Essential Data, Budget Sets and Rationalization

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Abstract

According to a minimalist version of Afriat's theorem, a consumer behaves as a utility maximizer if and only if a feasibility matrix associated with his choices is cyclically consistent. An "essential experiment" consists of observed consumption bundles $(x_1, \cdots, x_n)$ and a feasibility matrix $\boldsymbol{\alpha}$. Starting with a standard experiment, in which the economist has specific budget sets in mind, we show that the necessary and sufficient condition for the existence of a utility function rationalizing the experiment, namely, the cyclical consistency of the associated feasibility matrix, is equivalent to the existence, for any budget sets compatible with the deduced essential experiment, of a utility function rationalizing them (and typically depending on them). In other words, the conclusion of the standard rationalizability test, in which the economist takes budget sets for granted, does not depend on the full specification of the underlying budget sets but only on the essential data that these budget sets generate. Starting with an essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$, we show that the cyclical consistency of $\boldsymbol{\alpha}$, together with a further consistency condition involving both $(x_1, \cdots, x_n)$ and $\boldsymbol{\alpha}$, guarantees that the essential experiment is rationalizable almost robustly, in the sense that there exists a single utility function which rationalizes at once almost all budget sets which are compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$. The conditions are also trivially necessary.

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Key words: Afriat's theorem, budget sets, cyclical consistency, rational choice, revealed preference.

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1 Introduction

Afriat (1967)'s theorem has been revisited in a few recent papers, which propose new proofs (Fostel *et al.*, 2004; Chung-Piaw and Vohra, 2003), extensions (Forges and Minelli, 2009) or new interpretations (Ekeland and Galichon, 2010) of the result. In all these papers, as already in the classical one (see, e.g., Varian, 1982), information on the choices of a given consumer at various dates $j = 1, \cdots, n$ is summarized by an $n \times n$ feasibility matrix. The (j, k) entry of this matrix takes the value -1, 0 or 1 and indicates to which extent the item (e.g., a consumption bundle) that has been chosen by the consumer at date k is affordable or not at date j.[1] According to (a minimalist version of) Afriat's theorem, the consumer behaves as a utility maximizer if and only if the feasibility matrix satisfies a tractable property, referred to as "cyclical consistency". This version of Afriat's theorem is recalled in Section 2 as Proposition 1.

In a standard framework, the observed choices of the consumer are bundles $x_1, \cdots, x_n \in \mathbb{R}^{\ell}_{+}$, which define, together with the associated feasibility matrix, what we call in this paper an "essential experiment". To test the consumer's rationality, the economist basically has to check whether the feasibility matrix is cyclically consistent. When performing this test, the economist typically has precise budget sets in mind for every date. As shown by Forges and Minelli (2009), even if the budget sets are quite general (namely, just compact and comprehensive), Afriat's original constructive approach applies: if the feasibility matrix is cyclically consistent, the economist can derive an explicit utility function rationalizing the data. This is another version of Afriat's theorem, which is stated in Section 3 as Proposition 2.

Not surprisingly, the above utility function depends on the economist's budget sets. But, especially if these budget sets are complex, e.g., involve tariffs or taxes, the consumer's budget sets (namely, the ones over which he possibly optimizes) might not fully coincide with the economist's ones. For instance, if the consumer buys small quantities of some good at every date $j = 1, ..., n$, he may not understand that a low unit price is charged to large quantities of that good.

We are thus led to the following question:

Given an essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ in which the feasibility matrix $\boldsymbol{\alpha}$ is cyclically consistent, can we construct a utility function v which *robustly* rationalizes $(x_1, \cdots, x_n; \boldsymbol{\alpha})$, in the sense that $v(x_j)$ maximizes v over B_j, for *any* family (B_j) of budget sets compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$?

[1] Denoting the feasibility matrix as $\boldsymbol{\alpha} = (\alpha_{jk})$, $\alpha_{jk} = -1$ if item k is affordable at date j without exhausting the consumer's revenue, $\alpha_{jk} = 0$ if item k is affordable at date j and exhausts the consumer's revenue, $\alpha_{jk} = +1$ if item k is not affordable at date j.

The motivation for such a utility function v is clear: v would not be sensitive to those specific aspects of the budget sets that the consumer might not perceive.

First of all, we observe that the previous question is not meaningful unless the essential experiment satisfies some basic consistency requirement guaranteeing that there indeed exists (compact, comprehensive) budget sets that are compatible with it. We introduce the property that the essential experiment "contains no contradictory statement" in order to capture such a requirement.

Next, we construct an essential experiment $(x_1, x_2; \boldsymbol{\alpha}) \in \mathbb{R}^2_+$ which contains no contradictory statement, where $\boldsymbol{\alpha}$ is cyclically consistent, and which cannot be rationalized robustly. This simple example is by no means pathological and shows that, formulated exactly as above, the question cannot be answered positively.

Nonetheless, we prove that every essential experiment $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ which contains no contradictory statement and where $\boldsymbol{\alpha}$ is cyclically consistent can be rationalized in an almost robust way, in the sense that for every sufficiently small ϵ, there exist an almost largest family $(\mathbf{B}^\epsilon)$ of budget sets compatible with $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ and a utility function v^ϵ rationalizing $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ over $(\mathbf{B}^\epsilon)$. It is not difficult to prove that, conversely, if $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ can be rationalized in an almost robust way, then $(x_1, \cdots, x_n; \boldsymbol{\alpha})$ contains no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent. This is the main content of the theorem given in Section 4.

Our results can be interpreted in the standard framework where the economist starts with a priori given budget sets. From these and the observed consumption bundles, one can deduce an essential experiment. A by-product of the theorem (already contained in Proposition 2) is that the necessary and sufficient condition for the existence of a utility function rationalizing the economist's budget sets (namely, the cyclical consistency of the feasibility matrix or the Generalized Axiom of Revealed Preference - GARP -) is also equivalent to the existence, for any budget sets compatible with the deduced essential experiment, of a utility function rationalizing them. In other words, the conclusion of the standard rationalizability test, in which the economist takes budget sets for granted, does not depend on the full specification of the underlying budget sets but only on the essential data that these budget sets generate; the economist's conclusion automatically applies to a whole family of budget sets. This also means that there is no way to test whether standard data - involving a full description of budget sets - are rationalizable without testing at the same time that a whole class of data, based on a variety of different budget sets, are also rationalizable.

2 Essential data

Let $\boldsymbol{\alpha} = (\alpha_{jk})_{j,k \in N}$ be a **feasibility matrix** as described in Introduction, i.e. an $n \times n$ matrix with diagonal terms equal to 0 and remaining terms equal to -1,0 or 1, which summarize the affordability of observed choices at each step. Given this essential data which can be identified with a *restricted choice* experiment, a traditional question is to verify in which extent the choices are consistent with the data, namely if there exists a rationalization. This amounts to finding numbers v_j, for every item j, such that $v_j \geq v_k$ for every affordable item k at date j, with strict inequality if k does not exhaust entirely the revenue of the agent.

Definition 1 *Utils $(v_j)_{j \in N}$ are said to rationalize the feasibility matrix $\boldsymbol{\alpha}$, if, for every $j \in N$, $v_j \geq v_k$ for every $k \in N$ such that $\alpha_{jk} \leq 0$, and $v_j > v_k$ for every $k \in N$ such that $\alpha_{jk} < 0$.*

The following tractable condition of cyclical consistency is the usual test to verify whether or not an experiment can be rationalized.

Definition 2 *An $n \times n$ real matrix $A = (a_{jk})_{j,k \in N}$ is cyclically consistent if for every chain $j, k, \ell, ..., r$, $a_{jk} \leq 0, a_{k\ell} \leq 0, ..., a_{rj} \leq 0$ implies all terms are 0.*

In the framework of revealed preference analysis, the use of basic data $\boldsymbol{\alpha}$ is not new, and is the key ingredient to derive Afriat's inequalities in the consumer problem. More precisely, the role of the feasibility matrix is identified in the next result, which is actually implicit in the classical Afriat (1967)'s theorem. For recent proofs see also Fostel *et al.* (2004); Chung-Piaw and Vohra (2003); Ekeland and Galichon (2010).

Proposition 1 *The following conditions are equivalent:*

1. *There exist utils $(v_i)_{i \in N}$ rationalizing the feasibility matrix $\boldsymbol{\alpha}$.*
2. *The feasibility matrix $\boldsymbol{\alpha}$ is cyclically consistent.*

Proof [1. $\Rightarrow$ 2.] is proved in Ekeland and Galichon (2010, replacing R_{ij} by α_{ij} in the proof of 3. $\Rightarrow$ 1. in Theorem 0). [2. $\Rightarrow$ 1.] is proved in Fostel *et al.* (2004, replacing A' by $\boldsymbol{\alpha}$ page 215).■

Remark 1 *Ekeland and Galichon (2010) derive a dual interpretation of matrix $\boldsymbol{\alpha}$ in terms of a market with n traders and an indivisible good (house) to be traded (see also Shapley and Scarf (1974)). In the autarky allocation, each trader j owns house j. Matrix $\boldsymbol{\alpha}$ summarizes then preferences of traders in the initial autarky allocation: $\alpha_{jk} = 1$ is strict preference of his own house over house k; $\alpha_{jk} = -1$ is strict preference of house k over his own house; $\alpha_{jk} = 0$*

is indifference of trader j between house k and his own house. In this dual interpretation, Proposition 1 actually amounts to: the autarky allocation is a no trade equilibrium allocation supported by prices $\pi_j = -v_j$ (condition 1.) if and only if it is Pareto optimal (condition 2.).

3 Budget sets

From now on we turn to the single consumer problem with ℓ consumption goods, where utility is now defined by a function $v : \mathbb{R}^\ell_+ \to \mathbb{R}$. Hence, the economist observes consumption bundles in addition to the essential data. This leads to the following notion of experiment which is hereafter the basic data in our revealed preference analysis.

Definition 3 *An essential (consumer) experiment $(\mathbf{x}, \boldsymbol{\alpha})$ consists of observed consumption bundles $(x_j)_{j\in N}$, $x_j \in \mathbb{R}^\ell_{++}$, and a feasibility matrix $\boldsymbol{\alpha}$.*

We adopt a standard approach to model *general* budget sets. The formulation encompasses the following cases: classical linear budget sets; budget sets defined by the intersection of linear inequalities, as in Yatchew (1985); convex but non-linear budget sets, as in Matzkin (1991). Therefore the budget of the consumer can result from the application of quantity constraints, taxes and other sources of non convexities.

Besides compactness, the crucial requirement is monotonicity (condition A.2 in the definition below).

Definition 4 *A set B_j is a budget set if*

A.1. B_j is a compact subset of $\mathbb{R}^\ell_+$,

A.2. B_j is comprehensive from below in $\mathbb{R}^\ell_+$; and if $x \in \mathrm{Fr}B_j$ then, for all $k \in [0,1)$, $kx \in B_j \setminus \mathrm{Fr}B_j$.[2]

The next definition is the natural extension of the classical notion of experiment with linear budget sets. First, the budget sets B_j are implicitly assumed to be known by the economist, who will make inferences over the consumer's choices. Second, consumption choices exhaust entirely the available revenue, at each given date. Note that the latter fact is also implicitly assumed in the classical theory, with linear budget sets defined by prices and the consumption choices at each date.

[2] Given a set $C \in \mathbb{R}^\ell_+$, let $\mathrm{Fr}\, C$ be the set $\{x \in C \mid \{x_k\} + \mathbb{R}^\ell_{++}\} \cap C = \emptyset\}$.

Definition 5 *An experiment* $(\mathbf{x}, \mathbf{B})$ *consists of observed consumption bundles* $x_j \in \mathbb{R}^{\ell}_{++}$ *and of budget sets* B_j*, such that* $x_j \in \mathrm{Fr}B_j$ *for every* $j \in N$.[3]

In the standard approach of revealed preference analysis, an experiment $(\mathbf{x}, \mathbf{B})$ is given. This formulation implicitly assumes that a rational consumer perfectly knows his budget set B_j for every $j \in N$. The economist is interested in testing whether the consumer chooses every consumption bundle "rationally" given the budget sets at each date.

Definition 6 *A utility function* v *is said to rationalize an experiment* $(\mathbf{x}, \mathbf{B})$ *if* $v(x_j) = \max_{x \in B_j} v(x)$ *for every* $j \in N$.

Next we we describe how to relate budget sets and the matrix $\boldsymbol{\alpha}$ in order to establish a rationalizability test of the consumer problem in terms of essential data only.

Definition 7 *Given an experiment* $(\mathbf{x}, \mathbf{B})$*, let* $A^{x,B}$ *denote the* $n \times n$ *matrix with entries* $a^{x,B}_{jk} = -1$ *if* $x_k \in \mathrm{int}B_j$*;* $a^{x,B}_{jk} = 0$ *if* $x_k \in \mathrm{Fr}B_j$*;* $a^{x,B}_{jk} = 1$ *if* $x_k \notin B_j$.

An essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ *admits a budget representation if there exists a family of budget sets* $(B_j)_{j \in N}$ *such that* $(\mathbf{x}, \mathbf{B})$ *is an experiment and* $A^{x,B} = \boldsymbol{\alpha}$*. A family* $(B_j)_{j \in N}$ *with this property is said compatible with* $(\mathbf{x}, \boldsymbol{\alpha})$.

Given an experiment $(\mathbf{x}, \mathbf{B})$, the economist can deduce the corresponding essential experiment by setting $\boldsymbol{\alpha} = A^{x,B}$. Alternatively, let us imagine that the essential experiment is the only available one (the full sample may be too complex to be fully memorized or the consumer privately knows his budget sets and the economist just obtains essential budgetary information from the consumer, in a "thought experiment"). Under this interpretation, the essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ does not necessarily admit a budget representation. In the next section, we introduce a tractable necessary and sufficient condition ("no contradictory statement") for this property to hold (see also Corollary 1 at the end of Section 4). For the time being, we just assume that $(\mathbf{x}, \boldsymbol{\alpha})$ admits a budget representation, as it is the case if the essential experiment is simply deduced from some experiment $(\mathbf{x}, \mathbf{B})$.[4]

The next result can be deduced from Proposition 3 in Forges and Minelli (2009).

Proposition 2 *Let* $(\mathbf{x}, \boldsymbol{\alpha})$ *be an essential experiment which admits a budget representation. The following conditions are equivalent:*

[3]Note also that the definitions of budget set and experiment imply altogether that any budget set considered hereafter has a nonempty interior.

[4]The reason why we postpone the introduction of the condition is simply to avoid repetitive arguments in the proofs of Proposition 2 and Theorem 1.

1. *For any* $(B_j)_{j\in N}$ *compatible with* $(\mathbf{x},\boldsymbol{\alpha})$*, there exists a locally non satiated, continuous utility function* v^B *rationalizing the experiment* $(\mathbf{x},\mathbf{B})$*.*

2. *The matrix* $\boldsymbol{\alpha}$ *is cyclically consistent.*

Proof [1. $\Rightarrow$ 2.] Since $(\mathbf{x},\mathbf{B})$ admits a budget representation, there exists a locally non satiated, continuous utility function v^B rationalizing an experiment $(\mathbf{x},\mathbf{B})$ where $A^{x,B}=\boldsymbol{\alpha}$. Hence, $v(x_j)\geq v(x_k)$ for every k such that $\alpha_{jk}\leq 0$; with strict inequalities if $\alpha_{jk}<1$, using local non satiation. Then Proposition 1 gives the result.

[2. $\Rightarrow$ 1.] Since $(\mathbf{x},\boldsymbol{\alpha})$ admits a budget representation, it holds that $\boldsymbol{\alpha}$ is cyclically consistent iff $(\mathbf{x},\mathbf{B})$ satisfies GARP, for every family $(B_j)_{j\in N}$ compatible with $(\mathbf{x},\boldsymbol{\alpha})$ using straightforward arguments. Then apply Proposition 3 in Forges and Minelli (2009) to conclude the proof.[5] In particular, the construction of the utility functions relies on the following arguments: for every compatible family $(B_j)_{j\in N}$, construct continuous, monotone mappings $(g_j^B)_{j\in N}$ to describe the budget sets as $B_j=\left\{x\in\mathbb{R}_+^\ell : g_j^B(x)\leq 0\right\}$; use cyclical consistency of the matrix with entries $(g_j^B(x_k))_{j,k\in N}$ to derive inequalities *à la* Afriat; and finally, thanks to these inequalities, construct an explicit a utility function v^B depending on the mappings $(g_j^B)_{j\in N}$.[6] ■

The previous proposition sheds further light on the standard rationalizability test, which is performed on the basis of the full experiment $(\mathbf{x},\mathbf{B})$, but only uses the matrix $A^{x,B}$, equal here to $\boldsymbol{\alpha}$. The economist designs the test with specific budget sets $(B_j)_{j\in N}$ in mind but ends up checking the cyclical consistency (or rationalization) of the matrix $\boldsymbol{\alpha}$, which is equivalent to the rationalization of a whole class of budget sets. By proceeding in this way, we get a different utility function for every family of compatible budget sets. One can therefore question the predictiveness of such a utility function, defined up to a family of budget sets. This motivates the next section.

4 Robust rationalization

Let us start with an essential experiment $(\mathbf{x},\boldsymbol{\alpha})$. The following definition of robust rationalization naturally emerges from the discussion in Section 3: the utility function v *rationalizes robustly* the experiment $(\mathbf{x},\boldsymbol{\alpha})$ if v rationalizes the experiment $(\mathbf{x},\mathbf{B})$ for every family $(B_j)_{j\in N}$

[5] The experiment $(\mathbf{x},\mathbf{B})$ satisfies GARP if, for every $j,k\in N$, x_kHx_j implies $x_k\notin \mathrm{int}B_j$, where H is the transitive closure of the direct revealed preference relation R: x_kRx_j if $x_j\in B_k$. For easy constructive proofs of the equivalence between GARP and the existence of a rationalization, see, e.g., Varian (1982) in the linear case and Forges and Minelli (2009) in the general case.

[6] The matrix with entries $(g_j^B(x_k))_{j,k\in N}$ is cyclically consistent iff the matrix $A^{x,B}$ is cyclically consistent.

compatible with $(\mathbf{x}, \boldsymbol{\alpha})$. The existence of a robust rationalization amounts therefore to the existence of a largest family of budget sets compatible with the essential experiment. Unfortunately, even if $(\mathbf{x}, \boldsymbol{\alpha})$ is well behaved (in particular, $\boldsymbol{\alpha}$ is cyclically consistent), such a family may not exist as the next simple example illustrates.

Example Let $\left[(x_1, (\alpha_{11}, \alpha_{12})), (x_2, (\alpha_{21}, \alpha_{22}))\right]$ be an essential experiment such that $\alpha_{12} = 1$, $\alpha_{21} = -1$ and $x_1 \notin x_2 + \mathbb{R}^\ell_+$. First, it is an easy matter to verify that the experiment admits a budget representation (actually, $x_1 \notin x_2 + \mathbb{R}^\ell_+$ guarantees that there is no contradictory statement). For instance define a compatible family as follows: $B_1 = \left[\{x_1\} - \mathbb{R}^\ell_+\right]_+$ and $B_2 = \left[\{x_2\} - \mathbb{R}^\ell_+\right]_+ \cup \left[\{x_1 + \nu\mathbf{1}\} - \mathbb{R}^\ell_+\right]_+$ for some $\nu > 0$ sufficiently small.[7] [8]

Suppose now that $x_2 \notin x_1 + \mathbb{R}^\ell_+$, we can add a piece to the budget set B_1 without modifying the resulting matrix $A^{x,B}$. More precisely, there exists $\eta > 0$ such that, for all $\epsilon \in (0, \eta)$, $\frac{1}{1+\epsilon}x_2 \notin B_1$. Thus the family $(B_1^\epsilon, B_2^\epsilon)$, where $B_1^\epsilon = B_1 \cup \left[\{(\frac{1}{1+\epsilon}x_2\} - \mathbb{R}^\ell_+\right]_+$ and $B_2^\epsilon = B_2$ is compatible with the essential experiment. Suppose that there exists a well-behaved v rationalizing robustly the essential experiment, then v rationalizes the experiments $\left((x_1, B_1^\epsilon), (x_2, B_2^\epsilon)\right)$ for all $\epsilon \in (0, \eta)$. It follows that $v(x_1) \geq v(\frac{1}{1+\epsilon}x_2)$ since $\frac{1}{1+\epsilon}x_2 \in B_1^\epsilon$ and $v(x_2) \geq v(x_1)$ since $x_1 \in B_2^\epsilon$. From local non satiation, $v(x_2) > v(x_1)$ since $x_1 \in \text{int}B_2^\epsilon$ but this contradicts the continuity of v as ϵ tends to 0.

The previous experiment, which satisfies consumer's rationality for any compatible family of budget sets, is by no means pathological. Hence, we cannot hope for a robust rationalization. To obtain a contradiction in the above construction we assumed that $x_2 \notin \{x_1\} + \mathbb{R}^\ell_+$. One can define an analogue impossibility result in general provided that the essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$ satisfies the equivalent requirement.

The previous example also shows that, by enlarging gradually a family of budget sets compatible with a given essential experiment $(\mathbf{x}, \boldsymbol{\alpha})$, we get at the limit budget sets which are well-behaved but are not compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ anymore. We will nevertheless achieve almost robust rationalization, which we define precisely below.

Definition 8 *Let $(\mathbf{x}, \boldsymbol{\alpha})$ be an essential experiment. Let $\epsilon > 0$, the pair (B^ϵ, v^ϵ) where B^ϵ is a family of budget sets and v^ϵ is a utility function, is said to ϵ-robustly rationalize $(\mathbf{x}, \boldsymbol{\alpha})$ if:*

(i). The family B^ϵ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$,

(ii). The function v^ϵ rationalizes the experiment $(\mathbf{x}, \mathbf{B}^\epsilon)$,

[7] Note that the essential experiment satisfies cyclical consistency and therefore $(B_1^\epsilon, B_2^\epsilon)$ satisfies GARP.

[8] For any set $A \subset \mathbb{R}^\ell$, let $[A]_+$ denote the non negative subset of A, $[A]_+ = A \cap \mathbb{R}^\ell_+$.

(iii). For every family $(B_j)_{j\in N}$ compatible with $(\mathbf{x},\boldsymbol{\alpha})$, $B_j \subseteq (1+\epsilon)B_j^\epsilon$ for every $j \in N$.

The justification for the terminology is that *(ii)* implies that v^ϵ rationalizes experiment $(\mathbf{x},\mathbf{B})$, for every compatible family $(B_j)_{j\in N}$ included in $\mathbf{B}^\epsilon$ and, by *(iii)*, every compatible family is almost included in $\mathbf{B}^\epsilon$. To show the former statement, note that x_j is such that $v^\epsilon(x_j) \geq v^\epsilon(x)$ for all $x \in B_j^\epsilon$ then a fortiori $v^\epsilon(x_j) \geq v^\epsilon(x)$ for all $x \in B_j$; and since $x_j \in B_j$, it follows that v^ϵ rationalizes the experiment $(\mathbf{x},\mathbf{B})$.

Taking again the essential experiment $(\mathbf{x},\boldsymbol{\alpha})$ as basic data, Propositions 1 or 2 tell us which conclusion we can draw from the cyclical consistency of the matrix $\boldsymbol{\alpha}$ but does not question the compatibility between the consumer's choices $(x_j)_{j\in N}$ and $\boldsymbol{\alpha}$ viewed as budgetary information, namely whether *there exists* a family of budget sets $(B_j)_{j\in N}$ compatible with $(\mathbf{x},\boldsymbol{\alpha})$. This is the purpose of the following tractable condition which will be used in the final result, jointly with cyclical consistency.

Definition 9 *An essential experiment $(\mathbf{x},\boldsymbol{\alpha})$ admits a contradictory statement if there exist $j,k,k' \in N$ such that either $[\alpha_{jk} < \alpha_{jk'}$ and $x_k \geq x_{k'}]$ or $[\alpha_{jk} = \alpha_{jk'} = 0$ and $x_k \gg x_{k'}]$.*

We are now in position to state our main result which provides the existence of a (ϵ-robust) rationalization and a budget representation on the basis of essential experiment only, by putting together the properties of cyclical consistency and (no) contradictory statement.[9]

Theorem 1 *Let $(\mathbf{x},\boldsymbol{\alpha})$ be an essential experiment. The following conditions are equivalent:*

1. *There exist $(B_i)_{i\in N}$ compatible with $(\mathbf{x},\boldsymbol{\alpha})$ and a locally non satiated, continuous utility function v^B rationalizing the experiment $(\mathbf{x},\mathbf{B})$.*

2. *The essential experiment $(\mathbf{x},\boldsymbol{\alpha})$ admits no contradictory statement and $\boldsymbol{\alpha}$ is cyclically consistent.*

3. *There exists $\eta > 0$ such that, for all $\epsilon \in (0,\eta)$, there exists a locally non satiated, continuous utility function v^ϵ rationalizing ϵ-robustly the experiment $(\mathbf{x},\boldsymbol{\alpha})$.*

Proof [1. $\Rightarrow$ 2.] To show cyclical consistency of $\boldsymbol{\alpha}$ proceed as in the proof of Proposition 2 (1. $\Rightarrow$ 2.). To show the property of non contradictory statement, suppose, first, on the contrary that there exist $j,k,k' \in N$ such that $[\alpha_{jk} < \alpha_{jk'}$ and $x_k \in \{x_{k'}\} + \mathbb{R}^\ell_+]$. Since $(B_i)_{i\in N}$ is compatible with $(\mathbf{x},\boldsymbol{\alpha})$ we have either $[x_k \in \mathrm{int}B_j$ and $x_{k'} \notin \mathrm{int}B_j]$ or $[x_k \in \mathrm{Fr}B_j$ and $x_{k'} \notin B_j]$

[9]There is no hope to obtain testable restrictions in the consumer problem if one considers poorer information than the one contained in essential experiments.

together with $x_k \in \{x_{k'}\} + \mathbb{R}^\ell_+$, but this contradicts A.2. Second, suppose on the contrary that there exist $j, k, k' \in N$ such that $[\alpha_{jk} = \alpha_{jk'} = 0$ and $x_k \in \{x_{k'}\} + \mathbb{R}^\ell_{++}]$. Since $(B_i)_{i\in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$ we have $x_k \in \mathrm{Fr}B_j$ and $x_{k'} \in \mathrm{Fr}B_j$ with $x_k \in \{x_{k'}\} + \mathbb{R}^\ell_{++}$, but this contradicts again A.2.

[2. $\Rightarrow$ 3.] Let $m > 0$ be such that $x_j \leq m\mathbf{1}$ for every $j \in N$ and define the following family $B^\epsilon = (B^\epsilon_j)_{j\in N}$ (see also Figure 1):[10]

$$B^\epsilon_j = \left[\left(\mathrm{int}\left((\cup_{k\in N,\alpha_{jk}=0}(\{x_k\} + \mathbb{R}^\ell_+))\bigcup(\cup_{k\in N,\alpha_{jk}=1}(\{\frac{1}{1+\epsilon}x_k\} + \mathbb{R}^\ell_+))\right)\right)^c \cap \left(\{m\mathbf{1}\} - \mathbb{R}^\ell_+\right)\right]_+$$

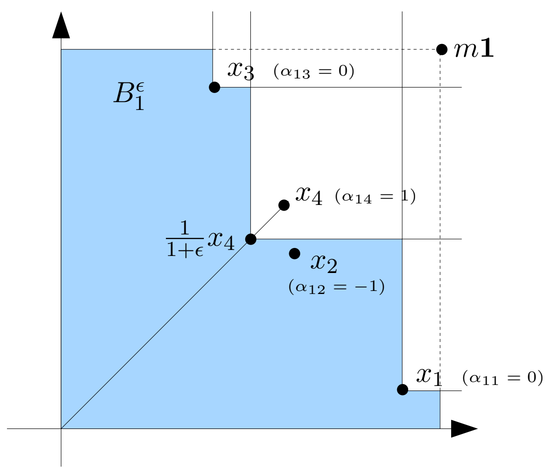

Figure 1: Construction of the family $(B^\epsilon_j)_{j\in N}$ (here B^ϵ_1)

By construction, each B^ϵ_j is a budget set. Suppose now that there exists $j \in N$ such that $x_j \notin \mathrm{Fr}B^\epsilon_j$ for all $\epsilon > 0$. Since $x_j \in \mathrm{int}B^\epsilon_j$ implies $\alpha_{jj} < 0$, there exists necessarily k such that either $\alpha_{jk} = 0$ and $x_k \ll x_j$ or $\alpha_{jk} = 1$ and $\frac{1}{1+\epsilon}x_k \ll x_j$, for all $\epsilon > 0$. Since $\alpha_{jj} = 0$ this contradicts the fact that $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement, (using ϵ tends to 0 if necessary). We have thus demonstrated that $(\mathbf{x}, \mathbf{B}^\epsilon)$ is an experiment for a sufficiently small ϵ.

We show now that $A^{x,B^\epsilon} = \boldsymbol{\alpha}$ for a sufficiently small ϵ.

Let $j, k \in N$ be such that $\alpha_{jk} = -1$. Suppose that there exists k' such that $x_k \in \{x'_k\} + \mathbb{R}^\ell_+$ with $\alpha_{jk'} = 0$. But it is then a contradictory statement. Suppose that there exists k' such

[10] The vector $\mathbf{1}$ is the vector of $\mathbb{R}^\ell$ whose components are equal to 1.

that $x_k \in \{\frac{1}{1+\epsilon}x'_k\} + \mathbb{R}^\ell_+$ with $\alpha_{jk'} = 1$ for all $\epsilon > 0$. As ϵ tends to 0, this contradicts again the fact that $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement. Therefore there exists $\epsilon > 0$ such that $x_k \notin (\cup_{k' \in N, \alpha_{jk'}=0}(\{x'_k\} + \mathbb{R}^\ell_+)) \bigcup (\cup_{k' \in N, \alpha_{jk'}=1}(\{\frac{1}{1+\epsilon}x'_k\} + \mathbb{R}^\ell_+))$. Thus $x_k \in \text{int}B^\epsilon_j$, that is $a^{x,B^\epsilon}_{jk} = -1$.

Let $j, k \in N$ such that $\alpha_{jk} = 0$. Suppose that there exists k' such that $x_k \in \{x'_k\} + \mathbb{R}^\ell_{++}$ with $\alpha_{jk'} = 0$. Then it is then a contradictory statement. Suppose then that there exists k' such that $x_k \in \{\frac{1}{1+\epsilon}x'_k\} + \mathbb{R}^\ell_+$ with $\alpha_{jk'} = 1$ for all $\epsilon > 0$. As ϵ tends to 0, this contradicts again the fact that $(\mathbf{x}, \boldsymbol{\alpha})$ admits no contradictory statement. Therefore there exists $\epsilon > 0$ such that $x_k \notin \text{int}\Big((\cup_{k' \in N, \alpha_{jk'}=0}(\{x'_k\} + \mathbb{R}^\ell_+)) \bigcup (\cup_{k' \in N, \alpha_{jk'}=1}(\{\frac{1}{1+\epsilon}x'_k\} + \mathbb{R}^\ell_+))\Big)$ but clearly belongs to $(\cup_{k' \in N, \alpha_{jk'}=0}(\{x'_k\} + \mathbb{R}^\ell_+)) \bigcup (\cup_{k' \in N, \alpha_{jk'}=1}(\{\frac{1}{1+\epsilon}x'_k\} + \mathbb{R}^\ell_+))$. Thus $x_k \in \text{Fr}B^\epsilon_j$, that is $a^{x,B^\epsilon}_{jk} = 0$.

Finally, let $j, k' \in N$ such that $\alpha_{jk'} = 1$. Then clearly, for all $\epsilon > 0$, $x_{k'} \in \text{int}\Big((\cup_{k \in N, \alpha_{jk}=0}(\{x_k\} + \mathbb{R}^\ell_+)) \bigcup (\cup_{k \in N, \alpha_{jk}=1}(\{\frac{1}{1+\epsilon}x_k\} + \mathbb{R}^\ell_+))\Big)$, that is to say $x_{k'} \notin B^\epsilon_j$, i.e. $a^{x,B^\epsilon}_{jk'} = 1$.

It follows that there exists $\eta > 0$ such that, for all $\epsilon \in (0, \eta)$, $(B^\epsilon_j)_{j \in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$, as was to be proved.

Let η be the threshold as constructed above and let $\epsilon \in (0, \eta)$. The construction of $(B^\epsilon_j)_{j \in N}$ is such that for every compatible family $(B_j)_{j \in N}$ with $(\mathbf{x}, \boldsymbol{\alpha})$, it holds that $B_j \subseteq (1 + \epsilon)B^\epsilon_j$ for every $j \in N$. It remains to prove that one can construct a well behaved utility function v^ϵ with the desired properties. Using 2. and the fact that $(B^\epsilon_j)_{j \in N}$ is compatible with $(\mathbf{x}, \boldsymbol{\alpha})$, Proposition 2 establishes the existence of a locally non satiated, continuous utility function v^ϵ rationalizing $(\mathbf{x}, \mathbf{B}^\epsilon)$.

[3. $\Rightarrow$ 1.] Consider the pair $(v^{\frac{\eta}{2}}, B^{\frac{\eta}{2}})$ which rationalizes the experiment $(\mathbf{x}, \boldsymbol{\alpha})$ $\frac{\eta}{2}$-robustly, which is given by condition 3. Then a fortiori the well behaved function $v^{\frac{\eta}{2}}$ rationalizes the experiment $(\mathbf{x}, \mathbf{B}^{\frac{\eta}{2}})$ as required by condition 1. ■

Finally, using the proof of the previous result we also obtain the following corollary which clarifies the role of contradictory statement:

Corollary 1 *The two following conditions are equivalent:*

1. *The essential experiment* $(\mathbf{x}, \boldsymbol{\alpha})$ *admits a budget representation*

2. *The essential experiment* $(\mathbf{x}, \boldsymbol{\alpha})$ *admits no contradictory statement.*

References

Afriat, S. (1967). The construction of a utility function from demand data. *International Economic Review*, **8**, 67–77.

Chung-Piaw, T. and Vohra, R. (2003). Afriat's theorem and negative cycles. mimeo, Northwestern University.

Ekeland, I. and Galichon, A. (2010). Pareto indivisible allocations, revealed preference and duality. mimeo, Ecole Polytechnique, Paris.

Forges, F. and Minelli, E. (2009). Afriat's theorem for general budget sets. *Journal of Economic Theory*, **144**(1), 135–145.

Fostel, A., Scarf, H., and Todd, M. (2004). Two new proofs of Afriat's theorem. *Economic Theory*, **24**(1), 211–219.

Matzkin, R. L. (1991). Axioms of revealed preference for nonlinear choice sets. *Econometrica*, **59**(6), 1779–86.

Shapley, L. and Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, **1**, 23–37.

Varian, H. R. (1982). The nonparametric approach to demand analysis. *Econometrica*, **50**(4), 945–73.

Yatchew, A. J. (1985). A note on non-parametric tests of consumer behaviour. *Economics Letters*, **18**(1), 45–48.