, in that all but two of the runs show the same fund evolution over time. Only in the case when the risky asset performs very well does the selfish manager start to excel. An interesting point to note is there seems to be little smoothing of the fund value by the selfish manager as one might expect. The manager's payoff is dependent on the fund value throughout the investment period, therefore it would be expected that there would be some incentive to attempt to raise the fund value during the investment period.

Now moving on to a more stochastic scenario the strategies and distributions of the two managers when $\sigma = 0.4$ is shown in Figure 12.

The selfish manager's strategy is slightly more aggressive and as a result shows a better probability of achieving the target. Whether the selfish manager's distribution is superior to that of the unselfish manager is again a matter of preferences. However, the probability of achieving a fund that is within 5% of the target has improved slightly. The probabilities of loss are only slightly higher (less than 1%) for the selfish manager for approximately a 2% increase in the probability of

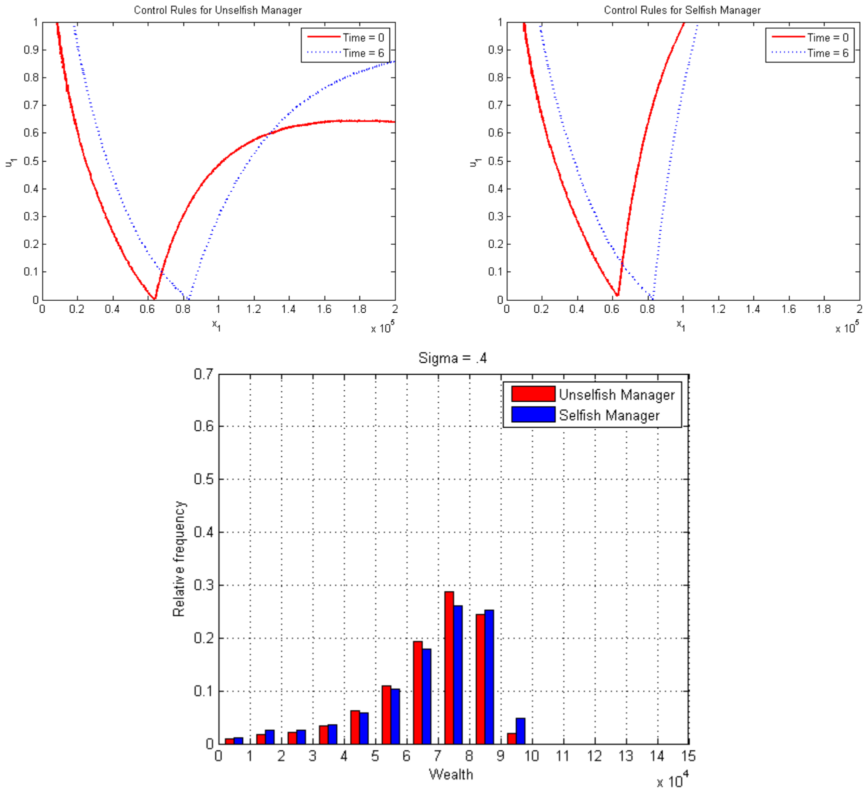

Figure 12: Control rule and final distributions for selfish and unselfish manager when $\sigma = 0.4$

achieving over \$75000. Again, this could lead one to conclude that an improved distribution might be the result of a manager acting selfishly when the client has a cautious-relaxed utility measure. It is very difficult to make such claims because it is highly dependent on the choices of the client, however the manager acting in his own interest has not caused a significant decline in the quality of the distribution. The selfish manager's payoff is now \$3,104 compared to \$3,091 for the unselfish manager. This again shows a very minimal reward for the manager implementing a more aggressive strategy. The time profiles are shown in Figure 13 and comparative statistics are plotted in Figure 14.

In this scenario the difference between the two strategies is a lot easier to see. The increased volatility is much more obvious here in that when the risky asset is performing strongly in the upper section of the plots, the selfish manager starts to perform better. In the lower section of the plot it is the unselfish manager who is more cautious, with the red lines lying above the black. This gives a good

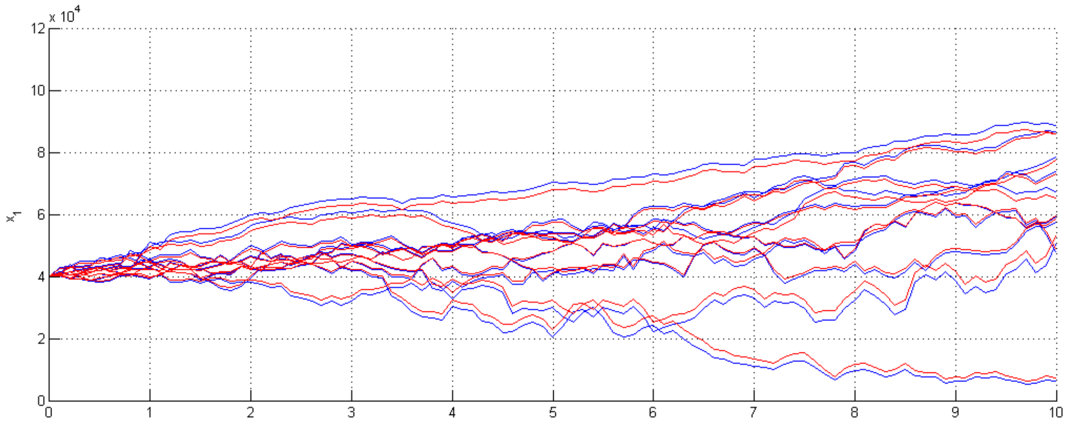

Figure 13: Time profiles for selfish and unselfish manager when $\sigma = 0.4$

illustration of the increased volatility brought out by the fund manager thinking of his own payoffs.

The plots shown in Figure 14 provide a summary of the above findings, with other levels of σ added to provide extra insight. The graphs show that there are not strong relationships between the measures and the two manager's performances. It is rare that one manager clearly outperforms the other in a measure for all levels of volatility. The selfish manager's mean and median is superior for all tested values of sigma .6 and below, above this level the unselfish manager's are better. The utility(the mean value of $s(x(T))$) of the unselfish manager is always higher as expected. The probabilities of loss (i.e. the probability of making less than \$40,000 or \$62,732) seem to be consistently higher for the selfish manager. An interesting characteristic of these plots is that the probability of having a final fund value less than \$40,000 starts to decrease for higher values of σ. This is at first an unexpected result as you would expect with a higher volatility would come a higher probability of loss. However, looking at the investment strategies of both managers with high values of σ the strategies are very cautious. For example the manager's both have approximately 20% invested in the risky asset at time 0 when σ is 0.4. This compared to around 60% for a sigma of 0.2. Finally, the probability of obtaining a fund value over \$75,000 appears better for an unselfish manager with a less volatile asset, but this is reversed when σ is at a higher level.

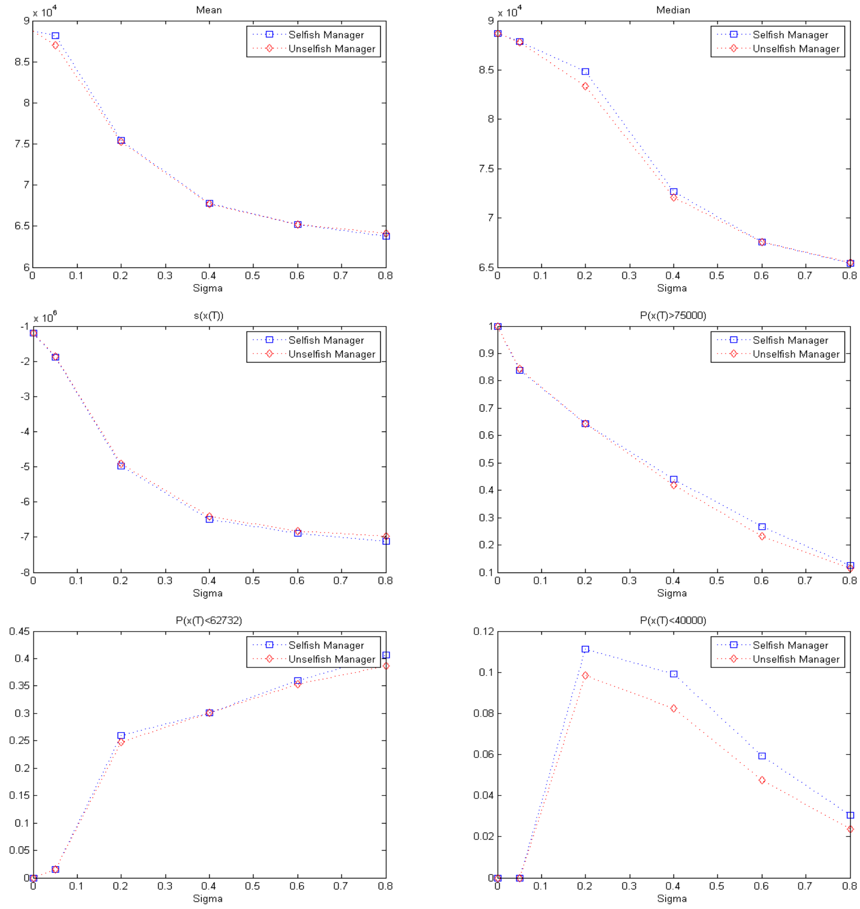

Figure 14: Comparisons of managers' statistics with various volatilities

This analysis shows that a manager acting selfishly on his own behalf does not have large ramifications in terms of the performance of the fund. In some cases the performance of the fund seems to improves slightly. This seems to be especially true when the risky asset is less volatile. In this case perhaps the unselfish manager would be well advised to choose a higher target.

6 Conclusion

This paper performs a further analysis of issues brought up in Krawczyk (2008) involving a cautious-relaxed utility measure used to solve stochastic optimal control problem. Due to the kinked nature of this utility function an analytic solution is not available. To deal with this, a program called SOCSol is used which discretises and solves using Markov chains. In Krawczyk (2008) it is shown that a client with a target seeking utility measure such as the one shown in Section 2.2 can lead to left skewed payoff distributions. This is different to previous findings such as those in Merton (1971), where using risk averse utility measures leads to distributions that are right skewed. The level of robustness of these findings is assessed in this paper. The performance of the measure is analysed when the target is varied, by measuring the quality of the obtained fund distributions. The next section deals with a manager who might be tempted to consider his own payoff against the wishes of the customer.

Increasing the target causes investment strategies to shift outwards and become more flat. Simulations of final fund values shows an increase in the mean and median of the fund value. The cautious-relaxed manager becomes more comparable with the Merton manager in these respects. However, the disadvantages of the Merton manager such as high probabilities of loss and high VaRs also become more evident in these strategies, but to a lesser extent. The preferable left skewness of the distributions also begins to diminish the higher the target becomes. This is expected as the higher the target of the fund, the more risk the manager must take on to achieve it. The effect of setting a target below the secure amount is also assessed, the resulting distributions possess some favourable qualities in that there is no value below the initial outlay. However, this advantage is outweighed by too high a frequency of fund values at or around the target.

Section 4 takes a slightly different tact by attempting to model the incentives of the manager. As the manager is paid continuously throughout the investment period, it is not always true that the manager will be content with the utility measure decided on by the client. It is shown that a risk - neutral manager who puts some weighting on his own payoff introduces some slight improvements in some performance criteria of the distribution. Despite slightly increasing the volatility of the distributions the manager improves the mean and medians of fund distributions for most levels of volatility, without significantly affecting the probability of loss.

A clear weakness of the model in this paper is the lack of transaction costs. As much as the Merton model does not perform as well in a frictionless framework it is likely to have much lower transaction costs. This opposed to the investment strategies discussed in this paper, which will involve constant re-balancing of the portfolio. This would most likely induce very high costs and affect significantly the quality of the final distribution. Incorporating transaction costs could potentially be implemented in this model by introducing a fee that is charged every time $u(t)$ changes size. This would change the form of $dx(t)$ in (3). Another further adjustment that could be made, is to model the risky asset using a jump-diffusion model. This would involve altering SOCSol in order to have the ability to deal with an added Poisson process. Finally, an interesting extension of this model would be to assess the performances of the new investment strategies with actual asset prices. These are left for further research.

This paper has shown that findings in Krawczyk (2008) are robust in that any increases in size of the target still provide distributions that are still arguably better than those found by the manager in Merton (1971). This finding also appears to be true to some extent, when a manager is tempted to deviate towards his own payoff.

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