Motivated Sellers Predatory Buyers

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ABSTRACT: We develop an equilibrium search model of the housing market where sellers may become distressed as they are unable to sell. A unique steady state equilibrium exists where distressed sellers attempt *fire sales* by accepting prices that are substantially below fundamental values. During periods where a large number of sellers are forced to liquidate customers exhibit *predation*: they hold off purchasing and strategically slow down the speed of trade, which in turn causes more sellers to become distressed. From sellers' point of view liquidity disappears when it is needed the most. The model naturally suggests several proxies of liquidity. Interestingly, the expected time on the market, one of the most frequently used statistics in the literature, does a poor job within the context of fire sales and predation.

Keywords: housing, random search, fire sales, predation, liquidity
JEL: D39, D49, D83

1 Introduction

Selling a house involves a long and non-trivial search process where the home seller faces a trade-off between the price and the time to sale. With sufficient time and no pressure to sell immediately, a seller can afford to wait to receive a price commensurate with the market value. However, due to factors such as bankruptcy, job loss, foreclosure, relocation, divorce etc. some sellers become *distressed* and attempt a *fire sale* in an effort to quickly sell and exit the market.

What is more, the presence of distressed sellers seems to affect buyers' purchasing behavior. During the recent housing crisis, for instance, where presumably a large number of sellers became distressed, buyers did exhibit what we call *predatory buying*. Despite the falling prices and the rising number of fire sales customers were reluctant to purchase—appearing to be strategically delaying purchasing in an effort to obtain better deals.

Based on these observations we develop an equilibrium search model of the housing market with two distinctive features. First, buyers' willingness to pay is private information and more importantly, second, sellers may become distressed as they wait to sell. Sellers enter the market in a regular state, or regular mood; though, as they are unable to sell they may be hit by an exogenous shock (job-loss, bankruptcy etc.) and become distressed. Regular and distressed sellers differ from each other in terms of their time preferences: once distressed, a seller discounts future utility more heavily.

What do we find? First, we show that in equilibrium financially distressed sellers indeed pursue fire sales. They accept prices that are substantially below fundamental values and consequently sell faster than regular

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sellers. How low the sale price can go depends on how severe the shock is. We show that the more painful the shock the lower the sale price and the quicker the sale.

A fire sale is costly not only because it is associated with significant price drops, but also it opens the door for predatory buying. We show that during periods where a large number of sellers are likely to encounter financial distress (e.g. a crisis or recession) the followings occur. First, the number of fire sales rises. Second, all sellers, regular and distressed, drop their prices. And most importantly, third, customers exhibit predation: they become more selective and hold off purchasing despite the abundance of distressed sales and lower prices. By doing so customers strategically slow down the speed of trade causing more sellers to become distressed, which in turn, exerts more pressure on sellers forcing them for further price cuts, and so on. From the buyers' point of view such behavior is optimal as it allows them to acquire better houses at lower prices, but from a seller's point of view it is the worst possible outcome. Indeed for distressed sellers liquidity disappears when it is mostly needed.

The model naturally suggests several proxies measuring liquidity from different angles. A first proxy is the expected time on the market TOM, which provides useful information regarding the speed and the volume of trade. A second proxy deals with the loss of profits in fire sales. We construct an index z (see Section 5) that captures a distressed seller's percentage-wise profit loss in a fire sale. Curiously, though, TOM—one of the most frequently used and referred statistics in the literature—does a poor job in the context of fire sales. We show that TOM falls during periods of predatory buying, which, if interpreted on the face value, indicates that the market becomes more liquid and more efficient with predation. In our framework the index z appears to be a more robust proxy of liquidity than TOM is.

Finally, the model provides simple and intuitive answers to two puzzles raised by Merlo and Ortalo-Magné [12]. Based on a unique data set of individual residential property transactions in England, the authors document that about 2/3 of sellers do not change the listing price at all, while remaining sellers revise the listing prices at least once (typically once). The facts that some seller revise the listing price while others do not and that price revisions are infrequent (on average, once in 11 weeks) and sizable (about 10% of the initial listing price) are in stark contrast to the predictions of most existing theories in the housing market. Based on the same data set the authors document a negative correlation between the sale price and the duration of the sale—the longer the time on the market the lower the sale price. This fact, again, is inconsistent with most of the existing theoretical models.

According to our model some sellers revise the listing price while others do not simply because some sellers become distressed while others do not. The ones who become distressed revise their listing price; the revision occurs only once (when the shock hits) and it can be sizeable if the shock is severe—see the simulation in Figure 5c. The negative correlation between the sale price and the duration is also easy to explain. Properties sold soon after the listing date are most likely "regular sales". Sellers of such properties cannot possibly become distressed within a short period of time. Sales taking place long after the listing date are most likely "distressed", because the longer a seller waits the more likely he is to become distressed. Since distressed sales occur at lower prices, the aforementioned negative correlation follows.

This paper belongs to a literature that studies the housing market using search theory, e.g. see Yavas and Yang [17], Krainer [11], Wheaton [15] and Albrecht et al. [1], among others. We differ from the aforementioned papers in that we focus on distressed sales and predation. The paper by Albrecht et al. is perhaps the closest to our model in terms of motivation and setup; however it is based on complete information. This difference is crucial because incomplete information is key in obtaining the predation result.

The paper is organized as follows. Section 2 outlines the model and explains how to solve it; Section 3 discusses fire sales and predation; Section 4 discusses list price trajectories and price dispersion; Section 5 is devoted to liquidity; Section 6 concludes.

2 Model

Time is continuous and infinite. The economy consists of a continuum of risk neutral buyers and sellers. Each seller is endowed with a house and each buyer seeks to purchase one. Buyers and sellers differ in terms of their intrinsic preferences towards ownership of a house, which creates the incentive to trade. For simplicity we assume that the utility to the seller from keeping the house is zero. Buyers on the other hand receive periodic dividends (housing services) starting the period after the purchase of the house and continuing forever. Following the asset pricing interpretation we assume that the value of a house is captured by the discounted sum of the future dividends.

Sellers' personal circumstances may change for worse if they are unable to sell for too long. All sellers enter the market in regular circumstances, though, eventually as they are unable to sell they may be hit by an idiosyncratic shock and become distressed ('motivated' in real estate parlance). The adverse shock arrives at an exogenous Poisson rate $\mu > 0$ and may be associated with financial difficulties forcing sellers into early liquidation, e.g. foreclosure, bankruptcy, divorce, etc. It is sensible to think that μ rises during crises or recessions where sellers are more likely to encounter financial distress. Regular and distressed sellers differ in terms of their time preferences. Buyers and regular sellers discount future utility by $\frac{1}{1+\delta} > 0$ whereas distressed sellers are more impatient and discount the future by $\frac{1}{1+\bar{\delta}} < \frac{1}{1+\delta}$, which means that $\bar{\delta} > \delta$. Sellers do not exit the market until they sell and a distressed seller remains distressed. The parameters of interest are the frequency of the shock, μ, and the severity of the shock, $\bar{\delta}$.

Transactions are bilateral and involve a non-trivial search process. At any point in time buyers and sellers meet each other at a constant Poisson rate $\alpha > 0$.[2] Upon meeting a seller and inspecting the house, a buyer realizes his own valuation of the house $v \in [0, 1]$, which is a random draw from a distribution with cdf $F(v)$. Buyers are identical in the sense that their valuations are generated by the same random process, however they may differ in their valuations for any particular house. This specification captures the notion that different buyers have different tastes and preferences and therefore will have different reservation prices. The realization of $v \in [0, 1]$ is match specific, so when buyers search they in fact search for a high v. We assume that v is time invariant; so, once a buyer finds and purchases a house with a sufficiently high v then he continues to enjoy the same v forever. We impose log-concavity on the survival function, which is a crucial technical assumption to obtain several key results in the paper.[3]

Assumption 1. *The density function $F'(v)$ is strictly positive whereas the survival function $1 - F$ is log-concave, that is*

$$F^{'2}(v) + F^{''}(v)[1 - F(v)] > 0, \ \forall v.$$

The realization of v is unobservable to the seller. The seller only knows the cdf F generating v, so, he advertises a *list price l* trading off the probability of sale with revenue. The *sale price $p(l)$* depends on the

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[2] What we have in mind is a Pissarides style random matching function where arrival rates are functions of the market tightness (buyer-seller ratio). Typically one assumes different measures of buyers and sellers so that arrival rates for buyers and sellers vary. To avoid excessive parameterization we simply assume equal measures, which means that agents meet each other with the same rate α.

[3] Log-concavity of the survival function is equivalent to the ratio of the density to the survival being monotone increasing and many well known distributions including Uniform, Normal, Exponential, χ^2 satisfy this property. See [3] for more details.

list price but may involve a non-trivial renegotiation process (more on this later). If agents agree to trade at price p then the seller receives payoff p; the buyer receives dividends v starting at the beginning of the next period and continuing forever; both agents leave the search market and are replaced by a buyer and a *regular* seller. The replacement assumption is standard in the literature; it is needed to maintain stationarity. Agents who do not trade receive a period payoff of zero and continue to the next round to play the same game.

SALE PRICES. In the housing market transactions rarely occur at the list price; the sale price typically involves a hard bargain between the buyer and the seller. To model negotiations the literature traditionally makes use of Nash bargaining or Rubinstein bargaining frameworks. Instead, we take the following approach. We are not particularly interested how buyers and sellers interact with each other as they negotiate, so we treat the renegotiation mechanism (be it Nash bargaining, strategic bargaining or even some esoteric price formation procedure) as a black box; however, we specify some mild properties that the resulting sale price ought to satisfy. As long as the renegotiation mechanism satisfies these properties our results go through.

More formally, let $G \langle l, \alpha \rangle$ denote an extensive form game that induces some expected sale price $p(l) : [0,1] \to [0,1]$ for any given list price l and contact frequency α.

Assumption 2. *The sale price $p(l) : [0,1] \to [0,1]$ is an increasing and differentiable function of l. The differential dp/dl is uniquely valued at any $l \in [0,1]$, i.e. $p(\cdot)$ is a "smooth" function with no kinks.*

If $G \langle l, \alpha \rangle$ has multiple equilibria and therefore generates multiple sale prices (which typically is the case with bargaining models with private information, e.g. see the survey by Kennan and Wilson [9] and the references therein), then we assume that there is an equilibrium selection device that uniquely pins down $p(l)$. The game, the selection device and the resulting sale price function $p(l)$ are all common knowledge.
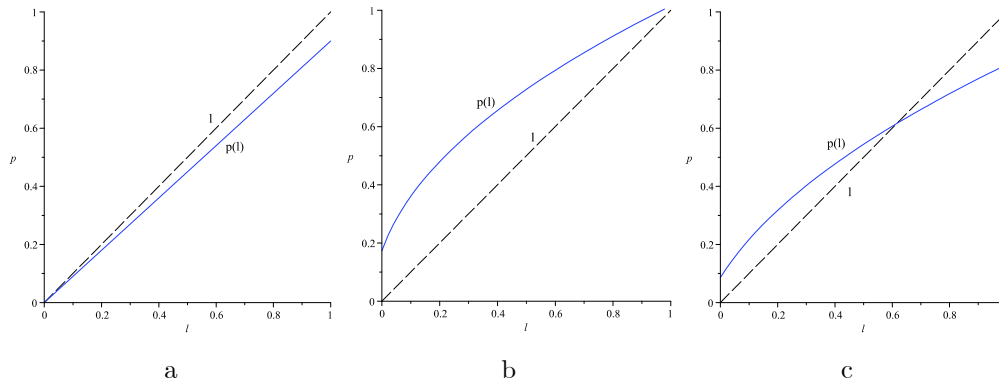

Figure 1: Sale Prices

The restrictions are indeed mild; hence, the model admits a continuum of sale price functions. The figure illustrates some examples. Figure 1a depicts an environment where the transaction takes place 10% below the list price, i.e. $p = 0.9l$. In 1b the sale price almost always exceeds the list price—much as the real estate market in Santa Monica, CA. In 1c the sale price is above or below the list price depending on how much sellers ask for. The house is sold above the list price if the list price is low and it is sold below the list price otherwise

We admit that the shape of the sale price function should be endogenous and should depend on the fundamentals of the market. The paper's purpose, however, is not to explain why a certain pricing practice emerges in this market and not in the other. Instead we want to investigate what happens to prices, the volume of trade, and above all, buyers purchasing behavior when some sellers become distressed. For this purpose the shape of the sale price function can be as it may; all we need is that it satisfies Assumption 2.

We move on to discuss buyers' and sellers' problems. We denote a seller's type by $j = r, d$ where r refers to regular sellers and d refers to distressed sellers. We focus on a symmetric steady state equilibrium where identical agents follow identical strategies. In particular, a type j seller advertise a list price l_j corresponding to the sale price $p(l_j) := p_j$. Buyers, upon meeting a type j seller, purchase if their private valuation v (willingness to pay) of the house exceeds an endogenous threshold v_j.

2.1 Buyers' Problem

The problem of a representative buyer has a recursive formulation. We use a dynamic programming approach letting Ω denote the value of search to a buyer. In a symmetric pure strategy equilibrium the distribution of list prices $\mathbf{p}^* = (p_r^*, p_d^*)$ is degenerate. Clearly Ω is a function of $\mathbf{p}^*$ however we omit the argument when understood. We have

$$\delta\Omega = \alpha\theta \int_0^1 \max\left[\frac{v}{\delta} - p_s - \Omega, 0\right] dF(v) + \alpha(1 - \theta) \int_0^1 \max\left[\frac{v}{\delta} - p_r - \Omega, 0\right] dF(v).$$

A buyer's lifetime utility from owning a house that yields v per period equals to v/δ. The parameter θ is the endogenous fraction of distressed sellers; so with probability $\alpha\theta$ a buyer meets a distressed seller who sells for p_d. If the consumer surplus $v/\delta - p_d$ exceeds the value of search Ω then the buyer purchases, otherwise he walks away. Similarly with probability $\alpha(1 - \theta)$ the buyer encounters a regular seller who sells for p_r. Again if the consumer surplus exceeds the value of search then the buyer purchases otherwise he keeps searching.

For any given sale price p_j we conjecture an associated *reservation value*

$$v_j = \delta(p_j + \Omega) \tag{1}$$

such that the customer purchases only if $v \geq v_j$. The implication is that a buyers' search process amounts to finding a house with a sufficiently high v. Obviously not all meetings result in trade; for trade to occur the house must turn out to be a good match for the buyer, which happens with probability $F(1 - v_j)$. A high v_j means that buyers are unlikely to purchase (they are selective).

Observe that there are two types of trading frictions in the model. The first is locating a vacant house, which is captured by the meeting probability α, and the second is whether the house, once found, is a good match, which is captured by the probability $F(1 - v_j)$. Clearly, in our model liquidity is endogenous and it is derived from the maximization behavior of buyers and sellers.[4]

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[4] The fact that some meetings do not result in trade is in line with the empirical observation by Merlo and Ortalo-Magné [12]. Analyzing transaction histories of residential properties sold in England between 1995 and 1998 they find that about a third of all meetings resolve with no agreement. Most of the existing theoretical models of the housing market are in clear contradiction with this empirical observation. e.g. Arnold [2], Chen and Rosenthal [6], Yavas [16], Yavas and Yang [17]. Assuming complete information, these models imply that a match necessarily results in trade.

Inserting the reservation values into Ω we have

$$
\begin{aligned}
\Omega &= \frac{\alpha\theta}{\delta^2} \int_{v_d}^1 (v - v_s)\, dF(v) + \frac{\alpha(1-\theta)}{\delta^2} \int_{v_r}^1 (v - v_r)\, dF(v) \\
&= \frac{\alpha\theta}{\delta^2} \int_{v_d}^1 [1 - F(v)]\, dv + \frac{\alpha(1-\theta)}{\delta^2} \int_{v_r}^1 [1 - F(v)]\, dv,
\end{aligned}
\tag{2}
$$

where in the second step we use integration by parts.

THE FRACTION OF DISTRESSED SELLERS. The steady state fraction of distressed sellers is endogenous and can be obtained by equating the inflow into the pool of distressed sellers to the outflow from the pool. The inflow equals to $(1 - \theta)\,\mu$ whereas the outflow is $\theta\alpha F(1 - v_d)$. Therefore

$$
\theta = \frac{\mu}{\mu + \alpha F(1 - v_d)} \in (0, 1).
\tag{3}
$$

Observe that θ depends on the arrival rate of the adverse shock, μ, and the meeting probability α. It is easy to see that θ rises in μ and falls in α. More importantly θ depends on the probability of trade $F(1 - v_d)$ which is endogenous and controlled by buyers. Observe that buyers can squeeze the outflow and raise θ by becoming more selective (i.e. by raising the threshold v_d). Put differently, buyers can strategically slow down the speed of trade and thereby cause more sellers to become distressed. This observation is essential in understanding the predation result.

Lemma 1 *We have $\frac{\partial\Omega}{\partial v_d} < 0$ and $\frac{\partial\Omega}{\partial v_r} < 0$.*

The Lemma has two implications. First, buyers' value of search falls as the market becomes less liquid, i.e. Ω falls as v_r and v_d go up. The other implication is that sellers face a trade-off between revenue and liquidity. Indeed the indifference condition (1) implies that

$$
\frac{dv_j}{dp_j} = \frac{\delta}{1 - \delta\partial\Omega/\partial v_j} > 0,
$$

which basically means that the higher the price the higher the threshold v_j. From the seller's perspective, raising the sale price p_j (by advertising a higher l_j) brings in a larger revenue but lowers the chance of a sale. The seller's task is to find a balance between these two effects, which we discuss next.

2.2 The Seller's Problem

A type j seller advertises a list price l_j taking as given the sale price function $p(\cdot)$ and buyers' search decisions. The value functions are given by

$$
\begin{aligned}
\delta\Pi_d &= \alpha[1 - F(v_d)]\max(p_d - \Pi_d, 0) \\
\delta\Pi_r &= \alpha[1 - F(v_r)]\max(p_r - \Pi_r, 0) + \mu(\Pi_d - \Pi_r).
\end{aligned}
\tag{4}
\tag{5}
$$

An interpretation is this. A distressed seller who lists l_d (and consequently sells for p_d) meets a buyer with probability α, who purchases with probability $1 - F(v_d)$. The seller agrees to trade only if the price exceeds his continued value of search i.e. if $p_d - \Pi_d \geq 0$. The second line can be interpreted similarly.

Conjecturing that $p_j \geq \Pi_j$, a type j seller solves

$$\max_{l_j} \Pi_j \quad \text{s.t.} \quad v_j = \delta \left(p_j + \Omega_s \right)$$

taking Ω as given[5]. The value functions are linked to each other and therefore it requires some algebra to solve the maximization problems. A complete analysis is provided in Appendix I; here we simply record some key steps. The FOC of seller j is given by

$$p_j - \Pi_j = \frac{1 - F(v_j)}{\delta F'(v_j)}, \quad j = r, d.$$

Using the FOC and manipulating the value functions with straightforward algebra one can obtain the profit

maximizing sale prices for regular and distressed sellers

$$p_r = \frac{1 - F(v_r)}{\delta F'(v_r)} + \frac{\alpha \left[1 - F(v_r)\right]^2}{\delta \left(\mu + \delta\right) F'(v_r)} + \frac{\alpha \mu \left[1 - F(v_d)\right]^2}{\delta \bar{\delta} \left(\mu + \delta\right) F'(v_d)} \tag{6}$$

$$p_d = \frac{1 - F(v_d)}{\delta F'(v_d)} + \frac{\alpha \left[1 - F(v_d)\right]^2}{\bar{\delta} \delta F'(v_d)}. \tag{7}$$

Lemma 2 *We have $\frac{\partial p_d}{\partial v_d} < \frac{\partial p_r}{\partial v_d} < 0$ and $\frac{\partial p_r}{\partial v_r} < \frac{\partial p_d}{\partial v_r} = 0$.*

The Lemma has two implications. First, the negative relationship between prices and reservation values reflect the aforementioned trade-off between revenue and liquidity. For low values of v_j the probability of a sale is high, so sellers can afford to charge high prices; however as v_j rises, liquidity concerns start to kick in and prices fall. Second, a type j seller is more sensitive to his probability of sale than the other type is, which is why $\frac{\partial p_d}{\partial v_d} < \frac{\partial p_r}{\partial v_d}$ and $\frac{\partial p_r}{\partial v_r} < \frac{\partial p_d}{\partial v_r}$.

Now we are ready to close down the model.

Definition 3 *A steady-state symmetric equilibrium is characterized by the sale price function $p(l)$ and the pairs $\mathbf{v}^* = (v_r^*, v_d^*)$ and $\mathbf{l}^* = (l_r^*, l_d^*)$ that simultaneously satisfy*

$$\Delta_r := p_r + \Omega - \frac{v_r}{\delta} = 0 \text{ and } \Delta_d := p_d + \Omega - \frac{v_d}{\delta}. \tag{8}$$

The fraction of distressed sellers θ, given by (3) and also implicitly part of the equilibrium, can be easily recovered from the conditions above.

Existence (and uniqueness) of an equilibrium amounts to showing that there exists an interior pair $\mathbf{v}^* = (v_r^*, v_d^*)$ satisfying (8). To do so one needs to demonstrate that the loduses of Δ_r and Δ_d intersect once in the $v_r - v_d$ space. Lemma 7 in Appendix II establishes that the loduses look as in Figure 2. The fact that κ_r is steeper than κ_d and the specific locations of the boundaries ($\underline{v}_d, \overline{v}_r$ etc.) guarantee a unique intersection.

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[5] From the seller's point of view, cutting the price directly improves the buyer's willingness to trade, but the seller fails to take into account how this drop changes equilibrium prices and the buyer's value of search. This "large market approach" is used in directed search models as well, e.g. see Camera and Selcuk [5].
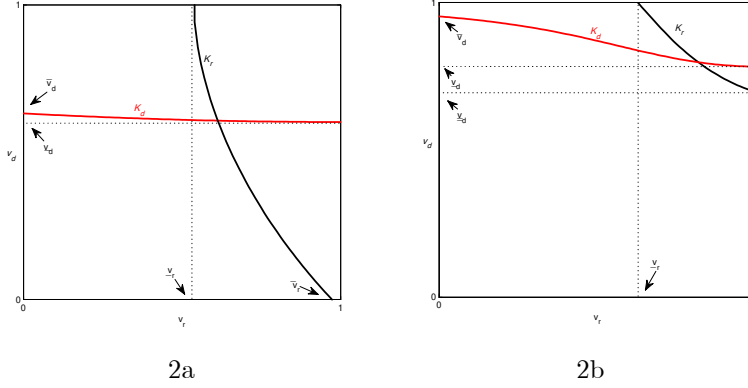
2a   2b

$Figure\ 2$: Locuses[6]

3   Fire Sales and Predation

Proposition 4 *A steady state symmetric equilibrium exists and it is unique. The equilibrium is characterized by "fire sales": desperate sellers accept lower prices and sell faster than regular sellers i.e. $p_d^* < p_r^*$ and $v_d^* < v_r^*$. If the adverse shock starts to arrive more often then "predatory buying" emerges: prices fall yet buyers hold off purchasing and strategically slow down the trade i.e. $dp_j^*/d\mu < 0$ and $dv_j^*/d\mu > 0$. Finally, if the shock becomes more severe then prices fall and trade speeds up i.e. $dp_j^*/d\bar{\delta} < 0$ and $dv_j^*/d\bar{\delta} < 0$.*

The proposition has a number of implications, so we move in steps.

Fire Sales. A distressed home owner is impatient to sell. To achieve his goal he undercuts his competitors, which we dub as attempting a fire sale. The price cut produces the desired outcome. Indeed $v_d^* < v_r^*$ implies that distressed home owners are more likely to sell than regular home owners. How low the price can go depends on fast the seller needs to unload the property, which in turn depends on how dire the situation of the seller is. The facts that $dp_d^*/d\bar{\delta} < 0$ and $dv_d^*/d\bar{\delta} < 0$ imply that the more painful the shock the lower the price and the quicker the trade. The simulation in Figure 5c provides further insight on this, where we plot a distressed seller's (percentage wise) profit loss against the severity of the shock $\bar{\delta}$. The profit loss is measured by the index

$$z = \frac{p_r^* - p_d^*}{p_r^*} \in (0,1).$$

Had the seller not become distressed he would have been able to sell at p_r^* but in a fire sale he can only get p_d^*, so the difference $p_r^* - p_d^*$ equals to his forgone profits. A high value of z means that distressed sellers need to offer substantial price cuts in order to sell quickly. Observe that in Fig 5c if the shock is mild ($\bar{\delta} \approx \delta$) then there is not much difference between what regular and distressed sellers charge, however as the shock starts to bite ($\bar{\delta} \gg \delta$), then distressed sellers face considerable losses. We will come back to this point later in Section 5 when we discuss liquidity.

There is a particular study by Glower et al. [8] that we would like to mention here. The paper's goal is to determine the effects of seller motivation on prices, the time on the market, the speed of trade, etc. To do so the authors survey sellers in Columbus, OH area to obtain information on their motivation by asking whether they have a planned date to move out or accepted a job offer elsewhere or bought another

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[6]Both figures are drawn for $F(v) = v$, $p(l) = l$, and $\alpha = 1$. In 2a we have $\delta = 0.9$, $\bar{\delta} = 1.2$ and $\mu = 2$. In 2b we have $\delta = 0.05$, $\bar{\delta} = 0.2$ and $\mu = 0.5$.

house. The conclusion is that motivated sellers accept lower prices and sell more quickly. This seems to be consistent with the preceding discussion.

Predatory Buying. The proposition says that during periods where an increasing number of sellers become distressed i.e. when μ goes up the followings happen (it is sensible to think that the adverse shock, μ, arrives more often during financial crises or recessions):

- All sellers, regular and distressed, accept lower prices, i.e. $dp_j^*/d\mu < 0$. Regular sellers face a higher likelihood of becoming distressed in the future. So, they accept lower prices to sell quickly before being hit by the shock, which is why $dp_r^*/d\mu < 0$. We call this the "spill-over effect" of distressed sales on regular sales. Desperate sellers, on the other hand, face stiffer competition. The fraction of desperate sellers θ rises with μ, so, realizing that there many other sellers in the same dire situation, distressed sellers are forced to cut their already low prices. This is why $dp_d^*/d\mu < 0$.

- Customers exhibit what we call *predatory buying*; they delay purchasing and strategically slow down the speed of trade (i.e. $dv_j^*/d\mu > 0$) despite the falling prices. The reason is that, unlike sellers, buyers benefit from the rising θ.[7] Realizing that there are plenty of good deals in the market (higher θ) buyers find it optimal to search longer, which means that they increase the thresholds v_r^* and v_d^*. This response has a spiral effect. By raising v_d^* buyers strategically slow down the speed of trade causing more sellers to become distressed (recall that θ rises with v_d^*). The growing θ puts additional downward pressure on prices and the speed of trade, and so on.

From the buyers' point of view predatory buying is optimal as it allows them to acquire better houses at lower prices, but from the sellers' point of view predation is the worst possible outcome; they are forced to lower their prices yet buyers are still reluctant to purchase. Indeed liquidity disappears when it is needed the most.

Predatory trading is well documented in financial markets; see, for instance, Brunnermeier and Pedersen [4] and the references therein.[8] Casual observations suggest that in the real estate market, too, various forms of predation take place including strategically delaying purchasing to pressurize distressed home owners. Newspaper stories are abound about potential buyers delaying their purchase and waiting for the 'right time' to enter the market. The number of such stories seems to have escalated during the recent housing crisis, where presumably the arrival rate of the adverse shock μ went up. Although somewhat casual, these observations seem to be consistent with the implications of the model. To the best of our knowledge predation is not empirically documented in the real estate market.

4   Prices

4.1   Price Trajectories

According to the model, for any given property the trajectory of the list price is either flat or looks like a step function with a sizeable jump-down at the time of the price reduction. To see why note that all sellers enter the market in the regular state; so all properties are initially listed at high prices (l_r^* in the model). Some

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[7]Buyers' value of search Ω increases in θ. To see why note that in the proof of Proposition 4 we establish that $\Omega_\mu > 0$. Since $\theta_\mu > 0$ it follows that $\Omega_\theta > 0$.

[8]Brunnermeier and Pedersen [4] who show that in financial markets if a distressed trader is forced to liquidate, other strategic traders initially sell in the same direction—driving down the price even faster—and then buy it back at dirt cheap prices.

sellers manage to sell without being distressed; so for those properties the trajectory remains flat throughout. Others, however, are hit by the adverse shock while still waiting to sell, so they reduce their prices from l_r^* to l_d^* when the shock hits. These properties have list price trajectories that look like a step function.

Interestingly, this is exactly what Merlo and Ortalo-Magné [12] observe empirically. Based on home sale transaction data from England, they find that 2/3 of sellers do not change their list price, 1/4 reduce only once, and the rest reduce twice or more. The individual list price trajectories are either flat or piecewise flat with typically one discontinuous jump-down (see Figure 2.1 in Merlo et al. [13]). Sellers wait, on average, 11 weeks to change their prices and the price reductions can be as high as 10%. These sizeable and infrequent price revisions are inconsistent with most of the existing theoretical literature. Existing models imply that in equilibrium, either the seller never revises the price (e.g., Arnold [2], Chen and Rosenthal [6], Yavas and Yang [17]), or gradually lowers the price in a continuous fashion (Coles [7]).

Our model, however, naturally accounts for the preceding empirical observations. Regular sellers do not modify the listing price whereas distressed sellers reduce the price only when the shock hits. The drop can be sizeable ($\sim 8\%$) if the shock is severe, see Fig. 5c. Finally, observe that in our setup the shock hits only once, hence the price drop occurs only once. One can easily cast the model in a setting where there are multiple levels of being distressed, so price drops may occur once, twice or more, as Merlo and Ortalo-Magné observe.

4.2   The Negative Relationship between Duration and the Expected Sale Price

Merlo and Ortalo-Magné [12] document a negative correlation between the sale price and the duration of the sale (the longer the time on the market the lower the sale price). This fact, again, is inconsistent with most of the existing theoretical models. Our setup, instead, provides a simple explanation: If a property is sold shortly after the listing date then it is most likely a "regular sale". The owner of the property cannot possibly become distressed within this short period of time. On the other hand if the sale occurs long after the listing date then most likely it is a "distressed sale". Indeed, the longer the wait the more likely is a seller to be hit by the shock. The negative correlation between the sale price and duration immediately follows from the fact that distressed sales occur at lower prices. Below we make these arguments more precise.

Consider a seller who enters the market at time 0 (wlog). The probability that he remains non-distressed without a sale until time t is given by

$$r(t) = e^{-(\mu + \alpha_r)t}. \tag{9}$$

The probability that he becomes distressed at some time $y \leq t$ while he is still unable to sell at t equals to

$$d(t) = \int_0^t \mu e^{-\mu y} e^{-\alpha_r y} e^{-\alpha_d(t-y)} dy, \tag{10}$$

where $\mu e^{-\mu y}$ is the density of transition time y (exponential pdf). Now, consider all sales completed with a duration t. The fraction of distressed sales equals to

$$g(t) = \frac{d(t)}{r(t) + d(t)}.$$

One can easily verify that g rises in t (see the proof of Proposition 5), i.e., the longer the duration, the more likely the sellers are to be distressed. An immediate corollary is that the expected sale price falls with the

duration. To see this more precisely define the expected sale price

$$\overline{p}(t) = gp_d^* + (1 - g)p_r^*$$

and the variance

$$\sigma^2(t) = g(p_d^* - \overline{p}(t))^2 + (1 - g)(p_r^* - \overline{p}(t))^2.$$

Proposition 5 $\overline{p}(t)$ *is monotone decreasing and* $\sigma^2(t)$ *is hump-shaped in* t.

Figure 3a and 3b illustrate $\overline{p}$ and σ. Observe that $\lim_{t \to 0} \overline{p}(t) \to p_r^* \approx 0.43$ and $\lim_{t \to \infty} \overline{p}(t) \to p_d^* \approx$ 0.41. Indeed, as stated above, if a house is sold very soon after it was put on sale then most likely it is not a distressed sale, because there is not sufficient time where the seller could be possibly hit by the shock. However the longer it stays on the market, the more likely the seller is to become distressed. The continuously downward slope in $\overline{p}$ may be somewhat misleading and create an illusion that the transaction price continuously falls with the duration. We emphasize that an individual transaction price trajectory is piecewise flat with a discontinuous drop from p_r^* to p_d^* at the time the seller is hit by the shock. It is the *expected* price that falls monotonically; the transaction price is either p_r^* or p_d^*.
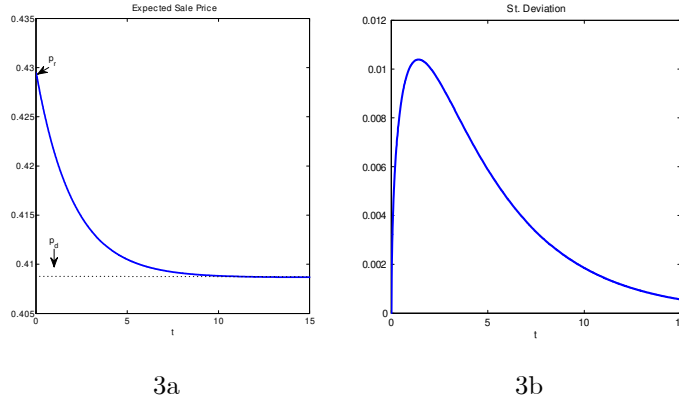

| 3a | 3b |

$Figure\ 3$: Exp. Sale Price and the St. Dev. wrt Duration[9]

The shape of the standard deviation is also intuitive. For very short or very long durations the sale is either non-distressed or distressed. Only for intermediate values of t there is ambiguity; hence the hump shape.

4.3   Price Dispersion

The model implies equilibrium price dispersion (in list prices as well as in sale prices). The source of price dispersion is the fact that sellers are heterogenous in terms of their time preferences: distressed sellers cannot wait, so they accept lower prices; regular sellers, on the other hand, are more patient, hence they fetch higher prices. Below we obtain the distribution of sale prices and discuss how it responds to the parameters of interest, μ and $\overline{\delta}$.

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[9]Parameters: $F(v) = v$, $p(l) = l$, $\alpha = 1$, $\mu = 0.5$, $\delta = 0.05$, $\overline{\delta} = 0.2$.

The distribution of list and sale prices follows from the steady state fraction of distressed sellers, θ. At any given point in time we have

$$\Pr\left(l = l_d^*\right) = \theta \quad \text{and} \quad \Pr\left(l = l_r^*\right) = 1 - \theta.$$

The distribution of sale prices is same as above. Now we can obtain the average sale price and the variance

$$\bar{p} = \theta p_d^* + (1-\theta)\, p_r^* \quad \text{and} \quad \sigma^2 = \theta\left(p_d^* - \bar{p}\right)^2 + (1-\theta)\left(p_r^* - \bar{p}\right)^2.$$

One can easily verify that $\bar{p}$ falls in μ and $\bar{\delta}$ (see also Figures 4a and 4b).[10] There are two channels through which these parameters affect $\bar{p}$. First, p_d^* and p_r^* fall in μ and $\bar{\delta}$ (Proposition 4). Second, θ rises in μ and $\bar{\delta}$. Both of these effects work in the same direction; hence the average sale price unambiguously falls if the shock becomes more severe or more frequent.
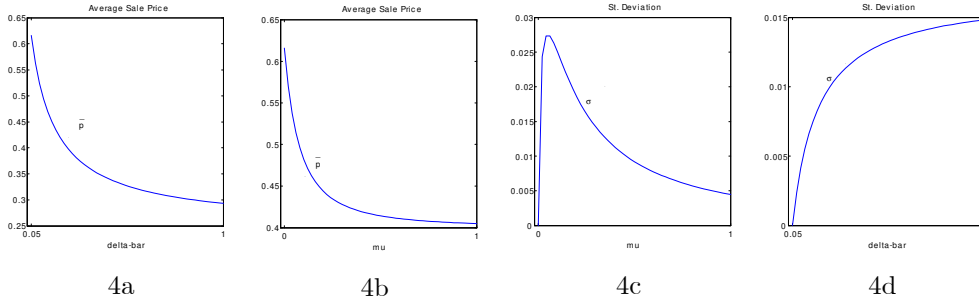

| 4a | 4b | 4c | 4d |

Figure 4 : Av. Sale Price and St. Deviation – Cross Section

The shape of the standard deviation is also intuitive (Figure 4c and 4d). If μ is too low then almost all sales are regular; if μ is too high then almost all sales are distressed; in either case there is little dispersion. Only for moderate values of μ we have price dispersion. Figure 4d illustrates σ with respect to $\bar{\delta}$. If $\bar{\delta} = \delta = 0.05$ then sellers are identical and charge the same price; hence there is no dispersion. However as $\bar{\delta}$ rises, the gap between p_r^* and p_d^* widens and price dispersion starts to appear, which is why σ rises in $\bar{\delta}$.

5 Liquidity

The working definition of *liquidity* in this paper is the capacity of how fast one can sell his property without any 'loss in value'. There are two aspects of liquidity that we are interested in: the speed of trade and the profit loss in fire sales. The former can be measured either by the probability of sale

$$\alpha_j = \alpha\left[1 - F\left(v_j^*\right)\right]$$

or the expected time on the market TOM. The probability of sale measures the speed of trade from an individual seller's point of view, whereas TOM is a market-wide weighted average taking into account all sellers, regular and distressed.

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[10] Observe that

$$\frac{dp}{d\mu} = -\left(p_r^* - p_d^*\right)\frac{d\theta}{d\mu} + \theta\frac{dp_d^*}{d\mu} + (1-\theta)\frac{dp_r^*}{d\mu}.$$

The first term is negative because $\theta_\mu > 0$ and $p_r^* > p_d^*$. The second and the third terms are also negative because $dp_d^*/d\mu < 0$ and $dp_r^*/d\mu < 0$. Similarly one can show that $d\bar{p}/d\bar{\delta} < 0$.

The second aspect of liquidity is the loss of value in fire sales. To measure it we use the index

$$z = \frac{p_r^* - p_d^*}{p_r^*} \in (0,1), \tag{11}$$

which is a distressed seller's percentage-wise profit loss compared to a regular seller. The index z has been introduced earlier; we rewrite it here for clarity. Recall that a high value of z means that distressed sales occur far below regular sales, which indicates illiquidity. Below we discuss the performance of these proxies within the context of fire sales and predation.

5.1 Time on the Market: TOM

The time on the market is one of the most frequently used and referred statistics in the literature. Low values of TOM is typically interpreted as an indication of high liquidity and market efficiency (e.g. Krainer [11], Knight [10]) or, in a slightly different context, as a sign of the quality of the property (Taylor [14]). In our model, though, TOM does a poor job as a proxy of liquidity. We show that TOM *falls* as the adverse shock μ arrives more frequently. When interpreted on the face value, the fall in TOM indicates that the market becomes more liquid and more efficient during times of financial distress where a rising number of sellers become distressed and attempt fire sales.

To see why, notice that during periods when μ goes up customers exhibit predation, so the probability of trade α_j falls for both types of sellers. In such a setting one naturally expects TOM to go up because sellers wait longer until a sale occurs; but things are more subtle than that. Because of the rising μ more sellers become distressed and distressed sellers trade faster than regular sellers. This *transition effect* puts a downward pressure on TOM and blurs the picture. Simulations suggest that the transition effect is in fact dominant; that is TOM *falls* as the adverse shock arrives more frequently; see Figure 5b. Below we make these arguments more precise.

Using $r(t)$ and $d(t)$, which are given by (9) and (10), one can obtain the density of time to sale γ and the expected time to sale TOM. We have

$$TOM = \int_0^\infty [r(t) + d(t)] \, dt \quad \text{and} \quad \gamma = -\frac{d[r(t) + d(t)]}{dt}.$$

Basic algebra reveals that TOM and γ are given by the expressions below.

Proposition 6 *The density of the time on the market is given by*

$$\gamma = \frac{\mu \alpha_d e^{-\alpha_d t} - (\alpha_d - \alpha_r)(\mu + \alpha_r) e^{-(\mu + \alpha_r)t}}{\mu - \alpha_d + \alpha_r}. \tag{12}$$

The pdf is hump shaped if $\frac{\mu}{\alpha} > \frac{[1 - F(v_r^*)]^2}{F(v_r^*) - F(v_d^*)}$ and monotone decreasing otherwise. The expected time on the market is given by

$$TOM = \frac{\mu + \alpha[1 - F(v_d^*)]}{\alpha[1 - F(v_d^*)](\mu + \alpha[1 - F(v_r^*)])}.$$

The pdf γ and the expected time on the market TOM are both endogenous and derived from the maximization behavior of buyers and sellers. Both expressions depend on the parameters of the model as well as the equilibrium objects v_r^* and v_d^*.[11] Now we can analyze how TOM responds to a change in μ. We

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[11]The shape of the density function γ is indeed realistic (Figure 5a illustrates γ). Merlo et al. [13], based on transaction data

have

$$\frac{dTOM}{d\mu} = \frac{\partial TOM}{\partial v_r^*}\frac{dv_r^*}{d\mu} + \frac{\partial TOM}{\partial v_d^*}\frac{dv_d^*}{d\mu} + \frac{\partial TOM}{\partial \mu}.$$

The first two terms are positive because $\partial TOM/dv_j^* > 0$ and $dv_j^*/d\mu > 0$ (Proposition 4). When μ rises sellers are less likely to trade, so they wait longer until a sale occurs. The last term, however, is negative because

$$\frac{\partial TOM}{\partial \mu} \propto F\left(v_d^*\right) - F\left(v_r^*\right) < 0 \text{ since } v_d^* < v_r^*,$$

confirming our intuition about the aforementioned *transition effect*. Analytically it is difficult to sign $dTOM/d\mu$ but simulations suggest that the transition effect is in fact more dominant; see Figure 5b.

As discussed above, if one relies on TOM as a proxy of liquidity, then the fall in TOM means that the market becomes more liquid during times of financial distress. It appears that in this particular setting the equilibrium probability of trade $\alpha_j = \alpha\left[1 - F\left(v_j^*\right)\right]$ is a better proxy of liquidity than TOM. In data α_j is the frequency of meetings resulting in a sale, which clearly falls with μ[12], indicating that during recessions or crises liquidity dries up as meetings are less likely to result in trade.
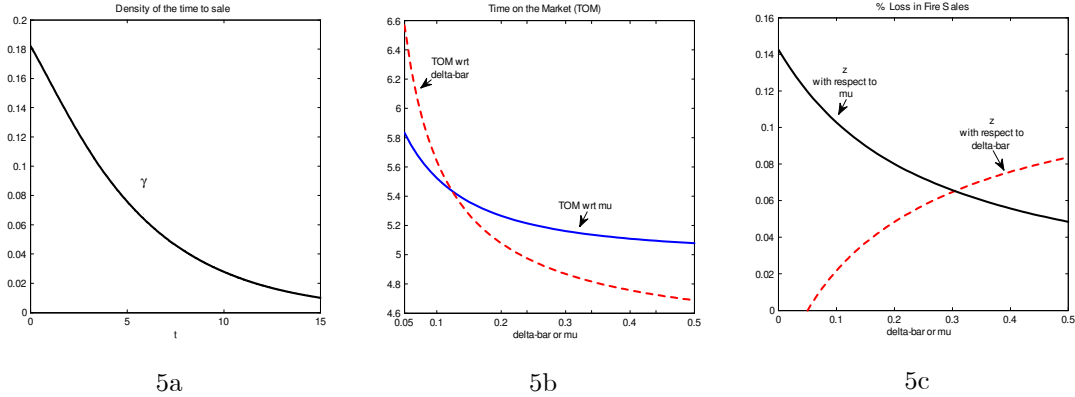

5a   5b   5c

Figure 5: The pdf γ, the time on the market TOM and the profit loss z[13]

5.2   Profit Loss: z

Figure 5c illustrates z against the frequency (μ) and the severity $\left(\overline{\delta}\right)$ of the adverse shock. Note that when $\mu \approx 0$ distressed sales take place about 14% below regular sales, whereas when $\mu \approx 0.5$ the difference is less than 5%. In general the simulation suggests that attempting a fire sale is, in fact, *less* costly when times are bad (when μ is high). The reason is simple. During bad times regular sellers, afraid of becoming distressed, substantially lower their prices to sell quickly and exit the market. This is the aforementioned spillover effect of fire sales onto the regular sales. Regular sellers are more sensitive to a rise in μ than distressed sellers are. Distressed sellers do not worry about being hit by the shock because they are already distressed. So, although both prices fall, the drop in p_r^* is sharper than the one in p_d^* which is why z declines.

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from England, obtain the empirical distribution of times to sale, which is right skewed and hump-shaped with a mean 10.27 weeks and median of 6 weeks (see figure 2.3 therein). The theoretical pdf is skewed to the right because of the Poisson arrivals and it may be hump shaped if the ratio μ/α is sufficiently large, i.e. if buyers are scarce and the adverse shock is frequent.

[12] Observe that $\frac{d\alpha_j}{d\mu} = -\alpha F\left(v_j^*\right)\frac{dv_j^*}{d\mu} < 0.$

[13] All figures are drawn for $F\left(v\right) = v$, $p\left(l\right) = l$, $\alpha = 1$, $\delta = 0.05$, $\overline{\delta} = 0.2$, $\mu = 0.5$. In 5b and 5c the x-axis is shared by $\overline{\delta}$ and μ. When plotting with respect to $\overline{\delta}$ we fix $\mu = 0.5$ and let $\overline{\delta}$ range between 0.05 and 0.5 on the x-axis (recall that we need $\overline{\delta} > \delta = 0.05$). Similarly when plotting with respect to μ we fix $\overline{\delta} = 0.2$.

Again, one has to be careful when interpreting this rather positive-looking result. In absolute terms all sellers are worse off (all prices fall in μ). Only in relative terms distressed sellers appear to be better off.

The relationship between z and $\overline{\delta}$ is more straightforward. The simulation in Figure 5c suggests that if the shock is mild ($\overline{\delta} \approx \delta$) then a fire sale is not too costly; however as the shock starts to bite ($\overline{\delta} \gg \delta$), then distressed sellers face considerable losses. Indeed when $\overline{\delta} = \delta = 0.05$ sellers are identical and charge the same price, so $z = 0$; but when $\delta = 0.5$ the price difference exceeds 8%. The reason is that desperate sellers are directly affected by a rise in $\overline{\delta}$ whereas regular sellers worry about $\overline{\delta}$ only because they may become desperate in the future. The fall in p_d^* is sharper than the one in p_r^*, which is why z goes up.

6   Conclusion

We have presented an equilibrium search model with three distinctive characteristics: (i) trade is decentralized; agents search for a counterparty to trade (ii) a buyer's willingness to pay is private information and (iii) sellers may become financially distressed as they are unable to sell. We have found that, once distressed, sellers attempt fire sales by accepting prices that are substantially below fundamental values. In addition, during periods where a large number of sellers are forced to liquidate customers strategically hold off purchasing and slow down the speed of trade in an effort to obtain better deals—an outcome which we call predatory or vulture buying.

When constructing the model what we had in mind was the housing market, however we think that the results should hold in other markets featuring the aforementioned characteristics. For instance the model is also relevant to the over the counter (OTC) markets; in particular markets for mortgage-backed securities, bank loans and derivatives among others. Indeed, search is a fundamental feature in many OTC markets as it is difficult to identify a counterparty with whom there are likely gains from trade. In these markets buyers valuations are private information; it is not uncommon at all for parties to simply walk away without trading. Finally, anecdotal evidence suggests that traders may become financially distressed due to, for instance, pressing debt obligations, nearing margin calls, hedging motives or being caught in a "short squeeze". The labor market is another setting where the model is applicable.

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Appendix I: Omitted Proofs

Proof of Lemma 1. Observe that θ does not depend on v_r. Hence

$$\frac{\partial \Omega}{\partial v_r} = -\frac{\alpha (1-\theta)}{\delta} \left[1 - F(v_r)\right],$$

which clearly is negative. Now consider

$$\frac{\partial \Omega}{\partial v_d} = \frac{\alpha \theta'}{\delta} \int_{v_s}^{v_r} \left[1 - F(v)\right] dv - \frac{\alpha \theta}{\delta} \left[1 - F(v_d)\right], \tag{13}$$

where

$$\theta' = \frac{\partial \theta}{\partial v_d} = \frac{\theta \alpha F'(v_d)}{\mu + \alpha \left[1 - F(v_d)\right]} > 0.$$

To show that $\partial \Omega / \partial \underline{v}_d < 0$ it suffices to demonstrate that

$$\eta(v_d) := \int_{v_d}^{1} \left[1 - F(v)\right] dv - \frac{\mu}{\alpha} \frac{1 - F(v_d)}{F'(v_d)} - \frac{\left[1 - F(v_d)\right]^2}{F'(v_d)} < 0.$$

Omitting the argument and differentiating with respect to $\underline{v}_d$ we have

$$\eta' = \frac{F'^2 + F''(1-F)}{F'^2} \left[\frac{\mu}{\alpha} + 1 - F\right]$$

which is positive under Assumption 1. Since η increases in v_d and $\eta(1) = 0$, it follows that $\eta(v_d) < 0$, $\forall v_d \in [0, 1)$. ∎

Maximization Problems of Distressed and Regular Sellers

Distressed Sellers. We start with the distressed sellers' problem. Note the followings:

- The sale price p is a function of l; p_j simply stands for $p(l_j)$. Because of Assumption 2 we have $p_j' = \partial p / \partial l_j > 0$. Since p does not have any kinks the derivative p' is uniquely valued for all l_j.

- The indifference constraint $v_d = \delta(p_d + \Omega)$ implies that $v_d' = \partial v_d / \partial l_d = \delta p_d' > 0$.

- Sellers take Ω as given, thus $\Omega' = \partial \Omega / \partial l_d = 0$.

Conjecturing that $p_d \geq \Pi_d$ rewrite (4) as follows

$$\bar{\delta} \Pi_d = \alpha \left[1 - F(v_d)\right] (p_d - \Pi_d).$$

Keeping the preceding points in mind differentiate Π_d with respect to l_d to obtain

$$\bar{\delta} \Pi_d' = -\alpha \delta F'(v_d) p_d' (p_d - \Pi_d) + \alpha \left[1 - F(v_d)\right] (p_d' - \Pi_d').$$

The FOC is given by

$$\Pi_{s,d}' = 0 \Rightarrow p_d - \Pi_d = \frac{1 - F(v_d)}{\delta F'(v_d)}. \tag{14}$$

Substitute the FOC into the expression for Π_d to obtain

$$\Pi_d = \frac{\alpha \left[1 - F\left(v_d\right)\right]^2}{\bar{\delta} \delta F'\left(v_d\right)}. \tag{15}$$

Expressions (14) and (15) together imply that p_d equals to the expression on display at (7). To verify the second order condition, note that

$$\frac{\bar{\delta}}{\alpha} \Pi_d'' = -\delta^2 F''\left(v_d\right) p_d'^2 \left(p_d - \Pi_d\right) - \delta F'\left(v_d\right) p_d'' \left(p_d - \Pi_d\right) - \delta F'\left(v_d\right) p_d' \left(p_d' - \Pi_d'\right)$$
$$- \delta p_d' F'\left(v_d\right) \left(p_d' - \Pi_d'\right) + \left[1 - F\left(v_d\right)\right]\left(p_d'' - \Pi_d''\right).$$

Substitute $\Pi_d' = 0$ and use (14) to obtain (we omit the argument v_d where understood):

$$\Pi_d'' = -\frac{\delta p_d'^2}{\frac{\bar{\delta}}{\alpha} + 1 - F} \times \frac{2F'\left[1 - F\right] F''}{F'}.$$

The first multiplicative term is clearly positive, whereas the second term is positive because of log-concavity. It follows that $\Pi'' < 0$; thus the solution to the first order condition yields a maximum.

Regular Sellers. The problem of a regular seller is similar. We have

$$\delta \Pi_r = \alpha \left[1 - F\left(v_r\right)\right]\left(v_r - \Omega - \Pi_r\right) + \mu \left[\Pi_d - \Pi_r\right]. \tag{16}$$

Differentiate Π_r with respect to l_r to obtain the first-order condition (imposing $\Omega' = \Pi_d' = 0$ since they are taken as given):

$$\Pi_r' = 0 \Leftrightarrow p_r - \Pi_r = \frac{1 - F\left(v_r\right)}{\delta F'\left(v_r\right)}. \tag{17}$$

The second order condition can be verified similarly.

Use (15), (16) and (17) to get

$$\Pi_r = \frac{\alpha \left[1 - F\left(v_r\right)\right]^2}{\delta \left(\mu + \delta\right) F'\left(v_r\right)} + \frac{\alpha \mu \left[1 - F\left(v_d\right)\right]^2}{\bar{\delta} \delta \left(\mu + \delta\right) F'\left(v_d\right)}.$$

Substitute Π_r into (17) to obtain the offer curve of a regular seller, given by (6). ∎

Proof of Lemma 2.

Differentiate p_r and p_d, given by (6) and (7), with respect to v_r and v_d to obtain:

$$\frac{\partial p_r}{\partial v_r} = -\frac{F_r'^2 + F_r''\left(1 - F_r\right)}{\delta F_2'^2} - \frac{\alpha \left(1 - F_r\right)}{\delta \left(\mu + \delta\right)} \left[\frac{2F_r'^2 + F_r''\left(1 - F_r\right)}{F_r'^2}\right] < 0, \tag{18}$$

$$\frac{\partial p_r}{\partial v_d} = -\frac{\alpha \mu \left(1 - F_d\right)}{\delta \bar{\delta} \left(\mu + \delta\right)} \left[\frac{2F_d'^2 + F_d''\left(1 - F_d\right)}{F_d'^2}\right] < 0, \quad \frac{\partial p_d}{\partial v_r} = 0, \tag{19}$$

$$\frac{\partial p_d}{\partial v_d} = -\frac{F_d'^2 + F_d''\left(1 - F_d\right)}{\delta F_d'^2} - \frac{\alpha \left(1 - F_d\right)}{\delta \bar{\delta}} \left[\frac{2F_d'^2 + F_d''\left(1 - F_d\right)}{F_d'^2}\right] < 0, \tag{20}$$

where $F_j := F\left(v_j\right)$. The expression $\frac{\partial p_r}{\partial v_r}$, $\frac{\partial p_r}{\partial v_d}$ and $\frac{\partial p_d}{\partial v_d}$ are negative because of log-concavity. The fact that $\frac{\partial p_d}{\partial v_d} < \frac{\partial p_r}{\partial v_d}$ is immediate after comparing them term by term. ∎

Proof of Proposition 4. The proof of existence and uniqueness is relegated to Appendix II. Below we prove the rest of the claims in the proposition.

Fire Sales. We will show that $v_r^* > v_d^*$ and $p_r^* > p_d^*$. Suppose $v_r^* = v_d^* = v$ and notice that

$$\Delta_r(v,v) - \Delta_d(v,v) = \frac{\alpha\left(\bar{\delta} - \delta\right)}{\bar{\delta}\delta(\mu+\delta)} \times \frac{[1 - F(v)]^2}{F'(v)} > 0,$$

which contradicts the equilibrium condition $\Delta_r(v_r^*, v_d^*) - \Delta_d(v_r^*, v_d^*) = 0$. Observe that

$$\frac{\partial(\Delta_r - \Delta_d)}{\partial v_r} = \frac{\partial p_r}{\partial v_r} - \frac{1}{\delta} - \frac{\partial p_d}{\partial v_r} < 0$$

because $\frac{\partial p_r}{\partial v_r} < 0$ and $\frac{\partial p_d}{\partial v_r} = 0$ (Lemma 2). It follows that $\Delta_r(v_r^*, v_d^*) = \Delta_d(v_r^*, v_d^*)$ is satisfied only when $v_r^* > v_d^*$. The inequality $p_r^* > p_d^*$ is immediate since $p_r^* - p_d^* = (v_r^* - v_d^*)/\delta > 0$.

For future reference note that

$$p_r^* - p_d^* = \frac{1 - F_r}{\delta F_r'} - \frac{1 - F_d}{\delta F_d'} + \frac{\alpha}{(\mu+\delta)}\left[\frac{(1 - F_r)^2}{\delta F_r'} - \frac{(1 - F_d)^2}{\bar{\delta} F_d'}\right] > 0. \tag{21}$$

Comparative Statics. The equilibrium values of v_r^* and v_d^* simultaneously satisfy

$$\Delta_r(v_r^*, v_d^*) = 0 \text{ and } \Delta_d(v_r^*, v_d^*) = 0.$$

Omit the superscript * and note that (General Implicit Function Theorem)

$$\frac{dv_j}{du} = \frac{\det B_j(u)}{\det A}, \text{ for any } u = \bar{\delta}, \mu \text{ and } j = r, d,$$

where

$$B_r(u) = \begin{bmatrix} -\frac{\partial \Delta_r}{\partial u} & \frac{\partial \Delta_r}{\partial v_d} \\ -\frac{\partial \Delta_d}{\partial u} & \frac{\partial \Delta_d}{\partial v_d} \end{bmatrix}, \quad B_d(u) = \begin{bmatrix} \frac{\partial \Delta_r}{\partial v_r} & -\frac{\partial \Delta_r}{\partial u} \\ \frac{\partial \Delta_d}{\partial v_r} & -\frac{\partial \Delta_d}{\partial u} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial \Delta_r}{\partial v_r} & \frac{\partial \Delta_r}{\partial v_d} \\ \frac{\partial \Delta_d}{\partial v_r} & \frac{\partial \Delta_d}{\partial v_d} \end{bmatrix}.$$

Note that

$$\det A = \frac{\partial \Delta_r}{\partial v_r}\frac{\partial \Delta_d}{\partial v_d} - \frac{\partial \Delta_d}{\partial v_r}\frac{\partial \Delta_r}{\partial v_d} > 0$$

since

$$\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_d}{\partial v_r} < 0 \text{ and } \frac{\partial \Delta_d}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} < 0 \text{ (see (22) and (23)).}$$

It follows that

$$sign\left(\frac{\partial v_j}{\partial u}\right) = sign\left(\det B_j(u)\right).$$

Below we investigate the signs of the determinants. To do so we need the following partial derivatives.

Partial Derivatives. Here we obtain the partial derivatives of Ω, p_d and p_r with respect to $\bar{\delta}$ and μ. To start, differentiate Ω, given by (2), to obtain

$$\frac{\partial \Omega}{\partial \bar{\delta}} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = \frac{\theta^2 \alpha [1 - F_d]}{\delta \mu^2}\int_{v_d}^{v_r}[1 - F(v)]\,dv > 0.$$

Notice that $\frac{\partial \Omega}{\partial \mu}$ is positive since $v_r^* > v_d^*$. Now differentiate p_r and p_d, given by (6) and (7), to obtain

$$\frac{\partial p_d}{\partial \overline{\delta}} = -\frac{\alpha \left[1 - F_d\right]^2}{\overline{\delta}^2 \delta F_d'} < 0, \qquad \qquad \frac{\partial p_r}{\partial \overline{\delta}} = \frac{\mu}{(\mu + \delta)} \frac{\partial p_d}{\partial \overline{\delta}} < 0$$

$$\frac{\partial p_d}{\partial \mu} = 0, \qquad \frac{\partial p_r}{\partial \mu} = -\frac{\alpha}{(\mu + \delta)^2} \left[\frac{(1 - F_r)^2}{\delta F_r'} - \frac{(1 - F_d)^2}{\overline{\delta} F_d'}\right] < 0.$$

Note that F_j' and F_j stand for $F'\left(v_j^*\right)$ and $F\left(v_j^*\right)$. The signs of the first three expressions are obvious. To see why $\frac{\partial p_r}{\partial \mu} < 0$ focus on the inequality above in (21) and notice that $\frac{\partial p_r}{\partial \mu}$ is negative if in (21) the expression in square brackets is positive. The term

$$\frac{1 - F_r}{F_r'} - \frac{1 - F_d}{F_d'}$$

in (21) is negative because $(1 - F)/F'$ decreases in v (log-concavity) and $v_r^* > v_d^*$. Therefore the expression in square brackets in (21) must be positive.

Reserve Values v_r^* and v_d^*. Now we can investigate the signs of $dv_j^*/d\overline{\delta}$ and $dv_j^*/d\mu$. To do so we need to determine the signs of $\det B_j \left(\overline{\delta}\right)$ and $\det B_j \left(\mu\right)$.

- Since $\partial \Omega/\partial \overline{\delta} = 0$ we have

$$\det B_d \left(\overline{\delta}\right) = \frac{\partial \Delta_d}{\partial v_r} \frac{\partial p_r}{\partial \overline{\delta}} - \frac{\partial \Delta_r}{\partial v_r} \frac{\partial p_d}{\partial \overline{\delta}}.$$

  Furthermore, since

$$\frac{\partial p_d}{\partial \overline{\delta}} < \frac{\partial p_r}{\partial \overline{\delta}} < 0 \text{ and } \frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_d}{\partial v_r} < 0$$

  it follows that $\det B_d \left(\overline{\delta}\right) < 0 \implies dv_d^*/d\overline{\delta} < 0.$

- Because $\partial p_d/\partial \mu = 0$ we have

$$\det B_d \left(\mu\right) = \frac{\partial \Delta_d}{\partial v_r} \frac{\partial p_r}{\partial \mu} + \frac{\partial \Omega}{\partial \mu} \left[\frac{\partial \Delta_d}{\partial v_r} - \frac{\partial \Delta_r}{\partial v_r}\right].$$

  Since

$$\frac{\partial p_r}{\partial \mu} < 0, \; \frac{\partial \Omega}{\partial \mu} > 0 \text{ and } \frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_d}{\partial v_r} < 0$$

  it follows that $\det B_d \left(\mu\right) > 0 \implies dv_d^*/d\mu > 0.$

- Since

$$\frac{\partial p_r}{\partial \overline{\delta}} = \frac{\mu}{\mu + \delta} \frac{\partial p_d}{\partial \overline{\delta}} < 0 \text{ and } \frac{\partial \Omega}{\partial \overline{\delta}} = 0$$

  it is easy to verify that

$$\det B_r \left(\overline{\delta}\right) = \frac{\partial p_d}{\partial \overline{\delta}} \left[\frac{\partial p_r}{\partial v_d} - \frac{\mu}{\mu + \delta} \frac{\partial p_d}{\partial v_d} + \frac{\delta}{\mu + \delta} \frac{\partial \Omega}{\partial v_d} + \frac{\mu}{\mu + \delta}\right].$$

  The expressions for $\frac{\partial \Omega}{\partial v_d}$, $\frac{\partial p_r}{\partial v_d}$ and $\frac{\partial p_d}{\partial v_d}$ are given by (13), (18) and (20). Using these, one can show that the expression inside the square brackets equals to

$$\frac{\mu}{\mu + \delta} \frac{F_d'^2 + F_d'' \left(1 - F_d\right)}{F_d'^2} + \frac{\alpha \theta'}{\mu + \delta} \int_{v_d}^{v_r} \left[1 - F\left(v\right)\right] dv + \frac{\mu \theta}{\mu + \delta} > 0.$$

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The first term is positive because of log-concavity and the second term is positive since $\theta' > 0$ and $v_r^* > v_d^*$. It follows that $\det B_r\left(\overline{\delta}\right) < 0 \Longrightarrow dv_r^*/d\overline{\delta} < 0$.

- Recalling $\partial p_d/\partial\mu = 0$ we obtain

$$\det B_r\left(\mu\right) = \frac{\partial\Omega}{\partial\mu}\left[\frac{\partial\Delta_r}{\partial v_d} - \frac{\partial\Delta_d}{\partial v_d}\right] + \frac{\partial\Delta_d}{\partial v_d}\frac{\partial p_r}{\partial\mu}.$$

The first term is positive since

$$\frac{\partial\Delta_d}{\partial v_d} < \frac{\partial\Delta_r}{\partial v_d} < 0 \text{ and } \frac{\partial\Omega}{\partial\mu} > 0.$$

The second term is also positive since $\partial\Delta_d/\partial v_d < 0$ and $\partial p_r/\partial\mu < 0$. It follows that $\det B_r\left(\mu\right) > 0 \Longrightarrow dv_r^*/d\mu > 0$.

Finally, we investigate the signs of $dp_j^*/d\overline{\delta}$ and $dp_j^*/d\mu$.

Prices. Totally differentiating p_j with respect to μ one obtains

$$\frac{dp_j}{d\mu} = \frac{\partial p_j}{\partial\mu} + \frac{\partial p_j}{\partial v_r}\frac{dv_r}{d\mu} + \frac{\partial p_j}{\partial v_d}\frac{dv_d}{d\mu}.$$

Recall that

$$\frac{\partial p_r}{\partial\mu} < \frac{\partial p_d}{\partial\mu} = 0, \ \frac{\partial p_j}{\partial v_r} \leq 0, \ \frac{\partial p_j}{\partial v_d} < 0 \text{ and } \frac{dv_j}{d\mu} > 0.$$

Hence $dp_j/d\mu < 0$. To show $dp_j/d\overline{\delta} < 0$, recall that $p_j = v_j/\delta - \Omega$ in equilibrium. Differentiation with respect to $\overline{\delta}$ yields

$$\frac{dp_j}{d\overline{\delta}} = \frac{dv_j}{\delta d\overline{\delta}} - \frac{\partial\Omega}{\partial v_r}\frac{dv_r}{d\overline{\delta}} - \frac{\partial\Omega}{\partial v_d}\frac{dv_d}{d\overline{\delta}},$$

which is negative since $dv_j/d\overline{\delta} < 0$ and $\partial\Omega/\partial v_j < 0$. ∎

Proof of Proposition 5. Notice that

$$\frac{d\overline{p}\left(t\right)}{dt} = -\frac{dg\left(t\right)}{dt}\left(p_r^* - p_d^*\right)$$

One can verify that

$$\frac{dg\left(t\right)}{dt} \propto \mu e^{-(\alpha_d+\alpha_r+\mu)} > 0.$$

It follows that $\overline{p}' < 0$ since $p_r^* > p_d^*$. Finally note that

$$\frac{d\sigma^2}{dt} = \left(p_r^* - p_d^*\right)\left[g'\left(2\overline{p} - p_r^* - p_d^*\right) + 2g\overline{p}'\right].$$

Clearly $d\sigma^2/dt$ shares the sign of the expression in the square brackets, since $p_r^* > p_d^*$. One can verify that $\lim_{t\to 0} g\left(t\right) = 0$ and $\lim_{t\to\infty} g\left(t\right) = 1$ so that $\lim_{t\to 0}\overline{p}\left(t\right) = p_r^*$ and $\lim_{t\to\infty}\overline{p}\left(t\right) = p_d^*$. It follows that $d\sigma^2/dt$ is positive for t small and negative for t large because $g' > 0$ and $\overline{p}' < 0$. In other words σ^2 first rises and subsequently falls with t. ∎

Proof of Proposition 6. Using $r(t)$ and $d(t)$, given by (9) and (10), one can obtain the density of time

to sale and the expected time to sale. Evaluating the integral in (10) we have

$$d\left(t\right) = \frac{\mu e^{-\alpha_d t} - \mu r\left(t\right)}{\mu - \alpha_d + \alpha_r}.$$

Note that

$$TOM = \int_0^\infty \left[r\left(t\right) + d\left(t\right)\right] dt \quad \text{and} \quad \gamma = -\frac{d\left[r\left(t\right) + d\left(t\right)\right]}{dt}.$$

Basic algebra reveals that TOM and γ are given by the expressions on display in Proposition 6. It is easy to verify that γ is positive and that

$$\int_0^\infty \gamma dt = -\left[r\left(t\right) + d\left(t\right)\right]|_0^\infty = 1.$$

To analyze the shape of γ note that

$$\gamma' = \frac{-\mu \alpha_d^2 e^{-\alpha_d t} + \left(\alpha_d - \alpha_r\right)\left(\mu + \alpha_r\right)^2 e^{-(\mu + \alpha_r)t}}{\mu - \alpha_d + \alpha_r},$$

where

$$\alpha_d - \alpha_r = \alpha \left[F\left(v_r^*\right) - F\left(v_d^*\right)\right] > 0 \text{ since } v_r^* > v_d^*.$$

Notice that the denominator could be either positive or negative. It follows that:

$$\text{If } \mu > \alpha_d - \alpha_r \text{ then } \gamma'\left(t\right) > 0 \Leftrightarrow \frac{\left(\alpha_d - \alpha_r\right)\left(\mu + \alpha_r\right)^2}{\mu \alpha_d^2} > e^{(\mu + \alpha_r - \alpha_d)t}.$$

$$\text{If } \mu < \alpha_d - \alpha_r \text{ then } \gamma'\left(t\right) > 0 \Leftrightarrow \frac{\left(\alpha_d - \alpha_r\right)\left(\mu + \alpha_r\right)^2}{\mu \alpha_d^2} < e^{(\mu + \alpha_r - \alpha_d)t}.$$

First note that $\lim_{t \to \infty} \gamma' < 0$, i.e., γ is monotone decreasing for t large. Now evaluate $\lim_{t \to 0} \gamma$. Note that in the first line the exponential term is minimum when $t = 0$ whereas in the second line it is maximum when $t = 0$. Hence

$$\gamma'\left(0\right) > 0 \text{ if } \frac{\mu}{\alpha} > \frac{\left[1 - F\left(v_r^*\right)\right]^2}{F\left(v_r^*\right) - F\left(v_d^*\right)}.$$

Clearly if $\gamma'\left(0\right) > 0$ then γ first rises and then falls (hump-shape). Otherwise if $\gamma'\left(0\right) < 0$ it falls monotonically. ∎

APPENDIX II: EXISTENCE (not intended for publication)

Let

$$\kappa_j\left(v_r\right) := \{v_d \in [0,1] \mid \Delta_j\left(v_r, v_d\right) = 0\}$$

be the locus of $\Delta_j\left(v_r, v_d\right)$. The following Lemma guarantees that the κ_d and κ_r intersect once in the $v_r - v_d$ space. Then using standard arguments we complete the proof of existence.

Lemma 7 *The simultaneous equations*

$$\Delta_r\left(v_r, v_d\right) = p_r - \frac{v_r}{\delta} + \Omega \quad and \quad \Delta_d\left(v_r, v_d\right) = p_d - \frac{v_d}{\delta} + \Omega$$

define κ_r and κ_d as implicit and strictly decreasing functions of v_r with $\frac{d\kappa_r}{dv_r} < \frac{d\kappa_d}{dv_r} < 0$. Furthermore there exists some $0 < \underline{v}_d < \overline{v}_d < 1$ and $\underline{v}_r \in (0,1)$ such that $\kappa_d\left(0\right) = \overline{v}_d$, $\kappa_d\left(1\right) = \underline{v}_d$ and $\kappa_r\left(\underline{v}_r\right) = 1$. Last either there exists some $\overline{v}_r \in (\underline{v}_r, 1)$ such that $\kappa_r\left(\overline{v}_r\right) = 0$ as in Figure 2a or there exists some $\underline{\underline{v}}_d \in (0, \underline{v}_d)$ such that $\kappa_r\left(1\right) = \underline{\underline{v}}_d$ as in Figure 2b.

Proof. We will first demonstrate that $\frac{d\kappa_r}{dv_r} < \frac{d\kappa_d}{dv_r}$ and then we will focus on the existence of boundaries $\underline{v}_j$, $\overline{v}_j$. Notice that

$$\frac{\partial \Delta_r}{\partial v_r} = \frac{\partial p_r}{\partial v_r} - \frac{1}{\delta} + \frac{\partial \Omega}{\partial v_r} < \frac{\partial \Delta_d}{\partial v_r} = \frac{\partial p_d}{\partial v_r} + \frac{\partial \Omega}{\partial v_r} < 0, \tag{22}$$

$$\frac{\partial \Delta_d}{\partial v_d} = \frac{\partial p_d}{\partial v_d} - \frac{1}{\delta} + \frac{\partial \Omega}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} = \frac{\partial p_r}{\partial v_d} + \frac{\partial \Omega}{\partial v_d} < 0. \tag{23}$$

These inequalities follow from the facts that $\frac{\partial \Omega}{\partial v_j} < 0$ (Lemma 1) and $\frac{\partial p_j}{\partial v_j} < \frac{\partial p_{\tilde{j}}}{\partial v_j} < 0$ (Lemma 2). Therefore $\Delta_j\left(v_r, v_d\right) = 0$ defines $v_d = \kappa_j\left(v_r\right)$ as an implicit function of v_r (Implicit Function Theorem) with

$$\frac{d\kappa_j}{dv_r} = -\frac{\partial \Delta_j/\partial v_r}{\partial \Delta_j/\partial v_d} < 0.$$

Since $\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_d}{\partial v_r} < 0$ and $\frac{\partial \Delta_d}{\partial v_d} < \frac{\partial \Delta_r}{\partial v_d} < 0$ it is obvious that $\frac{d\kappa_r}{dv_r} < \frac{d\kappa_d}{dv_r} < 0$.

Boundaries. Start by evaluating $\Delta_d\left(v_r, v_d\right)$ at end points. Observe that

$$\Delta_d\left(0,0\right) > \Delta_d\left(1,0\right) = p_d\left(1,0\right) + \Omega\left(1,0\right) > 0.$$

In addition

$$\Delta_d\left(0,1\right) = \Delta_d\left(1,1\right) = -1/\delta < 0,$$

because $\theta\left(1\right) = 1$. Since $\Delta_d\left(1,0\right) > 0$ and $\Delta_d\left(1,1\right) < 0$ and Δ_d decreases in v_d the Intermediate Value Theorem guarantees existence of some $\underline{v}_d \in (0,1)$ such that $\Delta_d\left(1, \underline{v}_d\right) = 0$, i.e., $\kappa_d\left(1\right) = \underline{v}_d$. Similarly $\Delta_d\left(0,0\right) > 0$ and $\Delta_d\left(0,1\right) < 0$ implies existence of some $\overline{v}_d \in (0,1)$ such that $\Delta_d\left(0, \overline{v}_d\right) = 0$, i.e., $\kappa_d\left(0\right) = \overline{v}_d$. Note that $\kappa_d\left(1\right) < \kappa_d\left(0\right)$ and since κ_d decreases in v_r we have $\underline{v}_d < \overline{v}_d$.

Now evaluate $\Delta_r\left(v_r, v_d\right)$ at end points. Similar to above, one can show that $\Delta_r\left(0,0\right) > \Delta_r\left(0,1\right) > 0$ and $\Delta_r\left(1,1\right) = -1/\delta < 0$. However $\Delta_r\left(1,0\right)$ can be positive or negative.

The existence of $\underline{v}_r \in (0,1)$ follows from the facts that $\Delta_r\left(0,1\right) > 0$, $\Delta_r\left(1,1\right) < 0$ and that Δ_r decreases in v_r. The Intermediate Value Theorem guarantees that there is some $\underline{v}_r \in (0,1)$ such that $\Delta_r\left(\underline{v}_r, 1\right) = 0$ which is equivalent to $\kappa_r\left(\underline{v}_r\right) = 1$. Existence of $\overline{v}_r$ or $\underline{\underline{v}}_d$ hinges on the sign of $\Delta_r\left(1,0\right)$, as we show below.

- Suppose $\Delta_r(1,0) < 0$: Since $\Delta_r(0,0) > 0$ there exists some $\overline{v}_r \in (0,1)$ such that $\Delta_r(\overline{v}_r, 0) = 0$ or equivalently $\kappa_r(\overline{v}_r) = 0$, and since κ_r is a decreasing function of v_r we have $\underline{v}_r < \overline{v}_r$.

- Suppose $\Delta_r(1,0) > 0$: First we will show that $\Delta_r(1, \underline{v}_d) < 0$. Notice that

$$\Delta_d(1, \underline{v}_d) - \Delta_r(1, \underline{v}_d) = \frac{1 - F(\underline{v}_d)}{\delta F'(\underline{v}_d)} + \frac{\alpha}{\overline{\delta}(\mu + \delta)} \frac{[1 - F(\underline{v}_d)]^2}{F'(\underline{v}_d)} + \frac{1 - \underline{v}_d}{\delta} > 0.$$

Since $\Delta_d(1, \underline{v}_d) = 0$ it is must be that $\Delta_r(1, \underline{v}_d) < 0$. Now, since $\Delta_r(1,0) > 0$ there exists some $\underline{\underline{v}}_d \in (0, \underline{v}_d)$ such that $\Delta_r\left(1, \underline{\underline{v}}\right) = 0$ or equivalently $\kappa_r(1) = \underline{\underline{v}}_d$. $\blacksquare$

Existence and Uniqueness. Below we argue that there exists a unique interior v_r^* satisfying $\kappa_r(v_r^*) = \kappa_d(v_r^*)$. Define $\varkappa(v_r) := \kappa_r(v_r) - \kappa_d(v_r)$ and notice that it decreases in v_r since

$$\frac{d\varkappa}{dv_r} = \frac{d\kappa_r}{dv_r} - \frac{d\kappa_d}{dv_r} < 0.$$

Now we will verify that $\varkappa(\underline{v}_r) > 0$ and $\varkappa(1) < 0$. Indeed $\varkappa(\underline{v}_r) = \kappa_r(\underline{v}_r) - \kappa_d(\underline{v}_r) = 1 - \kappa_d(\underline{v}_r) > 0$ since $\kappa_d(\underline{v}_r) < \kappa_d(0) = \overline{v}_d < 1$. Similarly $\varkappa(1) = \kappa_r(1) - \kappa_d(1) = \kappa_r(1) - \underline{v}_d < 0$ since $\kappa_r(1)$ is either negative or equals to $\underline{\underline{v}}_d$ both of which are smaller than $\underline{v}_d$. Consequently the Intermediate Value Theorem guarantees existence of a unique $v_r^* \in (\underline{v}_r, 1)$ such that $\kappa_r(v_r^*) = \kappa_d(v_r^*) = v_d^*$.