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ESSENTIAL STABILITY FOR LARGE GENERALIZED GAMES

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Abstract. We address the essential stability of Cournot-Nash equilibria for generalized games with a continuum of players, where only a finite number of them are atomic. Given any set of generalized games continuously parameterized by a complete metric space, we analyze the robustness of equilibria to perturbations on parameters.

Keywords. Essential equilibria - Essential sets and components - Generalized games
JEL Classification. C62, C72, C02.

1. Introduction

In this study we focus on stability properties of Cournot-Nash equilibria for large generalized games. We analyze how the set of equilibrium allocations changes when some characteristics of the generalized game are perturbed. We allow for any kind of perturbation, provided that it can be defined through a continuous parameterization over a complete metric space of parameters.

The concept of essential stability has its origins in the mathematical analysis literature, where was introduced as a natural property for fixed points of functions and correspondences. In a seminal paper, Fort (1950) introduces the concept of essential fixed point of a continuous function: a fixed point is essential if it can be approximated by fixed points of functions closed to the original. In addition, a continuous function is essential if it has only essential fixed points. Considering the set of continuous functions from a compact metric space to itself, Fort (1950) ensures that the set of essential functions is dense. He also proves that continuous functions that have only one fixed point are essential. These concepts have natural extensions to multivalued mappings and analogous properties hold, as shown by Jia-He (1962). However, not all mappings are essential and, therefore, it is natural to analyze the stability of subsets of fixed points. Thus, Kinoshita

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(1952) introduces the concept of essential component of the set of fixed points of a function: a maximal connected set that is stable to perturbations on the characteristics of the function. He proves that any continuous mapping has at least one essential component. Jia-He (1963) and Yu and Yang (2004) extend these results to multivalued mappings. They prove that compact-valued upper hemicontinuous correspondences have at least one essential component, although fixed points of these correspondences may not be essential. These results are complemented by Yu, Yang, and Xiang (2005) to analyze not only the existence of essential components, but also how they change when mappings are perturbed.

This literature motivates the study of stability of equilibria in games. Indeed, since in many non-cooperative games the set of Nash equilibria coincides with the set of fixed points of a correspondence, techniques described above allow to analyze stability of solutions in games when payoffs and action sets are perturbed. In this direction, the essential stability of Nash equilibria for games with finitely many players is studied by Wu and Jia-He (1962), Yu (1999), Yu, Yang, and Xiang (2005), Zhou, Yu and Xiang (2007), Yu (2009), and Carbonell-Nicolau (2010).

More precisely, Wu and Jia-He (1962) address the stability of the set of Nash equilibria for finite games. They ensure that any game can be approximated arbitrarily by a game whose equilibria are all essential. Yu (1999) formalizes and extends these results for convex games with a finite number of players, analyzing perturbations in payoffs, in sets of actions, and in correspondences of admissible strategies. Jia-He (1963), Yu, Yang, and Xiang (2005) and Yu (2009) supplement these results to analyze the existence of essential components of the set of Nash equilibria for games and generalized games. Zhou, Yu and Xiang (2007) study the notion of essential stability for mixed-strategy equilibria in games with compact sets of pure strategies and finitely many players. They also compare the concept of essential stability with strategic stability, a notion studied by Kohlberg and Mertens (1986), Hillas (1990), and Al-Najjar (1995). Recently, allowing for discontinuities on objective functions, Carbonell-Nicolau (2010) analyzes the essential stability of Nash equilibria for convex games with finitely many players, .

We contribute to this growing literature by address the essential stability of Cournot-Nash equilibria in large generalized games. We consider generalized games with two types of players: (i) a continuum set of non-atomic players, characterized by continuous objective functions, compact sets of actions, and continuous correspondences of admissible strategies; and (ii) a finite number of atomic players, with quasi-concave and continuous objective functions, compact and convex sets of actions, and continuous and convex-valued correspondences of admissible strategies. We assume that the profile of actions of non-atomic players is codified, and induces messages that may affect decisions of any player. Thus, while the actions of an atomic player may directly affect the decisions of other players, the decisions of non-atomic players only affect others participants through messages. These messages are obtained by the integration of codes that reveals information about the action of non-atomic players.

The previous results of essential stability for games take advantage of the fact that the set of (pure strategy) equilibria is compact and non-empty. Actually, with these properties, to obtain some of the main results of essential stability it is sufficient to ensure that the correspondence that associates games with equilibrium allocations has closed graph. In our case, under mild conditions on the characteristics of the generalized game, a Cournot-Nash equilibrium always exists, as was proved by Schmeidler (1973)—for the case of large games—and by Balder (1999, 2002)—for the case of generalized games. However, the set of pure strategy equilibria is not necessarily compact (see footnote 4). Therefore, the traditional analysis of essential stability can not be directly implemented in our context.

Nevertheless, associated to any Cournot-Nash equilibrium of a large generalized game there is a vector of messages (generated by the actions of non-atomic players) and a vector of optimal actions of atomic players. On the one hand, these vectors of messages-actions constitute all the relevant information that any player takes into account to make optimal decisions. On the other hand, when there is a compact set of non-atomic players with compact sets of actions, the set of equilibrium messages-actions coincides with the fixed points of a compact-graph correspondence and, therefore, it is a compact set too.[1] Therefore, we will analyze the stability of messages-actions to perturbations on the characteristics of the generalized game.

Allowing perturbations on objective functions, sets of actions, and correspondences of admissible strategies, we prove that there is a dense set of generalized games for which any Cournot-Nash equilibrium is essentially stable (Theorem 1). Also, unicity of equilibrium messages and actions for atomic players is a sufficient condition for stability. We also analyze the stability of subsets of equilibrium messages and actions. We obtain results analogous to those ensured for convex games with finitely many players: for any generalized game there are essential connected subsets of Cournot-Nash equilibria (Theorem 2).

In the main contribution of our paper, we extend these results of stability to allow specific perturbations, that we capture through parameterizations of the set of generalized games. We assume that the set of parameters constitutes a complete metric space, and that the mapping which associates parameters with generalized games is continuous. The stability results previously described still holds (Theorem 3) and, also, we prove that essential sets varies continuously (Theorem 4). As particular cases, we obtain stability results for large games and convex (generalized) games with finitely many players, extending results of the literature to allow for a great variety of admissible perturbations.

To obtain most of our results about essential stability, we prove that the correspondence that associates generalized games with sets of equilibrium messages-actions, referred as Cournot-Nash correspondence, is closed. We use the fact that the set of non-atomic players has finite measure and their actions are transformed into finite-dimensional codes (which are integrated to obtain messages). Indeed, under these conditions, we can ensure the closed graph property of the Cournot-Nash correspondence applying the multidimensional Fatou's Lemma (see Hildenbrand (1974, page 52), Theorems 1 and 2).

The rest of the paper is organized as follows: In Section 2 we describe the space of large generalized games, and we introduce the Cournot-Nash correspondence. In Section 3 we state stability results for

[1]This property was proved by Riascos and Torres-Martínez (2012), which extend to generalized games the proof of equilibrium existence in large games due to Rath (1992).

generalized games, when perturbations on objective functions, sets of strategies, and correspondences of admissible strategies are allowed. In Section 4 we extend these results to perturbations induced by continuous parameterizations over complete metric spaces, and we also analyze the stability of essential sets and components.

2. The space $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ of generalized games

In this section we introduce large generalized games, as those studied by Balder (1999, 2002) and Riascos and Torres-Martínez (2012). Through our model we fix some characteristics of the generalized game, which are summarized by a vector $(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$. The set T_1 is a non-empty and compact metric space of players. Also, for some σ-algebra $\mathbb{B}$ of subsets of T_1 there is a measure μ such that, $(T_1, \mathbb{B}, \mu)$ is a finite atomless measure space. T_2 is a non-empty and finite set of atomic players, $\widehat{K}$ is a non-empty and compact metric space and, for any $t \in T_2$, $\widehat{K}_t$ is a non-empty, convex and compact metric space. Finally, $H : T_1 \times \widehat{K} \to \mathbb{R}^m$ is a $\mathbb{B}$-measurable function, which is also continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$.

In a game $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ among players in $T_1 \cup T_2$, each $t \in T_1$ has associated a closed and non-empty action space $K_t \subset \widehat{K}$, while each $t \in T_2$ has a closed, convex and non-empty action space $K_t \subset \widehat{K}_t$. A profile of actions for players in T_1 is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Any vector $a = (a_t; t \in T_2) \in \prod_{t\in T_2} K_t$ constitutes a profile of actions for players in T_2. For each $i \in \{1, 2\}$, let $\mathcal{F}^i((K_t)_{t\in T_i})$ be the space of profiles of actions for agents in T_i. In addition, for any $t \in T_2$, let $\mathcal{F}^2_{-t}((K_j)_{j\in T_2\setminus\{t\}})$ be the set of vectors $a_{-t} \in \Pi_{j\in T_2\setminus\{t\}} K_j$.

Each participant considers aggregated information about the actions taken by players in T_1. Thus, if non-atomic players choose a profile of actions $f \in \mathcal{F}^1((K_t)_{t\in T_1})$, then the relevant characteristics of this actions are coded by the function H. Also, each player only take into account, for strategic purposes, aggregated information about these available characteristics through a message $m(f) := \int_{T_1} H(t, f(t))d\mu$. For this reason, we concentrate our attention only on those action profiles for which messages are well defined. That is, we only consider actions profiles $f \in \mathcal{F}^1((K_t)_{t\in T_1})$ such that $H(\cdot, f(\cdot))$ is a measurable function from T_1 to $\mathbb{R}^m$.[2]

Therefore, $M((K_t)_{t\in T_1}) = \left\{ \int_{T_1} H(t, f(t))d\mu : \quad f \in \mathcal{F}^1((K_t)_{t\in T_1}) \ \wedge \ H(\cdot, f(\cdot)) \text{ is measurable} \right\}$ is the set of messages associated with action profiles of non-atomic players. To ensure that $M((K_t)_{t\in T_1})$ is non-empty, we assume that there is at least one measurable profile of actions.[3]

Let $\widehat{M} = M((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^2_{-t} = \mathcal{F}^2((\widehat{K}_s)_{s\in T_2\setminus\{t\}})$, and $\widehat{\mathcal{F}}^i = \mathcal{F}^i((\widehat{K}_t)_{t\in T_i})$, where $i \in \{1, 2\}$. Messages and profiles of actions may restrict players admissible strategies. Indeed, the set of strategies that are available for a player $t \in T_1$ is determined by a continuous correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ with non-empty and compact values. Analogously, the set of strategies that a player $t \in T_2$ can

[2]That is, for any Borelian set $E \subset \mathbb{R}^m$, the set $\{t \in T_1 : H(t, f(t)) \in E\}$ belongs to $\mathbb{B}$.

[3]Indeed, given a measurable profile of actions f, the function $t \to H(t, f(t))$ is measurable. Since H is continuous and $\{T_1, \widehat{K}\}$ are compact, it follows that $t \to H(t, f(t))$ is bounded. Thus, as T_1 has finite measure, we conclude that the message $\int_{T_1} H(t, f(t))d\mu$ is well defined. Note that, when $\bigcap_{t\in T_1} K_t$ is non-empty, $\mathcal{F}^1((K_t)_{t\in T_1})$ always has measurable elements.

choose is determined by a continuous correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2_{-t} \twoheadrightarrow K_t$, which has non-empty, compact and convex values.

Given a metric space S, let $\mathcal{U}(S)$ be the set of continuous functions $u : S \to \mathbb{R}$ endowed with the sup norm topology. We assume that each player $t \in T_1$ has a objective function $u_t \in \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$, while each atomic player $t \in T_2$ has a objective function $u_t \in \mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ which is quasi-concave in its own strategy a_t (for convenience of notations, we refer to this subset of $\mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ as $\mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$). Finally, we require the mapping $U : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ defined by $U(t) = u_t$ to be measurable.

Definition 1. *A Cournot-Nash equilibrium of $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ is given by action profiles $(f^*, a^*) \in \mathcal{F}^1((K_t)_{t \in T_1}) \times \mathcal{F}^2((K_t)_{t \in T_2})$ such that,*

$$u_t(f^*(t), m^*, a^*) \geq u_t(f(t), m^*, a^*), \quad \forall t \in T_1,\ \forall f(t) \in \Gamma_t(m^*, a^*),$$
$$u_t(m^*, a^*) \geq u_t(m^*, a_t, a^*_{-t}), \quad \forall t \in T_2,\ \forall a_t \in \Gamma_t(m^*, a^*_{-t}),$$

where the message $m^* := \int_{T_1} H(t, f^*(t)) d\mu$.

Balder (1999, 2002) (see also Riascos and Torres-Martínez (2012)) ensure that, for any generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ that satisfies the assumptions described above, the set of Cournot-Nash equilibria , denoted by $\mathrm{CN}(\mathcal{G})$, is non-empty.

2.1. **The Cournot-Nash correspondence.** We want to analyze the stability of Cournot-Nash equilibria of a generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ when parameters $(K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}$ change. To attempt this objective, we will introduce the Cournot-Nash correspondence, which associates the parameters that define the generalized game $\mathcal{G}$ with the set of messages and actions $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ such that, for some $f^* \in \widehat{\mathcal{F}}^1$, we have $m^* = m(f^*)$ and $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$.

Since action profiles are coded using the function H, there may exist several Cournot-Nash equilibria that induce a same message. Despite this, to analyze stability of equilibria we concentrate in messages and actions of atomic players, since the set of Cournot-Nash equilibria is not necessarily compact,[4] a required property to analyze essential stability. Notice that, given any Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, the pair $(m(f^*), a^*)$ contains all the information that players require to take their decisions.

Given a metric space S, let $A(S)$ be the collection of compact and non-empty subset of S and $A_c(S) \subset A(S)$ the sub-collection that considers only convex sets. Denote by d_H the Hausdorff

[4]For instance, consider a large electoral game with a continuum of non-atomic players, $T_1 = [0, 1]$, which vote for a party in $\{a, b\}$. Let x_t be the action of player $t \in T_1$, and assume that his objective function, u_t, only takes into account the benefits that he receives for any party $\{v_t(a), v_t(b)\}$ weighted by the support that each party has in the population, i.e. $u_t \equiv v_t(a)\mu(\{s \in T_1 : x_s = a\}) + v_t(b)(1 - \mu(\{s \in T_1 : x_s = a\}))$, where μ denotes the Lebesgue measure in $[0, 1]$. That is, his own action does not affect the utility level of a player $t \in T_1$ and, therefore, any measurable profile of actions $x : [0, 1] \to \{a, b\}$ constitutes a Nash equilibrium of the game. As a consequence, the set of Nash equilibria is not compact. However, if we consider that each player receives as a message the support that party a has in the population, $m = \mu(\{s \in T_1 : x_s = a\})$, then the set of equilibrium messages is equal to $[0, 1]$, which is a compact set.

metric induced by the metric of S. If S is compact (resp. compact and convex), then $(A(S), d_H)$ (resp. $(A_c(S), d_H)$) is a complete metric space.[5] Let $\Xi(S)$ be the collection of correspondences $\Gamma : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow S$ that are continuous, non-empty and compact valued. In addition, denote by $\Xi_c(S)$ the subset of $\Xi(S)$ composed by convex valued correspondences. Consider the multivalued mapping $\mathbb{B}_S : A(S) \twoheadrightarrow \Xi(S)$ (resp. $\mathbb{B}_{S,c} : A_c(S) \twoheadrightarrow \Xi_c(S)$) that associates to any $S_1 \in A(S)$ (resp. $S_1 \in A_c(S)$) the set of correspondences $\Gamma \in \Xi(S)$ (resp. $\Gamma \in \Xi_c(S)$) whose values are in S_1.

With the notations above, we can recast a generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ by a vector $(U, F, (\eta_t)_{t \in T_2})$, where $U : T_1 \to \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ is a measurable function, $F : T_1 \to \text{Graph}(\mathbb{B}_{\widehat{K}})$ is a function and, for any $t \in T_2$, $\eta_t := (u_t, K_t, \Gamma_t) \in \mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2) \times \text{Graph}(\mathbb{B}_{\widehat{K}_t, c})$.[6]

Let $\mathbb{G} = \mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}, H))$ be the collection of generalized games.

Given two generalized games $\mathcal{G}_1 = (U^1, F^1, (\eta_t^1)_{t \in T_2})$ and $\mathcal{G}_2 = (U^2, F^2, (\eta_t^2)_{t \in T_2})$, we define the distance between $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$
\begin{aligned}
\rho(\mathcal{G}_1, \mathcal{G}_2) \;=\; & \max_{t \in T_1} \max_{(x,m,a) \in \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2} |u_t^1(x, m, a) - u_t^2(x, m, a)| \\
& + \max_{t \in T_1} \max_{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2} d_H(\Gamma_t^1(m, a), \Gamma_t^2(m, a)) + \max_{t \in T_1} d_H(K_t^1, K_t^2) \\
& + \max_{t \in T_2} \max_{(m,x,a_{-t}) \in \widehat{M} \times \widehat{K}_t \times \widehat{\mathcal{F}}^2_{-t}} |u_t^1(m, x, a_{-t}) - u_t^2(m, x, a_{-t})| \\
& + \max_{t \in T_2} \max_{(m,a_{-t}) \in \widehat{M} \times \widehat{\mathcal{F}}^2_{-t}} d_H(\Gamma_t^1(m, a_{-t}), \Gamma_t^2(m, a_{-t})) + \max_{t \in T_2} d_H(K_t^1, K_t^2),
\end{aligned}
$$

where, given $i \in \{1, 2\}$, $(U^i(t), F^i(t))_{t \in T_1} \equiv (u_t^i, (\Gamma_t^i, K_t^i))_{t \in T_1}$ and $(\eta_t^i)_{t \in T_2} \equiv (u_t^i, K_t^i, \Gamma_t^i)_{t \in T_2}$. Since $(T_1, (\widehat{K}, (\widehat{K}_t)_{t \in T_2}))$ are compact sets, T_2 is finite and $\widehat{M}$ is a compact set (see Riascos and Torres-Martínez (2012)), it follows that $(\mathbb{G}, \rho)$ is a complete metric space.[7]

Given a game $\mathcal{G} = (U, F, (\eta_t)_{t \in T_2}) \in \mathbb{G}$, consider the correspondence $\Phi_{\mathcal{G}} : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ defined by $\Phi_{\mathcal{G}}(m, a) = \left(\Omega^{\mathcal{G}}(m, a), (B_t^{\mathcal{G}}(m, a_{-t}))_{t \in T_2}\right)$ where

$$
\begin{aligned}
\Omega^{\mathcal{G}}(m, a) &= \int_{T_1} H(t, B_t^{\mathcal{G}}(m, a)) d\mu; \\
B_t^{\mathcal{G}}(m, a) &= \text{argmax}_{x_t \in \Gamma_t(m,a)} u_t(x_t, m, a), \quad \forall t \in T_1; \\
B_t^{\mathcal{G}}(m, a_{-t}) &= \text{argmax}_{x_t \in \Gamma_t(m,a_{-t})} u_t(x_t, m, a_{-t}), \quad \forall t \in T_2.
\end{aligned}
$$

It follows from Riascos and Torres-Martínez (2012, Theorem 1) that, (f^*, a^*) is a Cournot-Nash equilibrium of $\mathcal{G}$ if and only if $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ is a fixed point of $\Phi_{\mathcal{G}}$, where $m^* = \int_{T_1} H(t, f^*(t)) d\mu$.

[5] Since S is a compact metric spaces, it is complete. It follows from Ok (2005, page 227) that $A(S)$ is a complete metric space under the Hausdorff metric induced by the metric of S. When the space is restricted to $A_c(S)$, $(A_c(S), d_H)$ remains a complete metric space, since the Hausdorff limit of a sequence of compact and convex sets is still a compact and convex set.

[6] That is, any $t \in T_1$ is characterized by a vector $(U(t), F(t)) = (u_t, K_t, \Gamma_t) \in \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2) \times \text{Graph}(\mathbb{B}_{\widehat{K}})$.

[7] Since T_1 is a measurable space and $\mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ is a metric space, if a sequence of games $\left\{(U^n, F^n, (\eta_t^n)_{t \in T_2})\right\}_{n \in \mathbb{N}} \in \mathbb{G}$ converges to a game $(\overline{U}, \overline{F}, (\overline{\eta}_t)_{t \in T_2})$, it follows from the definition of ρ that $\{U^n\}_{n \in \mathbb{N}}$ uniformly converges to $\overline{U}$ and, therefore, $\overline{U}$ is measurable (see Aliprantis and Border (1999, page 139)).

We denote by $\mathrm{FP}(\Phi_{\mathcal{G}})$ the set of fixed points of $\Phi_{\mathcal{G}}$.

Definition 2. *The Cournot-Nash correspondence of* $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ *is given by the multivalued function* $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ *that associates to any* $\mathcal{G} \in \mathbb{G}$ *the set* $FP(\Phi_{\mathcal{G}})$.

Since for any generalized game $\mathcal{G} \in \mathbb{G}$ the correspondence $\Phi_{\mathcal{G}}$ is closed, it follows that $\Lambda(\mathcal{G})$ is a compact subset of $\widehat{M} \times \widehat{\mathcal{F}}^2$.

3. Essential stability of equilibria in $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$

We analyze as the set of Cournot-Nash equilibria of a generalized game changes when the parameters that define the game are modified. Our analysis is based in the concept of *essential stability*, that was introduced in the literature by Fort (1950), for single valued mappings, and by Jia-He (1962), for the case of correspondences.

Definition 3. *An equilibrium* $(f^*, a^*) \in CN(\mathcal{G})$ *is essential if for any open set* $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ *such that* $(m(f^*), a^*) \in O$*, there exists* $\epsilon > 0$ *such that* $\Lambda(\mathcal{G}') \cap O \neq \emptyset$*, for any* $\mathcal{G}' \in \mathbb{G}$ *that satisfies* $\rho(\mathcal{G}, \mathcal{G}') < \epsilon$*. A generalized game* $\mathcal{G} \in \mathbb{G}$ *is essential if all its Cournot-Nash equilibrium are essential.*

Hence, $\mathcal{G}$ is an essential generalized game if, and only if, for any $(m^*, a^*) \in \Lambda(\mathcal{G})$ and for each $\delta > 0$, there exists $\epsilon > 0$ such that, if $\rho(\mathcal{G}', \mathcal{G}) < \epsilon$, then $d((m^*, a^*), \Lambda(\mathcal{G}')) < \delta$.[8] In other words, $\mathcal{G}$ is essential if, and only if, messages and atomic players actions associated to a Cournot-Nash equilibrium of $\mathcal{G}$ can be approximated by equilibrium messages and actions of generalized games that are closed to $\mathcal{G}$. Unfortunately, as the following example illustrate, not all generalized games are essential.

Example. Consider the generalized game $\mathcal{G}$ characterized by $T_1 = [0, 1]$, $T_2 = \{\alpha\}$, $\widehat{K} = \{0, 1\}$, $\widehat{K}_\alpha = [0, 1]$, for any $t \in T_1 \cup T_2$, $(K_t, \Gamma_t) \equiv (\widehat{K}_t, \widehat{K}_t)$, and $H(\cdot, x) \equiv x$. In addition, $u_\alpha(m, x) = -\|m - x\|^2$ and, for any $t \in T_1$, $(u_t(0, m, a_\alpha), u_t(1, m, a_\alpha)) = (0.5, 0.5)$.

Then, there is a continuum of Cournot-Nash equilibria and $\Lambda(\mathcal{G}) = \{(\lambda, \lambda) \in \mathbb{R}^2 : \lambda \in [0, 1]\}$. On the other hand, given $\epsilon > 0$, let $\mathcal{G}_\epsilon$ be the generalized game obtaining from $\mathcal{G}$ by only change the objective functions of non-atomic players to $(u^\epsilon_t(0, m, a_\alpha), u^\epsilon_t(1, m, a_\alpha)) = (0.5(1 + \epsilon), 0.5)$, for any $t \in T_1$. It follows that $\mathcal{G}_\epsilon$ has only one Cournot-Nash equilibrium and $\Lambda(\mathcal{G}_\epsilon) = \{(0, 0)\}$. Since $\rho(\mathcal{G}, \mathcal{G}_\epsilon) < \epsilon$, we conclude that $\mathcal{G}$ is not essential. □

The following results shows that, despite the example above, essentiality of equilibrium is a generic property on $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$.

[8] Given a metric space (S, σ), fix $a \in S$ and $E \subseteq S$ non-empty and compact. Then, we define $d(a, E) = \min_{x\in E} \sigma(a, x)$.

Theorem 1. *Given a generalized game $\mathcal{G} \in \mathbb{G}$, for any $\epsilon > 0$ there exists an essential generalized game $\mathcal{G}' \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}') < \epsilon$. Moreover, if $\Phi_{\mathcal{G}}$ has only one fixed point, then $\mathcal{G}$ is essential.*

Proof. The proof of the theorem is a direct consequence of the following steps.

Step 1. *The correspondence $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ is upper hemicontinuous with compact values.*

Since $\widehat{M} \times \widehat{\mathcal{F}}^2$ is compact and non-empty, we only need to prove that Graph(Λ) is closed in $\mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\text{Graph}(\Lambda) = \Big\{(\mathcal{G}, (m, a)) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2 : (m, a) \in \text{FP}(\Phi_{\mathcal{G}})\Big\}$. Let $\{(\mathcal{G}_n, (m_n, a_n))\}_{n \in \mathbb{N}} \subset$ Graph(Λ) such that $(\mathcal{G}_n, (m_n, a_n)) \to (\overline{\mathcal{G}}, (\overline{m}, \overline{a})) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\mathcal{G}_n = (U^n, F^n, (\eta_t^n)_{t \in T_2})$ and $(U^n(t), F^n(t)) \equiv (u_t^n, (\Gamma_t^n, K_t^n))$. To prove that Graph($\Lambda$) is closed is sufficient to ensure that $(\overline{m}, \overline{a}) \in \text{FP}(\Phi_{\overline{\mathcal{G}}})$.

Since $(m_n, a_n) \in \Phi_{\mathcal{G}_n}(m_n, a_n)$, for any $t \in T_1$ there exists $f_n(t) \in \Gamma_t^n(m_n, a_n)$ such that,

$$m_n = \int_{T_1} H(t, f_n(t)) d\mu, \qquad\qquad u_t^n(f_n(t), m_n, a_n) = \max_{x \in \Gamma_t^n(m_n, a_n)} u_t^n(x, m_n, a_n),$$

and the function $g_n(\cdot) = H(\cdot, f_n(\cdot))$ is measurable.

Claim A. For any $t \in T_1$ there exists $\overline{f}(t) \in \widehat{K}$ such that $\overline{m} = \int_{T_1} H(t, \overline{f}(t)) d\mu$.
Proof. Since H is continuous, T_1 is compact and, for each $t \in T_1$, $f_n(t) \in \widehat{K}$, it follows that $\{g_n\}_{n \in \mathbb{N}}$ is a uniformly integrable sequence (see Hildenbrand (1974, page 52)). In addition, $\{\int_{T_1} g_n(t) d\mu\}_{n \in \mathbb{N}} \subset \mathbb{R}^m$ converges to $\overline{m}$ as n goes to infinity. Therefore, as a consequence of Fatou's Lemma in m-dimension (see Hildenbrand (1974, page 69)), there is an integrable function $g : T_1 \to \mathbb{R}^m$ such that,[9]

(1) $\lim_{n \to \infty} \int_{T_1} g_n(t) d\mu = \int_{T_1} g(t) d\mu$
(2) There exists $\tilde{T}_1 \subseteq T_1$ such that, for any $t \in \tilde{T}_1$, $g(t) \in L_S(g_n(t))$, where $L_S(g_n(t))$ is the set of cluster points of $\{g_n(t)\}_{n \in \mathbb{N}}$ and $T_1 \setminus \tilde{T}_1$ has zero measure.

Fix $t \in \tilde{T}_1$. Then there is a subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $g_{n_k}(t) \to g(t)$. Since $\{f_{n_k}(t)\}_{k \in \mathbb{N}} \subseteq \widehat{K}$, taking a subsequence again if it is necessary, we can assure that there exists $f(t) \in \widehat{K}$ such that both $f_{n_k}(t) \to f(t)$ and $g(t) = \lim_{k \to \infty} H(t, f_{n_k}(t)) = H(t, f(t))$.

Let $\overline{f} : T_1 \to \widehat{K}$ such that

$$\overline{f}(t) \in \begin{cases} \{f(t)\} & \text{if } t \in \tilde{T}_1, \\ argmax_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})} \, \overline{u}_t(x, \overline{m}, \overline{a}) & \text{if } t \notin \tilde{T}_1, \end{cases}$$

where $\overline{\mathcal{G}} = (\overline{U}, \overline{F}, (\overline{\eta}_t)_{t \in T_2})$ and $(\overline{U}(t), \overline{F}(t)) \equiv (\overline{u}_t, (\overline{\Gamma}_t, \overline{K}_t))$. Thus, $\overline{m} = \lim_{n \to \infty} \int_{T_1} H(t, f_n(t)) d\mu = \int_{T_1} H(t, \overline{f}(t)) d\mu$. □

Claim B. For any $t \in T_1$, $\overline{f}(t) \in \overline{\Gamma}_t(\overline{m}, \overline{a})$.
Proof. The results follows by definition for any $t \in T_1 \setminus \tilde{T}_1$. Thus, fix $t \in \tilde{T}_1$ and let $\{f_{n_k}(t)\}_{k \in \mathbb{N}}$ the sequence that was obtained in the previous claim and that converges to $\overline{f}(t)$. We known that, for

[9] Although functions $\{g_n\}_{n \in \mathbb{N}}$ can take negative values, they are uniformly bounded from below (since H is continuous and $\{\widehat{K}, T_1\}$ are compact sets). Thus, as T_1 has finite Lebesgue measure, we can apply the Fatou's Lemma.

any $k \in \mathbb{N}$, $f_{n_k}(t) \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})$ and, therefore,

$$\begin{aligned} d(\overline{f}(t), \overline{\Gamma}_t(\overline{m}, \overline{a})) &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + d(f_{n_k}(t), \Gamma_t^{n_k}(m_{n_k}, a_{n_k})) + d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(m_{n_k}, a_{n_k})) \\ &\qquad + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \\ &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})), \end{aligned}$$

where $\widehat{d}$ denotes the metric of the compact metric space $\widehat{K}$. Since $\overline{\Gamma}_t$ is continuous, by taking the limit as k goes to infinity, we obtain the result. □

Claim C. For any $t \in T_1$, $\overline{f}(t) \in argmax_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})}\ \overline{u}_t(x, \overline{m}, \overline{a})$.
Proof. As in the previous claim, the case $t \in T_1 \setminus \tilde{T}_1$ follows from definition. With the same notation used in the previous claim, we have that, for any $t \in \tilde{T}_1$,

$$d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \quad \le \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})).$$

Then $\Gamma_t^{n_k}(m_{n_k}, a_{n_k}) \longrightarrow_k \overline{\Gamma}_t(\overline{m}, \overline{a})$. Since $u_t^{n_k}$ converges uniformly to $\overline{u}_t$, it follows from Yu (1999, Lemma 2.5) and Aubin (1982, Theorem 3, page 70) that,

$$u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) = \max_{x \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})} u_t^{n_k}(x, m_{n_k}, a_{n_k}) \longrightarrow_k \max_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$$

On the other hand,

$$|u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})| \quad \le \quad \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + |\overline{u}_t(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})|.$$

Taking the limit as k goes to infinity, we conclude that $u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) \to \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})$. Therefore, as a consequence of Claim B, we conclude that $\overline{f}(t) \in argmax_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$. □

Claim D. For any $t \in T_2$, $\overline{a}_t \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$.
Proof. We known that, for any $(t, n) \in T_2 \times \mathbb{N}$, $a_{n,t} \in \Gamma_t^n(m_n, a_{n,-t})$ and, therefore,

$$\begin{aligned} d(\overline{a}_t, \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + d(a_{n,t}, \Gamma_t^n(m_n, a_{n,-t})) + d_H(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(m_n, a_{n,-t})) \\ &\qquad + d_H(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \\ &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + \rho(\mathcal{G}_n, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})), \end{aligned}$$

where $\widehat{d}_t$ denotes the metric of the compact metric space $\widehat{K}_t$. Taking the limit as n goes to infinity, we obtain the result. □

Claim E. For any $t \in T_2$, $\overline{a}_t \in argmax_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$.
Proof. Following the same arguments of Claim C, we have that

$$d_H(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \quad \le \quad \rho(\mathcal{G}_n, \mathcal{G}) + d_H(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t}))$$

and, therefore, $\Gamma_t^n(m_n, a_{n,-t})$ converges to $\overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$ as n goes to infinity. Hence, Yu (1999, Lemma 2.5) ensures that,

$$u_t^n(m_n, a_n) = \max_{x \in \Gamma_t^n(m_n, a_{n,-t})} u_t^n(m_n, x, a_{n,-t}) \longrightarrow \max_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t}).$$

Since $\lim_{n \to +\infty} u_t^n(m_n, a_n) = \overline{u}_t(\overline{m}, \overline{a})$,[10] it follows that $\overline{a}_t \in \text{argmax}_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$. □

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[10] It is a direct consequence of the fact that, for any $n \in \mathbb{N}$, we have

$$|u_t^n(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})| \quad \le \quad \rho(\mathcal{G}_n, \mathcal{G}) + |\overline{u}_t(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})|.$$

It follows from Claims A, C and E that $(\overline{m}, \overline{a})$ is a fixed point of $\Phi_{\overline{\mathcal{G}}}$. Thus, we ensure that Λ is an upper hemicontinuous correspondence with compact values.

Step 2. *There is a dense G_δ set $Q \subset \mathbb{G}$ such that Λ is lower hemicontinuous at every point of Q.*

As $(\mathbb{G}, \rho)$ is a complete metric space, $\mathbb{G}$ is a Baire space. Since the correspondence Λ is compact-valued and upper hemicontinuous with $\Lambda(\mathcal{G}) \neq \emptyset$ for all $\mathcal{G} \in \mathbb{G}$, it follows from Lemma 6 in Carbonell-Nicolau (2010) (see also Fort (1949) and Jia-He (1962)) that there exists a dense G_δ subset Q of $\mathbb{G}$ in which Λ is lower hemicontinuous.

Step 3. *If $\mathcal{G}$ is a point lower hemicontinuity of Λ, then $\mathcal{G}$ is essential.*

Fix a game $\mathcal{G}_1$ that is a point of lower hemicontinuity of Λ. We know that for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $\Lambda(\mathcal{G}_1) \subset O$ we have $\Lambda(\mathcal{G}_1) \cap O \neq \emptyset$ and, therefore, the set $\Lambda^-(O) := \{\mathcal{G} \in \mathbb{G} : \Lambda(\mathcal{G}) \cap O \neq \emptyset\}$ contains a neighborhood of $\mathcal{G}_1$. Then, there exists $\epsilon > 0$ such that, for any $\mathcal{G} \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}_1) < \epsilon$, we have that $\Lambda(\mathcal{G}) \cap O \neq \emptyset$. That is, all Cournot-Nash equilibrium of $\mathcal{G}_1$ are essential.

It follows from Steps 2 and 3 that any generalized game in the dense set Q is essential. Therefore, we can found essential generalized games arbitrarily near of any $\mathcal{G} \in \mathbb{G}$.

Finally, suppose that for a game $\mathcal{G} \in \mathbb{G}$, the correspondence $\Phi_{\mathcal{G}}$ has only one fixed point. Then, Λ is upper hemicontinuous and single valued at $\mathcal{G}$ and, therefore, it is continuous at this point. Using Step 3, we conclude that $\mathcal{G}$ is an essential generalized game. Q.E.D.

We continue with the characterization of essential stability proving that, even unessential generalized games may have subsets of Cournot-Nash equilibria that are stable. To do this, we introduce concepts of stability for subsets of equilibrium points. Indeed, we adapt to our context the concepts of essential set and essential component that were introduced, in the context of stability of fixed point of multivalued mappings, by Jia-He (1963) and Yu and Yang (2004), These concepts were also addressed by Zhou, Yu, and Xiang (2007), to study stability of mixed strategy equilibria in non-convex games with finitely many players.

Definition 4. *Given $\mathcal{G} \in \mathbb{G}$, $e(\mathcal{G}) \subseteq \Lambda(\mathcal{G})$ is an essential set if it is non-empty, compact, and for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ with $e(\mathcal{G}) \subset O$ there is $\epsilon > 0$ such that, for any $\mathcal{G}' \in \mathbb{G}$ with $\rho(\mathcal{G}, \mathcal{G}') < \epsilon$, $\Lambda(\mathcal{G}') \cap O \neq \emptyset$. An essential subset of $\Lambda(\mathcal{G})$ is minimal if it is a minimal element ordered by set inclusion.*

Definition 5. *Given $\mathcal{G} \in \mathbb{G}$, a set Λ_α is a component of $\Lambda(\mathcal{G})$ if there exists $(m^*, a^*) \in \Lambda(\mathcal{G})$ such that, Λ_α is equal to the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains (m^*, a^*).*

Since Λ is upper hemicontinuous, for any generalized game $\mathcal{G} \in \mathbb{G}$ the set $\Lambda(\mathcal{G})$ is essential. Also, given $A \subset B \subseteq \Lambda(\mathcal{G})$, if A is essential and B is compact, then B is essential too. Also, as for any

$\mathcal{G} \in \mathbb{G}$ the set $\Lambda(\mathcal{G})$ is compact, it follows that any component of $\Lambda(\mathcal{G})$ is non-empty, connected and compact.

Notice that, if $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$ is essential, then $\{(m(f^*), a^*)\}$ is an essential subset of $\Lambda(\mathcal{G})$. Thus, it follows from Theorem 1 that there exists a dense collection of generalized games $\mathcal{G} \in \mathbb{G}$ for which $\Lambda(\mathcal{G})$ has at least one minimal essential subset that is connected. Furthermore, for any $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$ the component associated to $\{(m(f^*), a^*)\}$[11] is compact and, therefore, it is an essential subset of $\Lambda(\mathcal{G})$. In other words, for a generic set of generalized games, there exists at least one essential component of the set of equilibrium messages and atomic players actions.

The following result ensures that these properties hold in fact for any generalized game.

Theorem 2. *For each generalized game $\mathcal{G} \in \mathbb{G}$, there exists a minimal essential set of $\Lambda(\mathcal{G})$. In addition, every minimal essential set of $\Lambda(\mathcal{G})$ is connected. Therefore, for each $\mathcal{G} \in \mathbb{G}$, there exists at least one essential component of $\Lambda(\mathcal{G})$.*

Proof. We will adapt to our context the arguments used by Yu and Yang (2004, Theorem 3.3) and Zhou, Yu and Xiang (2007, Theorem 2).

Fix $\mathcal{G} \in \mathbb{G}$. Let $\mathcal{S}$ be the family of essential sets of $\Lambda(\mathcal{G})$ ordered by set inclusion. Since $\Lambda(\mathcal{G}) \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. As any element of $\mathcal{S}$ is compact, any totally ordered subset of $\mathcal{S}$ has a lower bounded element. By Zorn's Lemma, $\mathcal{S}$ has a minimal element, and by definition of $\mathcal{S}$, its minimal element is an essential set of $\Lambda(\mathcal{G})$.

Fix a minimal essential set of $\Lambda(\mathcal{G})$, denoted by $m(\mathcal{G})$. We want to prove that $m(\mathcal{G})$ is connected. By contradiction, if $m(\mathcal{G})$ is not connected, then there are closed and non-empty subsets of $\Lambda(\mathcal{G})$, A_1 and A_2 such that $m(\mathcal{G}) = A_1 \cup A_2$. Also, there are open sets V_1, V_2 such that $A_1 \subset V_1$, $A_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since $m(\mathcal{G})$ is minimal, neither A_1 nor A_2 are essentials.

Fix $i \in \{1, 2\}$. It follows that there exists an open set O_i such that $A_i \subset O_i$ and for all $\epsilon > 0$ there exists $\mathcal{G}_i \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}_i) < \epsilon$ and $\Lambda(\mathcal{G}_i) \cap O_i = \emptyset$. Since A_i is compact, there exists an open set U_i such that $A_i \subset U_i \subset \overline{U}_i \subset V_i \cap O_i$. Therefore, the essential set $m(\mathcal{G}) \subset U_1 \cup U_2$.

Thus, there exists $\nu > 0$ such that for every $\mathcal{G}' \in \mathbb{G}$ with $\rho(\mathcal{G}, \mathcal{G}') < \nu$, we have $\Lambda(\mathcal{G}') \cap (U_1 \cup U_2) \neq \emptyset$. On the other hand, given $i \in \{1, 2\}$, as $U_i \subset O_i$, there exists $\mathcal{G}'_i \in \mathbb{G}$ such that $\rho(\mathcal{G}, \mathcal{G}'_i) < \frac{\nu}{3}$ and $\Lambda(\mathcal{G}'_i) \cap U_i = \emptyset$.

Define a correspondence $G : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \mathbb{G}$ such that

$$G(m, a) = \lambda(m, a)\mathcal{G}'_1 + (1 - \lambda(m, a))\mathcal{G}'_2, \quad \forall (m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2,$$

where $\lambda : \widehat{M} \times \widehat{\mathcal{F}}^2 \to [0, 1]$ is the continuous function given by,

$$\lambda(m, a) = \frac{d((m, a), \overline{U}_2)}{d((m, a), \overline{U}_1) + d((m, a), \overline{U}_2)}.$$

[11]That is, the set obtained by the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains $\{(m(f^*), a^*)\}$.

Notice that, $(m,a) \in U_i$ if and only if $G(m,a) = \mathcal{G}'_i$. In addition, for any $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, we have that,

$$\begin{aligned} \rho(G(m,a),\mathcal{G}'_1) &= \rho\left(\lambda(m,a)\,\mathcal{G}'_1 + (1-\lambda(m,a))\,\mathcal{G}'_2, \lambda(m,a)\,\mathcal{G}'_1 + (1-\lambda(m,a))\,\mathcal{G}'_1\right) \\ &\leq \rho(\mathcal{G}'_2,\mathcal{G}'_1) \leq \rho(\mathcal{G}'_2,\mathcal{G}) + \rho(\mathcal{G},\mathcal{G}'_1) < \frac{2\nu}{3}, \end{aligned}$$

which implies that,

$$\rho(\mathcal{G}, G(m,a)) \leq \rho(\mathcal{G},\mathcal{G}'_1) + \rho(\mathcal{G}'_1, G(m,a)) < \nu,$$

and, therefore, for each $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $\Lambda(G(m,a)) \cap (U_1 \cup U_2) \neq \emptyset$.

Claim. There exists $(\overline{m},\overline{a}) \in U_1$ *such that,* $(\overline{m},\overline{a}) \in \Lambda(G(\overline{m},\overline{a}))$.

Proof. Let $\tilde{A}_1 \subset U_1$ be a compact, convex and non-empty set. Define $\Theta : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1 \times \tilde{A}_1$ by $\Theta((m_1,a_1),(m_2,a_2)) = \left(\Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1\right) \times \{(m_1,a_1)\}$. If we ensure that the correspondence $\Theta_1 : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1$ given by $\Theta_1((m_1,a_1),(m_2,a_2)) = \Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1$ has closed graph, then the correspondence Θ is upper hemicontinous and has non-empty, compact and convex values. Thus, applying the Kakutani's Fixed Point Theorem we can find $(\overline{m},\overline{a}) \in \tilde{A}_1 \subset U_1$ such that, $(\overline{m},\overline{a}) \in \Lambda(G(\overline{m},\overline{a}))$.

Thus, let $\{(z_1^n, z_2^n, (m^n,a^n))\}_{n\in\mathbb{N}} \subset \mathrm{Graph}(\Theta_1)$ a sequence that converges to $(\tilde{z}_1,\tilde{z}_2,(\tilde{m},\tilde{a})) \in \tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1$. We want to prove that $(\tilde{m},\tilde{a}) \in \Theta_1(\tilde{z}_1,\tilde{z}_2)$.

Fix $t \in T_2$ and let $\gamma_t : (\widehat{M} \times \widehat{\mathcal{F}}^2_{-t}) \times \tilde{A}_1 \twoheadrightarrow \widehat{K}_t$ the correspondence characterized by

$$\gamma_t((m,a_{-t}),z) = argmax_{x\in\Psi((m,a_{-t}),z)} v_t(x,(m,a_{-t}),z),$$

where

$$\begin{aligned} \Psi((m,a_{-t}),z) &= \lambda(z)\Gamma_t^1(m,a_{-t}) + (1-\lambda(z))\Gamma_t^2(m,a_{-t}); \\ v_t(x,(m,a_{-t}),z) &= \lambda(z)u_t^1(m,x,a_{-t}) + (1-\lambda(z))u_t^2(m,x,a_{-t}), \end{aligned}$$

and, for each $i \in \{1,2\}$, $\mathcal{G}'_i = \mathcal{G}'_i((K_t^i,\Gamma_t^i,u_t^i)_{t\in T_1\cup T_2})$. Since $\mathcal{G}'_1,\mathcal{G}'_2 \in \mathbb{G}$ and λ is continuous, it follows that γ_t is upper hemicountinuous with non-empty and compact values. Therefore, the correspondence $\gamma : (\widehat{M}\times\widehat{\mathcal{F}}^2)\times\tilde{A}_1 \twoheadrightarrow \Pi_{t\in T_2}\widehat{K}_t$ given by $\gamma((m,a),z_2) = \Pi_{t\in T_2}\gamma_t((m,a_{-t}),z)$ is upper hemicontinuous with compact and non-empty values. In particular, γ has closed graph. Therefore, as for any $n \in \mathbb{N}$, $(z_1^n, z_2^n, a^n) \in \mathrm{Graph}(\gamma)$, it follows that $\tilde{a} \in \gamma(\tilde{z}_1,\tilde{z}_2)$.

On the other hand, for each $n \in \mathbb{N}$ there exists $f_n : T_1 \to \widehat{K}$ such that, $m_n = \int_{T_1} H(t,f_n(t))d\mu$ and, for any $t \in T_1$, $f_n(t) \in \rho_t(z_1^n,z_2^n) := argmax_{x\in\Psi(z_1^n,z_2^n)} v_t(x,z_1^n,z_2^n)$, where we use analogous notations to those described above. Thus, as in the case of γ_t, the correspondences $(\rho_t; t \in T_1)$ have closed graph.

Since $m^n \to \tilde{m}$, analogous arguments to those made in Claim A ensure that, applying the multidimensional Fatou's Lemma (see Hildenbrand (1974, page 69)), there exists a zero-measure set $\dot{T}_1 \subset T_1$ and a function $\overline{f} : T_1 \to \widehat{K}$ such that,

(i) For any $t \in \dot{T}_1$, $\overline{f}(t) \in \rho_t(\tilde{z}_1,\tilde{z}_2)$;
(ii) For any $t \in T_1 \setminus \dot{T}_1$, there existence a subsequence of $\{f_n(t)\}_{n\in\mathbb{N}}$ that converges to $\overline{f}(t)$;
(iii) $\tilde{m} = \int_{T_1} H(t,\overline{f}(t))d\mu$.

As for any $t \in T_1 \setminus \dot{T}_1$, the correspondence ρ_t is closed, it follows from item (ii) above that $\overline{f}(t) \in \rho_t(\tilde{z}_1, \tilde{z}_2)$. By items (i) and (iii), jointly with the fact that $\tilde{a} \in \gamma(\tilde{z}_1, \tilde{z}_2)$, we have that $(\tilde{m}, \tilde{a}) \in \Theta_1(\tilde{z}_1, \tilde{z}_2)$. This concludes the proof. □

The claim above ensures that $G(\overline{m}, \overline{a}) = \mathcal{G}_1'$ and, therefore, $\Lambda(G(\overline{m}, \overline{a})) \cap U_1 = \Lambda(\mathcal{G}_1') \cap U_1 = \emptyset$. A contradiction with the fact that $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$. Thus, the set $m(\mathcal{G})$ is connected.

We proved that there exists at least one minimal essential connected subset $m(\mathcal{G})$ of $\Lambda(\mathcal{G})$. Fix $(\widehat{m}, \widehat{a}) \in m(\mathcal{G})$ and consider the set $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$, defined as the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains $(\widehat{m}, \widehat{a})$. By definition, $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is a component of $\Lambda(\mathcal{G})$. Since $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is compact and $m(\mathcal{G}) \subset \Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$, the component $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is also an essential subset of $\Lambda(\mathcal{G})$. Q.E.D.

4. Essential stability for parametrized subsets of $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$

We know that $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ has a dense subset of essential generalized games. Moreover, for any generalized game there exist a minimal essential set and an essential component. To obtain these results, was assumed that all of the characteristic of the generalized game, namely $(K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2}$, can be perturbed. However, it is interesting to discuss stability of Cournot-Nash equilibria when only some of the characteristics that define the game are allowed to be perturbed.

Definition 6. *A parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space of generalized games $\mathbb{G}$ is characterized by a complete metric space $(\mathbb{X}, \tau)$ and a continuous function $\kappa : \mathbb{X} \to \mathbb{G}$.*

Definition 7. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space $\mathbb{G}$, fix $\mathcal{X} \in \mathbb{X}$.*

(i) A Cournot-Nash equilibrium $(f^*, a^*) \in CN(\kappa(\mathcal{X}))$ is $\mathcal{T}$-essential if for any open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $(m(f^*), a^*) \in O$, there exists $\epsilon > 0$ such that $\Lambda(\kappa(\mathcal{X}'))\cap O \neq \emptyset$, for any $\mathcal{X}' \in \mathbb{X}$ that satisfies $\tau(\mathcal{X}, \mathcal{X}') < \epsilon$. The generalized game $\kappa(\mathcal{X}) \in \mathbb{G}$ is essential with respect to the parametrization $\mathcal{T}$ if all its Cournot-Nash equilibrium are $\mathcal{T}$-essential.

(ii) A subset $E \subseteq \Lambda(\kappa(\mathcal{X}))$ is $\mathcal{T}$-essential if it is non-empty, compact, and for each open set $O \subset \widehat{M} \times \widehat{\mathcal{F}}^2$ there exists $\epsilon > 0$ such that, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \epsilon$, $\Lambda(\kappa(\mathcal{X}')) \cap O \neq \emptyset$. A $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is minimal if it is a minimal element ordered by set inclusion.

Note that, given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ with $\mathbb{X} \subseteq \mathbb{G}$, for any $\mathcal{X} \in \mathbb{X}$, the generalized game $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential if and only if $\mathcal{X}$ is essential in the sense of Definition 3. Moreover, if a generalized game $\mathcal{G} \in \mathbb{G}$ is essential, then $\mathcal{G}$ is essential with respect to any parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ such that, for some $\mathcal{X} \in \mathbb{X}$, $\mathcal{G} = \kappa(\mathcal{X})$.

The following result describes stability properties of Cournot-Nash equilibria when we allow for changes only in those variables that characterize a parametrization $\mathcal{T}$ of $\mathbb{G}$.

Theorem 3. *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of the space $\mathbb{G}$, for any $\mathcal{X} \in \mathbb{X}$ we have that,*

(i) For each $\epsilon > 0$, there exists $\mathcal{X}' \in \mathbb{X}$ such that, $\kappa(\mathcal{X}')$ is $\mathcal{T}$-essential and $\tau(\mathcal{X}, \mathcal{X}') < \epsilon$.

(ii) Given $\mathcal{X} \in \mathbb{X}$, if $\Lambda(\kappa(\mathcal{X}))$ is a singleton, then $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential.

(iii) There exists a minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ and any of such sets is connected.

(iv) Given a $\mathcal{T}$-essential and connected set $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$, there exists a $\mathcal{T}$-essential component of $\Lambda(\kappa(\mathcal{X}))$ that contains $m(\mathcal{X})$.

Proof. Since $\kappa : \mathbb{X} \to \mathbb{G}$ is continuous, $(\mathbb{X}, \tau)$ is a complete metric space, and Λ is a closed correspondence that has non-empty and compact values. Thus, it follows that the correspondence $\Lambda_\kappa : \mathbb{X} \twoheadrightarrow \widehat{M}$ given by $\Lambda_\kappa = \Lambda \circ \kappa$ has closed graph with non-empty and compact values. Therefore, items (i) and (ii) follow from identical arguments to those made after Step 1 of Theorem 1.

Furthermore, items (iii) and (iv) follow from the same arguments of the proof of Theorem 2, changing $(\mathbb{G}, \rho, \Lambda)$ by $(\mathbb{X}, \tau, \Lambda_\kappa)$. Q.E.D.

From Theorem 3 we can obtain stability results of Cournot-Nash equilibria when some but not all the characteristics that define a generalized game are allowed to change. For instance, when only the objective functions or the sets of admissible strategies can be perturbed.

More formally, fix a game $\mathcal{G} = (U, F, (\eta_t)_{t\in T_2}) \in \mathbb{G}$. Given $i \in \{1,2\}$, let $T_i^u, T_i^s, T_i^a \subseteq T_i$ be, respectively, the subset of players in T_i for which we allow perturbations in objective functions, perturbations in strategy sets, and perturbations in correspondences of admissible strategies.

Let $\mathbb{G}_{\mathcal{G}}((T_i^u, T_i^s, T_i^a)_{i\in\{1,2\}}) \subseteq \mathbb{G}$ be the set of generalized games $\widetilde{\mathcal{G}} = \left(\widetilde{U}, \widetilde{F}, (\widetilde{\eta}_t)_{t\in T_2}\right)$ such that:

(1) For any $t \in T_1 \setminus T_1^u$, $\widetilde{U}(t) = U(t)$.

(2) Given $t \in T_1$, $\widetilde{F}(t) = (\widetilde{K}_t, \widetilde{\Gamma}_t)$ and

(2.1) for any $t \in T_1 \setminus T_1^s$, $\widetilde{K}_t = K_t$,

(2.2) for any $t \in T_1 \setminus T_1^a$, $\widetilde{\Gamma}_t = \Gamma_t$.

(3) Given $t \in T_2$, $\eta_t = (\widetilde{u}_t, \widetilde{K}_t, \widetilde{\Gamma}_t)$ and

(3.1) for any $t \in T_2 \setminus T_2^u$, $\widetilde{u}_t = u_t$;

(3.2) for any $t \in T_2 \setminus T_2^s$, $\widetilde{K}_t = K_t$;

(3.3) for any $t \in T_2 \setminus T_2^a$, $\widetilde{\Gamma}_t = \Gamma_t$.

It follows that $(\mathbb{G}_{\mathcal{G}}((T_i^u, T_i^s, T_i^a)_{i\in\{1,2\}}), \rho)$ is a complete metric space. Also, since the inclusion $I : \mathbb{G}_{\mathcal{G}} \hookrightarrow \mathbb{G}$ is continuous, $((\mathbb{G}_{\mathcal{G}}, \rho), I)$ is a parametrization of $\mathbb{G}$. Hence, results of essential stability when only some perturbations are allowed follows from Theorem 3.

As a particular case of our analysis, we can obtain stability results for non-atomic games. Indeed, a non-atomic game is a generalized game where (i) there is only non-atomic players; and (ii) admissible strategies are independent of the actions chosen by the other players. Therefore, we can identify a non-atomic game with a generalized game where there is only one atomic player, whose actions have no effect on the decisions of the other players. This identification induce a parametrization of the space of generalized games and, therefore, the properties of essential stability can be obtained as a consequence of Theorem 3.[12]

[12] In particular, the existence of equilibria in non-atomic games is a consequence of the existence of equilibria in large generalized games. It is not surprising, since the results of equilibrium existence of Balder (1999, 2002) are a generalization of the results of Schmeidler (1973) for large games.

Analogously, if we fix a generalized game where actions chosen by non-atomic players have no effect on other agents decisions, then the equilibrium actions associated to atomic players are Cournot-Nash equilibria of a convex generalized game with finitely many players. That is, stability properties of equilibria in convex generalized games with a finite number of players can be obtained as a particular cases of our approach. Thus, we ensure that the properties about essential stability studied by Yu (1999), Yu (2009) and Yu, Yang and Xiang (2005) hold for more sophisticated types of admissible perturbations.

To complete our analysis, we state a result about stability of essential sets and components, which characterize how soft the components and minimal essential sets move when we perturb a generalized game. For convenience of notations, given $\epsilon > 0$ and $A \in \widehat{M} \times \widehat{\mathcal{F}}^2$, let $B[\epsilon, A] = \Big\{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \exists\, (m',a') \in A,\ \widehat{\sigma}((m,a),(m',a')) \leq \epsilon\Big\}$, where $\widehat{\sigma}$ is the metric associated to the product topology of $\mathbb{R}^m \times \prod_{t\in T_2} \widehat{K}_t$.

The following properties are a direct consequence of Theorem 3 jointly with the results of Yu, Yang and Xiang (2005, Theorems 4.1 and 4.2).[13]

Theorem 4. *Fix a parametrization* $\mathcal{T} = ((\mathbb{X},\tau),\kappa)$ *of* $\mathbb{G}(T_1, T_2, (\widehat{K}, (\widehat{K}_t)_{t\in T_2}, H))$ *and let* $\mathcal{X} \in \mathbb{X}$.

(i) If $m(\mathcal{X})$ *is a minimal* $\mathcal{T}$*-essential set of* $\Lambda(\kappa(\mathcal{X}))$*, then for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that, for any* $\mathcal{X}' \in \mathbb{X}$ *with* $\tau(\mathcal{X},\mathcal{X}') < \delta$*, there is a minimal essential set* $m(\mathcal{X}')$ *of* $\Lambda(\kappa(\mathcal{X}'))$ *for which* $m(\mathcal{X}') \subset B[\epsilon, m(\mathcal{X})]$.

(ii) Suppose that $A \subseteq \Lambda(\kappa(\mathcal{X}))$ *is an essential component and there is* $\pi > 0$ *such that* $B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$*. Then, for any* $\epsilon > 0$ *there is* $\delta > 0$ *such that, for any* $\mathcal{X}' \in \mathbb{X}$ *with* $\tau(\mathcal{X},\mathcal{X}') < \delta$*, there exists an essential component* $A' \subseteq \Lambda(\kappa(\mathcal{X}'))$ *such that* $A' \subset B[\epsilon, A]$.

Note that, when $\Lambda(\kappa(\mathcal{X}))$ has a finite number of components, there always exists $\pi > 0$ such that, $B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X})) \setminus A] = \emptyset$ for any component $A \subseteq \Lambda(\kappa(\mathcal{X}))$. Thus, in this particular case, Theorem 4(ii) constitutes a generic result of stability for essential components.

5. Concluding remarks

In this paper we analyzed the essential stability of Cournot-Nash equilibria in large generalized games with non-atomic players. Departing from the ideas of Rath (1992) and Riascos and Torres-Martínez (2012), that reduce the proof of equilibrium existence in non-atomic (generalized) games to find fixed points of correspondences, we use the stability theory of fixed points developed by Fort (1950) and Jia-He (1962) to address the essential stability of equilibria in non-atomic games.

[13] In the context of essential stability of components for some non-linear problems—which include convex (generalized) games with a finite number of players—Yu, Yang and Xiang (2005, page e2417) impose a technical condition, named *condition (c)*, to ensure that the set of solutions of the non-linear problem has a minimal essential sets and at least one essential component. In Theorem 3, we proved that in our context these two properties hold for any parametrization.

We guaranteed that essential stability is a generic property in the space of generalized games. Also, even unessential generalized games have essential components of the set of equilibria, which ensures that we always have local stability in a connected subset of Cournot-Nash equilibria. Also, these connected subsets of essential Cournot-Nash equilibria are locally stable too.

Our results are compatible with general types of perturbations on the characteristics of generalized games. Indeed, stability properties still hold when (i) admissible perturbation can be captured by a continuous parametrization of the set of generalized games; and (ii) the set of parameters constitutes a complete metric space. This generality about the type of admissible perturbations allow us to obtain, as byproducts of our analysis, extensions of the results of essential stability for non-atomic games and convex games with finitely many players.

References

[1] Aubin, J.P. (1982): *Mathematical Methods of Games and Economic Theory,* North-Holland, Amsterdam.

[2] Aliprantis, C. and K. Border (1999): *Infinite Dimensional Analysis,* Springer-Verlag, Berlin, Heidelberg.

[3] Al-Najjar, N. (1995): "Strategically stable equilibria in games with infinitely many pure strategies," *Mathematical Social Sciences*, volume 29, pages 151-164.

[4] Balder, E.J. (1999):"On the existence of Cournot-Nash equilibria in continuum games," *Journal of Mathematical Economics* 32, pages 207-223.

[5] Balder, E.J. (2002): "A unifying pair of Cournot-Nash equilibrium existence results," *Journal of Economic Theory* 102, pages 437-470.

[6] Carbonell-Nicolau, O. (2010): "Essential equilibria in normal-form games," *Journal of Economic Theory*, volume 145, pages 421-431.

[7] Fort, M.K. (1949): "A unified theory of semi-continuity," *Duke Mathematical Journal*, volume 16, pages 237-246.

[8] Fort, M.K. (1950): "Essential and non-essential fixed points," *American Journal of Mathematics*, volume 72, pages 315-322.

[9] Hildenbrand, W. (1974): *Core and Equilibria in a Large Economy,* Princeton University Press, Princeton, New Jersey.

[10] Hillas, J. (1990): "On the definition of the strategic stability of equilibria," *Econometrica*, volume 58, pages 1365-1390.

[11] Jia-He, J. (1962): "Essential fixed points of the multivalued mappings," *Scientia Sinica,* volume XI. pages 293-298.

[12] Jia-He, J. (1963): "Essential component of the set of fixed points of the multivalued mappings and its application to the theory of games," *Scientia Sinica,* volume XII. pages 951-964.

[13] Kinoshita, S. (1952): "On essential components of the set of fixed points," *Osaka Mathematical Journal,* volume 4, pages 19-22.

[14] Kohlberg, E. and J.F. Mertens (1986): "On the strategic stability of equilibrium point," *Econometrica,* volume 54, pages 1003-1037.

[15] Ok, E. (2005): *Real Analysis with Economic Applications,* Princeton University Press, Princeton, USA.

[16] Rath, K.P. (1992): "A direct proof of the existence of pure strategy equilibria in games with a continuum of players", *Economic Theory*, Volume 2, pages 427-433.

[17] Riascos, A.J. and J.P. Torres-Martínez (2012):"On the existence of pure strategy equilibria in large generalized games with atomic players," working paper, Department of Economics, University of Chile. Available at http://www.econ.uchile.cl/ficha/jutorres.

[18] Schmeidler, D. (1973): "Equilibrium point of non-atomic games," *Journal of Statistical Physics*, volume 17, 295-300.

[19] Yu, J. (1999): "Essential equilibrium of n-person noncooperative games," *Journal of Mathematical Economics*, volume 31, pages 361-372.

[20] Yu, J., and H. Yang (2004): "Essential components of the set of equilibrium points for set-valued maps," *Journal of Mathematical Analysis and Applications*, volume 300, pages 334-342.

[21] Yu, J., H. Yang, and S. Xiang (2005): "Unified approach to existence and stability of essential components," *Nonlinear Analysis*, volume 63, pages 2415-2425.

[22] Yu, X. (2009): "Essential components of the set of equilibrium points for generalized games in the uniform topological space of best reply correspondences," *International Journal of Pure and Applied Mathematics*, volume 55, pages 349-357.

[23] Zhou, Y., J.Yu, and S.Xiang (2007): "Essential stability in games with infinitely many pure strategies," *International Journal of Game Theory*, volume 35, pages 493-503.

[24] Wu, W., and J. Jia-He (1962): 'Essential equilibrium points of n-person noncooperative games," *Scientia Sinica,* volume XI, pages 1307-1322.