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Carboni, Oliviero and Russu, Paolo

University of Sassari

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Oliviero Carboni(University of Sassari)

Paolo Russu(University of Sassari) *

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Abstract

This paper studies the equilibrium dynamics of a growth model with public finance where two different allocations of public resources are considered. The model simultaneously determines the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utilities for a centralized economy. The analysis supplies a closed form solution. Moreover, with one restriction on the parameters ($\alpha = \sigma$) we fully determine the solutions path for all variables of the model and determine the conditions for a balanced growth.

Keywords: growth models, fiscal policy, public spending composition.

JEL classification: O40, H50, E13, H20

1 Introduction

In the last decades a vast literature has emerged on the relationship between fiscal policy and long-run economic growth. In their seminal contribution, Arrow and Kurz (1969) develop a neoclassical model of growth where aggregate production benefits from public capital services and government finances public capital by levying a proportional income tax, subtracting resources from private agents. Within the framework of growth models with constant returns to a 'broad concept' of capital Barro (1990) shows how the presence of a flow of public services as an input in the production function of the final good can affect long-run growth and welfare. Considering government spending implicitly productive his model determines the optimal level of public spending.(see Zagler and Dürnecker, 2003 for a comprehensive review).

* *Address of corresponding author*: Paolo Russu, Department of Economics, Business and Regulation, University of Sassari, via Torre Tonda 34, I-07100 Sassari, Italy. email: russu@uniss.it

Starting from this influential work the composition of public expenditures has become a central question in growth studies. Several papers distinguish between productive and unproductive public expenditures, and investigate how a country can ameliorate its economic performance by adjusting the share the two types of public spending. For instance, Lee (1992), Devarajan et al. (1996) expand on Barro's model, allowing different kinds of government expenditures to have different impacts on growth. Employing a simple analytical model Devarajan et al. (1996) consider two productive services (expressed as flow variables) with two different productivities in a CES production function and derive the conditions under which a change in the composition of expenditure leads to a higher steady-state growth rate of the decentralized economy. By using the distinction between productive and non-productive spending (see also Glomm and Ravikumar, 1997; Kneller et al., 1999), they are able to determine the optimal composition of different kinds of expenditures, based on their relative elasticities. Productive spending includes expenditures on infrastructure, the law system, education and training. Non-productive spending includes expenditures on national defence, national parks, social programs, etc.

Following a similar line, Chen (2006) investigates the optimal composition of public spending in an endogenous growth model with a benevolent government. He establishes the optimal productive public service share of the total government budget and the optimal public consumption share, determined by policy and structural parameters.

Also within an endogenous growth framework Ghosh and Roy (2004) introduce public capital and public services as inputs in the production of the final good. They show that optimal fiscal policy depends on the tax rate and on the share of spending for the accumulation of public capital and the provision of public services. Finally, employing a neoclassical framework, Carboni and Medda (2011, a,b) consider two different kinds of public capital accumulation and determine the government size and the mix of government expenditures which maximize the rate of growth and the long-run level of per capita income.

One of the characterizing feature of the Devarajan et al. (1996) model is that the economy's growth rate is expressed in terms of the tax rate and expenditure shares. These latter are both exogenous since the government's decisions are take as given. Ghosh and Gregoriu (2008) relax this latter hypothesis. Within a decentralized economy framework, they characterize the welfare-maximizing fiscal policy for a benevolent government, which chooses the fiscal policy to maximize the representative agent's utility. Their model solves for the three key endogenous variables: the optimal composition of public spending, the optimal tax rate, and the optimal growth. Furthermore, they derive the social optimum as an ideal benchmark, where the social planner chooses private consumption and private investment for the agent in addition to choosing the fiscal instruments.

2 Model Background

Following this strand of literature this paper studies the equilibrium dynamics of a growth model with public finance where two different allocations of public resources are considered. We consider the fiscal policy as a part of the aggregate economy by explicitly including the public sector in the production function. This generates a potential relationship between government and production. The introduction of government as a distinct input is based on the rationale that government services are not a substitute for private factors, and resources cannot be easily transferred from one sector to another.

The model developed here simultaneously determines the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utilities for a centralized economy. Moreover, under the condition $\alpha = \sigma$ (Uzawa, 1965; Smith, 2006; Chilarescu, 2008; Hiraguchi, 2009) the model supplies a closed form solution and determines the conditions for a balanced growth. This represents the main novelty of this paper.

It worth highlighting that Zhang (2011) provides an analytical expression of the balanced growth solution in a multi-sector model. He finds the optimal distribution coefficient of fixed capital investment and of labor hour, the proportion of production, the economic growth rate, the rate of change of the price index, and rental rates of different fixed capital. However, differently from our work his analysis does not consider optimal fiscal policy.

In line with Devarajan et al. (1996) and Ghosh and Gregoriu (2008) we consider the two types of public expenditures entering as flows in the production function. All government activities are considered as production-enhancing according to their respective elasticities. The reason for this is that the services offered by public expenditures to the private inputs is the result of a productive process in which some components of public and private investment take part together (e.g. improvements in the education system is likely to affect positively the productivity of private capital). Hence, the government can influence private production through investments in different types of public spending such as roads and highways, telecommunication systems, R & D capital stock, other infrastructures (Aschauer, 1989; Kneller et al., 1999) or simple services spending such as the maintenance of infrastructure networks and the maintenance of law and order. The different impact of each type of government spending on production makes it all the more necessary to disaggregate the public budget into its various components [1].

Differently from Devarajan et al. (1996) and in line with Ghosh and Gregoriu (2008), instead of taking the government's decisions as given, we consider fiscal policy endogenous.

[1] In his empirical analysis Aschauer (1989) finds that investment in infrastructure improves the productivity of private capital, leading to higher growth. Easterly and Rebelo (1993) support Aschauer in showing that public investment in transport and communication has a positive impact on growth.

Moreover, since our model considers a central planner optimal choice, also the level of private consumption is endogenized. We start from the case in Ghosh and Gregoriu (2008) where the social planner has the possibility to internalize the externalities. Differently from their work which considers four control variables (c, τ, g_1, g_2 in their terminology), we endogenize y so that the social planner directly accounts for the tax rate and the shares of the two public spending in the maximization decision. Employing a Cobb-Douglas production function our model ends up with three equations. Hence, the complexity of the dynamic system is reduced.

3 The Model

In this section we model the government expenditure composition as a part of the aggregate economy. Public capital provide flows of rival, non-excludable public services, which would not be provided by the market. Flows are proportional to the relative stocks and enter the production function together with private capital.

The model considers two different categories of public spending. The first (G_1) is traditional core productive spending. The second (G_2) is a broad concept of capital, namely "institutional" spending embracing all the activities which are designed to improve the environment in which firms can effectively operate (Glaeser et al, 2004). Both components of government expenditure are complementary with private production (e.g. private vehicles can be used more productively when the quality of the road network increases). Following Barro (1990) and most of the recent work in growth studies, in our specification productive government expenditure is introduced as a flow (Ireland, 1994; Glomm and Ravikumar, 1994; Turnowski and Fischer, 1995; Devarajan et al., 1996; Bruce and Turnovsky, 1999; Turnovsky, 2000; Eicher and Turnovsky, 2000; Ghosh and Gregoriu, 2008)[2].

We assume that there is a large number of infinitely lived households and firms that is normalized to one, that population growth is zero and that there is no entry or exit of firms. The representative firm produces a single composite good using private capital (k) which is broadly defined to encompass physical and human capital, and two public inputs, G_1 and G_2, based on CES technology:

$$y = (\theta k^{\nu} + \gamma_1 G_1^{\nu} + \gamma_2 G_2^{\nu})^{\nu} \tag{1}$$

[2]An alternative method is to allow the government also to accumulate stocks of durable consumption goods and physical infrastructure capital (Arrow and Kurz, 1969; Futagami et al. (1993); Fisher and Turnovsky, 1998; Carboni and Medda, 2011a,b; among others). Although attractive in terms of realism, this approach would substantially increase the dimensionality of the dynamic system. The introduction of two public capital stocks along with private capital would imply a macro dynamic equilibrium with three state variables which considerably complicate the formal analysis (Turnovsky and Fisher,1995). Thus, we believe that our current framework, which considers both types of government expenditures as flows, does not compromise the main target of this work.

where θ, γ_1 and γ_2 are distribution parameters with $\theta < 1 - \gamma 1 - \gamma 2$. The productivity of private capital used by the individual firm therefore positively depends on G_1 and G_2. G_1 and G_2 are non-rival and provided free of charge to the agents of the economy. Public capital is assumed not to depreciate whereas public services can be conceived to fully depreciate. Since we consider both extreme cases of depreciation, we do not consider intermediate cases (i.e. a rate of depreciation between 0 and 1) which would be unlikely to yield any additional insights. ν determines the elasticity of substitution, s, which corresponds to $s = \dfrac{1}{1-\nu}$. We assume that $0 \leq s \leq 0$ based on the fact that public and private inputs are often difficult to substitute against each other and therefore rather complements than substitutes. With $\nu = 0$, the production technology is Cobb-Douglas, and with $\nu < 0$, the inputs to private production are complements. Hence $\nu = 0$, and the production function can then be written as

$$y = k^{\theta} G_1^{\gamma_1} G_2^{\gamma_2} \tag{2}$$

where $\theta < 1-\gamma_1-\gamma_2$. The government finances total public expenditure, G_1+G_2, by levying a flat tax, τ, on income. In line with the main literature, we assume a permanent balanced government budget and rule out debt-financing of government spending (Barro (1990); Futugami, Morita, and Shibata (1993); Fisher and Turnovsky, (1998)). Public spending is financed by levying an average flat-rate tax on income τ $(0 < \tau < 1)$:

$$G_1 + G_2 = \tau y \tag{3}$$

$\phi(1-\phi)$ denotes the share of public revenue allocated to $G_1(G_2)$ so that

$$G_1 = \phi \tau y \tag{4}$$

$$G_2 = (1-\phi)\tau y \tag{5}$$

The households own the firms and therefore receive all their output net of taxation which they either reinvest in the firms to increase their capital stock or which they use for consumption depending on their preferences and the returns on private capital. Private investment by the representative household equals

$$\dot{k} = (1-\tau)y - c \tag{6}$$

The central planner maximizes lifetime utility U given by

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \tag{7}$$

where c represents per capita consumption, $r > 0$ is the constant rate of time preference, and σ is the inter-temporal elasticity of substitution. Replacing (4) and (5) in (2), we obtain

$$y = k^{\alpha}\Omega(\tau, \phi) \tag{8}$$

where $\Omega(\tau, \phi) := (\tau\phi)^{\beta_1}(\tau(1-\phi))^{\beta_2}$ and $\alpha = \frac{\theta}{1-\gamma_1-\gamma_2}$, $\beta_1 = \frac{\gamma_1}{1-\gamma_1-\gamma_2}$, $\beta_2 = \frac{\gamma_2}{1-\gamma_1-\gamma_2}$.
We assume that the central planner chooses the functions $c(t)$, $\tau(t)$ and $\phi(t)$ in order to solve the following problem

$$\underset{c,\tau,\phi}{MAX} \int_0^{\infty} \frac{c^{1-\sigma}-1}{1-\sigma} e^{-rt} dt \tag{9}$$

subject to

$$\dot{k} = (1-\tau)k^{\alpha}(\tau\phi)^{\beta_1}(\tau(1-\phi))^{\beta_2} - c$$

with $k(0)$ given, $k(t)$, $c(t)$, $\tau(t) \geq 0$ and $1 \geq \phi(t) \geq 0$ for every $t \in [0, +\infty)$; $r > 0$ is the discount rate.

4 Dynamics

The current value of the Hamiltonian function associated to problem (9) is

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda\left((1-\tau)k^{\alpha}\Omega(\tau, \phi) - c\right) \tag{10}$$

where λ is the co-state variable associated to k. By applying the Maximum Principle, the dynamics of the economy is described by the system

$$\dot{k} = \frac{\partial H}{\partial \lambda} = (1-\tau)k^{\alpha}\Omega(\tau, \phi) - c \tag{11}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = \lambda\left(r - \alpha(1-\tau)k^{\alpha-1}\Omega(\tau, \phi)\right) \tag{12}$$

with the constraint

$$H_c = c^{\sigma} - \lambda = 0 \tag{13}$$

$$H_{\tau} = (-k^{\alpha}\Omega + k^{\alpha}(1-\tau)\Omega_{\tau})\lambda = 0 \tag{14}$$

$$H_{\phi} = (1-\tau)k^{\alpha}\Omega_{\phi}\lambda = 0 \tag{15}$$

with $\Omega_\tau = \dfrac{\partial \Omega}{\partial \tau} = \dfrac{\Omega}{\tau}(\beta_1 + \beta_2)$ and $\Omega_\phi = \dfrac{\partial \Omega}{\partial \phi} = \dfrac{\Omega}{\phi}(\beta_1 - \beta_2)$.
By straight calculation, we can write the values of the control variables τ, ϕ which

$$\phi^\star = \frac{\beta_1}{\beta_1 + \beta_2} \tag{16}$$

$$\tau^\star = \frac{\beta_1 + \beta_2}{1 + \beta_1 + \beta_2} \tag{17}$$

By replacing the equations (16) and (17) in (9) and noting that from the equation (13) $\dfrac{\dot c}{c} = -\dfrac{1}{\sigma}\dfrac{\dot\lambda}{\lambda}$, one can write the following system, equivalent to (11)-(12)

$$\dot k = \Omega^\star k^\alpha - c \tag{18}$$

$$\frac{\dot c}{c} = \frac{1}{\sigma}(\alpha \Omega^\star k^{\alpha-1} - r) \tag{19}$$

where

$$\Omega^\star := \frac{\Omega(\tau^\star, \phi^\star)}{1 - \beta_1 - \beta_2} = \frac{\left(\dfrac{\beta_1}{1+\beta_1+\beta_2}\right)^{\beta_1}\left(\dfrac{\beta_2}{1+\beta_1+\beta_2}\right)^{\beta_2}}{1 - \beta_1 - \beta_2} \tag{20}$$

This condition is required in order to obtain a closed form solution and has been applied in Uzawa (1965) two-sector growth model, Smith(2006) while describing the Ramsey model, Chilarescu (2008) and Hiraguchi (2009) while describing the Lucas (1988) model.

Lemma 1 *If $\alpha = \sigma$ then the solution of the equation* (19) *is given by*

$$c(t) = \frac{\phi c_0}{c_0 + (\phi k_0 - c_0)e^{rt}} k(t) \tag{21}$$

Proof . If we consider the variable defined as $x = \dfrac{c}{k}$, we can write the following differential equation $\dfrac{\dot x}{x} = \dfrac{\dot c}{c} - \dfrac{\dot k}{k}$, replacing (18) and (19), we obtain

$$\frac{\dot x}{x} = \frac{\alpha}{\sigma}\Omega^\star k^{\alpha-1} - \frac{r}{\sigma} - \Omega^\star k^{\alpha-1} + \frac{c}{k} \tag{22}$$

under the hypothesis $\dfrac{\alpha}{\sigma} = 1$, we get $\dfrac{\dot x}{x} = -\dfrac{r}{\sigma} + x$, where for some $x(0) = x_0$ the solution is $x(t) = \dfrac{\phi}{1 + (\dfrac{\phi}{x_0} - 1)e^{\phi t}}$, where $\phi := \dfrac{r}{\sigma}$. But for some $x_0 = \dfrac{c_0}{k_0}$ the solution is given by (21).
■

Theorem 1 *Under the assumptions of the above lemma, the following statements are valid:*

1. *If $\phi k_0 - c_0 = 0$, then consumption per labor unit is always proportional to the capital per labor unit*

$$c(t) = \phi k(t) \tag{23}$$

2. If $\phi k_0 - c_0 > 0$, then

$$\frac{\dot{k}(t)}{k(t)} > \frac{\dot{c}(t)}{c(t)}, \qquad \forall t \tag{24}$$

3. If $\phi k_0 - c_0 < 0$, then

$$\begin{cases} \dfrac{\dot{k}(t)}{k(t)} < \dfrac{\dot{c}(t)}{c(t)}, & \forall t \in (0, \bar{t}) \\ \dfrac{\dot{k}(t)}{k(t)} > \dfrac{\dot{c}(t)}{c(t)}, & \forall t > \bar{t} \end{cases} \tag{25}$$

where $\bar{t} := \frac{1}{\phi} ln(\frac{c_0}{|\phi k_0 - c_0|})$

4. For $c_0 \neq \phi k_0$

$$\lim_{t\to\infty} (\frac{\dot{c}}{c} - \frac{\dot{k}}{k}) = -\phi \tag{26}$$

that is, there exist a t^*, such that $\frac{\dot{k}}{k} \approx \frac{\dot{c}}{c} + \phi \Leftrightarrow c(t) = \phi k(t) e^{-\phi(t-t^*)}$, $\forall\ t > t^*$

Proof . From 21, the first statement is obviously true. Differentiating $x(t)$, we obtain $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} = -\frac{r(\phi k_0 - c_0)}{c_0 e^{-\phi t} + (\phi k_0 - c_0)}$ thus the next three statements follow as consequence. ■

As it is well known, a macroeconomics model exhibits balanced growth if consumption and capital grow at a constant rate while hours of work per time period stay constant, that is if and only if $c_0 = rk_0$.

Theorem 2 *If model exhibits balanced growth, the dynamic of the state variable $k(t)$ is given by*

$$k(t) = \left(\frac{r}{\Omega^* + e^{r(\alpha-1)t}(k_0^{\alpha-1} r - \Omega^*)} \right)^{\frac{1}{\alpha-1}} \tag{27}$$

Proof . To prove the theorem, observe that, in the case $c_0 = \phi k_0$, $\dot{k(t)} = \Omega^* k^\alpha - \phi k$ is a differential equation of Bernoulli. ■

Theorem 1 shows the relation between growth and the variables c and k when varying the initial conditions (c_0, k_0).

- *Case 1.* realizes balanced growth.
- *Case 2.* tells us that if the ratio between initial conditions $(\frac{c_0}{k_0})$ is smaller than $\phi = \frac{r}{\sigma}$ (i.e. constant rate of time preference and constant elasticity of intertemporal substitution ratio) then the capital stock growth ratio $(\frac{\dot{k}}{k})$ is greater than the growth rate of consumption $(\frac{\dot{c}}{c})$ at any point in time.
- *Case 3.* implies that if the ratio between initial conditions $(\frac{c_0}{k_0})$ is larger than $\phi = \frac{r}{\sigma}$ then for a given initial period $(0; \bar{t})$ the growth rate of capital stock is larger than that of consumption while for the remaining time the opposite occurs.

- *Case 4.* if $c_0 \neq \phi k_0$ then for a significantly large period of time ($t \to \infty$) consumption goes to zero given $c(t) = \phi k(t) e^{-\phi(t-t^*)}$.

5 Conclusion

This paper studies the equilibrium dynamics of a growth model with public finance where two different allocations of public spending with two different elasticities are considered. Fiscal policy is part of the aggregate economy by explicitly including the public sector in the production function. This generates a potential relationship between government and production. The model analyzes the equilibrium dynamics and derives a closed form solution for the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utilities for a centralized economy. Finally, with one restriction on the parameters ($\alpha = \sigma$) we fully determine the solutions path for all variables of the model and determine the conditions for a balanced growth.

References

[1] Arrow, K. J., Kurz, M. (1969). Optimal public investment policy and controllability with fixed private savings ratio. Journal of Economic Theory, 1, pp. 141-77.

[2] Ascauer, D.A. (1989). Is public expenditure productive? Journal of Monetary Economics, vol. 23, pp. 177-200.

[3] Barro, R. J. (1990). Government spending in a simple model of endogenous growth. Journal of Political Economy, 98, pp. S103-25.

[4] Carboni, O.A.,Medda, G. (2011,a). Size and composition of public spending in a neoclassical growth model. Metroeconomica, 150-170 (2011)

[5] Carboni, O.A.,Medda, G. (2011,b). Government spending and growth in a neoclassical model. Mathematics and Financial Economics, 4: 269-285.

[6] Chen, B. (2006). Economic growth with an optimal public spending composition. Oxford Economic Papers, 58, 123-36.

[7] Chilarescu, C.(2008). An analytical solutions for a model of endogenous growth. Economic Modelling 25, 1175-1182.

[8] Devarajan, Shantayanan, Vinaya Swaroop, and Heng-fu Zou (1996). The Composition of Public Expenditure and Economic Growth. Journal of Monetary Economics, Vol. 37, 313-344.

[9] Easterly, William, and Sergio Rebelo (1993). Fiscal policy and economic growth: an empirical investigation. Journal of Monetary Economics 32, 417-58

[10] Futagami, K., Morita, H., Shibata, A. (1993). Dynamic analysis of an endogenous growth model with public capital. Scandinavian Journal of Economics, 95, pp. 607-25.

[11] Ghosh, S., Gregoriou, A. (2008). The composition of government spending and growth: is current or capital spending better? Oxford Economic Papers, 60, pp. 484-516.

[12] Eicher, Theo S., and Stephen J. Turnovsky (2000). Scale, Congestion, and Growth. Economica 67, 325-346.

[13] Fisher, W. H., Turnovsky, S. J. (1998). Public investment, congestion, and private capital accumulation. Economic Journal, 108, pp. 399-412.

[14] Lee, J. (1992). Optimal size and composition of government spending. Journal of the Japanese and International Economies, 6, pp. 423-39.

[15] Glomm, G., Ravikumar, B. (1997). Productive government expenditures and long-run growth. Journal of Economic Dynamics and Control, 21, pp. 183-204.

[16] Ghosh, Sugata and Udayan Roy (2004). Fiscal Policy, Long-Run Growth, and Welfare in a Stock-Flow Model of Public Goods. Canadian Journal of Economics, Vol. 37, 742-756.

[17] Hiraguchi. R.(2009). A note on the closed-form solution to the Lucas-Uzawa model with externality. mimeo.

[18] Kneller, R., Bleaney, M. F., Gemmell, N. (1999). Fiscal policy and growth: evidence from OECD countries. Journal of Public Economics, 74, pp. 171-90.

[19] Lucas, R. (1988). On the mechanics of economic development. Journal of Monetary Economics 22, 3-42.

[20] Smith, T.W.(2006). A closed form solution to the Ramsey model. Contributions to Macroeconomics 6, 1-27.

[21] Turnovsky, S.J., and Fisher, W.H. (1995). The composition of government expenditure and its consequences for macroeconomic performance. Journal of Economic Dynamics and Control, vol. 19, pp. 747-786.

[22] Turnovsky, S. J. (1997). Fiscal policy in a growing economy with public capital. Macroeconomic Dynamics, 1 (3), pp. 615-39.

[23] Uzawa, H.(1965). Optimum technical change in an aggregative model of economic growth. International Economic Review 6, 18-31.

[24] Zagler, M., Dürnecker, G.(2003). Fiscal policy and economic growth. Journal of Economin Survey 17(3) 397-418.

[25] Zhang J. S.(2011). The analytical solution of balanced growth of non-linear dynamic multi-sector economic model. Economic Modelling 28, 410-421