Incomplete financial participation: exclusive markets, investment clubs and credit risk

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INCOMPLETE FINANCIAL PARTICIPATION: EXCLUSIVE MARKETS, INVESTMENT CLUBS AND CREDIT RISK

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Abstract. We develop a general equilibrium model with incomplete financial participation and price dependent financial constraints. In our model, equilibrium exists even when agents do not have access to all financial contracts. For instance, exclusive credit lines and investment clubs are compatible with our framework. We also extend the literature of incomplete financial participation to include debts backed by price dependent collateral, and non-ordered preferences negatively affected by the amount of default.

Keywords: Incomplete financial participation - Exclusive credit lines - Collateralized loans

1. Introduction

There are several types of participation constraints in modern financial markets, restricting the access to credit or the availability of investment opportunities. For instance, debt constraints in the form of collateral requirements or margins calls are usually used by financial markets as a mechanism to reduce the aggregated risk of default. Also, barriers in the access to investment, that may depend on credit scores or on the amount of wealth, can allow traders to participate in investment clubs, obtaining exclusive access to financial opportunities with a reduced risk of default.

In this work we include these financial practices in an abstract model of general equilibrium, allowing for incomplete financial markets, price dependent financial constraints and non-ordered preferences. As byproducts of our analysis we obtain extensions of several models of general equilibrium to include incomplete financial participation. Particularly, we extend the traditional models

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of incomplete markets to allow for exclusive credit lines and investment clubs. Also, we can make financial participation constraints compatible with equilibrium in sophisticated financial markets, where debts are subject to credit risk and are securitized into investment assets.

Our results can be included into a growing literature of general equilibrium with financial participation constraints. This literature departs from equilibrium models with incomplete financial markets and unconstrained portfolio sets,[1] including additional restrictions in both debt and investment opportunities. In this direction, assuming that financial trades are restricted to a closed and convex set of portfolios with contains zero, equilibrium existence is analyzed by Siconolfi (1986) under a financial survival condition, which guarantee that independently of prices, all agents have access to a positive amount of resources through any credit line. With financial survival conditions and linear restrictions as a form of financial participation constraints, the existence of equilibrium is studied, for the case of nominal assets, by Balasko, Cass and Siconolfi (1990), Aouani and Cornet (2009) and Cornet and Ranjan (2011). The real asset market case is studied by Polemarchakis and Siconolfi (1997), when financial participation constraints are given by linear restrictions, and by Angeloni and Cornet (2006) for a more general case where portfolio sets are required to be convex and compact. Without financial survival, Aouani and Cornet (2009) proves that equilibrium can be ensured under a non-redundancy assumption over financial markets. Recently, Aouani and Cornet (2011) proves equilibrium existence with financial participation constraints in a model where either portfolio sets are defining by a finite number of linear inequalities or financial participation constraints satisfy a non-arbitrage condition. Alternatively, with a spanning condition over the set of admissible portfolios, which requires the closed cone generated by the union of portfolio sets to be a linear space, Cornet and Gopalan (2010) shows equilibrium existence for nominal assets markets. Assuming that agents are impatient, Seghir and Torres-Martínez (2011) address an incomplete markets model where financial participation constraints are endogenous, since depend on the amount of consumption. In that model, equilibrium exists without need to impose financial survival, non-redundancy assumptions or spanning conditions over admissible portfolios.

When financial participation constraints emerge endogenously due to regulatory, institutional or budgetary considerations, it is natural to allow this restrictions to depend on market prices. This possibility is considered by Cass, Siconolfi and Villanacci (2001), where financial constraints depends on asset prices; and by Carosi, Gori and Villanacci (2009), where restricted participation is given by functions that depend on commodity and asset prices.

[1]See, for instance, the real asset markets model of Duffie and Shafer (1985, 1986), where generic existence of equilibrium is analyzed. Also, equilibrium existence in unconstrained financial markets is addressed by Cass (2006), Werner (1985), Duffie (1987), and Geanakoplos and Polemarchakis (1986).

Analogously to Seghir and Torres-Martínez (2011), in our model we suppose that agents are impatient, in the sense that a reduction of consumption at the second period can be always compensated by an increment in the quantity of commodities consumed at the first period. Therefore, we ensure the existence of competitive equilibrium without need to include financial survival assumptions, spanning conditions over admissible portfolios, or require financial constraints to satisfy non-arbitrage conditions.

More precisely, we consider a two-period model where commodities may be durable, perishable or may transform into other goods through time. There is a finite number of agents, which demand commodities, negotiated credit contracts and invest in assets obtained by the securitization of debts. Agents are burden by participation constraints, that can bound the access to credit or investment opportunities. These constraints may depend on prices and on consumption allocations. Allowing preferences to be non-ordered, we prove the existence of equilibrium under technical conditions over the correspondences that determine financial constraints (continuity, convexity, compacticity). Also, we include credit risk by allow borrowers to choose, associated to a same amount of debt, different payments at the second period. The amount of default can negatively affect individual preferences. Our approach is compatible with markets where, independently of prices, only a subset of agents have access to some credit contract (exclusive credit lines). Also, we can restrict the access to investment as a function of the amount of wealth, inducing endogenous investment clubs.

As corollaries of our main result, we capture several extension of traditional models of general equilibrium. We extend the model of Dubey, Geanakoplos and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007) to allow for non-ordered preferences, price dependent physical and financial collateral requirements, exclusive credit lines and investment clubs. Also, we include incomplete financial participation and collateralized debts in the model of default and punishment of Dubey, Geanakoplos and Shubik (1990, 2005) and Zame (1993), allowing for non-ordered preferences negatively affected by amounts of default.

The remaining of the paper is organized as follows. In Section 2 we present our abstract model with endogenous financial participation. In Section 3 we introduce the required assumptions to ensure the existence of equilibrium and we state our main result. Section 4 is devoted to the discussion of the compatibility of our model with exclusive credit lines and investment clubs. In Section 5 we apply our main result to extend several models of the literature of general equilibrium to include incomplete financial participation. Technical proof are given in the Appendix.

2. An Abstract Model with Incomplete Financial Participation

We consider an economy with two periods $t \in \{0,1\}$ and uncertainty about which state of nature of a finite set $\mathcal{S} = \{1, ..., S\}$ will prevail at $t = 1$. There is only one state of nature at $t = 0$, denoted by $s = 0$. Let $\mathcal{S}^* = \{0\} \bigcup \mathcal{S}$.

Commodities are perfectly divisible and subject to transformations between periods. Let $\mathcal{L} = \{1, \ldots, L\}$ be the finite set of commodities in the economy, which are available for trade in spot markets at any state of nature $s \in \mathcal{S}^*$. The commodity space is $\mathbb{X} = \mathbb{R}_+^{L(S+1)}$ and $p = (p_s; s \in \mathcal{S}^*) \in \Delta^{S+1}$ denotes the vector of unitary commodity prices, where $\Delta = \left\{ v \in \mathbb{R}_+^L : \sum_{l \in \mathcal{L}} v_l = 1 \right\}$. There exists the possibility of depreciation, durability or transformation of commodities between periods. Thus, one unit of commodity l consumed at the first period is transformed in a bundle $(Y_s(l', l); l' \in \mathcal{L})$ at the state of nature $s \in \mathcal{S}$, where the matrix $Y_s = (Y_s(l', l); (l', l) \in \mathcal{L} \times \mathcal{L})$ has non-negative entries.

There is a finite set of agents $\mathcal{H} = \{1, \ldots, H\}$ which demand commodities at each state of nature. Each $h \in \mathcal{H}$ receives physical endowments $w^h = (w_s^h; s \in \mathcal{S}^*) \in \mathbb{X}$ and chooses a consumption plan $x^h = (x_s^h; s \in \mathcal{S}^*) \in \mathbb{X}$. Cumulated endowments of agent h are given by $W^h = (W_0^h, (W_s^h; s \in S)) := (w_0^h, (w_s^h + Y_s w_0^h; s \in S))$. Agents can trade financial contracts to smooth consumption between periods and their preferences can depend on the amount of financial default. Thus, before introduce the characteristics of individual's preferences, we will explain as financial markets works.

Financial markets are composed by debt contracts and assets obtained by the securitization of those debts. There is a finite set of debt contracts $\mathcal{J} = \{1, ..., J\}$. Each agent h that negotiates $\varphi_j^h \in \mathbb{R}_+$ units of a debt contract j receives a quantity of resources $\gamma_j \varphi_j^h$ at the first period, where $\gamma_j \in \mathbb{R}_+$ is the amount of resources obtained for each unit of debt contract j that was issued. After choose a position φ_j^h on j, agent h will choose to pay, at each state of nature $s \in \mathcal{S}$, an amount of resources $\delta_{s,j}^h \in \Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j^h)$, where the correspondence $\Omega_{s,j} : \Delta \times \Delta \times \mathbb{R}_+ \times \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+$ associates prices and financial positions on debt contract j with admissible payments at state of nature $s \in \mathcal{S}$.[2] For convenience of notations, denote by $\gamma = (\gamma_j; j \in \mathcal{J})$ the vector of unitary prices for loans, by $\varphi^h = (\varphi_j^h; j \in \mathcal{J})$ the positions of agent $h \in \mathcal{H}$ on debt contracts, and by $\delta^h = (\delta_{s,j}^h; (s,j) \in \mathcal{S} \times \mathcal{J})$ the payments made by agent h at the second period.

Financial positions on a debt contract $j \in \mathcal{J}$ are securitized into only one asset, which is also denoted by j. Thus, payments made by agents that negotiated debt contract j are pooled and distributed to investors on security j. Each unit of this security can be traded at the first period by a price $q_j \in \mathbb{R}_+$ and delivers, at each state of nature $s \in \mathcal{S}$, a unitary payment $R_{s,j} \in \mathbb{R}_+$. We denote by $q = (q_j; j \in \mathcal{J})$ the unitary prices of securities, by $R = (R_{s,j}; (s,j) \in \mathcal{S} \times \mathcal{J})$ the vector

[2] With this specification we can allow for models where promises associated to a debt contract are subject to default (see Sections 4 and 5 for specific applications).

of security payments, and by $\theta^h=(\theta^h_j; j\in\mathcal{J})\in\mathbb{R}^J_+$ the financial positions of an agent $h\in\mathcal{H}$ on securities $j\in\mathcal{J}$.

We allow financial positions to be restricted at the first period. That is, each agent $h\in\mathcal{H}$ is constrained to choose portfolios in the set $\Phi^h(p,x^h)$, where $\Phi^h:\Delta^{S+1}\times\,\mathbb{X}\twoheadrightarrow\mathbb{R}^J_+\times\mathbb{R}^J_+$. Thus, credit and investment opportunities can depend on commodity prices and on the purchases of commodities, as in the case of borrowing associated to collateralized loans.

In our specification of rules $(\Phi^h; h\in\mathcal{H})$, which determine the incomplete financial participation in our economy, is not necessary to impose a *financial survival assumption*, allowing agents to have access to credit in all debt contracts (see Angeloni and Cornet (2006), Aouani and Cornet (2009, 2011) and Cornet and Ranjan (2011)). Moreover, different to Cornet and Gopalan (2010), we do not need to require *financial accessibility*, a linear spanning condition which ensure that a fraction of any financial transfer among states of nature in $\mathcal{S}$ can be implemented by some agent. Therefore, as particular cases of our analysis, we guarantee equilibrium existence in economies with exclusive credit lines and/or investment clubs (see Section 4).

The absence of assumptions about financial survival or financial accessibility implies that budget set correspondences may have empty interior when commodity prices are equal to zero. For this reason, we restrict commodity prices to belong into the simplex. Therefore, to prove equilibrium existence in our economy, we need to found upper bounds on financial prices. We prove that these upper bounds exist as a consequence of two key assumptions. We assume that requirements of investment induced by financial constraints $(\Phi^h; h\in\mathcal{H})$ satisfy some regularity conditions, which avoid cycles in the process to bound financial prices using upper bounds on security cost (see Assumptions (A4) and (A5) below). Also, we suppose that agents are impatient, in the sense that they accept to move from any consumption plan to another physical allocation provided that they receive today an amount of consumption high enough (see Assumption (A2)).

For convenience of notations, it is useful to introduce the set of portfolios which are compatible with financial participation restrictions and give access to net credit on a debt contract j. That is, given $h\in\mathcal{H}$, we define $\Psi^h_j:\Delta^{S+1}\times\mathbb{X}\twoheadrightarrow\mathbb{R}^J_+\times\mathbb{R}^J_+$ by $\Psi^h_j(p,x):=\left\{(\theta,\varphi)\in\Phi^h(p,x):\varphi_j>\theta_j\right\}$.

With the aim to determine the interdependence between debt and investment constraints, we consider the set of securities required to have access to credit through market k, denoted by $\mathcal{J}_k$. A security $j\in\mathcal{J}_k$ if and only if there exists $(p,x,h)\in\Delta^{S+1}\times\mathbb{X}\times\mathcal{H}$ and $(\theta,\varphi)\in\Psi^h_k(p,x)$ such that: (i) $\theta_j>0$, and (ii) for any $(\tilde\theta,\tilde\varphi)\in\Psi^h_k(p,x)$, either $\tilde\theta_j>0$ or $\tilde\varphi_k-\tilde\theta_k<\varphi_k-\theta_k$. It follows from definition above that, given $(\theta,\varphi)\in\Psi^h_k(p,x)$, there always exists a portfolio $(\tilde\theta,\tilde\varphi)\in\Psi^h_k(p,x)$ that gives at least the same access to debt contract k than (θ,φ), but without require any investment on securities $j\in\mathcal{J}\setminus\mathcal{J}_k$. Therefore, the relationship between credit and investment requirements can be summarized by the matrix $B=(b_{k,j})\in\mathbb{M}_J(\{0,1\})$, where $b_{k,j}=1$ if and only if $j\in\mathcal{J}_k$. To avoid cycles generated by the restrictions imposed by our financial constraints, we require regularity conditions over the matrix B (see Assumption (A5)).

Each agent $h \in \mathcal{H}$ is characterized by preferences that may depend of both consumption allocations and amounts of default, through a correspondence $P^h : \mathbb{X} \times \mathbb{R}_+^{SJ} \twoheadrightarrow \mathbb{X} \times \mathbb{R}_+^{SJ}$. More precisely, given a plan of consumption $x \in \mathbb{X}$ and amounts of default $d = (d_{s,j})_{(s,j)\in\mathcal{S}\times\mathcal{J}} \in \mathbb{R}_+^{SJ}$, the set of allocations (x', d') that agent h strictly prefer to (x, d) is given by the set $P^h(x, d)$.

For prices $(p, q, \gamma) \in \mathbb{P} := \Delta^{S+1} \times \mathbb{R}_+^J \times \mathbb{R}_+^J$ and security promises $R \in \mathbb{R}_+^{SJ}$, the budget set of agent h, denoted by $B^h(p, q, \gamma, R)$, is given by the set of plans $(x, \theta, \varphi, \delta) \in \mathbb{E} := \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{SJ}$ such that,

$$((\theta, \varphi), \delta_{s,j}) \in \Phi^h(p, x) \times \Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j), \quad \forall (s, j) \in \mathcal{S} \times \mathcal{J},$$

$$p_0 x_0 + \sum_{j\in\mathcal{J}} (q_j \theta_j - \gamma_j \varphi_j) \le p_0 w_0^h; \qquad p_s x_s \le p_s w_s^h + p_s Y_s x_0 + \sum_{j\in\mathcal{J}} (R_{s,j}\theta_j - \delta_{s,j}).$$

For any $h \in \mathcal{H}$, the financial default induced by $(x, \theta, \varphi, \delta) \in B^h(p, q, \gamma, R)$ is measured through a function $d^h : \Delta^{S+1} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{SJ} \to \mathbb{R}_+^{SJ}$, where $d^h(p, \gamma, \varphi, \delta) = (d_{s,j}^h(p_0, p_s, \gamma_j, \varphi_j, \delta_{s,j}); (s, j) \in \mathcal{S} \times \mathcal{J})$. For convenience of notations, let $D^h : \Delta^{S+1} \times \mathbb{R}_+^J \times \mathbb{E} \twoheadrightarrow \mathbb{X} \times \mathbb{R}_+^{SJ}$ be the function defined by $D^h(p, \gamma, (x, \theta, \varphi, \delta)) = (x, d^h(p, \gamma, \varphi, \delta))$.

Taking as given prices $(p, q, \gamma) \in \mathbb{P}$ and security promises $R \in \mathbb{R}_+^{SJ}$, each agent $h \in \mathcal{H}$ chooses the most preferred allocation on $B^h(p, q, \gamma, R)$. That is, any $h \in \mathcal{H}$ chooses a vector $(x^h, \theta^h, \varphi^h, \delta^h) \in \mathbb{E}$ such that $P^h(D^h(p, \gamma, (x^h, \theta^h, \varphi^h, \delta^h))) \bigcap D^h(p, \gamma, B^h(p, q, \gamma, R)) = \emptyset$.

Definition. *An equilibrium is given by a vector of prices and promises* $(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R}) \in \mathbb{P} \times \mathbb{R}_+^{SJ}$ *jointly with individual allocations* $((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\delta}^h); h \in \mathcal{H}) \in \mathbb{E}^H$ *such that*:

(i) *For each agent* $h \in \mathcal{H}, (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\delta}^h) \in B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R})$.

(ii) *Agents make optimal choices,*

$$P^h(D^h(\overline{p}, \overline{\gamma}, (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\delta}^h))) \cap D^h(\overline{p}, \overline{\gamma}, B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R})) = \emptyset, \quad \forall h \in \mathcal{H}.$$

(iii) *Physical and asset markets clearing conditions hold,*

$$\sum_{h\in\mathcal{H}} \overline{x}^h = \sum_{h\in\mathcal{H}} W^h \qquad \sum_{h\in\mathcal{H}} \overline{q}_j \overline{\theta}_j^h = \sum_{h\in\mathcal{H}} \overline{\gamma}_j \overline{\varphi}_j^h, \quad \forall j \in \mathcal{J}.$$

(iv) *For any* $j \in \mathcal{J}$, *security payments are non-trivial, i.e.* $(\overline{R}_{s,j}; s \in \mathcal{S}) \neq 0$, *and total payments must be equal the aggregated amount of deliveries,*

$$\overline{R}_{s,j} \sum_{h\in\mathcal{H}} \overline{\theta}_j^h = \sum_{h\in\mathcal{H}} \overline{\delta}_{s,j}^h, \quad \forall s \in \mathcal{S}.$$

3. EQUILIBRIUM EXISTENCE

In this section we impose hypotheses to ensure that equilibrium always exists in our abstract economy.

ASSUMPTION A1. *Correspondences $(P^h; h \in \mathcal{H})$ are open, irreflexive, strictly increasing on $\mathbb{X}$ and have convex values. If $(x', d') \in P^h(x,d)$, then $(x', 0) \in P^h(x,d)$. In addition, $(W^h; h \in \mathcal{H}) \gg 0$.*

The first requirements imposed on Assumption (A1), about continuity, monotonicity and convexity of preferences, are traditional in the literature of equilibrium (see Gale and Mas-Colell (1976, 1979)). In addition, we impose a monotonicity requirement on preferences' default dependence. Precisely, departing from any allocation (x,d), if we pick another one that is preferred than it by an agent h, then the allocation obtained from the later by the reduction of the amount of default to zero is still better than (x,d). Notice that, this requirement is different than assume that P^h is decreasing in d: $(x,d') \in P^h(x,d)$, for any $0 \leq d' < d$, because this last requirement is compatible only with models where all agent have preferences that effectively depend on the amount of default.

The following hypothesis is one of the keys to show equilibrium existence without need to impose financial survival assumptions, spanning conditions, or financial accessibility requirements.

ASSUMPTION A2. *There exists $\tau^h : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^L_+$ continuous, such that, for any $y = (y_s; s \in S^*) \gg 0$,*

$$\left(\left(y_0 + \tau^h(x,y), (y_s; s \in \mathcal{S})\right), d\right) \in P^h(x,d), \quad \forall (x,d) \in \mathbb{X} \times \mathbb{R}^{SJ}_+.$$

This hypothesis is a type of impatience condition: any individual can improve his level of satisfaction by change his consumption allocation to another one, provided that the latter delivers a sufficient higher amount of resources to be consumed at the first period. In the context of models with ordered preferences, Seghir and Torres-Martínez (2011, Assumption A3) was the first that impose this type of requirement to avoid financial survival conditions.

To prove the existence of equilibrium in our abstract economy, we will need that budget set correspondences be continuous and convex valued.

ASSUMPTION A3. *For any $h \in \mathcal{H}$, $\Phi^h : \Delta^{S+1} \times \mathbb{X} \twoheadrightarrow \mathbb{R}^J_+ \times \mathbb{R}^J_+$ has closed graph and satisfies:*

(i) for any $x \in \mathbb{X}$, $p \twoheadrightarrow \Phi^h(p,x)$ is continuous;

(ii) for any $p \in \Delta^{S+1}$, $x \twoheadrightarrow \Phi^h(p,x)$ has convex graph;

(iii) given $p \in \Delta^{S+1}$, we have $0 \in \Phi^h(p,x) \subseteq \Phi^h(p,\tilde{x})$, for any $x \leq \tilde{x}$.

If admissible financial payments are well behaved (see Assumption (A7)), then the upper hemicontinuity of budget set correspondences is ensured using the closed graph property above, jointly with the continuity of $\Phi^h(\cdot, x)$. Lower-hemicontinuity of budget set correspondences follows from (A3)(ii)-(iii).

Although we do not impose any financial survival assumption, we assume that for any price and for any consumption level, there exists at least one agent that have access to credit on a debt contract $j \in \mathcal{J}$. Indeed, in other case, a debt contract does not exists as a financial instrument. Also, it is always possible to have access to credit j without need to invest into the security j. This conditions are summarized in the following hypothesis.

Assumption A4. *For any* $(p, x, j) \in \Delta^{S+1} \times \mathbb{R}^{L(S+1)}_{++} \times \mathcal{J}$, $\bigcup_{h \in \mathcal{H}} \Psi^h_j(p, x) \neq \emptyset$.
Given $(\theta, \varphi) \in \Psi^h_j(p, x)$, *there is* $(\tilde{\theta}, \varphi) \in \Psi^h_j(p, x)$ *such that* $\tilde{\theta}_j = 0$.

We want to avoid cycles of financial constraints in our economy. Thus, the following requirement ensure that, if the possession of some securities is required to obtain access to credit through contract j, then investment in j can not be required to obtain credit from debt contracts associated to these securities.

Assumption A5. *There exists* $\underline{\alpha} > 0$ *such that, for any* $\alpha > \underline{\alpha}$, *if* $(I - \alpha B)y \geq 0$ *then* $y \geq 0$.

We need admissible debt positions to be bounded. Although it is a strong assumption in a variety of general equilibrium models, we will impose it in our abstract economy. However, in most of the applications that we make the boundedness of debt appears endogenously.

Assumption A6. *For any* $(p, x, h) \in \Delta^{S+1} \times \mathbb{X} \times \mathcal{H}$, *the set* $\left\{\varphi \in \mathbb{R}^J_+ : \exists \theta \in \mathbb{R}^J_+, (\theta, \varphi) \in \Phi^h(p, x)\right\}$ *is compact.*

We introduce some requirements on the set of admissible financial payments. The hypotheses below are essential to ensure that budget set correspondences are well behaved: continuous, with convex and compact values.

Assumption A7. *Correspondences* $(\Omega_{s,j}; (s, j) \in \mathcal{S} \times \mathcal{J})$ *are continuous and have non-empty values. Given* $(p_0, p_s, \gamma_j) \in \Delta \times \Delta \times \mathbb{R}_+$, $\varphi_j \twoheadrightarrow \Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j)$ *has convex graph.*

Assumption A8. *For some* $M > 0$, *given* $(p, q, \gamma) \in \mathbb{P}$, $\Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j) \subseteq [0, M\varphi_j]$, $\forall (s, j) \in \mathcal{S} \times \mathcal{J}$.

The next requirement ensure that each security is non-trivial. That is, it has a positive promise at least in some state of nature at the second period.

Assumption A9. *There exists* $\nu_{s,j} : \Delta\times\Delta \to \mathbb{R}_+$ *such that,* $\Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j) \subseteq [\nu_{s,j}(p_0, p_s)\varphi_j, +\infty]$.

For any $j \in \mathcal{J}$, *if* $p \gg 0$ *then* $(\nu_{s,j}(p_0, p_s); s \in \mathcal{S}) \neq 0$.

Finally, continuity and convexity of default functions are needed to obtain well defined individuals' best reply correspondences. Also, agents can pay their promises (no default is always feasible).

Assumption A10. *Functions* $(d^h; h \in \mathcal{H})$ *are continuous and convex on* (φ, δ).
For any $h \in \mathcal{H}$, *given* $(p, x, \gamma) \in \Delta^{S+1} \times \mathbb{X} \times \mathbb{R}^J_+$ *and* $(\theta, \varphi) \in \Phi^h(p, x)$,

$$0 \in d^h_{s,j}(p_0, p_s, \gamma_j, \varphi_j, \Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j)), \quad \forall (s, j) \in \mathcal{S} \times \mathcal{J}.$$

Our main result is the following,

Theorem. *Suppose that Assumption (A1)-(A10) hold. Then, there exists an equilibrium for our economy where* $(\overline{p}, \overline{q}, \overline{\gamma}) \gg 0$ *and* $\overline{q} = \overline{\gamma}$.

The following corollary extends the main result of Seghir and Torres-Martínez (2011) to allow for non-ordered preferences and financial participation constraints over investment. As byproducts of this corollary, we obtain results of existence of equilibrium with nominal, real or collateralized assets discussed by Seghir and Torres-Martínez (2011), allowing for more general types of preferences and investment restrictions.

Corollary. *Suppose that Assumptions (A1)-(A6) hold.*
In addition, assume that,

(i) *For any* $h \in \mathcal{H}$, *there is* $\tilde{\Phi}^h : \mathbb{X} \to \mathbb{R}^J_+ \times \mathbb{R}^J_+$ *such that, for any* $p \in \Delta^{S+1}$, $\Phi^h(p, \cdot) = \tilde{\Phi}^h(\cdot)$.
(ii) *For any* $(s, j) \in \mathcal{S} \times \mathcal{J}$, *there exists a continuous function* $V^s_j : \Delta \to \mathbb{R}_+$ *such that,* $\Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j) = \left\{V^s_j(p_s)\varphi_j\right\}$. *Given* $j \in \mathcal{J}$ *and* $(p_s; s \in \mathcal{S}) \gg 0$, $(V^s_j(p_s); s \in \mathcal{S}) \neq 0$.
(iii) *For any* $h \in \mathcal{H}$, $(d^h; h \in \mathcal{H}) = 0$.

Then, there exists an equilibrium where $(\overline{p}, \overline{q}, \overline{\gamma}) \gg 0$ *and* $\overline{q} = \overline{\gamma}$.

Proof. It is sufficient to ensure that Assumption (A7)-(A10) hold. First, hypothesis (ii) assures that $\Omega_{s,j}$ is continuous and that $\varphi_j \twoheadrightarrow \Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j)$ has convex graph, which implies that (A7) holds. Functions $\{V^s_j; (s, j) \in \mathcal{S}\times\mathcal{J}\}$ have a maximum value–because they are continuous–a property which guarantee that (A8) is satisfied. Since for any $p \gg 0$, $(V^s_j(p_s); s \in \mathcal{S}) \neq 0$, Assumption (A9) holds. The absence of default and the validity of Assumption (A10) are a consequence of hypotheses (ii)-(iii). From our main result, we conclude that this economy has a non-trivial equilibrium. $\square$

4. Equilibrium with Exclusive Credit Lines and Investment Clubs

In our model we do not impose financial survival assumptions and, therefore, we can ensure the existence of equilibrium even when agents have access to only a subset of credit contracts and/or investment opportunities. Thus, the degree of market incompleteness does not depends only of prices, but also on the access to financial contracts.

Given prices $p \in \Delta^{S+1}$, assume that agent h can only invests in securities that belong into a set $\mathcal{J}_\theta(p) \subset \mathcal{J}$, and he has access to credit only through contracts that are in $\mathcal{J}_\varphi(p) \subset \mathcal{J}$. That is, for any $x \in \mathbb{X}$ we have that,

$$(\theta,\varphi) \in \Phi^h(p,x) \Longrightarrow (\theta_j,\varphi_k) = 0, \quad \forall j \notin \mathcal{J}^h_\theta(p),\ \forall k \notin \mathcal{J}^h_\varphi(p),$$

$$\exists(\tilde{\theta},\tilde{\varphi}) \in \Phi^h(p,x) : (\theta_j,\varphi_k) \gg 0, \quad \forall (j,k) \in \mathcal{J}^h_\theta(p) \times \mathcal{J}^h_\varphi(p).$$

It follows that, at prices $p \in \Delta^{S+1}$, a security j is an investment opportunity only for agents in $\mathcal{H}_{\theta,j}(p) := \{h \in \mathcal{H} : j \in \mathcal{J}^h_\theta(p)\}$. Analogously, a credit line $k \in \mathcal{J}$ is available only for agents in $\mathcal{H}_{\varphi,k}(p) := \{h \in \mathcal{H} : k \in \mathcal{J}^h_\varphi(p)\}$.

We refer to a non-empty set of agents $\mathcal{H}' \subsetneq \mathcal{H}$ as an *investment club*—compatible with prices $p \in \Delta^{S+1}$—if there is a subset of securities $\mathcal{J}' \subset \mathcal{J}$ such that $\mathcal{H}' \subseteq \bigcap_{j\in\mathcal{J}'} \mathcal{H}_{\theta,j}(p)$. In addition, a credit contract j constitutes an *exclusive credit line* at prices p if $\mathcal{H}_{\varphi,j}(p) \notin \{\emptyset,\mathcal{H}\}$.

As we said above, the absence of financial survival assumptions allow us to prove equilibrium existence in economies where the set of investment clubs and exclusive credit lines is non-empty. That is, economies where Assumptions (A1)-(A10) are compatible with

$$\left\{\mathcal{J}' \subset \mathcal{J} : \exists p \in \Delta^{S+1}, \bigcap_{j\in\mathcal{J}'} \mathcal{H}_{\theta,j}(p) \neq \emptyset\right\} \neq \emptyset \quad \text{and} \quad \Big\{j \in \mathcal{J} : \exists p \in \Delta^{S+1}, \mathcal{H}_{\varphi,j}(p) \notin \{\emptyset,\mathcal{H}\}\Big\} \neq \emptyset.$$

We illustrate this possibility with the following example.

Example. Suppose that $(Y_s; s \in \mathcal{S}) = 0$ and $\mathcal{J} = \{j_1, j_2, j_3\}$. For any $h \in \mathcal{H}$, given $(p,x) \in \Delta^{S+1} \times \mathbb{X}$, the set $\Phi^h(p,x)$ is equal to the family of portfolios $(\theta,\varphi) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$ such that,

$$\theta_{j_1} \in \left[0, \min_{s\in\mathcal{S}} F_1\left(\left[p_s w^h_s - p_s\varrho\right]^+\right)\right], \qquad \varphi_{j_2} \in \left[0, \min_{s\in\mathcal{S}} F_2\left([p_s\varepsilon - p_s w^h_s]^+\right)\right],$$

$$\sum_{j\in\mathcal{J}} p_s A_j\, \varphi_j \in \left[0, \min_{s\in\mathcal{S}} F_3\left(p_s w^h_s\right)\right],$$

where $(A_j; j \in \mathcal{J}) \in \mathbb{R}^{L\times 3}_{++}$, $(\varrho,\varepsilon) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+$, and (F_1,F_2,F_3) are continuous and satisfy $(F_1(0),F_2(0)) = 0$. In addition, assume that $(d^h; h \in \mathcal{H}) = 0$ and, for any $(h,s,j) \in \mathcal{H}\times\mathcal{S}\times\mathcal{J}$, $\Omega^h_{s,j}(p_0,p_s,\gamma_j,\varphi_j) =$

$\{p_s A_j \varphi_j\}$. Thus, it follows that Assumptions (A3)-(A10) hold. Therefore, if we assume that preferences and endowments satisfy (A1)-(A2), always exists an equilibrium with strictly positive prices.

In this model, an agent can invest in j_1 if and only if, at any $s \in \mathcal{S}$, his physical wealth is greater than the market value of ϱ. Hence, the investment club associated to security j_1 at prices p is given by $\mathcal{H}_{j_1}(p) := \{h \in \mathcal{H} : p_s w_s^h > p_s \varrho, \forall s \in \mathcal{S}\}$, which is non-empty if $\max_{h\in\mathcal{H}} \min_{(s,l)\in\mathcal{S}\times\mathcal{L}} w_{s,l}^h > \max_{l\in\mathcal{L}} \varrho_l$.

On the other hand, we have a financial market where the bankruptcy law protects investors in case of default. Thus, debt constraints depend on the non-exempt resources that the borrower has at the state of nature $s \in \mathcal{S}$, $F_3(p_s w_s^h)$, and borrowers never defaults on their promises. Therefore, there is no loss of generality if we suppose that, for any asset $j \in \mathcal{J}$, there is only one admissible payment: the market value of the original promise, $p_s A_j$.

Note that line of credit j_2 is exclusive to low income borrowers. Indeed, an agent can borrow resources through debt contract j_2 if and only if the value of his physical endowments, at any state of nature at the second period, is lower than the value of ε.

Therefore, if we assume that

$$\max_{h\in\mathcal{H}} \min_{(s,l)\in\mathcal{S}\times\mathcal{L}} w_{s,l}^h > \max_{l\in\mathcal{L}} \{\varepsilon_l, \varrho_l\} > \min_{l\in\mathcal{L}} \{\varepsilon_l, \varrho_l\} > \min_{h\in\mathcal{H}} \max_{(s,l)\in\mathcal{S}\times\mathcal{L}} w_{s,l}^h$$

then, *independently of prices*, we have both an investment club and an exclusive credit line in our economy. □

5. Applications to Credit Risk Models

Our framework is general enough to capture a variety of general equilibrium models with incomplete financial markets and credit risk. We illustrate this possibility making applications of our main result to models with collateralized asset markets, and to economies where preferences are negatively affect by the amount of default.

Collateralized asset markets. Suppose that the set of debt contracts $\mathcal{J}$ is characterized by

$$\Phi^h(p,x) = \left\{(\theta,\varphi) \in Z^h : x_0 \geq \sum_{j\in\mathcal{J}} C_j(p_0)\varphi_j\right\}, \quad \forall h \in \mathcal{H},$$

$$\Omega_{s,j}(p_0, p_s, \gamma_j, \varphi_j) = \{\min\{p_s A_{s,j}, p_s Y_s C_j(p_0)\}\varphi_j\}, \quad \forall (s,j) \in \mathcal{S}\times\mathcal{J},$$

where, for each $(h,j) \in \mathcal{H}\times\mathcal{J}$, $Z^h \subset \mathbb{R}_+^J \times \mathbb{R}_+^J$, $C_j : \Delta \to \mathbb{R}_+^L$, and $(A_{s,j}; s \in \mathcal{S}) \in \mathbb{R}_+^{L\times S}$.

Then, we have a model with collateralized debt, where promises are real, $(A_{s,j}; (s,j) \in \mathcal{S}\times\mathcal{J})$, collateral coefficients may depend on prices, $(C_j(p_0); j \in \mathcal{J})$; and agent can also suffer additional restrictions on their financial participation through sets $(Z^h; h \in \mathcal{H})$.

In this model, the only payment enforcement mechanism in case of default is the seizure of collateral guarantees. That is, individual preferences do not depends on the level of default. Thus, for any $h \in \mathcal{H}$ there exists $Q^h : \mathbb{X} \twoheadrightarrow \mathbb{X}$ such that $P^h(x,d) = Q^h(x)$, for any $x \in \mathbb{X}$.

We assume that Assumptions (A1) and (A2) hold. Also, for any $j \in \mathcal{J}$, there exists a non-empty set $L_j \subseteq L$ such that, for any price $p_0 \in \Delta$, $C_{j,l}(p_0) \neq 0 \Leftrightarrow l \in L_j$. That is, independently of prices, collateral requirements associated to asset j are always constituted by positive quantities of the same commodities. Then, it is not difficult to verify that Assumptions (A3) and (A6) hold, provided that functions $\{C_j; j \in \mathcal{J}\}$ be continuous and sets $(Z^h; h \in \mathcal{H})$ be non-empty, closed and convex, with $0 \in \bigcap_{h\in\mathcal{H}} Z^h$.

Suppose that, for any $h \in \mathcal{H}$, there are sets $Z^h_\theta \subset \mathbb{R}^J_+$ and $Z^h_\varphi \subset \mathbb{R}^J_+$ such that $Z^h = Z^h_\theta \times Z^h_\varphi$. Also, for some $a > 0$, $(a,\dots,a) \in \bigcup_{h\in\mathcal{H}} Z^h_\varphi$.[3] Then, Assumption (A4) is satisfied. Under the hypotheses imposed above, the matrix $B = 0$ and, therefore, Assumption (A5) holds.

Assumptions (A7) and (A8) are satisfied by construction. If for any $j \in \mathcal{J}$ there is $s \in \mathcal{S}$ such that $A_{s,j} \neq 0$ and $\left(\sum_{l'\in L_j} Y_s(l,l'); l \in \mathcal{L}\right) \neq 0$, then Assumption (A9) holds too. Taking $(d^h; h \in \mathcal{H}) = 0$, we guarantee the validity of (A10).

Under these conditions, and as a direct consequence of our main result, we obtain the existence of equilibrium in a financial market composed by collateralized debt where: (i) collateral requirements may depends on prices; (ii) borrowers may suffer financial participation constraints in addition to the requirement of constitute collateral guarantees; and (iii) individual preferences may be neither ordered nor transitive. That is, we extend the seminal work of Geanakoplos and Zame (1997, 2002, 2007) in several dimensions.

FINANCIAL COLLATERAL. Since in our abstract model we allow investment opportunities to be restricted as a function of prices, we can use our main result to include financial collateral in the model of Geanakoplos and Zame (1997, 2002, 2007).

For instance, following the notation of the previous application, assume that there exists $j_1 \in \mathcal{J}$ such that, for any $h \in \mathcal{H}$,

$$\Phi^h(p,x) = \left\{ (\theta,\varphi) \in Z^h_\theta \times Z^h_\varphi : \, x_0 \geq C_{j_1}(p_0)\varphi_{j_1} \quad \wedge \quad \theta_{j_1} \geq \sum_{j\neq j_1} C_j(p_0)\varphi_j \right\},$$

and, for any $(s,j) \in \mathcal{S} \times \mathcal{J}$,

$$\Omega_{s,j}(p_0,p_s,\gamma_j,\varphi_j) = \begin{cases} \{\min\{p_s A_{s,j}, p_s Y_s C_j(p_0)\}\varphi_j\}, & \text{if } j = j_1, \\ \{\min\{p_s A_{s,j}, \min\{p_s A_{s,j_1}, p_s Y_s C_{j_1}(p_0)\} C_j(p_0)\}\varphi_j\}, & \text{if } j \neq j_1, \end{cases}$$

[3]Note that, under previous assumptions, for any $(h,\alpha) \in \mathcal{H} \times \{\theta,\varphi\}$, the set Z^h_α is non-empty, closed, convex and contains the vector zero.

where $C_{j_1} : \Delta \to \mathbb{R}^L_+$ and $\{C_j : \Delta \to \mathbb{R}_{++}; j \neq j_1\}$ are the functions that determine unitary collateral coefficients (physical or financial). With the specification above, the loan j_1 is backed by physical collateral, and to negotiate any other debt contract it is necessary to constitute a financial collateral in units of asset j_1. As in the previous application, we concentrate in the seizure of collateral guarantees as payment enforcement mechanism. Thus, preferences do not depend of the amount of default.

To prove equilibrium existence in this context, we need to guarantee that Assumptions (A1)-(A10) hold. We impose (A1)-(A2). Also, for any $h \in \mathcal{H}$, sets Z^h_θ and Z^h_φ satisfy the same hypotheses imposed in the previous application, where sets $(Z^h_\theta; h \in \mathcal{H})$ do not induce any restriction on the amount of investment on j_1. Suppose that $\{C_j; j \in \mathcal{J})$ are continuous functions and there is $L_1 \subseteq L$ such that, for each $p_0 \in \Delta$, $C_{j_1,l}(p_0) \neq 0 \Leftrightarrow l \in L_1$. Then, (A3)-(A5) and (A7)-(A8) are satisfied.[4] To obtain Assumption (A9) it is sufficient to assume that, for any $j \in \mathcal{J}$, there exists $s \in \mathcal{S}$ such that, $A_{s,j} \neq 0$, $A_{s,j_1} \neq 0$, and $\left(\sum_{l' \in L_1} Y_s(l,l'); l \in \mathcal{L}\right) \neq 0$. If $(d^h; h \in \mathcal{H}) = 0$, Assumption (A10) is satisfied.

Therefore, to ensure the existence of equilibrium, it is sufficient to guarantee that Assumption (A6) holds. However, this is not necessarily the case, because sets $(Z^h_\theta; h \in \mathcal{H})$ do not induce restrictions on θ_{j_1} and, hence, the quantity of debt on assets $j \neq j_1$ is not necessarily bounded. Since in our main result we ensure the existence of an equilibrium where $\sum_{h\in\mathcal{H}} (\overline{\theta}^h_{j_1} - \overline{\varphi}^h_{j_1}) = 0$, there are natural upper bounds for the amount of borrowing on any debt contract. Indeed, collateral constraints induce upper bounds for positions on debt contract j_1 (as a function of consumption and collateral coefficients); using the market feasibility condition referred above, we obtain upper bounds for $(\overline{\theta}^h_{j_1}; h \in \mathcal{H})$. Thus, using the continuity and positivity of collateral requirements, collateral constraints induce upper bounds for positions on debts contracts $j \neq j_1$. In short, we can restrict, without loss of generality, the space of admissible debt to a compact set. This process assure the validity of Assumption (A6), and the existence of equilibria, in an abstract market whose equilibria are equilibria of the original economy.

Utility penalties as additional payment enforcement. We want to extend the seminal model of Dubey, Geanakoplos and Shubik (1990, 2005) (see also Zame (1993)) to allow for incomplete financial participation, debts backed by physical or financial collateral, and non-ordered preferences negatively affected by amounts of default.

Following the notations of the previous applications, suppose that the set of debt contracts can be partitioned into a finite number of non-empty subsets $(\mathcal{A}_k; k \in \{1, \ldots, r\})$ in such form that, for any agent $h \in \mathcal{H}$ the correspondence Φ^h associates to any $(p, x) \in \Delta^{S+1} \times \mathbb{X}$ the portfolios

[4] Note that, in this application, the matrix B is characterized by $b_{k,j} = 1$ if and only if both $k \neq j_1$ and $j = j_1$.

$(\theta,\varphi)\in Z^h$ that satisfy

$$x_0 \geq \sum_{j\in\mathcal{J}_1} C_j(p_0)\varphi_j, \qquad \theta_j \geq \sum_{m=k+1}^{r}\sum_{j'\in\mathcal{A}_m} (C_{j'}(p_0))_j\,\varphi_{j'},\ \ \forall j\in\mathcal{A}_k,\ \forall k\in\{2,\dots,r\}$$

In addition, suppose that, for any $(s,j)\in\mathcal{S}\times\mathcal{J}$, $\Omega_{s,j}(p_0,p_s,\gamma_j,\varphi_j)=[T_{s,j}(p_0,p_s)\varphi_j, p_sA_{s,j}\varphi_j]$, where $T_{s,j}:\Delta\times\Delta\to\mathbb{R}$ is recursively defined by

$$T_{s,j}(p_0,p_s)=\begin{cases}\min\{p_sA_{s,j}, p_sY_sC_j(p_0)\}, & \text{if } j\in\mathcal{A}_1,\\ \min\left\{p_sA_{s,j}, \sum\limits_{m=1}^{k}\sum\limits_{j'\in\mathcal{J}_m}(C_j(p_0))_{j'}\,T_{s,j'}(p_0,p_s)\right\}, & \text{if } j\in\mathcal{A}_{k+1},\ k\in\{1,\dots,r-1\}.\end{cases}$$

Thus, we have a pyramidal financial structure. The first tier $\mathcal{A}_1$ is composed by debt contracts that are collateralized by physical goods and make real promises $(A_{s,j};(s,j)\in\mathcal{S}\times\mathcal{A}_1)$. In the next levels, $(\mathcal{A}_k)_{1<k\leq r}$, debt contracts are real assets backed by securities that belongs on the previous tiers. Since collateral guarantees are seized in case of default, borrowers always decide to pay, at least, the minimum between the value of the original promise and the collateral value.

However, additional payment enforcement mechanism may induce debtors to increase their payments. To capture this effect, assume that for any $(s,j)\in\mathcal{S}\times\mathcal{J}$ there is a threshold $a_{s,j}\geq 0$ such that, if the amount of default of a borrower of j at s in greater than $a_{s,j}$, then he can receives a punishment. More precisely, if agent $h\in\mathcal{H}$ negotiates φ_j units of debt contract j and decides to pay an amount of resources $\delta_{s,j}\in\Omega_{s,j}(p_0,p_s,\gamma_j,\varphi_j)$ at state of nature s, then the amount of default that can affect individuals preferences is given by

$$d^h_{s,j}(p_0,p_s,\gamma_j,\varphi_j,\delta_{s,j})=\frac{[p_sA_{s,j}\varphi_j-\delta_{s,j}-a_{s,j}]^+}{p_sv_s},$$

where $v_s\in\mathbb{R}^L_{++}$ is a fixed bundle. Also, we assume that agents can not be benefited if he increase the amount of default, i.e., for any $x\in\mathbb{X}$, $(x,d)\notin P^h(x,d')$ when $d'\leq d$. Particularly, we have the case where agents have preferences represented by utility functions and suffer linear utility penalties proportional to the total amount of default, i.e. $(a_{s,j};(s,j)\in\mathcal{S}\times\mathcal{J})=0$ (as in the original contribution of Dubey, Geanakoplos and Shubik (1990, 2005)).

To guarantee the existence of equilibrium, is enough to ensure the validity of Assumptions (A1)-(A10). With this objective, assume that sets $(Z^h;h\in\mathcal{H})$ satisfy the conditions imposed in the previous applications and $Z^h_\theta=\mathbb{R}^J_+$ for any $h\in\mathcal{H}$. Suppose that, for each $j\in\mathcal{A}_1$, $C_j:\Delta\to\mathbb{R}^L_+$ is a continuous function and there exist a non-empty set $L_j\subseteq L$ such that, for any $p_0\in\Delta$, $C_{j,l}(p_0)\neq 0\Leftrightarrow l\in L_j$. Also, for each $j\in\mathcal{A}_{k+1}$, where $k\in\{1,\dots,r-1\}$, the functions $C_j:\Delta\to\mathbb{R}^{n(k)}_+$, $n(k)=\#\bigcup\limits_{r=1}^{k}\mathcal{A}_r$, are continuous and there exist a $E_j\subset\bigcup\limits_{r=1}^{k}\mathcal{A}_r$ such that for each $p_0\in\Delta$, $(C_j(p_0))_{j'}\neq 0\Leftrightarrow j'\in E_j$. On the other hand, for each $(s,j)\in\mathcal{S}\times\mathcal{J}$, $T_{s,j}$ is continuous and that for any $j\in\mathcal{J}$, there exists $s\in\mathcal{S}$ such that, $A_{s,j}\neq 0$, and $\left(\sum\limits_{l'\in L_1}Y_s(l,l');l\in\mathcal{L}\right)\neq 0$. Thus, Assumptions (A3)-(A5) and (A7)-(A9) are satisfied.

It is easy to see that the functions $d_{s,j}^h$ are continuous and convex in (φ^h, δ^h) and that satisfies Assumption (A10). Therefore, if we impose hypotheses (A1)-(A2), our main theorem ensures that equilibrium exists, provided that Assumption (A6) holds.

As in the previous application, we can impose bounds on admissible debt as a consequence of the pyramiding structure of the financial market. Market feasibility conditions and collateral constraints ensure the existence of natural upper bounds for investment on securities in $\mathcal{A}_1$. Thus, using collateral constraints again, we obtain natural upper bounds for debt on contract that belongs to $\mathcal{A}_2$. By analogous arguments, we can found upper bounds on debt that do not change the set of equilibria of the economy.

6. Concluding Remarks

In this work we address a general equilibrium model with incomplete financial participation. Different to previous results on this topic, we do not include financial survival assumptions or financial accessibility conditions to prove the existence of a competitive equilibrium. Essentially, we extend the model of Seghir and Torres-Martínez (2011) to allow for non-ordered preferences, price dependent financial restrictions, and securitization of debts.

We introduce two key assumptions in our abstract model: impatience on consumption and the absence of cycles on the relationship between investment requirements and access to credit. With this conditions, we can prove the existence of equilibrium in a financial model that allow for exclusive credit contracts and investment clubs. We also obtain extensions of models of the literature of general equilibrium to allow for credit risk, collateralized loans and utility punishment for default. Particularly, we generalize the results of Dubey, Geanakoplos, and Zame (1995) and Geanakoplos and Zame (1997, 2002, 2007), including restricted financial participation, price dependent collateral (physical or financial) and non-ordered preferences. Also, we extend the model of default and utility punishment of Dubey, Geanakoplos and Shubik (1990, 2005) and Zame (1993), allowing for more general types of punishments.

As a matter of future research, it is interesting to analyze the role of restricted financial participation in sequential models with incomplete financial markets and infinite time horizon. How general financial constraints are related with no-Ponzi schemes conditions, or the relationship between financial survival, equilibrium existence and uniform impatience can be studied.

Appendix

To prove the existence of a non-trivial equilibrium in our abstract economy, we will start by redefine the condition that characterize an individual optimal plan. Thus, for any $h \in \mathcal{H}$, consider the correspondence $A^h : \Delta^{S+1} \times \mathbb{R}^J_+ \times \mathbb{E} \twoheadrightarrow \mathbb{E}$ given by

$$A^h(p,\gamma,(x,\theta,\varphi,\delta)) = [D^h_{p,\gamma}]^{-1}\left(P^h\left(D^h(p,\gamma,(x,\theta,\varphi,\delta))\right)\right),$$

where, for any $(p,\gamma) \in \Delta^{S+1} \times \mathbb{R}^J_+$, the function $D^h_{p,\gamma} : \mathbb{E} \rightarrow \mathbb{X} \times \mathbb{R}^{SJ}_+$ is defined by $D^h_{p,\gamma}(x,\theta,\varphi,\delta) = D^h(p,\gamma,(x,\theta,\varphi,\delta))$. It follows that, under prices $(p,q,\gamma) \in \mathbb{P}$ and payments $R \in \mathbb{R}^{SJ}_+$, a budget feasible plan $(x^h,\theta^h,\varphi^h,\delta^h) \in \mathbb{E}$ is optimal for agent h if and only if

$$A^h(p,\gamma,(x,\theta,\varphi,\delta)) \cap B^h(p,q,\gamma,R) = \emptyset.$$

Under Assumption (A1), for any $h \in \mathcal{H}$, the correspondence A^h has open graph and convex values.[5] Moreover, if we consider the correspondence $\hat{A}^h : \Delta^{S+1} \times \mathbb{R}^J_+ \times \mathbb{E} \twoheadrightarrow \mathbb{E}$ defined by

$$\hat{A}^h(p,\gamma,z) = \{(1-\lambda)z' + \lambda z : z' \in A(p,\gamma,z) \ \wedge \ \lambda \in [0,1)\},$$

then a sufficient condition for a plan $(x,\theta,\varphi,\delta) \in B^h(p,q,\gamma,R)$ be an optimal choice for agent h is that $\hat{A}^h(p,\gamma,(x,\theta,\varphi,\delta)) \cap B^h(p,q,\gamma,R) = \emptyset$. This result is a consequence of the fact that

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[5] Suppose that $((p,\gamma,(x,\theta,\varphi,\delta));(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})) \in \text{Graph}[A^h]$, then $\overline{a} = (D^h(p,\gamma,(x,\theta,\varphi,\delta)); D^h(p,\gamma,(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta}))) \in \text{Graph}[\ P^h]$. Since the graph of the correspondence P^h is an open set relative to $(\mathbb{X} \times \mathbb{R}^{SJ}_+)^2$ (Assumption (A1)), it follows that there is $\epsilon_1 > 0$ such that $B_{\epsilon_1}(\overline{a}) \cap (\mathbb{X} \times \mathbb{R}^{SJ}_+)^2 \subseteq \text{Graph}[P^h]$, where for any pair $(\epsilon, a) \in \mathbb{R}_{++} \times \mathbb{R}^k$ we have $B_\epsilon(a) := \{a' \in \mathbb{R}^k : \|a - a'\| < \epsilon\}$. Therefore, there exists $\epsilon_2 > 0$ such that

$$\left[B_{\epsilon_2}(D^h(p,\gamma,(x,\theta,\varphi,\delta))) \times B_{\epsilon_2}(D^h(p,\gamma,(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})))\right] \bigcap (\mathbb{X} \times \mathbb{R}^{SJ}_+)^2 \subseteq \text{Graph}[P^h].$$

The continuity of function D^h and $D^h_{p,\gamma}$ (Assumption (A10)) assures that both $[D^h]^{-1}\left(B_{\epsilon_2}(D^h(p,\gamma,(x,\theta,\varphi,\delta)))\right)$ and $[D^h_{p,\gamma}]^{-1}\left(B_{\epsilon_2}(D^h(p,\gamma,(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})))\right)$ are open sets, which implies that, for some $\epsilon_3 > 0$,

$$\left[B_{\epsilon_3}(p,\gamma,(x,\theta,\varphi,\delta)) \ \times \ B_{\epsilon_3}(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})\right] \bigcap \left[(\mathbb{X} \times \mathbb{R}^J_+ \times \mathbb{E}) \times \mathbb{E}\right] \subseteq \text{Graph}[A^h].$$

Finally, it follows that there is $\epsilon_4 > 0$ such that $B_{\epsilon_4}((p,\gamma,(x,\theta,\varphi,\delta));(\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})) \bigcap \left[(\mathbb{X} \times \mathbb{R}^J_+ \times \mathbb{E}) \times \mathbb{E}\right] \subseteq \text{Graph}[A^h]$. That is, the graph of the correspondence A^h is open relative to the set $(\Delta^{S+1} \times \mathbb{R}^J_+ \times \mathbb{E}) \times \mathbb{E}$.

On the other hand, if $z = (\widetilde{x},\widetilde{\theta},\widetilde{\varphi},\widetilde{\delta})$ and $z' = (\widetilde{x}',\widetilde{\theta}',\widetilde{\varphi}',\widetilde{\delta}')$ belongs to $A^h(p,\gamma,(x,\theta,\varphi,\delta))$, then $D^h(p,\gamma,z)$ and $D^h(p,\gamma,z')$ are both in $P^h(D^h(p,\gamma,(x,\theta,\varphi,\delta)))$. Since the correspondence P^h has convex values, we have that, for any $\lambda \in [0,1]$, $\lambda D^h(p,\gamma,z) + (1-\lambda)D^h(p,\gamma,z') \in P^h(D^h(p,\gamma,(x,\theta,\varphi,\delta))$. The convexity of the function d^h (Assumption (A10)), assures that for any $\lambda \in [0,1]$, we have that $D^h(p,\gamma,\lambda z + (1-\lambda)z') \leq \lambda D^h(p,\gamma,z) + (1-\lambda)D^h(p,\gamma,z')$. Therefore, Assumption (A1) and the definition of D^h guarantee that, independently of $\lambda \in [0,1]$, $D^h(p,\gamma,\lambda z + (1-\lambda)z'^h(D^h(p,\gamma,(x,\theta,\varphi,\delta))$. Equivalently, $\lambda z + (1-\lambda)z' \in A^h(p,\gamma,(x,\theta,\varphi,\delta))$, $\forall \lambda \in [0,1]$, which concludes the proof.

$A^h(p,\gamma,z) \subseteq \hat{A}^h(p,\gamma,z)$, for any $(p,\gamma,z) \in \Delta^{S+1} \times \mathbb{R}^J_+ \times \mathbb{E}$. In addition, for any $h \in \mathcal{H}$, the correspondence $\hat{A}^h$ has open graph and convex values.[6]

Given $(n,m) \in \mathbb{N} \times \mathbb{N}$, we consider an abstract game $\mathcal{G}(n,m)$ in which prices (p,q,γ) belongs to the set $\widehat{\mathbb{P}}_m := \Delta^{S+1} \times \mathcal{Q}_m$, where $\mathcal{Q}_m := \left\{(q,\gamma) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+ : q_j = \gamma_j \in [0,m],\ \forall j \in \mathcal{J}\right\}$.

Let $\mathcal{R} = \prod_{(s,j)\in\mathcal{S}\times\mathcal{J}} [0,2M]$ and $W = \max_{(s,l)\in S^*\times\mathcal{L}} \sum_{h\in\mathcal{H}} W^h_{s,l}$.

For any agent $h \in \mathcal{H}$, let $B^h_{n,m} : \widehat{\mathbb{P}}_m \times \mathcal{R} \twoheadrightarrow \mathbb{Y}_n$ be the correspondence defined by $B^h_{n,m}(p,q,\gamma,R) := B^h(p,q,\gamma,R) \bigcap \mathbb{Y}_n$, where $\mathbb{Y}_n := [0,n]^L \times [0,2W]^{SL} \times K_n \times [0,2M\,\kappa_n]^{SJ}$, and

$$K_n = \left\{(\theta,\varphi) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+ : \max_{j\in\mathcal{J}} \varphi_j \le 2\,\kappa_n,\ \max_{j\in\mathcal{J}} \theta_j \le 2\,\kappa_n\, H\right\},$$

$$\begin{aligned}\kappa_n &= \max_{h\in\mathcal{H}} \max_{x\in[0,n]^L\times[0,2W]^{SL}} \max_{p\in\Delta^{S+1}} \max_{\varphi\in\pi_\varphi(\Phi^h)(p,x)} \sum_{j\in J} \varphi_j \\ &= \max_{h\in\mathcal{H}} \max_{p\in\Delta^{S+1}} \max_{\varphi\in\pi_\varphi(\Phi^h)(p,((n,\dots,n),(2W,\dots,2W)))} \sum_{j\in J} \varphi_j.\end{aligned}$$

Note that, the last equality in the definition of κ_n follows from Assumption (A3)(iii). Moreover, κ_n is well defined, because Δ^{S+1} is compact and, for any $h \in \mathcal{H}$, the correspondence $\pi_\varphi(\Phi^h) : \Delta^{S+1} \times \mathbb{X} \twoheadrightarrow \mathbb{R}^J_+$ defined by $\pi_\varphi(\Phi^h)(p,x) = \left\{\varphi \in \mathbb{R}^J_+ : \exists\theta \in \mathbb{R}^J_+, (\theta,\varphi) \in \Phi^h(p,x)\right\}$ is continuous on p, with non-empty and compact values (see Assumptions (A3)(i), (A3)(iii) and (A6)).

In $\mathcal{G}(n,m)$, associated to any $h \in \mathcal{H}$, we consider a correspondence $\Gamma^h_{n,m} : \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R} \twoheadrightarrow \mathbb{Y}_n$, which is defined by

$$\Gamma^h_{n,m}(z,(p,q,\gamma),R) = \begin{cases} \mathring{B}^h_{n,m}(p,q,\gamma,R), & \text{if } z^h \notin B^h_{n,m}(p,q,\gamma,R), \\ \mathring{B}^h_{n,m}(p,q,\gamma,R) \bigcap \hat{A}^h(p,\gamma,z^h), & \text{if } z^h \in B^h_{n,m}(p,q,\gamma,R), \end{cases}$$

where $z := (z^k; k \in \mathcal{H})$ is a generic element of $\mathbb{Y}^H_n$, and the set $\mathring{B}^h_{n,m}(p,q,\gamma,R)$ is given by the collection of allocations $z^h = (x^h,\theta^h,\varphi^h,\delta^h) \in B^h_{n,m}(p,q,\gamma,R)$ which satisfy,

$$p_0 x^h_0 + \sum_{j\in\mathcal{J}} (q_j\theta^h_j - \gamma_j\varphi^h_j) < p_0 w^h_0; \qquad p_s x^h_s < p_s w^h_s + p_s Y_s x^h_0 + \sum_{j\in\mathcal{J}} (R_{s,j}\theta^h_j - \delta^h_{s,j}),\ \ \forall s \in \mathcal{S}.$$

Moreover, we introduce a correspondence $\Gamma^0_{n,m} : \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R} \twoheadrightarrow \Delta \times \mathcal{Q}_m$ which associates to any $(z,(p,q,\gamma),R) \in \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R}$ the set of prices $(p'_0,q',\gamma') \in \Delta \times \mathcal{Q}_m$ such that,

$$p'_0 \sum_{h\in\mathcal{H}} (x^h_0 - W^h_0) + \sum_{j\in\mathcal{J}} \sum_{h\in\mathcal{H}} (q'_j\theta^h_j - \gamma'_j\varphi^h_j) > p_0 \sum_{h\in\mathcal{H}} (x^h_0 - W^h_0) + \sum_{j\in\mathcal{J}} \sum_{h\in\mathcal{H}} (q_j\theta^h_j - \gamma_j\varphi^h_j).$$

[6] See Bich and Cornet (2003, page 12).

For each $s \in \mathcal{S}$, let $\Gamma^s_{n,m} : \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R} \twoheadrightarrow \Delta$ be the correspondence

$$\Gamma^s_{n,m}(z,(p,q,\gamma),R) = \left\{ p'_s \in \Delta : p'_s \sum_{h\in\mathcal{H}} (x^h_s - W^h_s) > p_s \sum_{h\in\mathcal{H}} (x^h_s - W^h_s) \right\}.$$

Finally, for any $(s,j) \in \mathcal{S}\times\mathcal{J}$, define $\Gamma^{s,j}_{n,m} : \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R} \twoheadrightarrow [0,2M]$ by

$$\Gamma^{s,j}_{n,m}(z,(p,q,\gamma),R) = \left\{ R'_{s,j} \in \left[\frac{\nu_{s,j}(p_0,p_s)}{2}, 2M\right] : \right.$$
$$\left. -\left(R'_{s,j}\sum_{h\in\mathcal{H}}\varphi^h_j - \sum_{h\in\mathcal{H}}\delta^h_{s,j}\right)^2 > -\left(R_{s,j}\sum_{h\in\mathcal{H}}\varphi^h_j - \sum_{h\in\mathcal{H}}\delta^h_{s,j}\right)^2 \right\}.$$

Definition. *Given $(n,m) \in \mathbb{N}\times\mathbb{N}$, an equilibrium for the abstract game $\mathcal{G}(n,m)$ is a vector*

$$\left((\overline{z}^h)_{h\in\mathcal{H}}; (\overline{p},\overline{q},\overline{\gamma}); \overline{R}\right) \in \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R}$$

such that, for any $k \in \mathcal{H}\cup\mathcal{S}^*\cup(\mathcal{S}\times\mathcal{J})$, $\Gamma^k_{n,m}\left((\overline{z}^h)_{h\in\mathcal{H}}; (\overline{p},\overline{q},\overline{\gamma}); \overline{R}\right) = \emptyset$.

Lemma 1. *Given $(n,m,h) \in \mathbb{N}\times\mathbb{N}\times\mathcal{H}$, under Assumptions (A2)-(A3) and (A6)-(A8) the correspondence $\mathring{B}^h_{n,m}$ is lower hemicontinuous.*

Proof. Fix $(p,q,\gamma,R) \in \widehat{\mathbb{P}}_m \times \mathcal{R}$. It follows from Assumptions (A2), (A3)(iii) and (A8) that, for any $h \in \mathcal{H}$, the plan $(0.5w^h_0, (0,25W^h_s; s\in\mathcal{S}), 0,0,0) \in \mathring{B}^h_{n,m}(p,q,\gamma,R)$. Consider a sequence $\{(p_k,q_k,\gamma_k,R_k)\}_{k\in\mathbb{N}} \subset \widehat{\mathbb{P}}_m \times \mathcal{R}$ that converges to (p,q,γ,R). To assure the lower-hemicontinuity of $\mathring{B}_{n,m}$ it is sufficient to prove that, for any $(x^h,\theta^h,\varphi^h,\delta^h) \in \mathring{B}^h_{n,m}(p,q,\gamma,R)$, there is $\overline{k} > 0$ and a sequence $\left\{(x^h_k,\theta^h_k,\varphi^h_k,\delta^h_k)\right\}_{k\geq\overline{k}}$ which converges to $(x^h,\theta^h,\varphi^h,\delta^h)$ such that, for any $k \geq \overline{k}$, $(x^h_k,\theta^h_k,\varphi^h_k,\delta^h_k) \in \mathring{B}^h_{n,m}(p_k,q_k,\gamma_k,R_k)$.

Take as given $(x^h,\theta^h,\varphi^h,\delta^h) \in \mathring{B}^h_{n,m}(p,q,\gamma,R)$. It follows from Assumption (A3)(i) that there exists a sequence $\{(\theta^h_k,\varphi^h_k)\}_{k\in\mathbb{N}} \subset K_n$ that converges to (θ^h,φ^h), such that $(\theta^h_k,\varphi^h_k) \in \Phi^h(p_k,x^h)$. Furthermore, since $\{(p_k,\varphi^h_k)\}_{k\in\mathbb{N}}$ converges to (p,φ^h), Assumption (A7) assures that there exist a sequence $\{\delta^h_k\}_{k\in\mathbb{N}}$ that converges to δ^h, such that $\delta^h_{k,s,j} \in \Omega_{s,j}(p_{k,0},p_{k,s},\gamma_{k,j},\varphi^h_{k,j})$ for each $(k,s,j) \in \mathbb{N}\times\mathcal{S}\times\mathcal{J}$. Therefore, since the sequence $\{(x^h,\theta^h_k,\varphi^h_k,\delta^h_k)\}_{k\in\mathbb{N}}$ converges to $(x^h,\theta^h,\varphi^h,\delta^h)$, it follows from the definition of $\mathring{B}^h_{n,m}$ that there exist a $\overline{k}\in\mathbb{N}$ such that $(x^h,\theta^h_k,\varphi^h_k,\delta^h_k) \in \mathring{B}^h_{n,m}(p_k,q_k,\gamma_k,R_k)$ for any $k \geq \overline{k}$. Thus, $\mathring{B}^h_{n,m}$ is a lower hemicontinuous correspondence. □

Lemma 2. *Given $(n,m) \in \mathbb{N}\times\mathbb{N}$, under Assumptions (A1)-(A3), (A6)-(A8) and (A10), for any $k \in \mathcal{H}\cup\mathcal{S}^*\cup(\mathcal{S}\times\mathcal{J})$ the correspondence $\Gamma^k_{n,m}$ is lower hemicontinuous and has convex values.*

Proof. It is not difficult to verify that $(\Gamma^s_{n,m})_{s\in\mathcal{S}^*}$ and $(\Gamma^{s,j}_{n,m})_{(s,j)\in\mathcal{S}\times\mathcal{J}}$ are lower hemicontinuous with convex values. Assumptions (A1), (A3)(ii), (A7) and (A10) assure that, for any $h \in \mathcal{H}$, the correspondence $\Gamma^h_{n,m}$ has convex values too.

To guarantee the lower hemicontinuity of $\Gamma^h_{n,m}$ at a point $(z,(p,q,\gamma,R))$, consider a sequence $\{(z_k,(p_k,q_k,\gamma_k,R_k))\}_{k\in\mathbb{N}} \subset \mathbb{Y}^H_n \times \widehat{\mathbb{P}}_m \times \mathcal{R}$ that converges to $(z,(p,q,\gamma,R))$ and fix an element $b\in\Gamma^h_{n,m}(z,(p,q,\gamma,R))$.

If $z^h \notin B^h_{n,m}(p,q,\gamma,R)$, then $b\in \mathring{B}^h_{n,m}(p,q,\gamma,R)$. Since the correspondence $\mathring{B}^h_{n,m}$ is lower hemicontinuous, it follows that there is a sequence $\{b_k\}_{k\in\mathbb{N}}\subset\mathbb{Y}_n$ such that, for any $k\in\mathbb{N}$, $b_k\in \mathring{B}^h_{n,m}(p_k,q_k,\gamma_k,R_k)$ and b_k converges to b as k goes to infinity. By Assumptions (A3) and (A7), $B^h_{n,m}$ has closed graph and, therefore, for k high enough $z^h_k\notin B^h_{n,m}(p_k,q_k,\gamma_k,R_k)$. That is, there is $\overline{k}\in\mathbb{N}$, such that $b_k\in\Gamma^h_{n,m}(z_k,(p_k,q_k,\gamma_k,R_k))$, for any $k\geq\overline{k}$. This proves that $\Gamma^h_{n,m}$ is lower hemicontinuous at $(z,(p,q,\gamma,R))$.

Alternatively, if $z^h\in B^h_{n,m}(p,q,\gamma,R)$ then $b\in \mathring{B}^h_{n,m}(p,q,\gamma,R)\cap \hat{A}^h(p,\gamma,z^h)$. Since the correspondence $\Lambda(\tilde{z}^h,\tilde{p},\tilde{q},\tilde{\gamma},\tilde{R}) := \mathring{B}^h_{n,m}(\tilde{p},\tilde{q},\tilde{\gamma},\tilde{R})\cap\hat{A}^h(\tilde{p},\tilde{\gamma},\tilde{z}^h)$ is non-empty at (z^h,p,q,γ,R), the correspondence $\hat{A}^h$ has open graph, and $\mathring{B}^h_{n,m}$ is lower hemicontinuous, it follows that Λ is lower hemicontinuous at (z^h,p,q,γ,R) (see Border (1985, Proposition 11.21.(c))). Thus, there is a sequence $\{b_k\}_{k\in\mathbb{N}}\subset\mathbb{Y}_n$ which converges to b and, for any $k\in\mathbb{N}$, satisfies $b_k\in\mathring{B}^h_{n,m}(p_k,q_k,\gamma_k,R_k)\cap \hat{A}^h(p_k,\gamma_k,z^h_k)\subset \mathring{B}^h_{n,m}(p_k,q_k,\gamma_k,R_k)$. This last property implies that $b_k\in\Gamma^h_{n,m}(z_k,(p_k,q_k,\gamma_k),R_k)$, for any $k\in\mathbb{N}$. Thus, $\Gamma^h_{n,m}$ is lower hemicontinuous at $(z,(p,q,\gamma,R))$. $\square$

Lemma 3. *Given $(n,m)\in\mathbb{N}\times\mathbb{N}$, under Assumptions (A1), (A3)-(A4) and (A7)-(A9), there exists an equilibrium for the abstract game $\mathcal{G}(n,m)$.*

Proof. The correspondences $\{\Gamma^k_{n,m}\}_{k\in\mathcal{H}\cup\mathcal{S}^*\cup(\mathcal{S}\times\mathcal{J})}$ have the same domain, which is equal to the cartesian product of image spaces. In addition, these correspondences do not have fixed points, they are lower hemicontinuous and have convex values. Applying Gale-Mas-Colell Fixed Point Theorem (see Gale and Mas-Colell (1975, 1979)), we obtain an equilibrium for $\mathcal{G}(n,m)$. $\square$

Lemma 4. *Under Assumptions (A1)-(A10), there is $(\overline{n},\overline{m})\in\mathbb{N}\times\mathbb{N}$ such that, for any $n>\overline{n}$, if $\left((\overline{z}^h)_{h\in\mathcal{H}},(\overline{p},\overline{q},\overline{\gamma}),\overline{R}\right)$ is an equilibrium of $\mathcal{G}(n,m)$ and $\max\limits_{(h,l)\in\mathcal{H}\times\mathcal{L}}\overline{x}^h_{0,l}\leq W$, then $\max\limits_{j\in\mathcal{J}}\overline{q}_j<\overline{m}$.*

Proof. Fix $(n,m)\in\mathbb{N}\times\mathbb{N}$. Given an equilibrium of $\mathcal{G}(n,m)$, $\left((\overline{z}^h)_{h\in\mathcal{H}},(\overline{p},\overline{q},\overline{\gamma},\overline{R})\right)$, for any $h\in\mathcal{H}$, let $\overline{d}^h=d^h(\overline{p},\overline{\gamma},\overline{\varphi}^h,\overline{\delta}^h)$. Then, it follows from Assumptions (A1)-(A2) that, for any $h\in\mathcal{H}$,

$$((\overline{y}_0+\tau,(\overline{y}_s;s\in\mathcal{S})),0)\in P^h(\overline{x}^h,\overline{d}^h),$$

where the plan $\overline{y}=(\overline{y}_s;s\in\mathcal{S}^*)\in\mathbb{R}^{L(S+1)}_{++}$ satisfies $\overline{y}_0=\sum\limits_{k\in\mathcal{H}}w^k_0$ and, for any state of nature $s\in\mathcal{S}$, $\overline{y}_s:=(\overline{y}_{s,l};l\in\mathcal{L})=\left(0.5\min\limits_{k\in\mathcal{H}}W^k_{s,l};l\in\mathcal{L}\right)$. In addition, as $\max\limits_{(h,l)\in\mathcal{H}\times\mathcal{L}}\overline{x}^h_{0,l}\leq W$, we can consider the bundle $\tau = (\tau_l; l \in \mathcal{L})$ given by

$$\tau_l = \sum_{k\in\mathcal{H}} \max_{x^k\in[0,W]^L\times[0,2W]^{SL}} \tau_l^k(x^k,\overline{y}), \qquad \forall l \in \mathcal{L}.$$

Thus, it follows from Assumption (A1) and the definition of $\hat{A}^h$ that, for any vector $(\tilde{\theta}^h,\tilde{\varphi}^h,\tilde{\delta}^h) \in K_n \times [0, M\kappa_n]^{SJ}$ which satisfies $d^h(\overline{p},\overline{\gamma},\tilde{\varphi}^h,\tilde{\delta}^h) = 0$,

$$((\overline{y}_0+\tau,(\overline{y}_s; s\in\mathcal{S})),\tilde{\theta}^h,\tilde{\varphi}^h,\tilde{\delta}^h) \in \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h), \quad \forall h \in \mathcal{H}.$$

Since for any $h \in \mathcal{H}$, $\mathring{B}^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) = \emptyset$, it follows that $B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) = \emptyset$.[7] Therefore, for any vector $(\tilde{\theta}^h,\tilde{\varphi}^h,\tilde{\delta}^h) \in K_n\times[0,M\kappa_n]^{SJ}$ such that $d^h(\overline{p},\overline{\gamma},\tilde{\varphi}^h,\tilde{\delta}^h) = 0$, we have that $((\overline{y}_0+\tau,(\overline{y}_s; s\in S)),\tilde{\theta}^h,\tilde{\varphi}^h,\tilde{\delta}^h) \notin B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$.

On the other hand, as a consequence of Assumptions (A3)(i), (A3)(iii), (A4) and (A6), we have that, for any $j \in \mathcal{J}$, the function $v_j : \Delta^{L(S+1)} \to \mathbb{R}$ defined by

$$v_j(p) = \sum_{h\in\mathcal{H}} \max_{\varphi^h\in\pi_\varphi(\Phi^h)(p,\overline{c})\cap[0,2\kappa_1]^J} \varphi^h_j$$

is continuous and strictly positive, where $\overline{c} := (\overline{y}_0+\tau,(\overline{y}_s; s\in S))$. Therefore, there is μ strictly positive, such that, $\mu \le v_j(p)$, for all $p \in \Delta^{L(S+1)}$.

For any $j \in \mathcal{J}$, it follows from Assumptions (A3)(ii), (A3)(iii) and (A4) that the set of agents h for which $\max_{\varphi^h\in\pi_\varphi(\Phi^h)(p,\overline{c})\cap[0,2\kappa_1]^J} \varphi^h_j$ is strictly positive, i.e.,

$$\mathcal{H}_j(\overline{p}) := \{h\in\mathcal{H} : \exists(\theta^h,\varphi^h)\in\Phi^h(\overline{p},\overline{c})\cap K_1,\ \ \varphi^h_j > 0\}$$

is non-empty. It follows from Assumption (A4) that we can always fix, for any $h \in \mathcal{H}_j(\overline{p})$, a portfolio $(\hat{\theta}^h(j),\hat{\varphi}^h(j)) \in \Phi^h(\overline{p},\overline{c})\cap K_1$ such that, $(\hat{\theta}^h_k(j); k\in\mathcal{J}\setminus\mathcal{J}_j) = 0$, and

$$\hat{\varphi}^h(j) \in \operatorname{argmax}_{\varphi^h\in\pi_\varphi(\Phi^h)(p,\overline{c})\cap[0,2\kappa_1]^J} \ \varphi^h_j.$$

Let $\hat{\rho} = \min_{h\in\mathcal{H}}\left\{\frac{\min_{(s,l)\in\mathcal{S}\times\mathcal{L}} 0.5W^h_{s,l}}{2J\kappa_1 M}, 1\right\}$. Then, for any $h \in \mathcal{H}_j(\overline{p})$, given admissible payments $\tilde{\delta}^h_{s,j'} \in \Omega_{s,j'}(\overline{p}_0,\overline{p}_s,\overline{\gamma},\hat{\rho}\hat{\varphi}^h_{j'}(j))$, with $j' \in \mathcal{J}$, we have that,

$$\sum_{j'\in\mathcal{J}} \tilde{\delta}^h_{s,j'} \le M\sum_{j'\in\mathcal{J}} \hat{\rho}\hat{\varphi}^h_{j'}(j) \quad \le \quad \min_{(s,l)\in\mathcal{S}\times\mathcal{L}} 0.5W^h_{s,l} \ \le\ \overline{p}_s(w^h_s + Y_s\overline{y}_0 - \overline{y}_s).$$

Thus, at any state of nature $s \in \mathcal{S}$, an agent $h \in \mathcal{H}(j)$ can choose any feasible plan of payments associated to the portfolio of debt $\hat{\rho}\hat{\varphi}^h(j)$, since the associated cost can be honored with the resources that become available after the consumption of the bundle $\overline{y}_s$ (without need to take into account

[7]Suppose that there is $\widehat{z} \in B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h)$. Then, it follows from Assumptions (A3)(ii), (A3)(iii) and (A7) that, for any $\lambda\in[0,1)$, $\lambda\widehat{z} \in \mathring{B}^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$. Moreover, since $\hat{A}^h$ has open graph, for λ high enough we have that $\lambda\widehat{z} \in \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h)$. That is, there exists $\overline{\lambda} \in (0,1)$ such that, for any $\lambda > \overline{\lambda}$, $\lambda\widehat{z} \in \mathring{B}^h_{n,\overline{m}}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h)$, which is a contradiction. Therefore, $B^h_{n,\overline{m}}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) = \emptyset$

the resources obtained by the depreciated value of the bundle τ, or any financial payment associated to an investment).

Assumption (A10) assures that, for any $h \in \mathcal{H}_j(\overline{p})$, there is a plan of payments $\hat{\delta}^h(j) \in [0, 2M\kappa_1]$ such that, for each $(s, j') \in \mathcal{S} \times \mathcal{J}$, $\hat{\delta}^h_{s,j'}(j) \in \Omega^h_{s,j'}(\overline{p}_0, \overline{p}_s, \overline{\gamma}_{j'}, \hat{\rho}\hat{\varphi}^h_{j'}(j))$, and $d^h(\overline{p}, \overline{\gamma}, \hat{\rho}\hat{\varphi}^h(j), \hat{\delta}^h(j)) = 0$.

It follows that, there is $\overline{n} > W$ such that, for any $n \geq \overline{n}$,

$$\Big((\overline{y}_0 + \tau, (\overline{y}_s; s \in \mathcal{S})), \hat{\rho}\hat{\theta}^h(j), \hat{\rho}\hat{\varphi}h(j), \hat{\delta}^h(j)\Big) \in \mathbb{Y}_n, \quad \forall h \in \mathcal{H}_j(\overline{p}).$$

Moreover, the arguments made at the beginning of the proof, guarantee that, for any $h \in \mathcal{H}_j(\overline{p})$, $\hat{z}^h(j) := \Big((\overline{y}_0 + \tau, (\overline{y}_s; s \in \mathcal{S})), \hat{\rho}\hat{\theta}^h(j), \hat{\rho}\hat{\varphi}h(j), \hat{\delta}^h(j)\Big) \notin B^h_{n,m}(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R})$. However, by construction, plans $(\hat{z}^h(j); h \in \mathcal{H}_j(\overline{p}))$ satisfy second period budget set constraints. In addition, the associated portfolios and debt payments are admissible.[8] Thus, we conclude that, for any agent $h \in \mathcal{H}_j(\overline{p})$, the plan $\hat{z}^h(j)$ does not satisfy the first period budget set constraint.

Since $\overline{q} = \overline{\gamma}$ and Assumption (A4) holds, we have

$$\overline{q}_j \hat{\rho}\, \hat{\varphi}^h_j(j) + \sum_{j' \neq j} \overline{q}_{j'} \hat{\rho}\hat{\varphi}^h_{j'}(j) - \sum_{j' \in \mathcal{J}_j} \overline{q}_{j'} \hat{\rho}\hat{\theta}^h_{j'}(j) < \overline{p}_0 \tau + \overline{p}_0 \sum_{h' \neq h} w^{h'}_0.$$

We know that, for any $(h, j') \in \mathcal{H}_j(\overline{p}) \times \mathcal{J}$, $\hat{\theta}^h_{j'}(j) \leq 2\kappa_1 H$. Hence, adding over agents $h \in \mathcal{H}_j(\overline{p})$ and dividing by $\mu > 0$, we obtain that,

$$\frac{\overline{q}_j}{\mu} \sum_{h \in \mathcal{H}_j(\overline{p})} \hat{\varphi}^h_j(j) - \alpha_1 \sum_{j' \in \mathcal{J}_j} \overline{q}_{j'} \leq \alpha_2 := \frac{H}{\mu} \left(\frac{\left\| \tau + \sum_{h \in \mathcal{H}} w^h_0 \right\|_\Sigma}{\hat{\rho}} \right),$$

where $\alpha_1 = \frac{2\kappa_1 H^2}{\mu}$. Note that, $v_j(\overline{p}) = \sum_{h \in \mathcal{H}_j(\overline{p})} \hat{\varphi}^h_j(j)$ and, for any $j \in \mathcal{J}$, $\sum_{j' \in \mathcal{J}_j} \overline{q}_{j'} = \sum_{j' \in \mathcal{J}} b_{j,j'} \overline{q}_{j'}$. Therefore, using vectorial notation,

$$\overline{q}' - \alpha_1 B\, \overline{q}' \leq \alpha_2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which implies that $\overline{q}$ belongs to the set $\Xi(\alpha_1, \alpha_2) := \left\{ q \in \mathbb{R}^J_+ : (I - \alpha_1 B) q \leq \alpha_2 (1, \ldots, 1)' \right\}$. Since the parameters (α_1, α_2) depend only on primitive variables of the economy, and the set $\Xi(\alpha_1, \alpha_2)$ is bounded,[9] it follows that there exists $\overline{m} \in \mathbb{N}$ such that $\max_{j \in \mathcal{J}} \overline{q}_j < \overline{m}$. □

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[8] Since the correspondence $\Phi^{h(j)}(\overline{p}, \cdot)$ has convex graph and $0 \in \Phi^{h(j)}(\overline{p}, \overline{c})$ (Assumptions (A3)(ii) and (A3)(iii)), for any $\rho \in (0, 1)$ the portfolio $(\rho\hat{\theta}^{h(j)}, \rho\hat{\varphi}^{h(j)})$ belongs to $\Phi^{h(j)}(\overline{p}, \overline{c})$.

[9] Since the matrix B only has coordinates in $\{0, 1\}$ and $\alpha_1 > 0$, for any $\alpha > \alpha_1$, $\Xi(\alpha_1, \alpha_2) \subseteq \Xi(\alpha, \alpha_2)$. Therefore, it is sufficient to assure that, for α high enough, the set $\Xi(\alpha, \alpha_2)$ is bounded. On the other hand, given $\alpha > 0$, $(I - \alpha B)$ is invertible if and only if α^{-1} is an eigenvalue of B. Since B has at most n real eigenvalues, it follows that there exists $\underline{\alpha} > 0$ such that, $(I - \alpha B)$ is invertible for any $\alpha > \underline{\alpha}$.

LEMMA 5. *Under Assumptions (A1)-(A11), and for any $(n,m) > (\overline{n},\overline{m})$, an equilibrium of $\mathcal{G}(n,m)$ is also an equilibrium for our economy.*

PROOF. Let $\left((\overline{z}^h)_{h\in\mathcal{H}}, (\overline{p},\overline{q},\overline{\gamma},\overline{R})\right) \in \mathbb{Y}_n^H \times \widehat{\mathcal{P}}_m \times \mathcal{R}$ be a equilibrium of the $\mathcal{G}(n,m)$, with $(n,m) > (\overline{n},\overline{m})$. We know from Lemma 3 and footnote 3 that, for any agent $h \in \mathcal{H}$,

$$\overline{z}^h \in B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}), \qquad\qquad B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) = \emptyset.$$

Furthermore, since $\Gamma^0_{n,m}(\overline{z}^h,(\overline{p},\overline{q},\overline{\gamma}),\overline{R}) = \emptyset$, for any $(p_0',q',\gamma') \in \Delta \times \mathcal{Q}_m$, we have

$$p_0' \sum_{h\in\mathcal{H}} (\overline{x}_0^h - W_0^h) + \sum_{j\in J}\sum_{h\in\mathcal{H}} (q_j'\overline{\theta}_j^h - \gamma_j'\overline{\varphi}_j^h) \leq \overline{p}_0 \sum_{h\in\mathcal{H}} (\overline{x}_0^h - W_0^h) + \sum_{j\in\mathcal{J}}\sum_{h\in\mathcal{H}} (\overline{q}_j\overline{\theta}_j^h - \overline{\gamma}_j\overline{\varphi}_j^h).$$

As for any agent h, the plan $\overline{z}^h$ belongs to $B^h_{n,\overline{m}}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$, it follows that the right term in the inequality above in non-positive. Thus, for any $(p_0', q'^J$,

$$p_0' \sum_{h\in\mathcal{H}} (\overline{x}_0^h - W_0^h) + \sum_{j\in\mathcal{J}} q_j' \sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) \leq 0.$$

Suppose that, for some $l \in \mathcal{L}$, $\sum_{h\in\mathcal{H}} (\overline{x}_{0,l}^h - W_{0,l}^h) > 0$. Then, setting $p_{0,l}' = 1$, $p_{0,l'}' = 0$ for all $l \neq l'$, and $q' = 0$, we obtain a contradiction with inequality above. Therefore, $\overline{x}_{0,l}^h \leq W$, for all $(h,l) \in \mathcal{H}\times\mathcal{L}$. Moreover, if $\sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) > 0$ then $\overline{q}_j = m > \overline{m}$, a contradiction. We obtain that, $\sum_{h\in\mathcal{H}} (\overline{\theta}^h - \overline{\varphi}^h) \leq 0$.

Since for any $(h,l) \in \mathcal{H}\times\mathcal{L}$, $\overline{x}_{0,l}^h \leq W < n$, if for some agent h the first period budget constraint on $B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$ is non-binding, then there is $\widehat{x}_0^h \gg \overline{x}_0^h$, such that $(\widehat{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\delta}^h) \in B^h_{n,\overline{m}}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$. From the strict monotonicity of $\hat{A}^h$ on x_0, we have that $\hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) \cap B^h_{n,\overline{m}}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \neq \emptyset$, which contradicts the fact that $\Gamma^h_{n,m}(\overline{z},(\overline{p},\overline{q},\overline{\gamma}),\overline{R}) = \emptyset$. Thus, for each agent h, the first period budget constraint holds as an equality.

It follows from arguments above that,

$$\overline{p}_0 \sum_{h\in\mathcal{H}} (\overline{x}_0^h - W_0^h) + \sum_{j\in\mathcal{J}}\sum_{h\in\mathcal{H}} (\overline{q}_j\overline{\theta}_j^h - \overline{\gamma}_j\overline{\varphi}_j^h) = 0;$$

$$\sum_{h\in\mathcal{H}} (\overline{x}_0^h - W_0^h) \leq 0, \qquad \text{and} \qquad \sum_{h\in\mathcal{H}} (\overline{\theta}^h - \overline{\varphi}^h) \leq 0.$$

If $\alpha > \underline{\alpha}$ and $\hat{q} \in \{q \in \mathbb{R}^J_+ : (I-\alpha B)q = \alpha_2(1,\ldots,1)'\}$, then $\hat{q}$ is an upper bound for $\Xi(\alpha,\alpha_2)$. Indeed, for any $q \in \Xi(\alpha,\alpha_2)$, we have that $(I-\alpha B)q \leq (I-\alpha B)\hat{q}$. Thus, $(I-\alpha B)(\hat{q}-q) \geq 0$. Hence, Assumption (A5) guarantees that $\hat{q} \geq q$.

Therefore, we want to prove that, for any $\alpha > \max\{\alpha_1,\underline{\alpha}\}$, the set $\{q \in \mathbb{R}^J_+ : (I-\alpha B)q = \alpha_2(1,\ldots,1)'\}$ is non-empty and bounded. Fix $\alpha > \max\{\alpha_1,\underline{\alpha}\}$. Since the matrix $(I-\alpha B)$ is non-singular (Assumption (A5)), there exists a unique $q_\alpha \in \mathbb{R}^J$ such that $(I-\alpha B)q_\alpha = \alpha_2(1,\ldots,1)'$. Moreover, as $\alpha_2 > 0$, it follows from (A5) that $q_\alpha \geq 0$.

Therefore, if for some $l \in \mathcal{L}$, we have that $\sum_{h\in\mathcal{H}} \overline{x}^h_{0,l} < \sum_{h\in\mathcal{H}} W^h_{0,l}$, then $\overline{p}_{0,l} = 0$. Because $\overline{x}^h_{0,l} < n$, the strictly monotonicity of $\hat{A}^h$ on first period consumption, implies that $B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R}) \cap \hat{A}^h(\overline{p},\overline{\gamma},\overline{z}^h) \neq \emptyset$, a contradiction. We conclude that, $\overline{p}_0 \gg 0$ and $\sum_{h\in\mathcal{H}} (\overline{x}^h_0 - W^h_0) = 0$.

On the other hand, given $(s,j) \in \mathcal{S}\times\mathcal{J}$, as $\Gamma^{s,j}_{n,m}(\overline{z},(\overline{p},\overline{q},\overline{\gamma}),\overline{R}) = \emptyset$, for each $R'_{s,j} \in \left[\frac{\nu_{s,j}(\overline{p}_s)}{2}, 2M\right]$ we have

$$-\left(R'_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j - \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}\right)^2 \leq -\left(\overline{R}_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j - \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}\right)^2$$

If $\overline{R}_{s,j} \sum_{h\in\mathcal{H}} \overline{\varphi}^h_j < \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}$, then $\sum_{h\in\mathcal{H}} \varphi^h_j > 0$ and Assumption (A8) implies that

$$\overline{R}_{s,j} < \frac{\sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}}{\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j} \leq M,$$

which is a contradiction, because in this case any $R'_{s,j} \in (\overline{R}_{s,j}, 2M]$ that satisfies $R'_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j \leq \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}$ belongs to $\Gamma^{s,j}_{n,m}(\overline{z},(\overline{p},\overline{q},\overline{\gamma}),\overline{R})$. Analogously, when $\overline{R}_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j > \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}$, we have that $\sum_{h\in\mathcal{H}} \varphi^h_j > 0$ and Assumption (A9) assures that,

$$\overline{R}_{s,j} > \frac{\sum_{h\in\mathcal{H}} \delta^h_{s,j}}{\sum_{h\in\mathcal{H}} \varphi^h_j} \geq \nu_{s,j}(\overline{p}_0,\overline{p}_s),$$

which implies that any $R'_{s,j} \in \left[\frac{\nu_{s,j}(\overline{p}_0,\overline{p}_s)}{2}, \overline{R}_{s,j}\right)$ that satisfies $R'_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j \geq \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}$belongs to $\Gamma^{s,j}_{n,m}(\overline{z},(\overline{p},\overline{q},\overline{\gamma}),\overline{R})$, a contradiction. We conclude that, at each state $s \in \mathcal{S}$ and for any $j \in \mathcal{J}$,

$$\overline{R}_{s,j}\sum_{h\in\mathcal{H}} \overline{\varphi}^h_j = \sum_{h\in\mathcal{H}} \overline{\delta}^h_{s,j}.$$

Using that for each h, $\overline{z}^h \in B^h_{n,m}(\overline{p},\overline{q},\overline{\gamma},\overline{R})$, it follows from $\sum_{h\in\mathcal{H}} (\overline{\theta}^h - \overline{\varphi}^h) \leq 0$ that, for any $s \in \mathcal{S}$,

$$\overline{p}_s \sum_{h\in\mathcal{H}} (\overline{x}^h_s - (W^h_s + Y_s\overline{x}^h_0)) \leq 0$$

Since for any $s \in \mathcal{S}$, $\Gamma^s_{n,m}(\overline{z}^h,(\overline{p},\overline{q},\overline{\gamma}),\overline{R}) = \emptyset$, it follows that for any $p'_s \in \Delta$,

$$p'_s \sum_{h\in\mathcal{H}} (\overline{x}^h_s - (W^h_s + Y_s\overline{x}^h_0)) \leq 0,$$

which implies that $\sum_{h\in\mathcal{H}} (\overline{x}^h_s - (W^h_s + Y_s\overline{x}^h_0)) \leq 0$. Thus, for any $(h,s,l) \in \mathcal{H}\times\mathcal{S}\times\mathcal{L}$, $\overline{x}^h_{s,l} < 2W$, which in turn implies both that $(\overline{p}_s; s \in \mathcal{S}) \gg 0$ and that second period budget constraints are satisfied as equalities (since preferences are strictly monotonic on consumption).

As commodity prices are strictly positive, using Assumption (A9) we obtain that, for each $j \in \mathcal{J}$ there is $s \in \mathcal{S}$ such that $\overline{R}_{s,j} > 0$.

If for some $j \in \mathcal{J}$, $\sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) < 0$ then $\overline{q}_j = 0$, because first period budget set constraints hold as an equality, and there is no excess of demand in financial markets. However, as unitary security payments are non-trivial, i.e. $(\overline{R}_{s,j}; s \in \mathcal{S}) \neq 0$, when $\overline{q}_j = \overline{\gamma}_j$ are equal to zero, for any $h \in \mathcal{H}$, $\overline{\theta}_j^h = 2H\kappa_n$, which contradicts the fact that $\sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) \leq 0$. Indeed, for any h, $\overline{\varphi}^h \in \pi_\varphi(\Phi^h)(\overline{p}, \overline{x}^h)$, which assures that $\sum_{h\in\mathcal{H}} \overline{\varphi}_j^h \leq H\kappa_n$.

We conclude that, $\overline{q} \gg 0$ and $\sum_{h\in\mathcal{H}} (\overline{\theta}_j^h - \overline{\varphi}_j^h) = 0$. This last property, implies that, for any $(s,j) \in \mathcal{S} \times \mathcal{J}$,

$$\overline{R}_{s,j} \sum_{h\in\mathcal{H}} \overline{\theta}_j^h = \sum_{h\in\mathcal{H}} \overline{\delta}_{s,j}^h.$$

Then $\overline{p}_s \sum_{h\in\mathcal{H}} (\overline{x}_s^h - (W_s^h + Y_s \overline{x}_0^h)) = 0$. Since $p_s \gg 0$, we obtain that $\sum_{h\in\mathcal{H}} (\overline{x}_s^h - (W_s^h + Y_s \overline{x}_0^h)) = 0$. Therefore, market clearing conditions are satisfied.

Finally, for each agent $h \in \mathcal{H}$, the plan $\overline{z}^h \in B_{n,m}^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R}) \subset B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R})$ and it also belongs in the interior of $\mathbb{Y}_n$ relative to $\mathbb{E}$. Therefore, if there is $\tilde{z}^h \in B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R}) \cap A^h(\overline{p}, \overline{\gamma}, \overline{z}^h)$, then for any $\lambda \in [0,1)$, the plan $(1-\lambda)\overline{z}^h + \lambda \tilde{z}^h \in \hat{A}^h(\overline{p}, \overline{\gamma}, \overline{z}^h)$. This assures that, for $\lambda \in [0,1)$ high enough, $(1-\lambda)\overline{z}^h + \lambda \tilde{z}^h \in B_{n,m}^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R}) \cap \hat{A}^h(\overline{p}, \overline{\gamma}, \overline{z}^h)$, a contradiction. That is, for any agent $h \in \mathcal{H}$, $B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R}) \cap A^h(\overline{p}, \overline{\gamma}, \overline{z}^h) = \emptyset$. This last property assures the optimality of the plan $\overline{z}^h$ for agent h, when he is restricted to choose allocations on $B^h(\overline{p}, \overline{q}, \overline{\gamma}, \overline{R})$.

Thus, we assure the existence of equilibrium in our economy. $\square$

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